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Abel's theorem for analytic mappings of an open Riemann surface into compact Riemann surfaces of genus one

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Introduction

Abel's theorem on general open Riemann surfaces was first investigated by Ahlfors [1, 3] and Kusunoki [5, 7] independently. After then, some generalizations were obtained ([10], [11], [20] and so on). Recently, Sainouchi [12] and Watanabe [16–18] studied the corresponding problems from view points of canonical semiexact differentials and behavior spaces respectively. Cf. also Minda [9].

All of these modern results thus far obtained concern with, just as the classical Abel's theorem, the existence of certain meromorphic functions, i.e., analytic mappings of a Riemann surface R into the Riemann sphere. It seems yet worth while to study how the situation varies if we replace the Riemann sphere by a general Riemann surface R'. The problem in the most general setting would involve much difficulties. So we shall treat in this paper a simple case where R' is a compact Riemann surface T of genus one (torus).

First we have to formulate the problem more precisely, and to do this we appeal to the following familiar yet profound idea: the problem of singularities may be changed into the problem of boundary behaviors (see, for example, [3] pp. 299–300 as well as p. 148 ff.). Correspondingly the notion of singularities may be generalized. Although there would be several distinct and equally reasonable ways to generalize "singularities", it turns out that the one which we have done in [14] is of use at least for our present purposes.

Our task is thus to give a necessary and sufficient condition for the existence of analytic mappings of an open Riemann surface R into a torus T which have prescribed (generalized) singularities at the ideal boundary of R. And, in fact, we shall be able to give such a condition (see Theorem 1; cf. also Theorem 3) which is very close to the classical case.

After such rather classical consideration, we then set about an investigation of the topological aspect of Abel's theorem. In general, a homomorphism of the (one-dimensional integral) homology group of R modulo dividing cycles into the homology group of T does not always arise from an analytic mapping of R into T, while every analytic mapping induces such a homomorphism. This was shown by Gerstenhaber ([4]) when R is compact. (For a compact surface R of positive genus there is a theorem of H. Hopf: every homomorphism between homology groups is induced by a *continuous* mapping of R onto T.)

We shall characterize all the homomorphisms induced by analytic mappings (Theorem 2), provided that we consider only particular homomorphisms called "of finite type". This condition is trivially satisfied if R is of finite genus. Analytic mappings with simple homological properties will be studied in some detail (Theorems 3, 3′ and 4 etc.). Finally we shall confine ourselves to surfaces of finite connectivity (see Theorems 5 and 6).

Applications of the present results to the problem of reduction (degeneration) of Abelian integrals to elliptic integrals will appear elsewhere together with some related topics.

I. Preliminaries

1. Throughout this paper R denotes an open Riemann surface of genus g $(\leq\infty)$, and ∂R denotes its Kerékjártó-Stoïlow ideal boundary[1]. We set $J=\{1, 2,\dots, g\}$. Denote by $\mathscr{R}=\{R_n\}_{n=1}^{\infty}$ a fixed canonical exhaustion of R, and take a canonical homology basis $\Xi(R)=\Xi(R, \mathscr{R})=\{A_j, B_j\}_{j\in J}$ of R modulo ∂R such that[2]

i) $\{A_j, B_j\}_{j\in J_n}$ forms a canonical homology basis of R_n modulo its border ∂R_n, and

ii) a certain subcollection of $\{A_j, B_j\}_{j\in J_{n+1}-J_n}$ forms a canonical homology basis of each component $R_n^{(k)}$ of $R_{n+1}-\bar{R}_n$ modulo $\partial R_n^{(k)}$, $k=1, 2,\dots, \kappa_n$,

where $J_n=\{1, 2,\dots, g_n\}$, g_n being the genus of R_n, and κ_n is the number of components of $R_{n+1}-\bar{R}_n$. Furthermore we set $\mathscr{E}(R)=\{U\subset R|U$ is a canonical end of R, i.e., $R-\bar{U}$ is a canonical region$\}$.

The following lemma is fundamental. For the proof, see [13], for instance.

Lemma 1. *Let φ_1, φ_2 be closed C^1-differentials on the closure $\bar{R}_n$ of $R_n\in\mathscr{R}$. Suppose, further, that φ_1 is semiexact. Let Φ_1 be a primitive function of φ_1 on $\bar{R}_n-\bigcup_{j\in J_n}(A_j\cup B_j)$. Then we have*

$$\iint_{R_n}\varphi_1\wedge\bar{\varphi}_2=\int_{\partial R_n}\Phi_1\bar{\varphi}_2-\sum_{j\in J_n}\left(\int_{A_j}\varphi_1\int_{B_j}\bar{\varphi}_2-\int_{B_j}\varphi_1\int_{A_j}\bar{\varphi}_2\right).$$

We denote by $\Lambda=\Lambda(R)$ the *real* Hilbert space of square integrable *complex* differentials on R with the following inner product:

$$<\lambda_1, \lambda_2>=\mathrm{Re}\iint_R\lambda_1\wedge\bar{\lambda}_2^*=\mathrm{Re}\iint_R(a_1\bar{a}_2+b_1\bar{b}_2)dxdy,$$

1) The results below would be valid for a compact R, if ∂R is then interpreted as an empty set. Cf. [14].

2) In the preceding papers [13, 14] we did not write down these conditions explicitly, but assumed them implicitly.

where $\lambda_j = a_j dx + b_j dy$, $j=1, 2$ ($z = x+iy$ is a local parameter). The norm induced by this inner product is denoted by $\|\cdot\|$. There are many important subspaces of Λ, among which we shall be mainly concerned with the following ones:

$$\Lambda_h = \Lambda_h(R) = \{\lambda \in \Lambda | \lambda \text{ is harmonic on } R\},$$

$$\Lambda_{hse} = \Lambda_{hse}(R) = \{\lambda \in \Lambda_h | \lambda \text{ is semiexact}\},$$

$$\Lambda_{e0}^{(1)} = \Lambda_{e0}^{(1)}(R) = \left\{ \lambda \in \Lambda \,\middle|\, \begin{array}{l} \exists f \in C^2(R) \text{ and } \exists f_n \in C_0^2(R) \\ \text{such that } df = \lambda \text{ and} \\ \|df - df_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{array} \right\}$$

Note that $\Lambda_{e0}^{(1)}$ is not a closed subspace of Λ.

Needless to say, elements of these spaces are square integrable. We shall yet need some spaces of differentials which are not necessarily square integrable. To define them, let $\mathscr{A}(D)$ be the family of all regular analytic differentials on an open set $D \subset R$ and set

$$\mathscr{A}(\partial R) = \{\varphi | \varphi \in \mathscr{A}(U) \text{ for some } U \in \mathscr{E}(R)\},$$

$$(P)\mathscr{A}_{se}(\partial R) = \{\varphi \in \mathscr{A}(\partial R) \,|\, \varphi \text{ is } (P)\text{semiexact}\},$$

where P stands for any regular partition of the ideal boundary ∂R (cf. [3]). By saying that $\varphi \in \mathscr{A}(\partial R)$ is (P)semiexact we mean that $\int_d \varphi = 0$ for any (P)dividing cycle d which lies in dom φ, the domain of definition of φ. As usual, we refer to (Q) semiexact differentials simply as semiexact differentials, Q being the canonical partition. The family $\mathscr{A}(\partial R)$ has a vector space structure over the reals provided that we identify its elements appropriately. The real vector space thus obtained is also denoted by $\mathscr{A}(\partial R)$. The same convention is applied to each subspace of $\mathscr{A}(\partial R)$.

2. A straight line L in the complex plane $\boldsymbol{C}$ passing through the origin will be called simply a *line* in $\boldsymbol{C}$. Also, we shall use the notation "$z \equiv 0 \mod L$" to indicate that a complex number z belongs to L. A subspace $\Lambda_0 = \Lambda_0(R, \mathscr{L})$ of Λ_{hse} is called a *behavior space* (on R) associated with $\mathscr{L} = \{L_j\}_{j \in J}$, a family of lines L_j in $\boldsymbol{C}$, if the following conditions (i) and (ii) are satisfied:

(i) $i\Lambda_0^* = \Lambda_0^\perp$,

(ii) for all $\lambda_0 \in \Lambda_0$, $\int_{A_j} \lambda_0 \equiv \int_{B_j} \lambda_0 \equiv 0 \mod L_j \qquad (j \in J)$,

where $\Lambda_0^\perp$ stands for the orthogonal complement of Λ_0 in Λ_h. This considerably simplified definition of a behavior space is due to Matsui [8]. The original definition [13] was rather complicated.

Let L_0 be a line in $\boldsymbol{C}$. Two behavior spaces $\Lambda_0 = \Lambda_0(R, \mathscr{L})$ and $\Lambda_0' = \Lambda_0(R, \mathscr{L}')$ are called mutually *dual* (cf. [13]) with respect to L_0, or simply L_0-dual, if

(1°) $\langle \lambda_0, \bar{\lambda}_0'^* \rangle + i \langle \lambda_0, i\bar{\lambda}_0'^* \rangle \equiv 0 \mod L_0$ for all $(\lambda_0, \lambda_0') \in \Lambda_0 \times \Lambda_0'$, and

(2°) $L_j \circ L'_j = L_0$ $(j \in J)$ where $L_j \circ L'_j = \{z \in \boldsymbol{C} | z = z_j z'_j,\ (z_j, z'_j) \in L_j \times L'_j\}$, $\mathscr{L}' = \{L'_j\}_{j \in J}$.

In this paper, we shall restrict ourselves to the case $L_0 = \boldsymbol{R}$, the real axis. Then we know that Λ_0 and $\bar{\Lambda}_0$, the complex conjugate of Λ_0, are always ($\boldsymbol{R}$-)dual (cf. [13]). Conversely, we have

Proposition 1. ([15]) *Let Λ_0 and Λ'_0 be behavior spaces which are $\boldsymbol{R}$-dual to each other. Then $\Lambda'_0 = \bar{\Lambda}_0$.*

In view of this proposition we shall always denote by Λ'_0 the (uniquely determined) dual behavior space of Λ_0 with respect to $\boldsymbol{R}$.

On the other hand, it is recently shown [8] that one can construct a behavior space associated with an arbitrarily prescribed family of lines. Thus we have

Proposition 2. ([8]) *There always exist a behavior space associated with a given $\mathscr{L}$ and its dual behavior space (with respect to $\boldsymbol{R}$).*

3. Let there be given a behavior space $\Lambda_0 = \Lambda_0(R, \mathscr{L})$, $\mathscr{L} = \{L_j\}_{j \in J}$. A (closed) C^1-differential φ defined near ∂R is said to have *Λ_0-behavior* ([13, 15]; see also [20]) if there are some $\lambda_0 \in \Lambda_0$, $\lambda_{e0} \in \Lambda_{e0}^{(1)}$ and $U \in \mathscr{E}(R)$ such that

$$\varphi = \lambda_0 + \lambda_{e0} \qquad \text{on} \quad U.$$

Suppose a semiexact C^1-differential φ on $V = R - \bar{R}_n$, $R_n \in \mathscr{R}$ such that $\int_{A_j} \varphi \equiv \int_{B_j} \varphi \equiv 0 \mod L_j$, $j \in J - J_n$ has Λ_0-behavior. Then φ admits a representation $\varphi = \lambda_0 + \lambda_{e0}$ on the whole of V. For the proof, see [15]. We say that a behavior space Λ_0 is *equivalent* to another $\tilde{\Lambda}_0$, if every C^1-differential with Λ_0-behavior has $\tilde{\Lambda}_0$-behavior and every C^1-differential with $\tilde{\Lambda}_0$-behavior has Λ_0-behavior (cf. [15]).

We are particularly interested in analytic differentials with Λ_0-behavior. Denote by $\mathscr{A}_{\Lambda_0}$ the real vector space consisting of all $\varphi \in \mathscr{A}(\partial R)$ with Λ_0-behavior, and set $\mathscr{A}_{\mathscr{L}}^P = \{\varphi \in (P)\,\mathscr{A}_{se}(\partial R) \mid \int_{A_j} \varphi \equiv \int_{B_j} \varphi \equiv 0 \mod L_j$ for every $j \in J$ such that $A_j, B_j \subset \operatorname{dom} \varphi\}$. Then, clearly $\mathscr{A}_{\Lambda_0} \subset \mathscr{A}_{\mathscr{L}}^P \subset \mathscr{A}(\partial R)$. The elements of the factor space $\mathscr{A}_{\mathscr{L}}^P / \mathscr{A}_{\Lambda_0}$ are called *$(P)\Lambda_0$-singularities* ([14]), or sometimes (generalized) singularities. Each $(P)\Lambda_0$-singularity σ is represented by a regular analytic differential near ∂R, which we often denote by the same letter σ. A differential $\varphi \in \mathscr{A}(R)$ is said to have a generalized singularity σ if $\varphi - \sigma$ has Λ_0-behavior. A more formal definition of the generalized singularities (described in terms of inductive limits) will be found in the recent paper ([18]) of Watanabe, who also studied the connection between these generalized singularities and the classical polar singularities.

The following propositions were proved in [13, 14].

Proposition 3. *A regular analytic differential φ on R which has Λ_0-behavior (i.e., $\varphi \in \mathscr{A}(R) \cap \mathscr{A}_{\Lambda_0}$) is identically zero if*

$$\int_{A_j} \varphi \equiv \int_{B_j} \varphi \equiv 0 \mod \hat{L}_j, \qquad j \in J$$

for some family of lines $\hat{L}_j$. (Of course, $\hat{L}_j = L_j$ except for a finite number of j.)

Proposition 4. *For any $\xi_j, \eta_j \in \boldsymbol{C}$, $\xi_j, \eta_j \not\equiv 0 \mod L_j$, there are regular analytic differentials $\phi_{\xi_j}(A_j) = \phi(A_j; \Lambda_0, \xi_j)$ and $\phi_{\eta_j}(B_j) = \phi(B_j; \Lambda_0, \eta_j)$ such that*

(i) *$\phi_{\xi_j}(A_j)$ and $\phi_{\eta_j}(B_j)$ have Λ_0-behavior,*

(ii)
$$\begin{cases} \int_{A_k} \phi_{\xi_j}(A_j) \equiv \xi_j(A_j \times A_k) = 0, & \int_{B_k} \phi_{\xi_j}(A_j) \equiv \xi_j(A_j \times B_k) = \xi_j \delta_{jk} \\ \int_{A_k} \phi_{\eta_j}(B_j) \equiv \eta_j(B_j \times A_k) = -\eta_j \delta_{jk}, & \int_{B_k} \phi_{\eta_j}(B_j) \equiv \eta_j(B_j \times B_k) = 0 \end{cases} \mod L_k.$$

Here $\gamma \times \delta$ means the intersection number of two (1-)cycles γ, δ (cf. [3], p. 67ff.).

Proposition 5. *For any $(P)\Lambda_0$-singularity σ there exists a unique differential ψ_σ with the following properties:*

(i) *ψ_σ is regular analytic on (the interior of) R,*

(ii) *ψ_σ has σ as its singularity, i.e., on some $U \in \mathscr{E}(R)$, ψ_σ is equal to σ modulo Λ_0-behavior,*

(iii) *the periods of ψ_σ are normalized:*

$$\int_{A_j} \psi_\sigma \equiv \int_{B_j} \psi_\sigma \equiv 0 \mod L_j, \qquad j \in J.$$

Roughly speaking, a differential with Λ_0-behavior which is regular analytic over R plays a quite similar role to an Abelian differential of the first kind in the classical theory. So, hereafter, such a differential will be called a *Λ_0-Abelian differential of the first kind*. Similarly, a regular analytic differential on R which has a generalized singularity σ will be called a *Λ_0-Abelian differential of the second (resp. third) kind* if σ is (resp. is not) (Q)semiexact. Under these observations, Propositions 3, 4 and 5 correspond to the classical uniqueness and existence theorems of elementary Abelian differentials of three kinds.

Now we remind that Λ_0 and Λ_0' are two behavior spaces which are dual to each other (w.r.t. $\boldsymbol{R}$). The following proposition may be considered a generalization of the Riemann's period relations (cf. [5, 7]).

Proposition 6. *Let $d\Phi'$ be a Λ_0'-Abelian differential of the first or second kind, and ω any Λ_0-Abelian differential. Suppose further that ω has a $(P)\Lambda_0$-singularity σ (maybe $\equiv 0$), P being a regular partition of ∂R. Then*

(i) $\operatorname{Res}_{\partial R} \Phi'\omega = \lim_{n\to\infty} \operatorname{Re}\left[-\frac{1}{2\pi i}\int_{\partial R_n} \Phi'\omega\right]$ *always exists (and is finite). Actually we have*

$$\operatorname{Res}_{\partial R} \Phi'\omega = -\frac{1}{2\pi} \sum_{j\in J} \operatorname{Im}\left(\int_{A_j} d\Phi' \int_{B_j} \omega - \int_{B_j} d\Phi' \int_{A_j} \omega\right).$$

(ii) $\operatorname{Res}_{\partial R} \Phi'\sigma$ *can be also defined. In particular, if $d\Phi'$ is of the first kind, then we have* $\operatorname{Res}_{\partial R} \Phi'\sigma = \operatorname{Res}_{\partial R} \Phi'\omega$.

Proof. Omitted (see [14][3]). Cf. Lemma 1. We also note that Φ' is considered a single-valued holomorphic function on the planar surface $R-\bigcup_{j\in J}(A_j \cup B_j)$.

II. Abel's theorem

4. Let T be a compact Riemann surface of genus one (torus). As is well known, there are two complex numbers π_0, π_1, $\operatorname{Im}(\pi_1/\pi_0)<0$, called the fundamental (or primitive) periods, such that T is biholomorphically homeomorphic to the factor space $\boldsymbol{C}/\Pi$, where $\Pi=[\pi_0, \pi_1]=\{z\in\boldsymbol{C}\,|\,z=m\pi_0+n\pi_1,\ m, n\in\boldsymbol{Z}\}$, the period module. We denote by ρ the natural projection mapping $\boldsymbol{C}\to T$. Set $L_k=\{z\in\boldsymbol{C}\,|\,z=t\pi_k,\ t\in\boldsymbol{R}\}$ and $L'_k=\bar{L}_k=\{z\in\boldsymbol{C}\,|\,z=t\bar{\pi}_k,\ t\in\boldsymbol{R}\}$ for $k=0, 1$. Let ε be a mapping of J into the set $\{0, 1\}$. We define the complementary mapping ε^* of ε by $\varepsilon^*(j)=1-\varepsilon(j)$, $j\in J$. Clearly ε^* is also a mapping of J into the set $\{0, 1\}$, and $(\varepsilon^*)^*=\varepsilon$. We now obtain a family of lines in $\boldsymbol{C}$: $\mathscr{L}=\{L_{\varepsilon(j)}\}_{j\in J}$. There is a behavior space $\Lambda_0=\Lambda_0(R, \mathscr{L})$ associated with the $\mathscr{L}$ (Prop. 2). [Set-theoretically $\mathscr{L}$ consists of only two elements L_0 and L_1. But as is easily seen from the definition of behavior spaces, it is very important how they are arranged. Different ε's (there are 2^g distinct ε's in all) give rise to different behavior spaces.] Such a Λ_0 will be called an *ε-allowable behavior space belonging to $\Pi=[\pi_0, \pi_1]$*. It should be noted that for each ε there generally exist infinitely many (distinct) ε-allowable behavior spaces. Cf. Proposition 7 in sec. 7. The dual behavior space of Λ_0 (w.r.t. $\boldsymbol{R}$) is denoted by Λ'_0. We know that $\Lambda'_0=\bar{\Lambda}_0=\{\lambda\in\Lambda_h|\bar{\lambda}\in\Lambda_0\}$, a behavior space associated with $\mathscr{L}'=\{L'_{\varepsilon(j)}\}_{j\in J}$.

A generalized singularity σ will be called *Π-admissible* if

$$\int_\gamma \sigma\equiv 0 \mod \Pi$$

for every dividing cycle γ (outside some compact set). This means, of course, that there are some integers $l_j(\gamma)$, $j=0, 1$ such that $\int_\gamma \sigma=l_0(\gamma)\pi_0+l_1(\gamma)\pi_1$. (There should be no confusion with the expression "$z\equiv 0 \mod L$".)

5. Suppose, first of all, that there is an analytic mapping $f: R\to T$ such that $d(\rho^{-1}\circ f)$ has a $(P)\Lambda_0$-singularity σ. Here P denotes an arbitrary regular partition of the ideal boundary (cf. Remark 1 in sec. 6). Then, there exist $U\in\mathscr{E}(R)$; $\lambda_0\in\Lambda_0$, $\lambda_{e0}\in\Lambda_{e0}^{(1)}$ such that

$$d(\rho^{-1}\circ f)=\sigma+\lambda_0+\lambda_{e0} \qquad \text{on} \quad U.$$

As was noted earlier, the σ in the above expression stands for a representative of the $(P)\Lambda_0$-singularity σ. Namely, σ is an element of $\mathscr{A}^P_{\mathscr{L}}$. Taking U smaller if neces-

3) Here we want to make a correction to [14]. Namely, the term $\int_{\gamma_\Omega}(\lambda_0+\lambda'_{e0})\tau_0$ in the eighth line from the bottom in p. 11 should be read as $\int_{\gamma_\Omega} f\tau_0$.

sary, we may assume that $\sigma \in \mathscr{A}(U)$. For this U, there is an $R_n \in \mathscr{R}$ such that $R - R_n \subset U$. Hence we have

$$\int_{A_j} \sigma \equiv \int_{B_j} \sigma \equiv 0 \mod L_{\varepsilon(j)}, \qquad j \in J - J_n.$$

We also note that λ_{e0} is exact and the A_j- and B_j-periods of λ_0 belong to $L_{\varepsilon(j)}$ for all $j \in J$. Therefore we see that

$$\int_{A_j} d(\rho^{-1} \circ f) \equiv \int_{B_j} d(\rho^{-1} \circ f) \equiv 0 \mod L_{\varepsilon(j)}, \qquad j \in J - J_n.$$

On the other hand, since f maps every closed curve on R onto a closed curve on T, we can find integers m_j, m_j^*, n_j, n_j^* ($j \in J$) satisfying

$$\begin{cases} \int_{A_j} d(\rho^{-1} \circ f) = \int_{f(A_j)} d\rho^{-1} = m_j \pi_{\varepsilon(j)} + m_j^* \pi_{\varepsilon^*(j)} \\ \int_{B_j} d(\rho^{-1} \circ f) = \int_{f(B_j)} d\rho^{-1} = n_j \pi_{\varepsilon(j)} + n_j^* \pi_{\varepsilon^*(j)} \end{cases} \qquad j \in J.$$

Because $\operatorname{Im}[\pi_{\varepsilon(j)}/\pi_{\varepsilon^*(j)}] \neq 0$, $j \in J$, we conclude that

$$m_j^* = n_j^* = 0 \qquad \text{for all} \quad j \in J - J_n.$$

This fact enables us to construct a regular analytic differential ψ_0 on R with Λ_0-behavior (i.e., a Λ_0-Abelian differential of the first kind) such that

$$\int_{A_j} (d(\rho^{-1} \circ f) - \psi_0) \equiv \int_{B_j} (d(\rho^{-1} \circ f) - \psi_0) \equiv 0 \mod L_{\varepsilon(j)}, \qquad j \in J.$$

Indeed, ψ_0 is obtained by making a (finite) linear combination of elementary differentials of the first kind (see Prop. 4) with real coefficients. The periods of ψ_0 can be written in the form

$$\begin{cases} \int_{A_j} \psi_0 = \alpha_j \pi_{\varepsilon(j)} + m_j^* \pi_{\varepsilon^*(j)} \\ \int_{B_j} \psi_0 = \beta_j \pi_{\varepsilon(j)} + n_j^* \pi_{\varepsilon^*(j)} \end{cases} \qquad j \in J,$$

where α_j and β_j are appropriate real numbers.

Now let $\phi'(A_j) = \phi(A_j;\, \Lambda_0',\, -2\pi i/\pi_{\varepsilon(j)})$ and $\phi'(B_j) = \phi(B_j;\, \Lambda_0',\, -2\pi i/\pi_{\varepsilon(j)})$ be regular analytic differentials on R with Λ_0'-behavior such that

$$\begin{cases} \int_{A_k} \phi'(A_j) \equiv \int_{B_k} \phi'(B_j) \equiv 0 \\ \int_{B_k} \phi'(A_j) \equiv -\int_{A_k} \phi'(B_j) \equiv -2\pi i \delta_{jk}/\pi_{\varepsilon(j)} \end{cases} \mod L'_{\varepsilon(k)}.$$

Such differentials surely exist, since $-2\pi i/\pi_{\varepsilon(j)} \not\equiv 0 \mod L'_{\varepsilon(j)}$ (cf. Prop. 4). These differentials evidently form a basis for Λ_0'-Abelian differentials of the first kind. We set further

$$\Phi'_{A_j}(p)=\int^p \phi'(A_j), \quad \Phi'_{B_j}(p)=\int^p \phi'(B_j), \qquad p\in R-\bigcup_{j\in J}(A_j\cup B_j).$$

Applying the generalized period relation (Prop. 6) for the differentials $\phi'(A_j)$ and $\psi_\sigma=d(\rho^{-1}\circ f)-\psi_0$, we obtain

$$\operatorname*{Res}_{\partial R}\Phi'_{A_j}\sigma=-\frac{1}{2\pi}\sum_{k\in J}\operatorname{Im}\left(\int_{A_k}\phi'(A_j)\int_{B_k}\psi_\sigma-\int_{B_k}\phi'(A_j)\int_{A_k}\psi_\sigma\right)$$

$$=\alpha_j-m_j,$$

since we can write the periods of $\phi'(A_j)$ as $\int_{A_k}\phi'(A_j)=a_k\overline{\pi_{\varepsilon(k)}}$, $\int_{B_k}\phi'(A_j)=b_k\overline{\pi_{\varepsilon(k)}}-2\pi i\delta_{jk}/\pi_{\varepsilon(k)}$ for suitable real numbers a_k, b_k and the periods of ψ_σ are $\int_{A_k}\psi_\sigma=(m_k-\alpha_k)\pi_{\varepsilon(k)}$, $\int_{B_k}\psi_\sigma=(n_k-\beta_k)\pi_{\varepsilon(k)}$. Consequently

$$\pi_{\varepsilon(j)}\operatorname*{Res}_{\partial R}\Phi'_{A_j}\sigma=(\alpha_j-m_j)\pi_{\varepsilon(j)}$$

$$=\int_{A_j}\psi_0-[m_j\pi_{\varepsilon(j)}+m_j^*\pi_{\varepsilon^*(j)}]$$

or

$$\pi_{\varepsilon(j)}\operatorname*{Res}_{\partial R}\Phi'_{A_j}\sigma\equiv\int_{A_j}\psi_0 \mod \Pi, \qquad j\in J. \tag{1}$$

Similarly we have

$$\pi_{\varepsilon(j)}\operatorname*{Res}_{\partial R}\Phi'_{B_j}\sigma\equiv\int_{B_j}\psi_0 \mod \Pi, \qquad j\in J. \tag{1'}$$

Next let γ be any dividing cycle on R. We assume that $\gamma\subset U$. Then, since the image $f(\gamma)$ is again a closed curve on T, we can find integers $l_0(\gamma)$, $l_1(\gamma)$ for which

$$\int_\gamma d(\rho^{-1}\circ f)=\int_{f(\gamma)}d\rho^{-1}=l_0(\gamma)\pi_0+l_1(\gamma)\pi_1.$$

The left hand side is equal to $\int_\gamma\sigma$, for λ_0 and λ_{e0} are both semiexact. Hence we have

$$\int_\gamma\sigma\equiv 0 \mod \Pi \text{ for every dividing cycle } \gamma. \tag{1''}$$

That is, σ is a Π-admissible singularity.

Equations (1), (1') and (1'') are necessary for the existence of an analytic mapping $f: R\to T$ such that the differential $d(\rho^{-1}\circ f)$ has the prescribed $(P)\Lambda_0$-singularity σ. We have thus proved the only if part of the following theorem.

Theorem 1. *Let R be an open Riemann surface of genus $g\ (\leq\infty)$ and $T=\boldsymbol{C}/\Pi$ be a compact Riemann surface of genus one, $\Pi=[\pi_0, \pi_1]=\{z\in\boldsymbol{C}\,|\,z=m\pi_0+n\pi_1,\ m,\ n\in\boldsymbol{Z}\}$. Let $\rho: \boldsymbol{C}\to T$ be the natural projection. Let ε be a mapping* *of $J=\{1, 2,\dots, g\}$ into the set $\{0, 1\}$. Let Λ_0 be an ε-allowable behavior space (on R) belonging to Π and σ a $(P)\Lambda_0$-singularity, P being a regular partition of ∂R.*

Then there exists an analytic mapping f of R into T such that $d(\rho^{-1}\circ f)$ has the singularity σ if and only if σ is Π-admissible and there is a Λ_0-Abelian differential ψ_0 of the first kind satisfying

$$\pi_{\varepsilon(j)} \operatorname*{Res}_{\partial R} \Phi'_{\substack{A_j \\ B_j}}\sigma \equiv \int_{\substack{A_j \\ B_j}} \psi_0 \mod \Pi, \qquad j \in J. \tag{2}$$

6. To prove the converse, we assume that there is such a differential ψ_0 as in the theorem. By Proposition 5 there is a unique differential ψ_σ on R, the normalized Λ_0-Abelian differential (of the second or third kind) whose generalized singularity is exactly σ. Since ψ_0 is a Λ_0-Abelian differential of the first kind, $\psi_0 + \psi_\sigma$ is obviously a Λ_0-Abelian differential with the singularity σ.

A use of the period relation (Prop. 6) for $\phi'(A_j)$ (resp. $\phi'(B_j)$) and ψ_σ yields that

$$\int_{\substack{A_j \\ B_j}} \psi_\sigma = -\pi_{\varepsilon(j)} \operatorname*{Res}_{\partial R} \Phi'_{\substack{A_j \\ B_j}}\sigma, \qquad j \in J,$$

and therefore we know that

$$\int_{A_j} (\psi_0 + \psi_\sigma) \equiv \int_{B_j} (\psi_0 + \psi_\sigma) \equiv 0 \mod \Pi, \qquad j \in J.$$

Moreover, for a dividing cycle γ we have

$$\int_\gamma (\psi_0 + \psi_\sigma) \equiv 0 \mod \Pi,$$

because σ is Π-admissible and $\psi_0 + \psi_\sigma - \sigma$ is semiexact.

The function $\Psi = \int^p (\psi_0 + \psi_\sigma)$ is thus multi-valued, but the composition $f = \rho\circ\Psi$ gives a well-defined analytic mapping of R into T. It is obvious that $d(\rho^{-1}\circ f) = d\Psi$ has the singularity σ. This completes the proof of Theorem 1.

Remarks. (1) In the above theorem, the regular partition P of ∂R is quite arbitrary; as a matter of fact, it is sufficient to consider the case $P = I$, the identity partition (cf. Theorem 2).

(2) The analytic mapping f in the above theorem is not necessarily uniquely determined. In fact, uniqueness is not assured if there is a non-constant analytic mapping f_0 such that $d(\rho^{-1}\circ f_0)$ is of the first kind. (In other words, f is not unique if there is a *reducible* Λ_0-Abelian integral of the first kind other than trivial ones. As for the properties of reducible Abelian integrals and some other related topics, see a forthcoming paper.)

Theorem 1 can be viewed as an Abel's theorem for analytic mappings of an open Riemann surface into tori. The classical Abel's theorem gives a necessary and sufficient condition for a divisor of degree zero to be principal, i.e., the divisor of a meromorphic function f on the surface considered. Namely, we are concerned with both of zeros and poles of f (cf. [7], [19] etc.). However, if we lift f to the universal covering surface $\boldsymbol{C}$ of $\hat{\boldsymbol{C}}-\{0, \infty\}$ and take its differential, the zeros and poles of f are equally transformed into the singularities of the third kind. Likewise, our Abel's theorem (Th. 1 above) deals with the preassignment of only the singularities of the differential $d(\rho^{-1}\circ f)$ of the lifting of an analytic mapping $f: R\rightarrow T$ to the universal covering surface $\boldsymbol{C}$ of T. (Notice that we have made good use of parabolicity of the universal covering surface of T.)

III. Homology groups

7. Let C_0, C_1 be 1-cycles on T corresponding to π_0, π_1 respectively. Then C_0, C_1 forms a canonical homology basis $\Xi(T)$ of T. We denote by $H_1(T)$ the 1-dimensional integral homology group of T.

Let $H_1^*(R)$ be the 1-dimensional integral homology group of R modulo dividing cycles (the relative homology group with respect to the ideal boundary). The group $H_1^*(R)$ is defined as the quotient group $H_1(R)/H_1\beta(R)$ where $H_1\beta(R)$ stands for the homology classes of dividing (singular) cycles (cf. [3]). The $\Xi(R)$ defined in sec. 1 gives rise to a basis of $H_1^*(R)$.

Every continuous mapping $h: R\rightarrow T$ induces a homomorphism of $H_1^*(R)$ into $H_1(T)$, which we denote by h_*. We also use the symbols $\approx$ to express "is homologous to" and $[\gamma]$ to indicate the homology class (modulo dividing cycles) determined by a 1-cycle γ.

From now on, a continuous mapping $f: R\rightarrow T$ will be called of *finite type* (*relative to* $(\Xi(R), \Xi(T))$) if

$$\sum_{j\in J}\prod_{k=0,1}[(f(A_j)\times C_k)^2+(f(B_j)\times C_k)^2]<\infty.$$

If a stronger condition

$$\sum_{j\in J}\prod_{k=0,1}[(f(A_j)\times C_k)^2+(f(B_j)\times C_k)^2]=0$$

is satisfied, we shall call f a mapping of *null type* (*relative to* $(\Xi(R), \Xi(T))$).

If f is a mapping of null type relative to $(\Xi(R), \Xi(T))$, then for each $j\in J$ $[f(A_j)]$ and $[f(B_j)]$ are both integral multiples of a single one of $[C_0]$, $[C_1]$ (i.e., there are integers $k=k(j)=0$ or 1 such that $f(A_j)\approx m_jC_k$, $f(B_j)\approx n_jC_k$ with m_j, $n_j\in\boldsymbol{Z}$) and vice versa. Similarly a mapping of finite type relative to $(\Xi(R), \Xi(T))$ is a mapping f with the following property: When we write $f(A_j)$, $f(B_j)$ as

$$\begin{cases} f(A_j)\approx m_{j0}C_0+m_{j1}C_1 \\ f(B_j)\approx n_{j0}C_0+n_{j1}C_1 \end{cases} \qquad (m_{jk},\ n_{jk}\in\boldsymbol{Z},\quad j\in J,\quad k=0,\ 1),$$

either $m_{j0}=n_{j0}=0$ or $m_{j1}=n_{j1}=0$ except for a finite number of $j\in J$. See Lemma 2 below.

The mapping f which we constructed in Theorem 1 is of finite type. If the surface R is of finite genus, every continuous mapping of R into T is trivially of finite type. A detailed study on analytic mappings of null type will be postponed to sec. 9. In this section we shall mainly consider analytic mappings of finite type.

Since the intersection number $\gamma\times\delta$ of two 1-cycles γ, δ is unaltered under the replacement of γ, δ with other 1-cycles γ', δ' such that $\gamma'\approx\gamma$, $\delta'\approx\delta$ (cf. [3], [7] etc.), it is also possible to ask whether a homomorphism of $H_1^*(R)$ into $H_1(T)$ is of finite type. The following lemma is almost trivial.

Lemma 2. *Let η: $H_1^*(R)\to H_1(T)$ be a homomorphism of finite type relative to ($\Xi(R)$, $\Xi(T)$). Then we can find a mapping $\varepsilon=\varepsilon_\eta$: $J\to\{0, 1\}$ such that*

$$(3)\qquad \begin{cases} \eta([A_j])=m_j[C_{\varepsilon(j)}]+m_j^*[C_{\varepsilon^*(j)}] \\ \eta([B_j])=n_j[C_{\varepsilon(j)}]+n_j^*[C_{\varepsilon^*(j)}] \end{cases} \qquad j\in J,$$

where m_j, m_j^*, n_j, $n_j^*\in\boldsymbol{Z}$ and $m_j^*=n_j^*=0$ for all but a finite number of $j\in J$. The converse is also true.

By Proposition 2 and Lemma 2 we have at once

Proposition 7. *For any η: $H_1^*(R)\to H_1(T)$, a homomorphism of finite type, there is an ε_η-allowable behavior space.* (We always denote such an ε as in Lemma 2 by ε_η when it is necessary to indicate η explicitly.)

As was already remarked (in sec. 4), an ε-allowable behavior space is not uniquely determined by ε. Furthermore, in Lemma 2, η does not determine ε_η uniquely. A fortiori, the behavior space in Proposition 7 is not the unique one for the preassigned η.

Examining the proof of Theorem 1, we arrive at

Theorem 2. *Let R, T and Π be the same as in Theorem 1. Suppose that η: $H_1^*(R)\to H_1(T)$ is a given homomorphism of finite type relative to ($\Xi(R)$, $\Xi(T)$). We may assume that η is expressed in* (3), *$\varepsilon=\varepsilon_\eta$.*

If f: $R\to T$ is an analytic mapping (of finite type) which induces η, then for "any" ε_η-allowable behavior space Λ_0 belonging to Π, we can find a Λ_0-Abelian differential ψ_0 of the first kind and a Π-admissible $(I)\Lambda_0$-singularity[4] *σ such that*

$$(4)\qquad \begin{cases} \int_{A_j}\psi_0=\pi_{\varepsilon(j)}\operatorname*{Res}_{\partial R}\Phi'_{A_j}\sigma+m_j\pi_{\varepsilon(j)}+m_j^*\pi_{\varepsilon^*(j)} \\ \int_{B_j}\psi_0=\pi_{\varepsilon(j)}\operatorname*{Res}_{\partial R}\Phi'_{B_j}\sigma+n_j\pi_{\varepsilon(j)}+n_j^*\pi_{\varepsilon^*(j)} \end{cases} \qquad j\in J.$$

Conversely, assume that for "some" ε_η-allowable behavior space Λ_0 (belonging to Π) we can find a Λ_0-Abelian differential ψ_0 of the first kind and a Π-admissible $(I)\Lambda_0$-singularity σ for which the system of equations (4) *is satisfied. Then there exists an analytic mapping f: $R\to T$ (of finite type) such that $f_*=\eta$ and the differ-*

4) In some cases, this σ may be identified with an appropriate $(P)\Lambda_0$-singularity, P being a regular partition of ∂R, $I>P>Q$. (See also Remark (1) to Theorem 1.)

ential $d(\rho^{-1}\circ f)$ *has the* $(I)\Lambda_0$*-singularity* σ.

Proof. Omitted.

As corollaries of this theorem, we have the following propositions as to the existence of an analytic mapping of finite type. Expediently we shall separate the results into two parts.

Proposition 8–1. *Suppose that* $f: R\to T\ (=C/\Pi)$ *is an analytic mapping of finite type. Then, for "any"* ε_{f_*}*-allowable behavior space* Λ_0 *(belonging to* Π*) there are a* Λ_0*-Abelian differential* ψ_0 *of the first kind and a* Π*-admissible* $(I)\Lambda_0$*-singularity* σ *such that* (2) *holds for* $\varepsilon=\varepsilon_{f_*}$.

Proposition 8–2. *Let* $\Pi=[\pi_0, \pi_1]$, $\pi_0, \pi_1\in \boldsymbol{C}$, $\operatorname{Im}(\pi_1/\pi_0)<0$ *and* $\rho: \boldsymbol{C}\to T$ $=\boldsymbol{C}/\Pi$ *be the projection mapping. Suppose that there is an* $\varepsilon: J\to\{0, 1\}$ *such that* (2) *holds for "some"* ε*-allowable behavior space* Λ_0 *belonging to* Π, *a* Λ_0*-Abelian differential* ψ_0 *of the first kind and a* Π*-admissible* $(I)\Lambda_0$*-singularity* σ. *Then there exists an analytic mapping* $f: R\to T$ *of finite type such that* $d(\rho^{-1}\circ f)$ *has the singularity* σ. *What is more, the* Λ_0 *is* ε_{f_*}*-allowable* $(\varepsilon=\varepsilon_{f_*})$.

Theorem 2 above gives a characterization of the boundary behavior of an analytic mapping (of finite type) $f: R\to T$ which induces a prescribed homomorphism η of finite type, $\eta: H_1^*(R)\to H_1(T)$. More generally it will make sense to ask the following problem:

Let R be the same as before and R' another. Let there be given a homomorphism $\eta: H_1^*(R)\to H_1(R')$ (or $H_1^*(R')$). How should an analytic mapping f of R into R' which induces η behave near the ideal boundary of R?

8. In case that R be of finite type[5] and R' be the punctured plane $S=\boldsymbol{C}-\{0\}$ $=\{0<|z|<\infty\}$, the classical Abel's theorem gives an answer to the preceding problem. Indeed, viewing the routine proof (see [7], [19], for instance), we can reformulate the Abel's theorem as follows.

Theorem 2′. *Let* $[C]$ *be the fixed generator of* $H_1(S)$ *represented by the positively oriented unit circle* $C: |z|=1$. *Then, for a homomorphism* $\eta: H_1^*(R)$ $\to H_1(S)$ *which is described with* $\eta([A_j])=m_j[C]$, $\eta([B_j])=n_j[C]$ $(m_j, n_j\in \boldsymbol{Z}$, $j\in J)$, *the following assertions* (I) *and* (II) *are equivalent.*

(I) *There exists an analytic mapping* $f: R\to S$ *which is algebraic at* ∂R *and induces* η.

(II) *We can find a polar singularity* σ *of the first order at* ∂R *with integral residues such that*

$$\operatorname*{Res}_{\partial R}\Phi_{A_j}\sigma=m_j,\quad \operatorname*{Res}_{\partial R}\Phi_{B_j}\sigma=n_j, \tag{5}$$

where $d\Phi_{A_j}$, $d\Phi_{B_j}$ *are square integrable holomorphic differentials on* R

5) This means that there is a compact Riemann surface R_0 such that R_0 contains R and R_0-R consists of only a finite number of points.

with the following properties: $\mathrm{Re}\int_{A_k} d\Phi_{A_j} = \mathrm{Re}\int_{B_k} d\Phi_{B_j} = 0$, $\mathrm{Re}\int_{B_k} d\Phi_{A_j} = -\mathrm{Re}\int_{A_k} d\Phi_{B_j} = \delta_{jk}$.

Notice that $H_1^*(R) \cong H_1(R_0)$ with R_0 explained in footnote 5), and that $d\Phi_{A_j}$, $d\Phi_{B_j}$ may be identified with Abelian differentials of the first kind (in the classical sense) on the surface R_0. We also note that the singularity σ can be more general if we allow f to be transcendental at ∂R (cf. [17]). Moreover, even if R is completely general, the equivalency of (I) and (II) is still valid with a few necessary but natural modifications. Refer to Kusunoki [5]; see also [7], [10], [20] etc..

For the completeness, we shall finally include another remark. The classical Abel's theorem as well as its generalizations to open Riemann surfaces ([1, 3], [5, 7] etc.) are, of course, concerned primarily with the assignment of zeros and poles with their multiplicities, i.e., the local degree of mapping at these points. In other words, the Abel's theorem, classical and modern alike, is also expressed in terms of a set of homomorphisms between 2-dimensional local homology groups. To be more precise, let $\delta = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r} / q_1^{n_1} q_2^{n_2} \cdots q_s^{n_s}$ be a divisor of degree zero, given on a Riemann surface R ($p_j \neq q_k$; m_j, $n_k > 0$, $\sum_{j=1}^r m_j = \sum_{k=1}^s n_k$), and let p, $q \in \hat{\boldsymbol{C}}$, $p \neq q$. Then the problem of finding a meromorphic function whose divisor is exactly the δ is nothing other than the problem of finding an analytic mapping of R into $\hat{\boldsymbol{C}}$ which induces the following set of homomorphisms η_j and η_k':

$$\eta_j: H_2(R, p_j) \longrightarrow H_2(\hat{\boldsymbol{C}}, p), \quad \eta_j(\Delta_j) = m_j D_p, \qquad j = 1, 2, \ldots, r,$$

$$\eta_k': H_2(R, q_k) \longrightarrow H_2(\hat{\boldsymbol{C}}, q), \quad \eta_k'(\Delta_k') = n_k D_q, \qquad k = 1, 2, \ldots, s.$$

Here $H_2(X, x_0)$ stands for the 2-dimensional local homology group of X at x_0 (with integral coefficients) and Δ_j, Δ_k', D_p, D_q are generators of $H_2(R, p_j)$, $H_2(R, q_k)$, $H_2(\hat{\boldsymbol{C}}, p)$, $H_2(\hat{\boldsymbol{C}}, q)$ respectively. The number m_j (resp. n_k) is what we call the local degree of mapping at p_j (resp. q_k).

IV. Analytic mappings of null type

9. So far we have dealt with analytic mappings of finite type. We shall now study analytic mappings of null type in some detail.

Let ε be a fixed mapping of J into the set $\{0, 1\}$ and L_0, L_1 be the lines in $\boldsymbol{C}$ as in sec. 4. With the family $\mathscr{L} = \{L_{\varepsilon(j)}\}_{j \in J}$ we associate a uniquely determined family $\mathscr{L}^* = \{L_{\varepsilon^*(j)}\}_{j \in J}$. For a finite subset J^* of J we define a mapping $\tilde{\varepsilon}: J \to \{0, 1\}$ by

$$\tilde{\varepsilon}(j) = \begin{cases} \varepsilon^*(j) & j \in J^* \\ \varepsilon(j) & j \in J - J^*, \end{cases}$$

and set $\tilde{\mathscr{L}} = \{L_{\tilde{\varepsilon}(j)}\}_{j \in J}$. We also set

$$\Lambda_0^{J^*} = \{\lambda \in \Lambda_h | \lambda \text{ has } \Lambda_0\text{-behavior and } \int_{A_j} \lambda \equiv \int_{B_j} \lambda \equiv 0 \mod L_{\tilde{\varepsilon}(j)},\ j \in J\}.$$

Later we shall make use of the following propositions whose proofs may be found in [15].

Proposition 9. *For any finite subset J^* of J and any ε-allowable behavior space Λ_0, $\Lambda_0^{J^*}$ is a behavior space which is equivalent to Λ_0 (i.e., every C^1-differential with Λ_0-behavior has $\Lambda_0^{J^*}$-behavior and every C^1-differential with $\Lambda_0^{J^*}$-behavior has Λ_0-behavior). $\Lambda_0^{J^*}$ is $\tilde{\varepsilon}$-allowable.*

Proposition 10. *Let $\tilde{\Lambda}_0$ be an $\tilde{\varepsilon}$-allowable behavior space which is equivalent to an ε-allowable Λ_0. Then there is a finite subset J^* of J such that $\tilde{\Lambda}_0=\Lambda_0^{J^*}$.*

Proposition 11. *If Λ_0 is an ε-allowable behavior space, then we have $(\Lambda_0^{J^*})'=(\Lambda_0')^{J^*}$ for any finite subset J^* of J. (We remind that Λ_0' denotes the ($\boldsymbol{R}$-)dual behavior space of Λ_0).*

As a corollary of Proposition 9 we have

Proposition 12. *Let Λ_0, J^* be as above and P a regular partition of the ideal boundary of R. Then a $(P)\Lambda_0$-singularity is a $(P)\Lambda_0^{J^*}$-singularity and vice versa.*

We are now ready to prove

Theorem 3. *Suppose that there is given a non-trivial Π-admissible (I) Λ_0-singularity σ. Then we can find an analytic mapping of null type (relative to $(\Xi(R), \Xi(T))$) $f\colon R\to T$ such that $d(\rho^{-1}\circ f)$ has the singularity σ if and only if*

$$(6)\qquad \operatorname*{Res}_{\partial R}\tilde{\Phi}'_{A_j}\sigma\equiv\operatorname*{Res}_{\partial R}\tilde{\Phi}'_{B_j}\sigma\equiv 0 \mod \boldsymbol{Z},\qquad j\in J$$

for some behavior space $\tilde{\Lambda}_0'$ which is equivalent to Λ_0'. Here, of course, $d\tilde{\Phi}'_{A_j}=\phi(A_j;\ \tilde{\Lambda}_0',\ -2\pi i/\pi_{\tilde{\varepsilon}(j)})$ and $d\tilde{\Phi}'_{B_j}=\phi(B_j;\ \tilde{\Lambda}_0',\ -2\pi i/\pi_{\tilde{\varepsilon}(j)})$.

Proof. We begin with the assumption that there is an analytic mapping $f\colon R\to T$ such that $d(\rho^{-1}\circ f)$ has σ as its generalized singularity. Since σ is an $(I)\Lambda_0$-singularity, it is easy to see that f is of finite type. Furthermore, by Theorem 1 we can find a Λ_0-Abelian differential ψ_0 of the first kind and integers m_j, m_j^*, n_j, n_j^* ($m_j^*=n_j^*=0$ for almost all $j\in J$) for which (4) holds. If f is of null type, $(m_j^2+n_j^2)(m_j^{*2}+n_j^{*2})=0$, $j\in J$. We can not, however, insist that $m_j^*=n_j^*=0$ for all $j\in J$; some of m_j^* and n_j^* may not vanish. If we set $J'=\{j\in J|m_j^*=n_j^*=0\}$, then we only know that $J^*=J-J'$ is a finite subset of J.

Now let $\tilde{\Lambda}_0=\Lambda_0^{J^*}$ and $\tilde{\Lambda}_0'=(\Lambda_0')^{J^*}$. By Proposition 11, $\tilde{\Lambda}_0'$ is the dual behavior space of $\tilde{\Lambda}_0$ (w.r.t. $\boldsymbol{R}$). Due to Proposition 12 we can regard σ as an $(I)\tilde{\Lambda}_0$-singularity. Thus, we have instead of (4)

$$(4')\qquad \left\{\begin{aligned} \int_{A_j}\tilde{\psi}_0&=\pi_{\tilde{\varepsilon}(j)}\operatorname*{Res}_{\partial R}\tilde{\Phi}'_{A_j}\sigma+m_j\pi_{\tilde{\varepsilon}(j)}\\ \int_{B_j}\tilde{\psi}_0&=\pi_{\tilde{\varepsilon}(j)}\operatorname*{Res}_{\partial R}\tilde{\Phi}'_{B_j}\sigma+n_j\pi_{\tilde{\varepsilon}(j)}\end{aligned}\right.\qquad j\in J,$$

for some $\tilde{\psi}_0$, a $\tilde{\Lambda}_0$-Abelian differential of the first kind, and integers m_j, n_j. By Proposition 9 $\tilde{\psi}_0$ can be considered a Λ_0-Abelian differential of the first kind as well. Hence we write ψ_0 for $\tilde{\psi}_0$. Equations (4′) then imply

$$(4'')\qquad \begin{cases} \int_{A_j} \psi_0 = (\operatorname*{Res}_{\partial R} \tilde{\Phi}'_{A_j}\sigma + m_j)\pi_{\tilde{\varepsilon}(j)} \equiv 0 \\ \int_{B_j} \psi_0 = (\operatorname*{Res}_{\partial R} \tilde{\Phi}'_{B_j}\sigma + n_j)\pi_{\tilde{\varepsilon}(j)} \equiv 0 \end{cases} \mod L_{\tilde{\varepsilon}(j)},\quad j\in J.$$

Proposition 3 is now applicable and we deduce that $\psi_0\equiv 0$. Turning to (4″) again, we obtain $\operatorname*{Res}_{\partial R} \tilde{\Phi}'_{A_j}\sigma \equiv \operatorname*{Res}_{\partial R} \tilde{\Phi}'_{B_j}\sigma \equiv 0 \mod \boldsymbol{Z},\ j\in J$.

Conversely, let σ be a non-trivial Π-admissible $(I)\Lambda_0$-singularity which satisfies the system of equations (6) for some $\tilde{\Lambda}'_0$, an $\tilde{\varepsilon}$-allowable behavior space equivalent to Λ'_0. It follows from Proposition 10 that there is a finite subset J^* of J such that $\tilde{\Lambda}'_0=(\Lambda'_0)^{J^*}$. We set $\tilde{\Lambda}_0=\Lambda_0^{J^*}$. Then (4′) is fulfilled by the Λ_0-Abelian differential $\tilde{\psi}_0\equiv 0$ and suitable integers m_j, n_j. If we take the normalized $\tilde{\Lambda}_0$-Abelian differential $\tilde{\psi}_\sigma$ with the singularity σ (cf. Prop. 5), the A_j- and B_j-periods of $\tilde{\psi}_\sigma$ are of the form $\int_{A_j}\tilde{\psi}_\sigma=\tilde{\alpha}_j\pi_{\tilde{\varepsilon}(j)}$ and $\int_{B_j}\tilde{\psi}_\sigma=\tilde{\beta}_j\pi_{\tilde{\varepsilon}(j)}$, $\tilde{\alpha}_j$, $\tilde{\beta}_j\in\boldsymbol{R}$. A use of Proposition 6 yields that $\tilde{\alpha}_j=m_j$ and $\tilde{\beta}_j=n_j$. It follows that $f=\rho\left(\int^p\tilde{\psi}_\sigma\right)$ $(p\in R)$ defines an analytic mapping of R into T (cf. Theorem 1). We also know that f is a mapping of null type relative to $(\Xi(R), \Xi(T))$ and $d(\rho^{-1}\circ f)$ has the singularity σ. q.e.d.

It should be noted that condition (6) is very similar to (5) in Theorem 2′ (the classical Abel's theorem). See Theorem 6.

10. Let σ, $d\Phi'_{A_j, B_j}$, $d\tilde{\Phi}'_{A_j, B_j}$ and $\tilde{\psi}_\sigma$ be the same as above. We may write

$$(7)\qquad \begin{cases} \int_{A_k} d\Phi'_{A_j}=a_{jk}\overline{\pi_{\varepsilon(k)}}, & \int_{A_k}\tilde{\psi}_\sigma=\tilde{\alpha}_k\pi_{\tilde{\varepsilon}(k)} \\ \int_{B_k} d\Phi'_{B_j}=b_{jk}\overline{\pi_{\varepsilon(k)}}-\dfrac{2\pi i}{\pi_{\varepsilon(j)}}\delta_{jk}, & \int_{B_k}\tilde{\psi}_\sigma=\tilde{\beta}_k\pi_{\tilde{\varepsilon}(k)} \end{cases}\qquad (k\in J)$$

for appropriate real numbers a_{jk}, b_{jk}, $\tilde{\alpha}_k$, $\tilde{\beta}_k$. These numbers are uniquely determined only by σ and J^*.

We may use Proposition 6 to compute

$$\operatorname*{Res}_{\partial R}\tilde{\Phi}'_{A_j}\sigma-\operatorname*{Res}_{\partial R}\Phi'_{A_j}\sigma=\operatorname*{Res}_{\partial R}(\tilde{\Phi}'_{A_j}-\Phi'_{A_j})\tilde{\psi}_\sigma,$$

since $d(\tilde{\Phi}'_{A_j}-\Phi'_{A_j})$ is regarded as a $\tilde{\Lambda}'_0$-Abelian differential of the first kind (cf. Prop. 9). We find out that

$$\operatorname*{Res}_{\partial R}\tilde{\Phi}'_{A_j}\sigma-\operatorname*{Res}_{\partial R}\Phi'_{A_j}\sigma$$
$$=\begin{cases} \mu\sum_{k\in J^*}(-1)^{\varepsilon(k)}(a_{jk}\tilde{\beta}_k-b_{jk}\tilde{\alpha}_k)+\tilde{\alpha}_j(\mathrm{Re}\,[\tau^{1-2\varepsilon(j)}]-1), & \text{if } j\in J^* \\ \mu\sum_{k\in J^*}(-1)^{\varepsilon(k)}(a_{jk}\tilde{\beta}_k-b_{jk}\tilde{\alpha}_k), & \text{if } j\in J-J^*. \end{cases}$$

where we set $\tau=\pi_1/\pi_0$ and $2\pi\mu=\operatorname{Im}(\pi_1\bar{\pi}_0)=|\pi_0|^2\operatorname{Im}\tau$ (<0). Similar formulae hold for $\operatorname{Res}_{\partial R}\tilde{\Phi}'_{B_j}\sigma-\operatorname{Res}_{\partial R}\Phi'_{B_j}\sigma$. Actually, if we set

$$(7')\qquad \begin{cases} \int_{A_k} d\Phi'_{B_j}=a'_{jk}\overline{\pi_{\varepsilon(k)}}+\dfrac{2\pi i}{\pi_{\varepsilon(k)}}\delta_{jk} \\ \int_{B_k} d\Phi'_{B_j}=b'_{jk}\overline{\pi_{\varepsilon(k)}} \end{cases}\qquad a'_{jk},\ b'_{jk}\in\boldsymbol{R},$$

then

$$\operatorname{Res}_{\partial R}\tilde{\Phi}'_{B_j}\sigma-\operatorname{Res}_{\partial R}\Phi'_{B_j}\sigma$$
$$=\begin{cases} \mu\sum_{k\in J^*}(-1)^{\varepsilon(k)}(a'_{jk}\tilde{\beta}_k-b'_{jk}\tilde{\alpha}_k)+\tilde{\beta}_j(\operatorname{Re}[\tau^{1-2\varepsilon(j)}]-1), & \text{if } j\in J^* \\ \mu\sum_{k\in J^*}(-1)^{\varepsilon(k)}(a'_{jk}\tilde{\beta}_k-b'_{jk}\tilde{\alpha}_k), & \text{if } j\in J-J^*. \end{cases}$$

In particular, when the fundamental periods π_0, π_1 of T are normalized so that $|\pi_0|=1$, we have

Lemma 3. *If we set*

$$(7'')\qquad \begin{cases} \hat{a}_{jk}=a_{jk}+(-1)^{\varepsilon(k)}\dfrac{2\pi}{\operatorname{Im}\pi}(1-\operatorname{Re}[\tau^{1-2\varepsilon(j)}])\delta_{jk} \\ \hat{a}'_{jk}=a'_{jk}-(-1)^{\varepsilon(k)}\dfrac{2\pi}{\operatorname{Im}\tau}(1-\operatorname{Re}[\tau^{1-2\varepsilon(j)}])\delta_{jk} \end{cases}\qquad j,\ k\in J,$$

then

$$\begin{cases} \operatorname{Res}_{\partial R}\tilde{\Phi}'_{A_j}\sigma-\operatorname{Res}_{\partial R}\Phi'_{A_j}\sigma=\dfrac{\operatorname{Im}\tau}{2\pi}\sum_{k\in J^*}(-1)^{\varepsilon(k)}(a_{jk}\tilde{\beta}_k-\hat{b}_{jk}\tilde{\alpha}_k) \\ \operatorname{Res}_{\partial R}\tilde{\Phi}'_{B_j}\sigma-\operatorname{Res}_{\partial R}\Phi'_{B_j}\sigma=\dfrac{\operatorname{Im}\tau}{2\pi}\sum_{k\in J^*}(-1)^{\varepsilon(k)}(\hat{a}'_{jk}\tilde{\beta}_k-b'_{jk}\tilde{\alpha}_k) \end{cases}\qquad j\in J.$$

Combining this lemma with Theorem 3, we obtain the following

Theorem 3′. *Let $\Pi=[\pi_0,\pi_1]$ and assume $|\pi_0|=1$. Let $\varepsilon\colon J\to\{0,1\}$ and Λ_0 be an ε-allowable behavior space belonging to Π. Then, for a non-trivial Π-admissible $(I)\Lambda_0$-singularity σ, the following two statements are equivalent.*

(I) *There exists an analytic mapping $f\colon R\to T$ such that f is of null type and $d(\rho^{-1}\circ f)$ has the singularity σ. (ρ is the projection mapping $\boldsymbol{C}\to T=\boldsymbol{C}/\Pi$.)*

(II) *There is a finite subset J^* of J such that*

$$\begin{cases} \operatorname{Res}_{\partial R}\Phi'_{A_j}\sigma+\dfrac{\operatorname{Im}\tau}{2\pi}\sum_{k\in J^*}(-1)^{\varepsilon(k)}(a_{jk}\tilde{\beta}_k-\hat{b}_{jk}\tilde{\alpha}_k)\equiv 0 \\ \operatorname{Res}_{\partial R}\Phi'_{B_j}\sigma+\dfrac{\operatorname{Im}\tau}{2\pi}\sum_{k\in J^*}(-1)^{\varepsilon(k)}(\hat{a}'_{jk}\tilde{\beta}_k-b'_{jk}\tilde{\alpha}_k)\equiv 0 \end{cases}\qquad \operatorname{mod}\boldsymbol{Z},\qquad j\in J,$$

where a_{jk}, $\hat{b}_{jk}$; $\hat{a}'_{jk}$, b'_{jk}; $\tilde{\alpha}_k$, $\tilde{\beta}_k$ are real numbers defined by equations

(7), (7′) and (7″).

In contrast with Theorems 3 and 3′, we now claim

Theorem 4. *A non-trivial analytic mapping of R into a torus T cannot be of null type without having singularity. More precisely, let $\rho: \boldsymbol{C}\to T=\boldsymbol{C}/\Pi$, $\Pi=[\pi_0, \pi_1]$ and ε a mapping of J into the set $\{0, 1\}$. Let $f: R\to T$ be an analytic mapping ($\not\equiv$const.) of null type. Then, for every ε-allowable behavior space Λ_0 belonging to Π, $d(\rho^{-1}\circ f)$ can never be a Λ_0-Abelian differential of the first kind.*

For the proof we need the following lemma which can be shown in a way quite similar to the compact case (cf. Prop. 6; see also [5, 7], [13] etc.).

Lemma 4. *Let Λ_0 be a behavior space and φ any Λ_0-Abelian differential of the first kind. Then we have*

$$2\sum_{j\in J}\mathrm{Im}\int_{A_j}\varphi\int_{B_j}\bar{\varphi}=-i\sum_{j\in J}\left(\int_{A_j}\varphi\int_{B_j}\bar{\varphi}-\int_{B_j}\varphi\int_{A_j}\bar{\varphi}\right)=\|\varphi\|^2\geq 0.$$

Proof of Theorem 4. Let Λ_0 be an arbitrary ε-allowable behavior space belonging to Π. We set $\Phi=\rho^{-1}\circ f$. Consider the case where $d\Phi$ can be regarded as a Λ_0-Abelian differential. Suppose that $d\Phi$ is a Λ_0-Abelian differential of the first kind. Then

$$\int_{A_j}d\Phi=m_j\pi_{\varepsilon(j)}+m_j^*\pi_{\varepsilon^*(j)},\quad \int_{B_j}d\Phi=n_j\pi_{\varepsilon(j)}+n_j^*\pi_{\varepsilon^*(j)}$$

for some integers m_j, m_j^*, n_j, n_j^*. Since f is non-constant, $d\Phi\not\equiv 0$. Lemma 4 now implies that

$$2\pi\mu\sum_{j\in J}(-1)^{\varepsilon(j)}(m_j^*n_j-m_jn_j^*)>0,$$

$\mu=\mathrm{Im}(\pi_1\bar{\pi}_0)/2\pi\neq 0$. It follows that

$$\{j\in J|m_j=n_j=0\}\cup\{j\in J|m_j^*=n_j^*=0\}\subsetneqq J.$$

Hence f cannot be of null type. q.e.d.

11. In this and the next sections we shall confine ourselves to surfaces of finite connectivity. If this is the case, every continuous mapping is of finite type and therefore our Theorem 1, for example, is concerned with analytic mappings without any homological restrictions. Here we shall be concerned with analytic mappings of null type.

For our purposes, let τ be a complex number with negative imaginary part and $L_0=i\boldsymbol{R}$, $L_1=i\tau\boldsymbol{R}=\{z\in\boldsymbol{C}|z=i\tau t,\ t\in\boldsymbol{R}\}$. Set $\Pi=[2\pi i, 2\pi i\tau]$ and $T=\boldsymbol{C}/\Pi$. Let $\rho: \boldsymbol{C}\to T$ be the natural projection mapping as before.

Let R be the interior of a compact bordered Riemann surface (of genus g). Suppose that the border ∂R consists of h contours $\beta_1, \beta_2,\dots, \beta_h$. With every β_i we associate a doubly connected subregion of R which is conformally equivalent to a ring domain $1<|z_i|<r_i\ (<\infty)$. We assume that β_i corresponds to the circle $|z_i|=1$. Note that z_i gives a local parameter near β_i.

We denote by $\Gamma_h=\Gamma_h(R)$ the (real) Hilbert space of square integrable *real* harmonic differentials on R (cf. [3]). The space Γ_h is naturally a subspace of Λ_h. Let $\Gamma_{hse}=\Gamma_{hse}(R)=\{\lambda\in\Gamma_h|\lambda \text{ is semiexact}\}=\Lambda_{hse}\cap\Gamma_h$ and $\Gamma_{hm}=\Gamma_{hm}(R)$ be the space of real harmonic measures on R. We know that $\Gamma_h=\Gamma_{hm}\oplus\Gamma^*_{hse}=\Gamma^*_{hm}\oplus\Gamma_{hse}$ ([3]). If we set

$$\Lambda_K=\Lambda_K(R)=\Gamma_{hm}+i\Gamma_{hse},$$

Λ_K is a behavior space associated with the family $\mathscr{L}_K=\{i\boldsymbol{R}\}$, where $i\boldsymbol{R}$ denotes the imaginary axis. We also know that Λ_K is dual to itself (see [5, 7], [13]). Meromorphic differentials on R which have Λ_K-behavior were first considered by Kusunoki ([5–7]) and termed *semiexact canonical differentials*. Cf. also Ahlfors' distinguished differentials ([1–3]) and Γ_χ-behavior in Yoshida [20].

For any mapping ε of J into the set $\{0, 1\}$ we set

$$\Lambda(\varepsilon)=\Lambda(R,\varepsilon)=\left\{\lambda\in\Lambda_{hse}\ \middle|\ \begin{array}{l}\operatorname{Re}\lambda \text{ has a harmonic extension}\\ \text{across } \partial R \text{ and vanishes along } \partial R;\\ \int_{A_j}\lambda\equiv\int_{B_j}\lambda\equiv 0 \mod L_{\varepsilon(j)},\ j\in J\end{array}\right\}$$

It can be shown that $\Lambda(\varepsilon)$ is a non-void closed subspace of Λ_{hse} and is a behavior space ([13]). We set as usual $\Lambda'(\varepsilon)=\overline{\Lambda(\varepsilon)}$, the dual behavior space of $\Lambda(\varepsilon)$. We note that $\Lambda_K=\Lambda(\varepsilon_0)$, $\varepsilon_0\equiv 0$. While $\Lambda_K\neq\Lambda(\varepsilon)$ in general, we have as a result of Propositions 9 and 10

Proposition 13. *A meromorphic differential on R is a semiexact canonical differential if and only if it has $\Lambda(\varepsilon)$-behavior. In other words, every $\Lambda(\varepsilon)$ defines the same boundary behavior.*

We set $\sigma_i=c_i dz_i/z_i$ on U_i[6], $i=1, 2,\ldots, h$, where c_i is a complex number (maybe $=0$). If we set furthermore

$$\sigma=\sigma_i \quad \text{on} \quad U_i, \tag{8}$$

σ is an analytic differential defined on $U=\bigcup_{i=1}^{h} U_i$, a neighborhood of ∂R. We may write c_i as $c_i=p_i+\tau q_i$, $p_i, q_i\in\boldsymbol{R}$. Since $\int_{\beta_i}\sigma=2\pi i(p_i+\tau q_i)$ and $\int_{\partial R}\sigma=\sum_{i=1}^{h}\int_{\beta_i}\sigma_i=2\pi i\sum_{i=1}^{h}(p_i+\tau q_i)$, σ determines a Π-admissible $(I)\Lambda(\varepsilon)$-singularity if and only if p_i, q_i are all integers and $\sum_{i=1}^{h}p_i=\sum_{i=1}^{h}q_i=0$, ε being any mapping $J\to\{0, 1\}$ (cf. Prop. 12).

We generally denote by $d\Phi'_{A_j,\varepsilon}$, $d\Phi'_{B_j,\varepsilon}$ the normalized $\Lambda'(\varepsilon)$-Abelian differentials of the first kind. The integrals $\Phi'_{A_j,\varepsilon}$, $\Phi'_{B_j,\varepsilon}$ are considered single-valued

6) Cf. Shiba, M.: Notes on the existence of certain slit mappings. Proc. Jap. Acad. **51** (1975), 687–690.

holomorphic functions on $R-\bigcup_{j\in J}(A_j\cup B_j)$. In particular, they can be expanded in a series on each U_i, $i=1, 2,\ldots, h$. Let $\sum_{n=-\infty}^{+\infty} a_{ji}^{(n)}z_i^n$ be the Laurant expansion of $\Phi'_{A_j,\varepsilon}$ on U_i. We set

$$\Phi'_{A_j,\varepsilon}(\beta_i)=a_{0i}^{(n)}. \tag{9}$$

Similarly for $\Phi'_{B_j,\varepsilon}(\beta_i)$. Note that $\Phi'_{A_j,\varepsilon}(\beta_i)$ does not stand for the value of $\Phi'_{A_j,\varepsilon}$ assumed actually on the contour β_i. They coincide only in their real parts.

As an immediate consequence of Theorem 3 (cf. Prop. 13), we have

Theorem 5. *Let R be the interior of a compact bordered Riemann surface, $\partial R=\bigcup_{i=1}^{h}\beta_i$. Let σ be an analytic differential defined by* (8), *where $c_i=p_i+\tau q_i$, p_i, $q_i\in \boldsymbol{Z}$, $\sum_{i=1}^{h}p_i=\sum_{i=1}^{h}q_i=0$.*

Then there exists an analytic mapping $f\colon R\to T$ such that i) *f is of null type and* ii) $\mathrm{Re}(d(\rho^{-1}\circ f)-\sigma)$ *has a harmonic extension across ∂R and vanishes along ∂R, if and only if*

$$\mathrm{Re}\sum_{i=1}^{h}c_i\Phi'_{A_j,\varepsilon}(\beta_i)\equiv \mathrm{Re}\sum_{i=1}^{h}c_i\Phi'_{B_j,\varepsilon}(\beta_i)\equiv 0 \mod \boldsymbol{Z},\quad j\in J \tag{10}$$

for some mapping $\varepsilon\colon J\to\{0, 1\}$.

According to Theorem 3′ we would be able to replace (10) by another condition which is described in terms of the periods of $d\Phi'_{A_j,\varepsilon_0}$, $d\Phi'_{B_j,\varepsilon_0}$ (the normalized semi-exact canonical differentials of the first kind).

12. In case that R be a surface of finite type, the above argument remains valid if a few modifications are made. In fact, let R_0 be a compact Riemann surface, P_i $(i=1, 2,\ldots, h)$ distinct points on R_0. Let (U_i, z_i) be a (fixed) parametric disk about P_i, $z_i(P_i)=0$. We also set $R=R_0-\{P_i\}_{i=1}^{h}$. We know that every holomorphic function on a punctured disk $U_i-\{P_i\}$ which has a finite Dirichlet integral is holomorphic on the whole U_i. Particularly, the integrals $\Phi'_{A_j,\varepsilon}$ and $\Phi'_{B_j,\varepsilon}$ (considered single-valued holomorphic functions on the planar surface $R-\bigcup_{j\in J}(A_j\cup B_j)$) can be expanded on U_i as follows:

$$\Phi'_{A_j,\varepsilon}(z_i)=\sum_{n=0}^{\infty}s_{j,\varepsilon,i}^{(n)}z_i^n,\quad \Phi'_{B_j,\varepsilon}(z_i)=\sum_{n=0}^{\infty}t_{j,\varepsilon,i}^{(n)}z_i^n. \tag{9′}$$

Then we can prove the following theorem without difficulty.

Theorem 6. *Let R, (U_i, z_i) be as above, $i=1, 2,\ldots, h$. Suppose that there is given an analytic singularity σ such that*

$$\sigma=\Big(\sum_{n=1}^{\infty}c_i^{(n)}/z_i^n\Big)dz_i\qquad \text{on}\quad U_i,\quad i=1, 2,\ldots, h.$$

Then the following two statements are equivalent.

(I) *There exists an analytic mapping $f\colon R\to T$ such that f is of null type and*

$d(\rho^{-1}\circ f)-\sigma$ is holomorphic on each U_i, $i=1, 2,\dots, h$.

(II) *There are $2h$ integers p_i, q_i such that $\sum_{i=1}^{h} p_i=\sum_{i=1}^{h} q_i=0$ and $c_i^{(1)}=p_i+\tau q_i$, $i=1, 2,\dots, h$. Furthermore, for some mapping ε: $J\to\{0, 1\}$*

$$\mathrm{Re}\sum_{i=1}^{h}\sum_{n=1}^{\infty}c_i^{(n)}s_{j,\varepsilon,i}^{(n-1)}\equiv \mathrm{Re}\sum_{i=1}^{h}\sum_{n=1}^{\infty}c_i^{(n)}t_{j,\varepsilon,i}^{(n-1)}\equiv 0 \quad \mathrm{mod}\ \boldsymbol{Z}, \quad j\in J.$$

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