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The Paley-Wiener type theorem for finite covering groups of $SU(1, 1)$

By

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Introduction

The purpose of the present paper is to characterize explicitly the image of the Fourier transform of C^∞-functions with compact support on an n-fold covering group G of $SU(1, 1) \simeq SL(2, \boldsymbol{R})$, that is, to establish an analogue of the classical Paley-Wiener theorem.

The classical Paley-Wiener theorem can be stated as follows. Let f be a C^∞-function on $\boldsymbol{R}$ vanishing for $|t| \geq T$, and define its ordinary Fourier transform by

$$(0.1) \qquad F(s) = \int_{-\infty}^{\infty} f(t) e^{-ist} dt \qquad (s \in \boldsymbol{C}).$$

Then, F is an entire function with the property that for every non-negative integer r, there exists a constant C_r such that

$$(0.2) \qquad |F(s)| \leq C_r (1 + |s|)^{-r} e^{T|\mathrm{Im}\, s|}.$$

We topologize the vector space $\mathscr{H}_T$ of all entire functions satisfying (0.2) by means of seminorms

$$|F|_{r,M} = \sup_{|\mathrm{Im}\, s| \leq M} (1 + |s|)^r |F(s)| \qquad (r, M = 0, 1, \ldots).$$

Then the ordinary Fourier transform gives a topological isomorphism between $\mathscr{D}_T(\boldsymbol{R})$ and $\mathscr{H}_T$, where $\mathscr{D}_T(\boldsymbol{R})$ stands for the topological vector space of C^∞-functions on $\boldsymbol{R}$ vanishing for $|t| \geq T$ equipped with the usual topology.

Our method basically follows Ehrenpreis and Mautner [3] in which they treated the group $SU(1, 1)/\{\pm 1\}$. In the present case, however, there arise some difficulties when we follow their method directly. Let us explain this point in more detail.

In Part I of [3], they dealt with K_0-bi-invariant functions, where K_0 is a maximal compact subgroup of $SU(1, 1)/\{\pm 1\}$. But there they failed to derive the exact correspondence between "support" and "type of exponential" mainly due to rough estimates. On the other hand, in Part II of [3], they succeeded in deriving the above correspondence, using the result on the Abel transform corresponding to the trivial one-dimensional representation of K_0. In other words, they established the Paley-

Wiener type theorem for K_0-bi-invariant C^∞-functions with compact support. Next, shifting "K_0-type" of functions by means of certain differential operators, they deduced the Paley-Wiener type theorem for arbitrary K_0-type from that for the trivial K_0-type. In our case, however, it is difficult to study at first the Abel transform corresponding to the non-trivial representation of a maximal compact subgroup K of G. Therefore we proceed in another way, that is, we improve the method of estimation in Part I of [3], and then prove the Paley-Wiener type theorem for G for any n. After that, we are able to study the Abel transform.

We can treat any finite covering group of $SU(1, 1)$ in a unified way. The only exception is the case of odd functions on $SU(1, 1)$. This case requires careful treatment because we must take account of some validity problem concerning the integral expression of hypergeometric functions (cf. (4.17) and Lemma 4.3).

The present paper consists of six sections. We introduce a parametrization on G in 1.1 and construct its representations in 1.3. In 1.4 we give intertwining operators which are important in characterizing the explicit image of the Fourier transform. Sally constructed in [13] intertwining operators for the universal covering group of $SU(1, 1)$ by a different method. In 1.6 we give the list of all irreducible unitary representations of G. Section 2 is devoted to the study of matrix elements of representations. In section 3 we derive the inversion formula (Plancherel formula) by an elementary method. In sections 4 and 5 we establish the Paley-Wiener type theorem for arbitrarily fixed K-type (Theorems 4.1 and 5.1). Let us outline these theorems. Denote by $\mathscr{D}^k_{pq,T}$ $(-n+1\leq k\leq n,\ p,\ q\in \boldsymbol{Z},\ T>0)$ the space of C^∞-functions with compact support satisfying

$$(0.3)\qquad f(ugv)=\overline{\chi^k_p(u)}f(g)\overline{\chi^k_p(v)}\qquad (u,\ v\in K\simeq \boldsymbol{R}/4n\pi\boldsymbol{Z}),$$

$$(0.4)\qquad f(ua_tv)=0\qquad \text{for}\quad u,\ v\in K,\ t\geq T,$$

where $\chi^k_p(u_\theta)=\exp\left(-i\left(p+\frac{k}{2n}\right)\theta\right)$ (as for u_θ, a_t see 1.1). In section 4 we treat the case $p=q=0$, and in section 5 the case of arbitrary p,q. The image of $\mathscr{D}^k_{pq,T}$ under the Fourier transform is the space $\mathscr{H}^k_{pq,T}$ of all entire functions with the property analogous as (0.2) (we replace $\operatorname{Im} s$ by $\operatorname{Re} s$ in (0.2)) and satisfying certain functional equation and the condition of zeros. As a consequence of Theorem 5.1, we can investigate the Abel transform mentioned above (Theorem 5.3). In section 6 we finally establish the Paley-Wiener type theorem for C^∞-functions with compact support (Theorem 6.3). The image of a function f which fulfills (0.4) under the Fourier transform is given as an operator-valued entire function with the properties analogous as those for $\mathscr{H}^k_{pq,T}$, and satisfying certain "rapidly decreasing" conditions.

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Notations

Here we give notations frequently used in the sequel. As usual $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Z}$, $\boldsymbol{N}$

stand for the sets of complex numbers, real numbers, integers, positive integers respectively. We denote by $\boldsymbol{T}$ the set of complex numbers with absolute value one. $\operatorname{Im} s$ (resp. $\operatorname{Re} s$) denotes the imaginary (resp. real) part of a complex number s. Let G be a real Lie group. Then $C^\infty(G)$ (resp. $C_0^\infty(G)$) denotes the space of C^∞-functions (resp. C^∞-functions with compact support) on G. When we topologize $C_0^\infty(G)$ as usual, we denote it by $\mathscr{D}(G)$. Let $\mathfrak{g}$ be the Lie algebra of G. For $X \in \mathfrak{g}$ we define a right invariant differential operator on G, denoted again by X, in such a way that

$$Xf(g) = \frac{d}{dt} f(\exp(-tX)g)\Big|_{t=0} \qquad (g \in G).$$

Similarly we define for $X \in \mathfrak{g}$ a left invariant differential operator on G, denoted by X', in such a way that

$$X' f(g) = \frac{d}{dt} f(g \exp tX)\Big|_{t=0} \qquad (g \in G).$$

Let $U(\mathfrak{g}^C)$ be the universal enveloping algebra of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. Then any element $X \in U(\mathfrak{g}^C)$ can be considered canonically as a right (resp. left) invariant differential operator on G. We denote it by X (resp. X'). For a Hilbert space $\mathfrak{H}$, we denote by $\boldsymbol{B}(\mathfrak{H})$ and $\boldsymbol{U}(\mathfrak{H})$ the sets of bounded operators defined everywhere on $\mathfrak{H}$ and unitary operators on $\mathfrak{H}$ respectively. For a linear operator T on $\mathfrak{H}$, we denote by T^* its adjoint. $\operatorname{Dom}(T)$ and $\operatorname{Ran}(T)$ stand for the domain and the range of T respectively.

§1. Preliminaries

1.1. Let G_1 be the group $SU(1, 1)$ consisting of all 2×2 complex matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{with} \quad |\alpha|^2 - |\beta|^2 = 1. \tag{1.1}$$

As in Bargmann [1, p. 594] put $\beta/\alpha = \gamma$, $\arg \alpha = \omega \in \boldsymbol{R}/2\pi\boldsymbol{Z}$, then G_1 is parametrized as

$$\{(\gamma, \omega);\ |\gamma| < 1,\ \omega \in \boldsymbol{R}/2\pi\boldsymbol{Z}\}.$$

In this system of coordinates group operation is written as follows: let $(\gamma, \omega)(\gamma', \omega') = (\gamma'', \omega'')$, then γ'' and ω'' are given by

$$\gamma'' = (\gamma e^{-2i\omega'} + \gamma')(1 + \gamma\overline{\gamma'} e^{-2i\omega'})^{-1}, \tag{1.2}$$

$$\omega'' \equiv \omega + \omega' + (2i)^{-1} \log (1 + \gamma\overline{\gamma'} e^{-2i\omega'})(1 + \bar{\gamma}\gamma' e^{2i\omega'})^{-1}, \tag{1.3}$$

where the latter is understood by congruence mod 2π. Here we take the principal branch of logarithm and this is possible because $\operatorname{Re}(1 + \gamma\overline{\gamma'} e^{-2i\omega'}) > 0$.

Let n be a positive integer and consider a manifold

$$G=G_n=\{(\gamma, \omega);\ |\gamma|<1,\ \omega\in\boldsymbol{R}/2n\pi\boldsymbol{Z}\}.$$

We introduce an operation in G by (1.2) and (1.3) with congruence $\bmod 2n\pi$, then G becomes a Lie group as is easily seen. The unit element e is $(0, 0)$ and $(\gamma, \omega)^{-1}=(-\gamma e^{2i\omega}, -\omega)$. G is actually an n-fold covering group of G_1. The natural covering map Φ of G onto G_1 is given as

$$\Phi(\gamma, \omega(\bmod 2n\pi))=(\gamma, \omega(\bmod 2\pi)).$$

We identify under Φ the Lie algebra of G_1 with that of G, which is denoted by $\mathfrak{g}$. Put

$$X_0=\frac{1}{2}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix},\quad X_1=\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad X_2=\frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad Y=\frac{1}{2}\begin{pmatrix}i & -i\\ i & -i\end{pmatrix}$$

and

$$u_\theta=\exp\theta X_0=(0, -\theta/2),\quad a_t=\exp tX_1=(\operatorname{th}(t/2), 0),$$

$$b_t=\exp tX_2=(-i\operatorname{th}(t/2), 0),\quad n_\xi=\exp\xi Y,$$

where exp denotes the exponential mapping from $\mathfrak{g}$ into G. We use the following subgroups:

$$K=\{u_\theta;\ \theta\in\boldsymbol{R}\},\quad A=\{a_t;\ t\in\boldsymbol{R}\},$$

$$B=\{b_t;\ t\in\boldsymbol{R}\},\quad N=\{n_\xi;\ \xi\in\boldsymbol{R}\}.$$

Each element g in G can be expressed uniquely by

$$g=u_\theta a_t n_\xi \qquad (0\le\theta<4n\pi,\ t\in\boldsymbol{R},\ \xi\in\boldsymbol{R}).$$

Also g can be expressed by

$$g=u_\varphi a_t u_\psi \qquad (0\le\varphi<4n\pi,\ t\ge 0,\ 0\le\psi<2\pi).$$

For $g\notin K$ this expression is unique.

1.2. Haar integral. We normalize Haar measure on K in such a way that the total mass is equal to one, and that on G as

$$\begin{aligned}\int f(g)dg&=\int_K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(ua_tn_\xi)e^t du\,dt\,d\xi\\ &=2\pi\int_K\int_0^{\infty}\int_K f(ua_tv)\operatorname{sh}t\,du\,dt\,dv\\ &=\frac{2}{n\pi}\int_0^{4n\pi}\iint_{|\gamma|<1}f(\gamma, \omega)\frac{d\gamma_1 d\gamma_2 d\omega}{(1-|\gamma|^2)^2} \qquad (\gamma=\gamma_1+i\gamma_2).\end{aligned}$$

We know that G is unimodular.

1.3. For a fixed integer k such that $-n+1\le k\le n$ we put

$$j(g, \zeta)=(1-|\gamma|^2)|1+\bar\gamma\zeta|^{-2}, \tag{1.4}$$

$$v^k(g,\zeta)=e^{-2i\omega\lambda_k}\left[\frac{1+\bar{\gamma}\zeta}{1+\gamma\bar{\zeta}}\right]^{\lambda_k}, \tag{1.5}$$

where

$$g=(\gamma,\omega)\in G,\quad \zeta=e^{i\theta}\in \boldsymbol{T},\quad \lambda_k=k/2n,$$

and as before we take the principal branch of the fractional power in the right hand side of (1.5). We make G act on $\boldsymbol{T}$ by

$$g\cdot\zeta=\Phi(g)\cdot\zeta=\frac{\alpha\zeta+\beta}{\bar{\beta}\zeta+\bar{\alpha}},$$

where $\Phi(g)$ is given by (1.1).

Let $d\mu(\zeta)$ be the ordinary normalized Haar measure on $\boldsymbol{T}$, and denote by $\mathfrak{H}$ the Hilbert space $L^2(\boldsymbol{T};d\mu(\zeta))$. For any fixed $s\in\boldsymbol{C}$, we define operators $U^k(g,s)$ $(g\in G)$ by

$$\begin{aligned}U^k(g,s)f(\zeta)&=v^k(g^{-1},\zeta)j(g^{-1},\zeta)^{1/2+s}f(g^{-1}\cdot\zeta)\\&=e^{-2i\omega\lambda_k}\left[\frac{1+\bar{\gamma}\zeta}{1+\gamma\bar{\zeta}}\right]^{\lambda_k}(1-|\gamma|^2)^{1/2+s}|1+\bar{\gamma}\zeta|^{-1-2s}f\Big(e^{2i\omega}\zeta\frac{1+\gamma\bar{\zeta}}{1+\bar{\gamma}\zeta}\Big),\end{aligned}$$

where $g^{-1}=(\gamma,\omega)$, $f\in\mathfrak{H}$. It is clear that $\{U^k(\cdot,s),\mathfrak{H}\}$ is a strongly continuous bounded representation of G for any fixed $s\in\boldsymbol{C}$. We put $e_p(\zeta)=\zeta^{-p}$ $(p\in\boldsymbol{Z})$. Then, $\{e_p;\ p\in\boldsymbol{Z}\}$ forms a complete orthonormal system in $\mathfrak{H}$.

Proposition 1.1.

(1) $\dfrac{d\mu(g\cdot\zeta)}{d\mu(\zeta)}=j(g,\zeta)$.

(2) *For* $v\in K$, $U^k(v,s)$ *is independent of* s, *and* $U^k(v,s)\in\boldsymbol{U}(\mathfrak{H})$ *for all* $v\in K$.

(3) $\|U^k(g,s)\|\le e^{t|\mathrm{Re}\,s|}$ *for* $g=u_\varphi a_t u_\psi$ $(t\ge 0)$.

(4) $U^k(g,s)^*=U^k(g^{-1},-\bar{s})$.

(5) $\{U^k(\cdot,s),\mathfrak{H}\}$ *is unitary if and only if* $s\in i\boldsymbol{R}$.

(6) *For any* $s\in\boldsymbol{C}$, $C^\infty(\boldsymbol{T})$ *is contained in* $\mathfrak{H}_\infty(U^k(\cdot,s))$, *the totality of* C^∞*-vectors for* $\{U^k(\cdot,s),\mathfrak{H}\}$.

Proofs are all elementary, so we omit the details.

Remark 1. For $k=0$ or n, $g\mapsto U^k(g,s)$ gives actually a representation of G_1.

We define for $f\in\mathfrak{H}_\infty(U^k(\cdot,s))$, $X\in\mathfrak{g}$,

$$U^k_\infty(X,s)f=\frac{d}{dt}U^k(\exp tX,s)f\Big|_{t=0}.$$

Then the mapping $X\mapsto U^k_\infty(X,s)$ gives rise to a representation of $\mathfrak{g}$, and is uniquely extended to that of the universal enveloping algebra $U(\mathfrak{g}^C)$ of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$.

Proposition 1.2.

(1) $U^k_\infty(X_0,s)e_p=-i(\lambda_k+p)e_p$.

(2) $U_\infty^k(X_+, s)e_p=\left(\lambda_k+p+\frac{1}{2}+s\right)e_{p+1}$,

$U_\infty^k(X_-, s)e_p=-\left(\lambda_k+p-\frac{1}{2}-s\right)e_{p-1}$, *where* $X_\pm=X_1\pm iX_2$.

(3) *Let* Ω *be the Casimir element in* $U(\mathfrak{g}^C)$, *that is*, $\Omega=(X_0)^2-(X_1)^2-(X_2)^2$, *then*

$$U_\infty^k(\Omega, s)e_p=\left(\frac{1}{4}-s^2\right)e_p.$$

Proofs are given by simple calculations.

1.4. Intertwining operators. We define for any integer p a rational function $\alpha_p^k(s)$ as

$$(1.6)\quad \alpha_p^k(s)=\begin{cases}\prod_{0\le j\le p-1}\left(\lambda_k+j+\frac{1}{2}-s\right)\left(\lambda_k+j+\frac{1}{2}+s\right)^{-1} & \text{for } p\geq 1\\ 1 & \text{for } p=0\\ \prod_{0\le j\le |p|-1}\left(-\lambda_k+j+\frac{1}{2}-s\right)\left(-\lambda_k+j+\frac{1}{2}+s\right)^{-1} & \text{for } p\leq -1.\end{cases}$$

Note that $\alpha_p^n(0)=-1$ for $p\leq -1$. We see easily that $|\alpha_p^k(s)|\leq 1$ for $\mathrm{Re}\, s\geq 0$, and then we can define for $\mathrm{Re}\, s\geq 0$, bounded operators $A^k(s)\in \boldsymbol{B}(\mathfrak{H})$ by $A^k(s)e_p=\alpha_p^k(s)e_p$.

Proposition 1.3.

(1) $A^k(s)^*=A^k(\bar{s})$ $(\mathrm{Re}\, s\geq 0)$.

(2) $A^k(s)\in \boldsymbol{U}(\mathfrak{H})$ *if and only if* $s\in i\boldsymbol{R}$.

(3) *The operator* $A^k(s)$ *intertwines the representations* $U^k(\cdot, s)$ *and* $U^k(\cdot, -s)$ *as follows*:

$$A^k(s)U^k(g, s)=U^k(g, -s)A^k(s)\qquad (g\in G).$$

The assertions (1) and (2) are immediately verified. We shall prove (3) after the next two lemmas.

Since $U^k(\exp tX_j, i\tau)$ is a one-parameter subgroup in t of unitary operators for fixed $\tau\in\boldsymbol{R}$, there exists by Stone's theorem a selfadjoint operator $H_j^k(\tau)$ such that

$$U^k(\exp tX_j, i\tau)=\exp(-itH_j^k(\tau)).$$

We denote by $H_j^k(\tau)^\sim$ the restriction of $H_j^k(\tau)$ to $\mathfrak{D}$, where $\mathfrak{D}$ is the totality of all finite linear combinations of e_p's. Note that $\mathfrak{D}\subset \mathrm{Dom}(H_j^k(\tau))$ by Proposition 1.1 (6).

Lemma 1.4. $iU_\infty^k(X_j, i\tau)$ *is essentially selfadjoint.*

This lemma is well-known although its proof is not so trivial (cf. e.g. [19]).

Lemma 1.5. $H_j^k(\tau)^\sim$ *is essentially selfadjoint.*

Proof. For simplicity we drop superscript k and parameter τ, fixing them. Since $\tilde{H}_0$ is "diagonal" (cf. Proposition 1.2 (1)), the lemma holds at least for $j=0$ as is seen without difficulty. Thus there exists for each $x\in \mathrm{Dom}(H_0)$ a sequence

$x_m \in \mathfrak{D}$ such that $x_m \to x$, $H_0 x_m \to H_0 x$. On the other hand, we have

$$-(H_0)^2 x_m + (H_1)^2 x_m + (H_2)^2 x_m = \left(\frac{1}{4} + \tau^2\right) x_m \qquad \text{(cf. Proposition 1.2 (3))}.$$

Since $((H_j)^2 y, y) = \|H_j y\|^2$ $(y \in \mathfrak{D})$, we see that $\{H_j x_m\}$ $(j=1, 2)$ are convergent. Since $\tilde{H}_j x_m = H_j x_m$ and H_j is closed, we get

$$\mathrm{Dom}(H_0) \subset \mathrm{Dom}(H_j'), \quad \mathrm{Dom}(H_0) \subset \mathrm{Dom}(H_j), \quad H_j x = H_j' x \quad \text{for} \quad x \in \mathrm{Dom}(H_0),$$

where H_j' denotes the closure of $\tilde{H}_j$. On the other hand, by Lemma 1.4 there exists for each $y \in \mathrm{Dom}(H_j)$ a sequence $y_m \in \mathfrak{H}_\infty(U^k(\cdot, i\tau))$ such that $y_m \to y$, $H_j y_m \to H_j y$. By the discussion above we have $H_j y_m = H_j' y_m$. (Note that $\mathfrak{H}_\infty(U^k(\cdot, i\tau)) \subset \mathrm{Dom}(H_0)$ as is seen from the definition of C^∞-vectors.) Thus we obtain $H_j \subset H_j'$. The converse inclusion is clear. Q.E.D.

Proof of Proposition 1.3 (3). First of all we shall prove it for $s = i\tau \in i\boldsymbol{R}$. By a simple calculation we obtain

$$A^k(i\tau) H_j^k(\tau) e_p = H_j^k(-\tau) A^k(i\tau) e_p \qquad (j=0, 1, 2).$$

This implies $A^k(i\tau) H_j^k(\tau)^\sim = H_j^k(-\tau)^\sim A^k(i\tau)$. Then by limiting procedure and Lemma 1.5 and noting that $A^k(i\tau) \in \boldsymbol{U}(\mathfrak{H})$, $A^k(i\tau)\mathfrak{D} = \mathfrak{D}$, we obtain

$$(1.7) \qquad A^k(i\tau) H_j^k(\tau) = H_j^k(-\tau) A^k(i\tau).$$

By a familiar argument, (1.7) gives us

$$A^k(i\tau) U^k(g, i\tau) = U^k(g, -i\tau) A^k(i\tau) \qquad \text{for} \quad g = u_\theta,\ a_t \text{ or } b_t.$$

Since G is generated by the one-parameter subgroups u_θ, a_t and b_t, the above equality leads us to the assertion (3) for $\mathrm{Re}\, s = 0$.

Now, consider an operator $T(s) \in \boldsymbol{B}(\mathfrak{H})$ defined by

$$T(s) = A^k(s) U^k(g, s) - U^k(g, -s) A^k(s).$$

As is seen above, $T(i\tau) = 0$ for $\tau \in \boldsymbol{R}$. On the other hand, it is clear that $T(\cdot)$ is an operator-valued holomorphic function for $\mathrm{Re}\, s > 0$ and continuous for $\mathrm{Re}\, s \geq 0$. Hence by the reflection principle of Schwarz we can continue $T(s)$ analytically across the imaginary axis, because $T(i\tau) = 0$ is a symmetric operator. By the theorem of identity, we get $T(s) = 0$ for all s with $\mathrm{Re}\, s \geq 0$. Q.E.D.

Proposition 1.6. *Suppose that there exists a non-zero closed linear operator L with the following properties* (i)~(iii).

(i) $\mathrm{Dom}(L) \supset C^\infty(\boldsymbol{T})$.

(ii) *L leaves $C^\infty(\boldsymbol{T})$ invariant.*

(iii) *For a pair (s, s'), $LU^k(g, s) f = U^k(g, s') Lf$* $(g \in G, f \in C^\infty(\boldsymbol{T}))$.

Then s' is equal to s or $-s$.

Remark 2. It is readily verified that $U^k(g, s)$ leaves $C^\infty(\boldsymbol{T})$ invariant for every $g\in G$, $s\in \boldsymbol{C}$. Hence the above condition (iii) makes sense.

Proof of Proposition 1.6. Taking account of Proposition 1.1 (6) and the closedness of L, we deduce easily that for $f\in C^{\infty}(\boldsymbol{T})$, $X\in \mathfrak{g}$,

$$U_{\infty}^{k}(X, s)f\in \mathrm{Dom}(L) \quad \text{and} \quad LU_{\infty}^{k}(X, s)f=U_{\infty}^{k}(X, s')Lf.$$

Thus $\left(\frac{1}{4}-s^{2}\right)Lf=LU_{\infty}^{k}(\Omega, s)f=U_{\infty}^{k}(\Omega, s')Lf=\left(\frac{1}{4}-s'^{2}\right)Lf$. Since L is non-zero, the assertion follows. Q.E.D.

1.5. Invariant subspaces. Here we investigate $U^{k}(\cdot, s)$-invariant subspaces. When $k=0$ or n, $g\mapsto U^{k}(g, s)$ defines a representation of G_{1} as was noted in Remark 1, so we shall omit here these well-known cases. Of course the following discussion also holds in the case $k=0$ or n after slight modifications.

Consider $\mathrm{Ran}\left(A^{k}\left(\lambda_{k}+j-\frac{1}{2}\right)\right)^{\perp}$ and $\mathrm{Ran}\left(A^{k}\left(-\lambda_{k}+j-\frac{1}{2}\right)\right)^{\perp}$ for $j\in \boldsymbol{N}$. It is easily seen that they are respectively equal to

$$\mathfrak{H}_{j}^{+}=\textstyle\sum_{p\geq j}^{\oplus}\boldsymbol{C}e_{p}, \quad \mathfrak{H}_{j}^{-}=\sum_{p\leq -j}^{\oplus}\boldsymbol{C}e_{p}.$$

Proposition 1.7.

(1) $\mathfrak{H}_{j}^{+}$ *is invariant under* $U^{k}\left(\cdot, \lambda_{k}+j-\frac{1}{2}\right)$ $(j\in \boldsymbol{N})$.

(2) $\mathfrak{H}_{j}^{-}$ *is invariant under* $U^{k}\left(\cdot, -\lambda_{k}+j-\frac{1}{2}\right)$ $(j\in \boldsymbol{N})$.

Proof. (1) Let $f\in \mathfrak{H}_{j}^{+}$, $h\in \mathrm{Ran}\left(A^{k}\left(\lambda_{k}+j-\frac{1}{2}\right)\right)$. We have $h=A^{k}\left(\lambda_{k}+j-\frac{1}{2}\right)h'$ for some $h'\in \mathfrak{H}$. Then

$$\begin{aligned}\left(U^{k}\left(g, \lambda_{k}+j-\tfrac{1}{2}\right)f, h\right)&=\left(U^{k}\left(g, \lambda_{k}+j-\tfrac{1}{2}\right)f, A^{k}\left(\lambda_{k}+j-\tfrac{1}{2}\right)h'\right)\\&=\left(f, U^{k}\left(g^{-1}, -\lambda_{k}-j+\tfrac{1}{2}\right)A^{k}\left(\lambda_{k}+j-\tfrac{1}{2}\right)h'\right)\\&=\left(f, A^{k}\left(\lambda_{k}+j-\tfrac{1}{2}\right)U^{k}\left(g^{-1}, \lambda_{k}+j-\tfrac{1}{2}\right)h'\right)=0.\end{aligned}$$

Thus $\mathfrak{H}_{j}^{+}$ is invariant under $U^{k}\left(\cdot, \lambda_{k}+j-\frac{1}{2}\right)$.

(2) The proof is completely similar to that of (1). Q.E.D.

We denote by $A_{j}^{k,\pm}(s)$ the restriction of $A^{k}(s)$ to $\mathfrak{H}_{j}^{\pm}$. Note that $A^{k}(s)$ is "diagonal". Define an operator $B_{j}^{k,+}\in \boldsymbol{B}(\mathfrak{H}_{j}^{+})$ by

$$B_{j}^{k,+}=\lim_{s\to 0}\frac{A_{j}^{k,+}\left(\lambda_{k}+j-\frac{1}{2}+s\right)}{\alpha_{j}^{k}\left(\lambda_{k}+j-\frac{1}{2}+s\right)} \quad \text{(in norm)}.$$

Then

$$B_{j}^{k,+}e_{p}=\prod_{0\leq l\leq p-j-1}\frac{l+1}{l+2(j+\lambda_{k})}e_{p} \quad \text{for} \quad p\geq j. \tag{1.8}$$

Here we understand $\prod_{0\leq l\leq -1}=1$. Define a hermitian form on $\mathfrak{H}_j^+$ by

$$(f_1, f_2)_j^{k,+}=(B_j^{k,+}f_1, f_2) \qquad (f_1, f_2\in\mathfrak{H}_j^+).$$

We see easily that $(\cdot,\ \cdot)_j^{k,+}$ is $U^k\left(\cdot,\ \lambda_k+j-\frac{1}{2}\right)$-invariant. It is clear from (1.8) that $(\cdot,\ \cdot)_j^{k,+}$ is positive definite. We denote by $\mathfrak{H}_j^{k,+}$ the completion of the pre-Hilbert space $(\mathfrak{H}_j^+,\ (\cdot,\ \cdot)_j^{k,+})$. Thus we see that $U^k\left(g,\ \lambda_k+j-\frac{1}{2}\right)$ is uniquely extended to a unitary operator on $\mathfrak{H}_j^{k,+}$ which we again denote by $U^k\left(g,\ \lambda_k+j-\frac{1}{2}\right)$. Hence $\left\{U^k\left(\cdot,\ \lambda_k+j-\frac{1}{2}\right),\ \mathfrak{H}_j^{k,+}\right\}$ is a unitary representation of G. Furthermore it is irreducible. We denote by $D_{\lambda_k+j}^+$ this irreducible unitary representation of G.

By the same reasoning as above, we get another irreducible unitary representation $\left\{U^k\left(\cdot,\ -\lambda_k+j-\frac{1}{2}\right),\ \mathfrak{H}_j^{k,-}\right\}$ of G. In this case we use the operator

$$B_j^{k,-}e_p=\prod_{0\leq l\leq|p|-j-1}\frac{l+1}{l+2(j-\lambda_k)}e_p \qquad \text{for } p\leq -j. \tag{1.9}$$

We denote by $D_{\lambda_k-j}^-$ this irreducible unitary representation of G.

Proposition 1.8. *For* $\mathrm{Re}\,s\geq 0$, *there is no* $U^k(\cdot,\ s)$*-invariant subspace other than those stated above.*

Proof can be given by the analogous method for Theorem 2.1 in [15, p. 218].

1.6. Classification of the irreducible unitary representations of *G*. There are other irreducible unitary representations of G. Concerning this point we give two propositions whose proofs are elementary and omitted here.

Proposition 1.9. $(A^k(\sigma)\cdot,\cdot)$ *is a positive definite* $U^k(\cdot,\ \sigma)$*-invariant hermitian form on* $\mathfrak{H}$ *for* $0<\sigma<\frac{1}{2}-|\lambda_k|$.

This proposition assures the existence of supplementary series, which we denote by $E^k(\sigma)$.

Proposition 1.10.

(1) $\mathrm{Ran}\left(A^k\left(-\lambda_k+j-\frac{1}{2}\right)\right)$ *is invariant under* $U^k\left(\cdot,\ \lambda_k-j+\frac{1}{2}\right)$ $(j\in\boldsymbol{N})$.

(2) $\mathrm{Ran}\left(A^k\left(\lambda_k+j-\frac{1}{2}\right)\right)$ *is invariant under* $U^k\left(\cdot,\ -\lambda_k-j+\frac{1}{2}\right)$ $(j\in\boldsymbol{N})$.

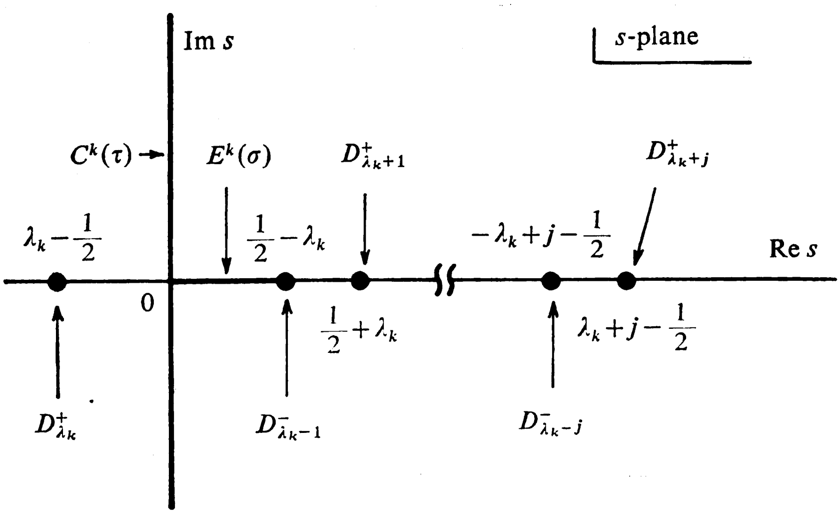
When $1\leq k\leq n-1$, it is readily verified that $\alpha_p^k\left(\frac{1}{2}-\lambda_k\right)>0$ for all $p\geq 0$. Hence $\alpha_p^k\left(\lambda_k-\frac{1}{2}\right)=\alpha_p^k\left(\frac{1}{2}-\lambda_k\right)^{-1}>0$ for $p\geq 0$. Thus we can define for every such k an unbounded positive definite selfadjoint operator A_+^k in $\mathfrak{H}^+=\sum_{p\geq 0}^{\oplus}\boldsymbol{C}e_p$ in such a way that $A_+^k e_p=\alpha_p^k\left(\lambda_k-\frac{1}{2}\right)e_p$ for $p\geq 0$. Note that $\mathrm{Dom}(A_+^k)=\mathrm{Ran}\left(A^k\left(\frac{1}{2}-\lambda_k\right)\right)$. $(A_+^k\cdot,\ \cdot)$ is a positive definite $U^k\left(\cdot,\ \lambda_k-\frac{1}{2}\right)$-invariant hermitian form. The closedness of A_+^k assures the completeness of $(\mathrm{Dom}(A_+^k), (A_+^k\cdot, \cdot))$. Thus we obtain a unitary representation of G, which is seen to be irreducible. We denote it by $D_{\lambda_k}^+$.

When $-n+1\leq k\leq -1$, we have $\alpha_p^k\left(-\frac{1}{2}-\lambda_k\right)>0$ for all $p\leq 0$. By a similar reasoning as above we also get an irreducible unitary representation of G for every such k. We denote it by $D_{\lambda_k}^-$.

It can be shown that except for the trivial one-dimensional representation there is no irreducible unitary representation of G other than those stated above. Here we give the list of all irreducible unitary representations of G.

1. $C^k(\tau)$ $(-n+1\leq k\leq n,\ \tau\in\boldsymbol{R})$ (excluding $(k, \tau)=(n, 0)$).
2. (1) $D_{\lambda_k+j}^+$ $(-n+1\leq k\leq n,\ j\in\boldsymbol{N})$ and $D_{1/2}^+$;
   (2) $D_{\lambda_k-j}^-$ $(-n+1\leq k\leq n,\ j\in\boldsymbol{N})$.
3. (1) $D_{\lambda_k}^+$ $(1\leq k\leq n-1)$;
   (2) $D_{\lambda_k}^-$ $(-n+1\leq k\leq -1)$.
4. $E^k(\sigma)$ $\left(-n+1\leq k\leq n-1,\ 0<\sigma<\frac{1}{2}-|\lambda_k|\right)$.
5. Trivial one-dimensional representation.

For $1\leq k\leq n-1$ the locations of the irreducible unitary representations in s-plane corresponding to λ_k are illustrated in the following figure. For $-n+1\leq k\leq -1$ the reader can easily draw the similar figure. The case $k=0$ or n is well-known.



In the following we do not need explicitly the representations in 3, 4 and 5.

§2. Matrix elements

For the later uses, we prepare the matrix elements $u_{pq}^k(g, s)$ of the representation $U^k(\cdot, s)$ as follows:

$$u_{pq}^k(g, s)=(U^k(g, s)e_q, e_p) \qquad (p, q \in \boldsymbol{Z}, -n+1 \leq k \leq n).$$

Also we prepare the matrix elements of $D^+_{\lambda_k+j}$ and $D^-_{\lambda_k-j}$ as

$$v_{pq}^{k,+}(g, j)=\left(U^k\left(g, \lambda_k+j-\frac{1}{2}\right)\gamma_q^{k,+}(j)e_q, \gamma_p^{k,+}(j)e_p\right)_j^{k,+} \qquad (p, q \geq j, j \in \boldsymbol{N}),$$

$$v_{pq}^{k,-}(g, j)=\left(U^k\left(g, -\lambda_k+j-\frac{1}{2}\right)\gamma_q^{k,-}(j)e_q, \gamma_p^{k,-}(j)e_p\right)_j^{k,-} \qquad (p, q \leq -j, j \in \boldsymbol{N}),$$

where

$$\gamma_p^{k,+}(j)=\prod_{0 \leq l \leq p-j-1}\left[\frac{l+2(j+\lambda_k)}{l+1}\right]^{1/2}, \tag{2.1}$$

$$\gamma_p^{k,-}(j)=\prod_{0 \leq l \leq |p|-j-1}\left[\frac{l+2(j-\lambda_k)}{l+1}\right]^{1/2}. \tag{2.2}$$

By an easy calculation we obtain

$$v_{pq}^{k,+}(g, j)=\omega_{pq}^{k,+}(j)u_{pq}^k\left(g, \lambda_k+j-\frac{1}{2}\right), \tag{2.3}$$

$$v_{pq}^{k,-}(g, j)=\omega_{pq}^{k,-}(j)u_{pq}^k\left(g, -\lambda_k+j-\frac{1}{2}\right), \tag{2.4}$$

where we put

$$\omega_{pq}^{k,\pm}(j)=\gamma_q^{k,\pm}(j)/\gamma_p^{k,\pm}(j).$$

In the following we extend this definition of $\omega_{pq}^{k,\pm}(j)$ by putting them equal to zero for any triplet (p, q, j) not appearing in the definitions of $v_{pq}^{k,\pm}(g, j)$.

Proposition 2.1.
(1) $u_{pq}^k(u_\varphi g u_\psi, s)=\chi_p^k(u_\varphi)u_{pq}^k(g, s)\chi_q^k(u_\psi)$,
where

$$\chi_p^k(u_\varphi)=e^{-i(\lambda_k+p)\varphi}. \tag{2.5}$$

The same relation holds for $v_{pq}^{k,\pm}(g, j)$.
(2) $|u_{pq}^k(g, s)| \leq e^{t|\mathrm{Re}\, s|}$ *for* $g=u_\varphi a_t u_\psi$ $(t \geq 0)$.
(3) $X_0 u_{pq}^k(g, s)=i(p+\lambda_k)u_{pq}^k(g, s)$,
$X'_0 u_{pq}^k(g, s)=-i(q+\lambda_k)u_{pq}^k(g, s)$,
$X_+ u_{pq}^k(g, s)=\left(\lambda_k+p+\frac{1}{2}-s\right)u_{p+1,q}^k(g, s)$,
$X_- u_{pq}^k(g, s)=-\left(\lambda_k+p-\frac{1}{2}+s\right)u_{p-1,q}^k(g, s)$.

Proof. The assertion (1) follows directly from the definition, (2) from Proposition 1.1 (3), and (3) from (1) and Proposition 1.2. Q.E.D.

Proposition 2.2. *Put* $A_{pq}^k(s)=\alpha_p^k(s)/\alpha_q^k(s)$. *Then, as meromorphic functions in* s, *we have the following identity.*

$$u_{pq}^k(g, -s)=A_{pq}^k(s)u_{pq}^k(g, s). \tag{2.6}$$

Proof. For $\operatorname{Re} s \geq 0$ with $\alpha_q^k(s) \neq 0$ we have

$$\begin{aligned} u_{pq}^k(g, -s) &= (U^k(g, -s)e_q, e_p) = \alpha_q^k(s)^{-1}(U^k(g, -s)A^k(s)e_q, e_p) \\ &= \alpha_q^k(s)^{-1}(A^k(s)U^k(g, s)e_q, e_p) = \Lambda_{pq}^k(s)u_{pq}^k(g, s). \end{aligned}$$

On the other hand, $u_{pq}^k(g, -s)$ on the left hand side is an entire function in s. Therefore by the uniqueness of analytic continuation, the equality in the proposition holds for all $s \in \mathbf{C}$. Q.E.D.

Remark 3. $\Lambda_{pp}^k(s) = 1$ for all $s \in \boldsymbol{C}$.

We define the hypergeometric function as follows:

$$(2.7) \quad F(a, b, c; z) = \sum_{j \geq 0} \frac{\Gamma(a+j)}{\Gamma(a)} \frac{\Gamma(b+j)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+j)} \frac{z^j}{j!} \qquad \text{for} \quad |z| < 1,$$

where $\Gamma(z)$ stands for the gamma function. In the sequel we consider $F(a, b, c; \cdot)$ as a meromorphic function on $\boldsymbol{C} \backslash \{1 \leq z \leq \infty\}$ obtained by the analytic continuation of the right hand side of (2.7).

Proposition 2.3.

$$u_{pp}^k(a_t, s) = (1 - \operatorname{th}^2(t/2))^{1/2+s} F\left(s - \lambda_k - p + \frac{1}{2}, s + \lambda_k + p + \frac{1}{2}, 1; \operatorname{th}^2(t/2)\right).$$

Proof. By definition

$$\begin{aligned} u_{pp}^k(a_t, s) &= \int_T \left[\frac{1 - \zeta \operatorname{th}(t/2)}{1 - \bar{\zeta} \operatorname{th}(t/2)}\right]^{\lambda_k + p} (1 - \operatorname{th}^2(t/2))^{1/2+s} |1 - \zeta \operatorname{th}(t/2)|^{-1-2s} d\mu(\zeta) \\ &= (1 - x^2)^{1/2+s} \int_T (1 - \zeta x)^{\lambda_k + p - 1/2 - s} (1 - \bar{\zeta} x)^{-\lambda_k - p - 1/2 - s} d\mu(\zeta), \end{aligned}$$

where we put $x = \operatorname{th}(t/2)$. Expanding $(1 - \zeta x)^{\lambda_k + p - 1/2 - s}$, $(1 - \bar{\zeta} x)^{-\lambda_k - p - 1/2 - s}$ into binomial series, we get

$$(1 - \zeta x)^{\lambda_k + p - 1/2 - s} = \sum_{j \geq 0} \frac{\Gamma\left(s - p - \lambda_k + \frac{1}{2} + j\right)}{j! \Gamma\left(s - p - \lambda_k + \frac{1}{2}\right)} x^j \zeta^j,$$

$$(1 - \bar{\zeta} x)^{-\lambda_k - p - 1/2 - s} = \sum_{j \geq 0} \frac{\Gamma\left(s + p + \lambda_k + \frac{1}{2} + j\right)}{j! \Gamma\left(s + p + \lambda_k + \frac{1}{2}\right)} x^j \zeta^{-j}.$$

Since $|x| < 1$, $|\zeta| = 1$, we can integrate them term by term. Noting that

$$\int_T \zeta^j d\mu(\zeta) = \delta_{j0},$$

we get the right hand side of (2.7) multiplied by $(1 - x^2)^{1/2+s}$ with $a = s - p - \lambda_k + \frac{1}{2}$, $b = s + p + \lambda_k + \frac{1}{2}$, $c = 1$, $z = x^2$. Q.E.D.

We define a function η_p^k by

$$\eta_p^k(g, s) = \chi_p^k(u_\theta)e^{-(s+1/2)t} \qquad \text{for} \quad g = u_\theta a_t n_\xi .$$

Proposition 2.4. $u_{pp}^k(g, s) = \int_K \eta_p^k(u^{-1}gu, s)du.$

Proof. Denote by $\beta_p^k(g, s)$ the right hand side. Then we see that

$$\beta_p^k(u_\varphi g u_\psi, s) = \int_K \eta_p^k(u^{-1}u_\varphi g u_\psi u, s)du$$

$$= \int_K \eta_p^k(u^{-1}u_{\psi+\varphi} gu, s)du = \chi_p^k(u_{\varphi+\psi})\beta_p^k(g, s).$$

Therefore it suffices to prove the equality only for $g = a_t$ by Proposition 2.1 (1). It is readily seen that $u_\pi a_t u_{-\pi} = a_{-t}$, and so $u_{pp}^k(a_{-t}, s) = u_{pp}^k(a_t, s)$. Note that u_π is a generator of the normalizer of A in K. Thus what we must prove is $u_{pp}^k(a_t, s) = \beta_p^k(a_{-t}, s)$. Write $a_{-t}u_\theta = u_{\theta'}a_{t'}n_{\xi'}$. Then we have

$$e^{-i\theta'/2} = (1 - \zeta\,\mathrm{th}\,(t/2))\,|1 - \zeta\,\mathrm{th}\,(t/2)|^{-1}e^{-i\theta/2},$$

$$e^{t'} = j(a_{-t}, \zeta)^{-1} \qquad (\zeta = e^{i\theta}).$$

Hence by (2.5), we get

$$\beta_p^k(a_{-t}, s) = \frac{1}{4n\pi}\int_0^{4n\pi} e^{i(p+\lambda_k)\theta}e^{-i(p+\lambda_k)\theta'}e^{-(s+1/2)t'}d\theta$$

$$= \int_T \left[\frac{1-\zeta\,\mathrm{th}(t/2)}{1-\bar{\zeta}\,\mathrm{th}(t/2)}\right]^{\lambda_k+p} j(a_{-t},\zeta)^{1/2+s}d\mu(\zeta) = u_{pp}^k(a_t, s).$$

Q.E.D.

§3. Inversion formula

In this section we derive the inversion formula which serves to prove the injectiveness of the Fourier transform. Here we do not appeal to the general results of Harish-Chandra [7] but to the direct calculations because the structure of the group is not complicated and so it would be exaggerated to apply the profound theory to the present case, and because the constant factor appearing in the inversion formula is automatically calculated.

We denote by $\mathscr{D}_{pq}^k$ the totality of functions $f \in C_0^\infty(G)$ satisfying

$$(3.1) \qquad f(u_\varphi g u_\psi) = e^{i(\lambda_k+p)\varphi}f(g)e^{i(\lambda_k+q)\psi} = \overline{\chi_p^k(u_\varphi)}f(g)\overline{\chi_q^k(u_\psi)}.$$

We do not consider any topology on $\mathscr{D}_{pq}^k$ in this section.

3.1. Put

$$(3.2) \qquad F_{pq}^k(s) = \int f(g)u_{pq}^k(g, s)dg \qquad \text{for} \quad f \in \mathscr{D}_{pq}^k,$$

(3.3) $$\varphi_f(t)=e^{t/2}\int_{-\infty}^{\infty} f(a_t n_\xi)d\xi \qquad \text{for } f\in\mathscr{D}^k_{pp}.$$

Then for $f\in\mathscr{D}^k_{pp}$, we have

(3.4) $$\begin{aligned} F^k_{pp}(s)&=\int f(g)\int_K \eta^k_p(u^{-1}gu,\ s)du\,dg=\int f(g)\eta^k_p(g,\ s)dg \\ &=\int_K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ua_t n_\xi)\chi^k_p(u)e^{-(s+1/2)t}e^t du\,dt\,d\xi \\ &=\int_{-\infty}^{\infty}\varphi_f(t)e^{-st}\,dt. \end{aligned}$$

In the following we use the method in Takahashi [16].

Lemma 3.1. *Put* $Z^k_p(s)=s\tan\pi(s-\lambda_k)F^k_{pp}(s)$. *Then it is a meromorphic function in* s *and*

$$\lim_{|\mathrm{Im}\, s|\to\infty} Z^k_p(s)=0 \quad \textit{uniformly in each strip } a\le \mathrm{Re}\, s\le b.$$

Proof. Note that φ_f is a C^∞-function with compact support on $\boldsymbol{R}$. Then by (3.4) the assertion is a direct consequence of the Riemann-Lebesgue's lemma for the ordinary Fourier transform. Q.E.D.

Now we integrate $Z^k_p(s)$ along the rectangle Γ_p having vertices $\pm iT$, $p+\lambda_k\pm iT$ $(T>0)$ counterclockwise. In the interior of Γ_p, Z^k_p has poles at

$$s=\lambda_k+j-\frac{1}{2} \quad (1\le j\le p) \qquad \text{if } p>0,$$

$$s=\lambda_k-j+\frac{1}{2} \quad (1\le j\le |p|) \qquad \text{if } p<0.$$

Note that Z^k_0 has no poles in the interior of Γ_0. The residues at these poles are

$$-\pi^{-1}\Big(\lambda_k+j-\frac{1}{2}\Big)F^k_{pp}\Big(\lambda_k+j-\frac{1}{2}\Big) \qquad \text{if } p>0,$$

$$\pi^{-1}\Big(-\lambda_k+j-\frac{1}{2}\Big)F^k_{pp}\Big(-\lambda_k+j-\frac{1}{2}\Big) \qquad \text{if } p<0.$$

From this we have for $p>0$,

$$\begin{aligned} &-2i\textstyle\sum_{1\le j\le p}\Big(\lambda_k+j-\frac{1}{2}\Big)F^k_{pp}\Big(\lambda_k+j-\frac{1}{2}\Big) \\ &=i\int_{-T}^{T} Z^k_p(p+\lambda_k+i\tau)\,d\tau-\int_0^{p+\lambda_k} Z^k_p(\sigma+iT)d\sigma \\ &\quad -i\int_{-T}^{T} Z^k_p(i\tau)d\tau+\int_0^{p+\lambda_k} Z^k_p(\sigma-iT)d\sigma. \end{aligned}$$

Letting $T\to\infty$, we see by Lemma 3.1 that the second and the fourth terms tend to zero. Then we obtain

(3.5)
$$-i\int_{-\infty}^{\infty}(p+\lambda_k+i\tau)F_{pp}^k(p+\lambda_k+i\tau)\operatorname{th}\pi\tau\,d\tau$$
$$=2\int_0^{\infty}F_{pp}^k(i\tau)\tau\operatorname{Re}\operatorname{th}\pi(\tau+i\lambda_k)d\tau$$
$$+2\sum\nolimits_{1\le j\le p}\left(\lambda_k+j-\frac{1}{2}\right)F_{pp}^k\left(\lambda_k+j-\frac{1}{2}\right).$$

In the same way we obtain for $p<0$

(3.6)
$$-i\int_{-\infty}^{\infty}(p+\lambda_k+i\tau)F_{pp}^k(p+\lambda_k+i\tau)\operatorname{th}\pi\tau\,d\tau$$
$$=2\int_0^{\infty}F_{pp}^k(i\tau)\tau\operatorname{Re}\operatorname{th}\pi(\tau+i\lambda_k)d\tau$$
$$+2\sum\nolimits_{1\le j\le |p|}\left(-\lambda_k+j-\frac{1}{2}\right)F_{pp}^k\left(-\lambda_k+j-\frac{1}{2}\right).$$

In case $p=0$ and $0\le k\le n$, we obtain (3.5) without the last sum-part. In case $p=0$ and $-n+1\le k\le -1$, we have (3.6) without the last sum-part.

Lemma 3.2.
$$4\pi f(e)=-i\int_{-\infty}^{\infty}(p+\lambda_k+i\tau)F_{pp}^k(p+\lambda_k+i\tau)\operatorname{th}\pi\tau\,d\tau.$$

Proof. Let I be the right hand side member. Then by (3.4),
$$I=-i\int_{-\infty}^{\infty}(p+\lambda_k+i\tau)\operatorname{th}\pi\tau\int_{-\infty}^{\infty}\varphi_f(t)e^{-(p+\lambda_k+i\tau)t}dt\,d\tau$$
$$=-i\int_{-\infty}^{\infty}\operatorname{th}\pi\tau\int_{-\infty}^{\infty}\varphi_f'(t)e^{-(p+\lambda_k)t}e^{-i\tau t}dt\,d\tau$$
$$=-i\int_{-\infty}^{\infty}\hat{\psi}_f(\tau)\operatorname{th}\pi\tau\,d\tau,$$
where $\psi_f(t)=\varphi_f'(t)e^{-(p+\lambda_k)t}$ and $\hat{\psi}_f$ is the ordinary Fourier transform of ψ_f (cf. (0.1)).

We need the following lemma (cf. [15, p. 341]).

Lemma 3.3. *Suppose that $\varphi\in C_0^\infty(\boldsymbol{R})$ satisfies $\varphi(0)=0$. Then*
$$\int_{-\infty}^{\infty}\varphi(t)\frac{dt}{\operatorname{sh}(t/2)}=i\int_{-\infty}^{\infty}\hat{\varphi}(\tau)\operatorname{th}\pi\tau\,d\tau.$$

For $f\in\mathscr{D}_{pp}^k$, φ_f is an even function, and so $\varphi_f'(0)=0$. Hence applying Lemma 3.3 to ψ_f, we get
$$I=-\int_{-\infty}^{\infty}\frac{\varphi_f'(t)}{\operatorname{sh}(t/2)}e^{-(p+\lambda_k)t}dt=-2\int_0^{\infty}\varphi_f'(t)\frac{\operatorname{ch}(p+\lambda_k)t}{\operatorname{sh}(t/2)}dt.$$

Since $f\in\mathscr{D}_{pp}^k$, we have $f(a_t)=f(a_{-t})$. Thus we can put $f(a_t)=f[\operatorname{ch}t]$. Write $a_tn_\xi=u_\varphi a_{t'}u_\psi$ $(t'\ge 0)$, then we get

$$\operatorname{ch} t' = \operatorname{ch} t + \frac{1}{2} e^t \xi^2, \tag{3.7}$$

$$e^{i(\varphi+\psi)/2} = (\operatorname{ch}(t/2) - (i/2)\xi e^{t/2})\left(\operatorname{ch}^2(t/2) + \frac{1}{4} e^t \xi^2\right)^{-1/2}. \tag{3.8}$$

Hence

$$\varphi_f(t) = e^{t/2} \int_{-\infty}^{\infty} f\left[\operatorname{ch} t + \frac{1}{2} e^t \xi^2\right] \left[\frac{\operatorname{ch}(t/2) - (i/2)\xi e^{t/2}}{\left(\operatorname{ch}^2(t/2) + \frac{1}{4} e^t \xi^2\right)^{1/2}}\right]^{2(p+\lambda_k)} d\xi.$$

Putting $x = \operatorname{sh}(t/2)$, $y = \frac{1}{2}\xi e^{t/2}$, we get

$$\varphi_f(t) = 4\int_0^\infty f[2(x^2+y^2)+1]\, T_{2(p+\lambda_k)}\left(\left[\frac{1+x^2}{1+x^2+y^2}\right]^{1/2}\right) dy,$$

where $T_\alpha(z)$ denotes the Tschebyscheff's function of the first kind defined by $T_\alpha(z) = F(-\alpha, \alpha, 1/2; (1-z)/2)$ (cf. [10]). The function T_α satisfies that $T_\alpha(\cos\theta) = \cos\alpha\theta$. Express (x, y) as $(\sqrt{r}\cos\theta, \sqrt{r}\sin\theta)$, then we get for $\alpha = 2(p+\lambda_k)$,

$$\begin{aligned}
\frac{d}{dt}\varphi_f(t) &= 4\int_0^\infty 2f'[1+2r]\, T_\alpha\left(\left[\frac{1+r\cos^2\theta}{1+r}\right]^{1/2}\right) 2x\frac{dx}{dt} dy \\
&+ 4\int_0^\infty f[1+2r]\left\{\frac{\partial}{\partial r} T_\alpha\left(\left[\frac{1+r\cos^2\theta}{1+r}\right]^{1/2}\right)\right\} 2x\frac{dx}{dt} dy \\
&+ 4\int_0^\infty f[1+2r]\, T'_\alpha\left(\left[\frac{1+r\cos^2\theta}{1+r}\right]^{1/2}\right) \frac{x\sin^2\theta}{(1+r)^{1/2}(1+r\cos^2\theta)^{1/2}} \frac{dx}{dt} dy.
\end{aligned}$$

On the other hand, we have $\operatorname{ch}(p+\lambda_k)t = T_\alpha(\operatorname{ch}(t/2)) = T_\alpha([1+r\cos^2\theta]^{1/2})$. Hence

$$\begin{aligned}
I &= -2\int_0^\infty \varphi'_f(t) x^{-1} T_\alpha([1+r\cos^2\theta]^{1/2}) dt \\
&= -8\int_0^\infty\int_0^{\pi/2} 2f'[1+2r]\, T_\alpha\left(\left[\frac{1+r\cos^2\theta}{1+r}\right]^{1/2}\right) T_\alpha([1+r\cos^2\theta]^{1/2})\, dr\, d\theta \\
&\quad - 8\int_0^\infty\int_0^{\pi/2} f[1+2r]\left\{\frac{\partial}{\partial r} T_\alpha\left(\left[\frac{1+r\cos^2\theta}{1+r}\right]^{1/2}\right)\right\} T_\alpha([1+r\cos^2\theta]^{1/2})\, dr\, d\theta \\
&\quad - 4\int_0^\infty\int_0^{\pi/2} f[1+2r]\, T'_\alpha\left(\left[\frac{1+r\cos^2\theta}{1+r}\right]^{1/2}\right) T_\alpha([1+r\cos^2\theta]^{1/2}) \\
&\qquad\qquad \times \frac{\sin^2\theta\, dr\, d\theta}{(1+r)^{1/2}(1+r\cos^2\theta)^{1/2}}.
\end{aligned}$$

Denote by I_1, I_2, I_3 the first, the second and the third term respectively in the last expression. Integrating by parts with respect to r, we obtain $I_1 = 4\pi f(e) - I_2 + J_3$, where

$$J_3 = 4\int_0^\infty\int_0^{\pi/2} f[1+2r]\, T_\alpha\left(\left[\frac{1+r\cos^2\theta}{1+r}\right]^{1/2}\right) T'_\alpha([1+r\cos^2\theta]^{1/2}) \frac{\cos^2\theta\, dr\, d\theta}{(1+r\cos^2\theta)^{1/2}}.$$

Let us prove $I_3+J_3=0$. It suffices to show that for all $r>0$,

$$(*)\qquad \int_0^{\pi/2} T'_\alpha\left(\left[\frac{1+r\cos^2\theta}{1+r}\right]^{1/2}\right)T_\alpha([1+r\cos^2\theta]^{1/2})\frac{\sin^2\theta\, d\theta}{(1+r)^{1/2}(1+r\cos^2\theta)^{1/2}}$$

$$=\int_0^{\pi/2} T_\alpha\left(\left[\frac{1+r\cos^2\theta}{1+r}\right]^{1/2}\right)T'_\alpha([1+r\cos^2\theta]^{1/2})\frac{\cos^2\theta\, d\theta}{(1+r\cos^2\theta)^{1/2}}\,.$$

It follows from the definition of T_α that

$$T_\alpha(z)=zF((1+\alpha)/2,\ (1-\alpha)/2,\ 1/2;\ 1-z^2),$$

$$T'_\alpha(z)=\alpha^2 F((1+\alpha)/2,\ (1-\alpha)/2,\ 3/2;\ 1-z^2).$$

Notice that both sides of $(*)$ are real analytic in $r>0$, so we have only to show $(*)$ for $0<r<1$. After a simple calculation our problem is reduced to show the equality:

$$\int_0^{\pi/2} F\left(\frac{1+\alpha}{2},\frac{1-\alpha}{2},\frac{3}{2};\ -s\sin^2\theta\right)F\left(\frac{1+\alpha}{2},\frac{1-\alpha}{2},\frac{1}{2};\ -r\cos^2\theta\right)\sin^2\theta\, d\theta$$

$$=\int_0^{\pi/2} F\left(\frac{1+\alpha}{2},\frac{1-\alpha}{2},\frac{1}{2};\ -s\sin^2\theta\right)F\left(\frac{1+\alpha}{2},\frac{1-\alpha}{2},\frac{3}{2};\ -r\cos^2\theta\right)\cos^2\theta\, d\theta,$$

where $s=-r/(1+r)$. Since $0<r<1$, $-1<s<0$, we can expand the hypergeometric functions in the above integrals into the hypergeometric series, which converge absolutely and uniformly in θ. Hence we can integrate them term by term, then simple calculations show us that the both sides are expressed by the same sum of a certain kind of infinite series. This completes the proof of Lemma 3.2. Q. E. D.

For an entire function F we formally define $I_p^k(F)$ by

$$I_p^k(F)=\int_0^\infty F(i\tau)\tau\,\mathrm{Re}\,\mathrm{th}\,\pi(\tau+i\lambda_k)d\tau$$

$$+\begin{cases}\sum_{1\le j\le p}\left(\lambda_k+j-\frac{1}{2}\right)F\left(\lambda_k+j-\frac{1}{2}\right) & \text{for}\quad p>0\\ \sum_{1\le j\le |p|}\left(-\lambda_k+j-\frac{1}{2}\right)F\left(-\lambda_k+j-\frac{1}{2}\right) & \text{for}\quad p<0,\\ 0 & \text{for}\quad p=0.\end{cases}$$

Combining Lemma 3.2 with the foregoing discussion, we obtain the following theorem.

Theorem 3.4. *Let k be an integer such that $-n+1\le k\le n, f\in\mathscr{D}_{pp}^k$, and put $\lambda_k=k/2n$. Then the operation I_p^k is well-defined for F_{pp}^k in* (3.2) *and*

$$2\pi f(e)=I_p^k(F_{pp}^k)\,.$$

Remark 4. For $k=0$ or n, the above formula has been given by Takahashi [16].

3.2. For $f\in C_0^\infty(G)$ we define

(3.9) $$P_{pq}^k f(g)=\int_K\int_K \chi_p^k(u)f(ugv)\chi_q^k(v)\,du\,dv.$$

Then $P_{pq}^k f\in\mathscr{D}_{pq}^k$. Put

(3.10) $$F_{pq}^k(s)=\int f(g)u_{pq}^k(g,s)dg,$$

then we have

$$F_{pq}^k(s)=\int P_{pq}^k f(g)u_{pq}^k(g,s)dg.$$

Theorem 3.5. *Let $f\in C_0^\infty(G)$. Then the operation I_p^k is well-defined for F_{pp}^k in* (3.10) *for any possible k, p and*

$$2\pi f(e)=\sum_{k,p} I_p^k(F_{pp}^k).$$

Proof. Applying Theorem 3.4 to $P_{pp}^k f\in\mathscr{D}_{pp}^k$, we first see that $I_p^k(F_{pp}^k)$ is well-defined and $2\pi P_{pp}^k f(e)=I_p^k(F_{pp}^k)$. On the other hand, we deduce easily

$$P_{pp}^k f(e)=\int_K f(u)\chi_p^k(u)du.$$

Noting that $\{\chi_p^k;\ p\in\boldsymbol{Z},\ -n+1\le k\le n\}$ forms a complete orthonormal system in $L^2(K)$, and that the function $u\mapsto f(u)$ is smooth, we get for each $u\in K$,

$$f(u)=\sum_{k,p} P_{pp}^k f(e)\overline{\chi_p^k(u)}.$$

In particular, putting $u=e$, we get the equality in the theorem. Q. E. D.

By a familiar argument we obtain the following inversion formula for $\mathscr{D}_{pq}^k$.

Theorem 3.6. *For any $f\in\mathscr{D}_{pq}^k$,*

(3.11) $$2\pi f(g)=\int_0^\infty F_{pq}^k(i\tau)\overline{u_{pq}^k(g,i\tau)}\,\tau\,\mathrm{Re\,th}\,\pi(\tau+i\lambda_k)d\tau$$

$$+\begin{cases}\displaystyle\sum_{1\le j\le\min(p,q)}\Big(\lambda_k+j-\frac{1}{2}\Big)\omega_{pq}^{k,+}(j)F_{pq}^k\Big(\lambda_k+j-\frac{1}{2}\Big)\overline{v_{pq}^{k,+}(g,j)} & \text{for } p,q>0,\\ \displaystyle\sum_{1\le j\le\min(-p,-q)}\Big(-\lambda_k+j-\frac{1}{2}\Big)\omega_{pq}^{k,-}(j)F_{pq}^k\Big(-\lambda_k+j-\frac{1}{2}\Big)\overline{v_{pq}^{k,-}(g,j)} & \text{for } p,q<0,\\ 0 & \text{otherwise.}\end{cases}$$

§4. Analogue of the Paley-Wiener theorem for $\mathscr{D}_{00}^k$

We defined the function space $\mathscr{D}_{pq}^k$ by (3.1). Denote by $\mathscr{D}_{pq,T}^k$ the subspace of $\mathscr{D}_{pq}^k$ consisting of functions f such that

$$f(ua_tv)=0 \qquad \text{for} \quad t\geq T,\ u,\ v\in K.$$

We denote by $D^k_{pq,T}$ when we topologize $\mathscr{D}^k_{pq,T}$ by means of seminorms

$$|f|_r=\sup_{g\in G}|\Delta^r f(g)| \qquad (r=0,\ 1,\dots), \tag{4.1}$$

where Δ is the Casimir operator on G normalized in such a way that

$$\Delta u^k_{pq}(g,\ s)=\Big(\frac{1}{4}-s^2\Big)u^k_{pq}(g,\ s) \qquad \text{(cf. Proposition 1.2 (3))}. \tag{4.2}$$

Let Ω be the Casimir element in $U(\mathfrak{g}^C)$, then the normalization above is equivalent to identifying Δ with $-\Omega$. It is not clear at this stage whether or not $D^k_{pq,T}$ is complete. But Theorems 4.1 and 5.1 answer this in the affirmative. We introduce another topology in $\mathscr{D}^k_{pq,T}$ by means of seminorms

$$|f|_X=\sup_{g\in G}|Xf(g)| \qquad (X\in U(\mathfrak{g}^C)). \tag{4.3}$$

This topological vector space, denoted again by $\mathscr{D}^k_{pq,T}$, is a Fréchet space.

4.1. Let $\mathscr{H}_{00,T}$ be the totality of all functions F on $\boldsymbol{C}$ which satisfy the following conditions (i)$\sim$(iii).

(i) F is an entire function.

(ii) For every non-negative integer r, there exists a constant C_r depending on F such that

$$|F(s)|\leq C_r(1+|s|)^{-r}e^{T|\mathrm{Re}\, s|}. \tag{4.4}$$

(iii) $F(s)=F(-s)$.

We topologize $\mathscr{H}_{00,T}$ by means of seminorms

$$|F|_{r,M}=\sup_{|\mathrm{Re}\, s|\leq M}(1+|s|)^r|F(s)| \qquad (r,\ M=0,\ 1,\dots). \tag{4.5}$$

The classical Paley-Wiener theorem mentioned in Introduction assures that $\mathscr{H}_{00,T}$ is a Fréchet space.

Theorem 4.1. *The linear mapping*

$$\mathscr{T}: f\longmapsto \int f(g)u^k_{00}(g,\ \cdot)dg$$

gives a topological isomorphism between $D^k_{00,T}$ *and* $\mathscr{H}_{00,T}$.

Corollary 4.2. $D^k_{00,T}$ *is a Fréchet space, and it coincides with the Fréchet space* $\mathscr{D}^k_{00,T}$.

Proof of Corollary 4.2. The second assertion is a direct consequence of the open mapping theorem. Q. E. D.

4.2. Proof of Theorem 4.1.

Step 1. Let $f\in D^k_{00,T}$. Here we show that $F=\mathscr{T}f\in\mathscr{H}_{00,T}$. We see easily that F is an entire function. By Proposition 2.2 and Remark 3, we have $F(s)=$

$F(-s)$.

Now let us prove (4.4). Since Δ is bi-invariant, we have $\Delta^r f \in D^k_{00,T}$ for each non-negative integer r. Noting (4.2), we obtain

$$\begin{aligned}\mathscr{T}(\Delta^r f)(s) &= 2\pi\int_0^T \Delta^r f(a_t) u^k_{00}(a_t, s)\,\mathrm{sh}\,t\,dt \\ &= 2\pi\int_0^T f(a_t)\Delta^r u^k_{00}(a_t, s)\,\mathrm{sh}\,t\,dt = \left(\frac{1}{4}-s^2\right)^r F(s).\end{aligned}$$

Therefore we get by Proposition 2.1 (2)

$$\left|\left(\frac{1}{4}-s^2\right)^r F(s)\right| \leq \mathrm{const.}\,|f|_r e^{T|\mathrm{Re}\,s|}. \tag{4.6}$$

For $|s|\geq 3/4$, we have

$$\begin{aligned}|F(s)| &\leq \mathrm{const.}\left|\frac{1}{4}-s^2\right|^{-r}|f|_r e^{T|\mathrm{Re}\,s|} \\ &\leq \mathrm{const}_r(1+|s|)^{-r}|f|_r e^{T|\mathrm{Re}\,s|},\end{aligned} \tag{4.7}$$

because $|s|^2-\frac{1}{4}\geq(5/28)(1+|s|)$. Here const_r stands for constant depending only on r. For $|s|\leq 3/4$, we get by (4.6)

$$\left|\left[1+\left(\frac{1}{4}-s^2\right)^r\right]F(s)\right| \leq \mathrm{const.}\,(|f|_0+|f|_r)e^{T|\mathrm{Re}\,s|}.$$

Since $\left|1+\left(\frac{1}{4}-s^2\right)^r\right| \geq 1-\left(\frac{1}{4}+|s|^2\right)^r \geq 3/16 \quad (r\geq 1)$, we get

$$|F(s)| \leq \mathrm{const.}\,(|f|_0+|f|_r)e^{T|\mathrm{Re}\,s|}.$$

It is obvious from this inequality that

$$|F(s)| \leq \mathrm{const}_r(1+|s|)^{-r}(|f|_0+|f|_r)e^{T|\mathrm{Re}\,s|} \quad \text{for} \quad |s|\leq 3/4. \tag{4.8}$$

By (4.7) and (4.8), we see that F satisfies (4.4) and the mapping $\mathscr{T}$ is continuous.

Step 2. Theorem 3.6 assures that $\mathscr{T}$ is injective.

Step 3. Let $F \in \mathscr{H}_{00,T}$ be given. We define a function f on G by

$$f(g) = \frac{1}{2\pi}\int_0^\infty F(i\tau)\overline{u^k_{00}(g, i\tau)}\tau\,\mathrm{Re}\,\mathrm{th}\,\pi(\tau+i\lambda_k)\,d\tau. \tag{4.9}$$

This f is well-defined, because $F \in \mathscr{H}_{00,T}$, $|u^k_{00}(g, i\tau)|\leq 1$ and

$$|\tau\,\mathrm{Re}\,\mathrm{th}\,\pi(\tau+i\lambda_k)| \leq \mathrm{const.}\,(1+|\tau|).$$

In order to prove that f is a C^∞-function, it is sufficient to show that the differential operators X_0, X_1 and X_2 are applicable indefinitely many times. This can be done by virtue of Proposition 2.1 (3). Thus we have $f \in C^\infty(G)$. Moreover it is clear from

(4.9) and Proposition 2.1 (1) that

$$f(ugv)=\overline{\chi_0^k(u)}f(g)\overline{\chi_0^k(v)}.$$

Step 4. In this step we show that $f(a_t)=0$ for $t\geq T$.

Noting that

$$\overline{u_{00}^k(a_t,\, i\tau)}=\overline{(U^k(a_t,\, i\tau)e_0,\, e_0)}=(U^k(a_{-t},\, i\tau)e_0,\, e_0)$$
$$=u_{00}^k(a_{-t},\, i\tau)=u_{00}^k(a_t,\, i\tau) \qquad (\text{by } a_{-t}=u_\pi a_t u_{-\pi}),$$

we have

$$f(a_t)=\frac{1}{2\pi}\int_0^\infty F(i\tau)u_{00}^k(a_t,\, i\tau)\tau\,\mathrm{Re\,th}\,\pi(\tau+i\lambda_k)d\tau \tag{4.10}$$
$$=\frac{i}{4\pi}\int_{-i\infty}^{i\infty}F(s)u_{00}^k(a_t,\, s)s\tan\pi(s-\lambda_k)ds.$$

Putting $y=\mathrm{sh}^2(t/2)$ in Proposition 2.3, we get

$$u_{00}^k(a_t,\, s)=(1+y)^{-(1/2+s)}F\Big(s-\lambda_k+\frac{1}{2},\, s+\lambda_k+\frac{1}{2},\, 1;\, y/(1+y)\Big)$$
$$=(1+y)^{-\lambda_k}F\Big(s-\lambda_k+\frac{1}{2},\, -s-\lambda_k+\frac{1}{2},\, 1;\, -y\Big)$$
$$\text{(by Kummer's formula)}$$
$$=\varphi_1(y,\, s)+\varphi_2(y,\, s) \qquad \text{(by Gauss' formula)},$$

where

$$\varphi_1(y,\, s)=\Gamma(-2s)\Gamma\Big(-s-\lambda_k+\frac{1}{2}\Big)^{-1}\Gamma\Big(-s+\lambda_k+\frac{1}{2}\Big)^{-1}\times \tag{4.11}$$
$$\times(1+y)^{-\lambda_k}y^{-(s-\lambda_k+1/2)}F\Big(s-\lambda_k+\frac{1}{2},\, s-\lambda_k+\frac{1}{2},\, 1+2s;\, -1/y\Big),$$

and $\varphi_2(y,\, s)=\varphi_1(y,\, -s)$.

In case "$k\neq n$, $\mathrm{Re}\, s\geq 0$" or "$k=n$, $\mathrm{Re}\, s\geq\delta$" ($\delta$ is a positive number), we can apply the Euler's integral expression to the hypergeometric function in (4.11). Hence in these cases we get for $y>1$,

$$\varphi_1(y,\, s)=$$
$$=\frac{\Gamma(-2s)}{\Gamma\Big(-s-\lambda_k+\frac{1}{2}\Big)}\frac{\Gamma(1+2s)}{\Gamma\Big(s-\lambda_k+\frac{1}{2}\Big)}\frac{(1+y)^{-\lambda_k}}{\Gamma\Big(-s+\lambda_k+\frac{1}{2}\Big)}\frac{y^{-(s-\lambda_k+1/2)}}{\Gamma\Big(s+\lambda_k+\frac{1}{2}\Big)}\psi(y,\, s),$$

where

$$\psi(y,\, s)=\int_0^1 x^{s-\lambda_k-1/2}(1-x)^{s+\lambda_k-1/2}\Big(1+\frac{x}{y}\Big)^{-s+\lambda_k-1/2}dx. \tag{4.12}$$

(It should be noticed here that the integral in (4.12) does not converge absolutely for

$\mathrm{Re}\, s=0$ when $k=n$. This is the reason why we make s apart from the imaginary axis when $k=n$.) Using the well-known formula

$$\Gamma(z)\Gamma(1-z)=\pi/\sin \pi z,$$

we have the following expression of $\varphi_1(y, s)$:

(4.13) $$\varphi_1(y, s)=$$
$$=-\frac{\cos \pi(s+\lambda_k) \cos \pi(s-\lambda_k)}{\pi \sin 2\pi s}(1+y)^{-\lambda_k} y^{-(s-\lambda_k+1/2)} \psi(y, s).$$

On the other hand, since $\varphi_2(y, s)=\varphi_1(y, -s)$, (4.10) is rewritten as

(4.14) $$f(a_t)=\frac{i}{4\pi}\int_{-i\infty}^{i\infty} F(s)\varphi_1(y, s)s[\tan \pi(s-\lambda_k)+\tan \pi(s+\lambda_k)]ds.$$

Then using (4.13), we obtain

(4.15) $$f(a_t)=-\frac{i}{4\pi^2}(1+y)^{-\lambda_k}\int_{-i\infty}^{i\infty} F(s) y^{-(s-\lambda_k+1/2)}\psi(y, s)s\, ds.$$

Now we estimate $y^{-(s-\lambda_k+1/2)}\psi(y, s)$ in (4.15). Since

$$y^{-(s-\lambda_k+1/2)}\psi(y, s)=\int_0^1\left[\frac{x(1-x)}{y+x}\right]^{s-1/2} x^{-\lambda_k}(1-x)^{\lambda_k}(y+x)^{\lambda_k-1}dx,$$

we have

$$|y^{-(s-\lambda_k+1/2)}\psi(y, s)|\leq\int_0^1\left[\frac{x(1-x)}{y+x}\right]^{\sigma-1/2} x^{-\lambda_k}(1-x)^{\lambda_k}dx,$$

where $\sigma=\mathrm{Re}\, s$. (Note that we assume $y>1$.) By a straightforward calculation we obtain

$$0\leq\frac{x(1-x)}{y+x}\leq e^{-t} \quad \text{for} \quad 0\leq x\leq 1 \qquad (y=\mathrm{sh}^2(t/2)).$$

Thus we have for $\sigma\geq\frac{1}{2}$,

(4.16) $$|y^{-(s-\lambda_k+1/2)}\psi(y, s)|\leq \text{const.}\, e^{-\sigma t}e^{(1/2)t}.$$

On the other hand, we have for "$k\neq n$, $0\leq\sigma\leq\frac{1}{2}$" or "$k=n$, $\delta\leq\sigma\leq\frac{1}{2}$"

$$|\psi(y, s)|\leq B\left(\sigma-\lambda_k+\frac{1}{2}, \sigma+\lambda_k+\frac{1}{2}\right)\leq \text{const.},$$

where $B(\cdot, \cdot)$ stands for the beta function. It is clear that

$$|y^{-(s-\lambda_k+1/2)}|\leq \text{const.} \qquad \text{for} \quad 0\leq\sigma\leq\frac{1}{2}.$$

Since $F\in\mathscr{H}_{00,T}$, we can shift the path of integration in (4.15) in case $k\neq n$. Hence for $\alpha>0$,

$$f(a_t) = -\frac{i}{4\pi^2}(1+y)^{-\lambda_k}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)y^{-(s-\lambda_k+1/2)}\psi(y, s)s\, ds.$$

For $\alpha \geq \frac{1}{2}$, we get by (4.16)

$$|f(a_t)| \leq \text{const.}\ (1+\text{sh}^2(t/2))^{-\lambda_k} e^{-\alpha t} e^{(1/2)t} \int_{\alpha-i\infty}^{\alpha+i\infty} |sF(s)|\,|ds|$$

$$\leq \text{const.}\,(1+\text{sh}^2(t/2))^{-\lambda_k} e^{(T-t)\alpha} e^{(1/2)t},$$

where const. does not depend on α. Making $\alpha \to \infty$, we see that $f(a_t)=0$ for $t>T$. By continuity we finally obtain the desired result so long as $k \neq n$.

It remains in this step to prove that in case $k=n$ we can shift the path of integration in (4.14) from the line $\text{Re}\, s=0$ to the line $\text{Re}\, s=\delta$. For this purpose we must evaluate the function $\varphi_1(y, s)$. To do so we need another integral expression for the hypergeometric function. By the formula in [4, p. 114, 2.12 (3)], we have for $0 \leq \text{Re}\, s \leq \delta < 1$ the following expression:

$$\text{(4.17)} \qquad sF(s, s, 1+2s; -1/y) = -\frac{ie^{-\pi i s} s\Gamma(1+2s)}{2\sin \pi s \Gamma(s)\Gamma(1+s)}\tilde{\psi}(y, s),$$

where

$$\text{(4.18)} \qquad \tilde{\psi}(y, s) = \int_C x^{s-1}(1-x)^s\left(1+\frac{x}{y}\right)^{-s} dx.$$

The path C of integration can be taken as follows. It starts from 1 and goes to ε $(0<\varepsilon<1)$ along the real axis, rounds 0 counterclockwise along the circle with radius ε, and returns to 1 along the real axis. We take the branch $\arg x=0$ at the starting point. Concerning other factors in (4.18) we take the principal branch. Here we assume $y>1$ for simplicity. We have the following lemma.

Lemma 4.3.

$$|\tilde{\psi}(y, s)| \leq 2|e^{\pi i s}|\,\{|\sin \pi s| \log 2(1+|s|) + 4|s^{-1}\sin \pi s|\}.$$

Proof. We decompose C into three parts C_1, C_2, C_3 at the point ε according to the order explained above. Denote by I_j the integration along the path C_j of the integrand in (4.18). We put for each fixed s, $\varepsilon = 2^{-1}(1+|s|)^{-1}$. By simple calculations we get

$$|I_1+I_3| \leq 2|e^{\pi i s}|\,|\sin \pi s| \int_\varepsilon^1 x^{\sigma-1} dx \qquad (\sigma = \text{Re}\, s)$$

$$\leq 2|e^{\pi i s}|\,|\sin \pi s| \log 2(1+|s|).$$

Concerning I_2, we first expand the integrand into power series on C_2:

$$x^{s-1}(1-x)^s(1+x/y)^{-s} =$$

$$= \sum_{m,n\geq 0} \varepsilon^{m+n+s-1} e^{i(m+n+s-1)\theta}(-1)^n \binom{s}{n}\binom{-s}{m} y^{-m} \qquad (x = \varepsilon e^{i\theta}).$$

We integrate it term by term and evaluate each term by using

$$y>1,\ 0\leq \operatorname{Re} s=\sigma\leq\delta,\ \left|\binom{s}{n}\right|\leq(1+|s|)^n,$$

then we get

$$|I_2|\leq\left|\frac{e^{2\pi is}-1}{s}\right|\varepsilon^\sigma\left[\sum\nolimits_{n\geq 0}\varepsilon^n(1+|s|)^n\right]^2\leq 8|e^{\pi is}|\,|s^{-1}\sin\pi s|.$$

This completes the proof of Lemma 4.3. Q.E.D.

The integrand in (4.14) is estimated as follows. First by Lemma 4.3 and (4.17) we get

$$|sF(s,\ s,\ 1+2s;\ -1/y)|\leq\{|s|\log 2(1+|s|)+4\}\left|\frac{\Gamma(1+2s)}{\Gamma(s)\Gamma(1+s)}\right|,$$

and therefore from (4.11) the final estimate

$$|s\varphi_1(y,\ s)\cot\pi s|\leq(2\pi)^{-1}\{|s|\log 2(1+|s|)+4\}(1+y)^{-1/2}y^{-\sigma}.$$

Using this estimate and taking account of (4.4), we can shift the path of integration in (4.14) from the line $\operatorname{Re} s=0$ to the line $\operatorname{Re} s=\delta$ as desired (in case $k=n$, $\lambda_k=\frac{1}{2}$). Once the path of integration is shifted to the line $\operatorname{Re} s=\delta>0$, the previous discussion applies to the case $k=n$ and we get the desired result.

Step 5. It remains to show the continuity of $\mathscr{T}^{-1}$. Let $F\in\mathscr{H}_{00,T}$ and put $f=\mathscr{T}^{-1}F$. Then by (4.9) we get

$$\Delta^r f(g)=\frac{1}{2\pi}\int_0^\infty\left(\frac{1}{4}+\tau^2\right)^r F(i\tau)\overline{u_{00}^k(g,\ i\tau)}\tau\operatorname{Re}\operatorname{th}\pi(\tau+i\lambda_k)d\tau.$$

Hence we have $|f|_r\leq\mathrm{const}_r|F|_{2r+3,0}$. This proves the desired continuity.

Step 1 to Step 5 complete the proof of Theorem 4.1. Q.E.D.

§5. Analogue of the Paley-Wiener theorem for $\mathscr{D}_{pq}^k$

5.1. First of all we define

$$N_{pq,1}^k=\left\{\lambda_k+j-\frac{1}{2};\ j\in\boldsymbol{N}\ \text{ such that }\ p<j\leq q\right\}\qquad\text{for}\quad k\neq n,$$

$$N_{pq,1}^n=\{j;\ j\in\boldsymbol{N}\cup\{0\}\ \text{ such that }\ p<j\leq q\},$$

$$N_{pq,2}^k=\left\{-\lambda_k+j-\frac{1}{2};\ j\in\boldsymbol{N}\ \text{ such that }\ q\leq -j<p\right\}\qquad\text{for}\quad k\neq n,$$

$$N_{pq,2}^n=\{j;\ j\in\boldsymbol{N}\cup\{0\}\ \text{ such that }\ q\leq -j-1<p\}.$$

By Proposition 1.7 we have

$$u_{pq}^k(\cdot,\ s)=0\qquad\text{for all}\quad s\in\bigcup\nolimits_{j=1,2}N_{pq,j}^k. \tag{5.1}$$

Remark 5. $N^k_{pp,j}=\phi$ for $j=1, 2$.

Let $\mathscr{H}^k_{pq,T}$ be the totality of all functions F on $\boldsymbol{C}$ which satisfy the following conditions (i)$\sim$(iv).

(i) F is an entire function.

(ii) For every non-negative integer r, there exists a constant C_r depending on F such that

$$|F(s)|\leq C_r(1+|s|)^{-r}e^{T|\mathrm{Re}\, s|}. \tag{5.2}$$

(iii) $F(-s)=\Lambda^k_{pq}(s)F(s)$ (cf. Proposition 2.2).

(iv) $F(s)=0$ for all $s\in\bigcup_{j=1,2}N^k_{pq,j}$.

Remark 6. By Remarks 3 and 5 we see that $\mathscr{H}^k_{pp,T}=\mathscr{H}_{00,T}$ for any $p\in\boldsymbol{Z}$.

We topologize $\mathscr{H}^k_{pq,T}$ by means of seminorms in (4.5), then it becomes a Fréchet space as is easily seen from the classical Paley-Wiener theorem.

Theorem 5.1. *The linear mapping*

$$\mathscr{T}: f\longmapsto \int f(g)u^k_{pq}(g, \cdot)dg$$

gives a topological isomorphism between $D^k_{pq,T}$ *and* $\mathscr{H}^k_{pq,T}$.

Corollary 5.2. $D^k_{pq,T}$ *is a Fréchet space, and it coincides with the Fréchet space* $\mathscr{D}^k_{pq,T}$.

5.2. Proof of Theorem 5.1.

[I] For $f\in D^k_{pq,T}$ we put $F=\mathscr{T}f$. In a completely similar way as in the proof of Theorem 4.1, we can prove that F satisfies (i), (ii) and that $\mathscr{T}$ is continuous and injective. So we omit the details. The equalities in (iii) and (iv) follow from the analogous ones (2.6) and (5.1) for $u^k_{pq}(g, s)$. Therefore the image of $D^k_{pq,T}$ under $\mathscr{T}$ is contained in $\mathscr{H}^k_{pq,T}$.

[II] We show that $\mathscr{T}$ is surjective. Let $F\in\mathscr{H}^k_{pq,T}$ be given. We assume at first $p\geq q\geq 0$. Moreover we assume that F satisfies

$$F\Big(\lambda_k+j-\frac{1}{2}\Big)=0 \quad (1\leq j\leq q) \qquad \text{if} \quad q\geq 1. \tag{5.3}$$

This assumption (5.3) means that the discrete parts in the formula for $\mathscr{T}^{-1}$ vanish. Put

$$\Xi^k_{pq}(s)=\prod\nolimits_{0\leq j\leq p-1}\Big(\lambda_k+j+\frac{1}{2}+s\Big)\cdot\prod\nolimits_{0\leq j\leq q-1}\Big(\lambda_k+j+\frac{1}{2}-s\Big).$$

Here we understand that $\prod_{0\leq j\leq -1}=1$. It is easily verified that

$$\Xi^k_{pq}(-s)=\Lambda^k_{pq}(s)\Xi^k_{pq}(s). \tag{5.4}$$

Notice that for $p\geq q\geq 0$, $\Lambda^k_{pq}(s)$ takes the form

$$\Lambda_{pq}^{k}(s)=\prod_{q\leq j\leq p-1}\left(\lambda_{k}+j+\frac{1}{2}-s\right)\left(\lambda_{k}+j+\frac{1}{2}+s\right)^{-1}.$$

Put $H(s)=F(s)/\Xi_{pq}^{k}(s)$. The assumption (5.3) together with the condition (iii) makes H entire. By (5.4) and the condition (iii) we have $H(-s)=H(s)$. Hence H belongs to $\mathscr{H}_{00,T}$. (The inequality (4.4) is obviously satisfied.) Thus by Theorem 4.1 there exists uniquely a function $h\in D_{00,T}^{k}$ satisfying

$$\int h(g)u_{00}^{k}(g,\ s)dg=H(s).$$

Let X_{-}, X_{-}' be as in Proposition 2.1 (3) and set

$$f_{0}=(X_{-}')^{q}(X_{-})^{p}h. \tag{5.5}$$

We have $f_{0}(ua_{t}v)=0$ for $t\geq T$, $u,\ v\in K$, because so does h. An easy calculation and Proposition 1.2 (2) lead us to the following: let P_{pq}^{k} be as in (3.9), then

$$\int P_{ab}^{k}f_{0}(g)u_{ab}^{k}(g,\ s)dg=\int f_{0}(g)u_{ab}^{k}(g,\ s)dg \tag{5.6}$$

$$=\delta_{ap}\delta_{bq}\Xi_{pq}^{k}(s)\int h(g)u_{00}^{k}(g,\ s)dg=\delta_{ap}\delta_{bq}F(s).$$

Since we already know the injectiveness of $\mathscr{T}$, this together with Lemma 6.2 in the succeeding section implies that $f_{0}\in D_{pq,T}^{k}$ and $\mathscr{T}f_{0}=F$.

Now we eliminate the assumption (5.3). Since $u_{pq}^{k}\left(\cdot,\ \lambda_{k}+j-\frac{1}{2}\right)(1\leq j\leq q)$ are eigenfunctions of the Casimir operator Δ corresponding to distinct eigenvalues $(\lambda_{k}+j)(1-\lambda_{k}-j)$, they are linearly independent each other. Moreover they are real analytic. Thus by the uniqueness of analytic continuation, they are linearly independent each other even if we consider them as functions on an arbitrarily small non-empty open subset of G. Hence we can find for arbitrarily small $\varepsilon>0$, functions $h_{i}\in D_{pq,\varepsilon}^{k}$ $(1\leq i\leq q)$ such that

$$\int h_{i}(g)u_{pq}^{k}\left(g,\ \lambda_{k}+j-\frac{1}{2}\right)dg=\delta_{ij}\qquad(1\leq i,\ j\leq q). \tag{5.7}$$

Put $H_{i}=\mathscr{T}h_{i}$ and

$$F_{0}(s)=F(s)-\sum_{1\leq i\leq q}F\left(\lambda_{k}+i-\frac{1}{2}\right)H_{i}(s).$$

Since we already know that $H_{i}\in\mathscr{H}_{pq,\varepsilon}^{k}\subset\mathscr{H}_{pq,T}^{k}$, we have $F_{0}\in\mathscr{H}_{pq,T}^{k}$. Moreover by (5.7), F_{0} satisfies the previous assumption (5.3). Therefore by the discussion above, we can find $f_{0}\in D_{pq,T}^{k}$ such that $F_{0}=\mathscr{T}f_{0}$. Put

$$f=f_{0}+\sum_{1\leq i\leq q}F\left(\lambda_{k}+i-\frac{1}{2}\right)h_{i}.$$

It is readily verified that $f\in D_{pq,T}^{k}$ and $\mathscr{T}f=F$. This proves the surjectiveness of $\mathscr{T}$ in case $p\geq q\geq 0$.

The case $q \geq p \geq 0$ can be treated in a similar way.

The case $-p \geq -q \geq 0$ and $k \neq n$ can be handled as the above two cases except that we must put

$$\Xi^k_{pq}(s) = \prod_{0 \leq j \leq |p|-1}\left(-\lambda_k + j + \frac{1}{2} + s\right) \cdot \prod_{0 \leq j \leq |q|-1}\left(-\lambda_k + j + \frac{1}{2} - s\right),$$

and in (5.5), $(X'_+)^{|q|}(X_+)^{|p|}h$ instead of $(X'_-)^q(X_-)^p h$.

On the other hand, for the case $-p \geq -q \geq 0$ and $k = n$, some remarks should be added. In this case we put

$$\Xi^n_{pq}(s) = \prod_{0 \leq j \leq |p|-1}(j+s) \cdot \prod_{0 \leq j \leq |q|-1}(j-s),$$

$$\Lambda^n_{pq}(s) = \begin{cases} \prod_{|q| \leq j \leq |p|-1}(j-s)(j+s)^{-1} & \text{for } q \neq 0, \\ -\prod_{1 \leq j \leq |p|-1}(j-s)(j+s)^{-1} & \text{for } q = 0 \text{ and } p \neq 0. \end{cases}$$

The differences consist in the point that when $q \neq 0$, Ξ^n_{pq} has a two-fold zero at $s = 0$, and when $q = 0$ and $p \neq 0$, Ξ^n_{pq} has a simple zero at $s = 0$. But in case $q \neq 0$, by differentiating the functional equation in (iii), we get $F'(0) = 0$ if $F(0) = 0$. Hence F has a two-fold zero at $s = 0$ provided $F(0) = 0$. Thus the preceding discussion holds. In case $q = 0$ and $p \neq 0$, noting that $0 \in N^n_{pq,1}$, we can also apply the preceding discussion.

The case $-q \geq -p \geq 0$ can be handled in the same way as the case $-p \geq -q \geq 0$.

Now we consider the case $p > 0 > q$. In this case no assumption such as (5.3) is necessary thanks to (iv). We put

$$\Xi^k_{pq}(s) = \prod_{0 \leq j \leq p-1}\left(\lambda_k + j + \frac{1}{2} + s\right) \cdot \prod_{0 \leq j \leq |q|-1}\left(-\lambda_k + j + \frac{1}{2} - s\right),$$

and in (5.5), $(X'_+)^{|q|}(X_-)^p h$ instead of $(X'_-)^q(X_-)^p h$. We omit the details.

The case $q > 0 > p$ can be treated in the same way as the case above.

[III] It remains to prove the continuity of $\mathscr{T}^{-1}$. Since we already know the bijectiveness of $\mathscr{T}$, the inverse transform $\mathscr{T}^{-1}$ is written in the form (3.11). Note that the point evaluation: $F \mapsto F(s)$ is continuous in $\mathscr{H}^k_{pq,T}$, and that the discrete part in the above inverse transform for $F \in \mathscr{H}^k_{pq,T}$ contains at most $|q|$-terms. Then the continuity of $\mathscr{T}^{-1}$ can be proved as in the proof of Theorem 4.1.

Thus Theorem 5.1 is completely proved. Q. E. D.

5.3. Using Theorem 5.1, we can investigate the linear mapping $\mathscr{A}: f \mapsto \varphi_f$ defined by (3.3).

Let $D_{ev,T}(\boldsymbol{R})$ be the totality of even functions $\varphi \in C_0^\infty(\boldsymbol{R})$ vanishing for $|t| \geq T$. The topology of $D_{ev,T}(\boldsymbol{R})$ is that induced by $\mathscr{D}(\boldsymbol{R})$ which we topologize as usual, then $D_{ev,T}(\boldsymbol{R})$ is a Fréchet space.

Theorem 5.3. *The linear mapping $\mathscr{A}$ gives a topological isomorphism between $D^k_{pp,T}$ and $D_{ev,T}(\boldsymbol{R})$.*

Proof. Let $f \in D^k_{pp,T}$. By (3.3) and (3.7) we easily deduce that $\varphi_f(t) = 0$ for

$|t|\geq T$. It is also easily verified that $\varphi_f(-t)=\varphi_f(t)$ and that $\varphi_f\in C^\infty(\boldsymbol{R})$. Hence $\varphi_f\in D_{\mathrm{ev},T}(\boldsymbol{R})$.

Conversely, let $\varphi\in D_{\mathrm{ev},T}(\boldsymbol{R})$ be given. Denote by $\hat{\varphi}$ its ordinary Fourier transform in (0.1). Since $\varphi(-t)=\varphi(t)$, we have $\hat{\varphi}(s)=\hat{\varphi}(-s)$. By the classical Paley-Wiener theorem we conclude that $\hat{\varphi}$ is an entire function with the property

$$|\hat{\varphi}(s)|\leq C_r(1+|s|)^{-r}e^{T|\mathrm{Im}\, s|}.$$

Put $F(s)=\hat{\varphi}(-is)$. We see from the above that $F\in\mathscr{H}_{00,T}=\mathscr{H}^k_{pp,T}$ (see Remark 6). Hence by Theorem 5.1 there exists uniquely a function $f\in D^k_{pp,T}$ such that $F=\mathscr{T}f$. Constructing φ_f from this f, we obtain by (3.4)

$$\hat{\varphi}(-is)=F(s)=\int f(g)u^k_{pp}(g,s)dg=\int_{-\infty}^{\infty}\varphi_f(t)e^{-st}dt=\hat{\varphi}_f(-is).$$

Therefore we have $\varphi=\varphi_f$ by the injectiveness of the ordinary Fourier transform. This proves that the mapping $\mathscr{A}$ is bijective.

It remains to prove the bi-continuity. Let $\varphi_j\to 0$ in $D_{\mathrm{ev},T}(\boldsymbol{R})$. Putting $F_j(s)=\hat{\varphi}_j(-is)$, we have $F_j\to 0$ in $\mathscr{H}^k_{pp,T}$ by the classical Paley-Wiener theorem. Let $f_j=\mathscr{A}^{-1}\varphi_j$. As is seen above, f_j coincides with $\mathscr{T}^{-1}F_j$. Hence by Theorem 5.1 we see that $f_j\to 0$ in $D^k_{pp,T}$. This proves the continuity of $\mathscr{A}^{-1}$. Since both $D_{\mathrm{ev},T}(\boldsymbol{R})$ and $D^k_{pp,T}$ are Fréchet spaces (Corollary 5.2), the continuity of $\mathscr{A}$ follows from the open mapping theorem.

Now Theorem 5.3 is completely proved. Q. E. D.

In the course of the above discussion, we get explicitly the inverse transform of $\mathscr{A}$. We do not write it down here.

§6. Analogue of the Paley-Wiener theorem for $\mathscr{D}(G)$

6.1. Let $\mathscr{D}_T$ be the space of functions $f\in C_0^\infty(G)$ satisfying

(6.1) $$f(ua_tv)=0 \qquad \text{for} \quad t\geq T,\ u,\ v\in K.$$

The topology of $\mathscr{D}_T$ is introduced by means of seminorms in (4.3). This topology of $\mathscr{D}_T$ coincides with the usual ones when we consider G as a C^∞-manifold. We denote by $\mathscr{D}^k_T$ the closed subspace of $\mathscr{D}_T$ consisting of functions such that

(6.2) $$f(u_{2\pi}g)=e^{ik\pi/n}f(g) \qquad (-n+1\leq k\leq n).$$

Notice that $u_{2\pi}$ is a generator of the center of G, and that $\chi^k_p(u_{2\pi})=e^{-ik\pi/n}$ for any $p\in\boldsymbol{Z}$.

Remark 7. $U^k(u_{2\pi},s)=e^{-ik\pi/n}$.

Lemma 6.1. $\mathscr{D}_T=\sum^{\oplus}_{-n+1\leq k\leq n}\mathscr{D}^k_T$.

Let P^k_{pq} be the projection defined by (3.9). It is clear that if f satisfies (6.1), so does $P^k_{pq}f$ too. It should be remarked here that P^k_{pq} is also applicable to those functions whose supports are not necessarily compact.

Lemma 6.2. *Suppose* $f \in C^\infty(G)$ *satisfies* (6.2). *Then*

$$f(g)=\sum_{p,q\in Z} P^k_{pq} f(g) \qquad (\textit{pointwise absolute convergence}).$$

Proof. Since f satisfies (6.2), we have

$$\int_K\int_K \chi_p^l(u) f(ugv)\chi_q^m(v)\,du\,dv=\delta_{lk}\delta_{mk}P^k_{pq}f(g).$$

Note that $\{\chi_p^l\chi_q^m;\ p,\ q\in \boldsymbol{Z},\ -n+1\le l,\ m\le n\}$ forms a complete orthonormal system in $L^2(K\times K)$ and that $K\times K\ni(u,\ v)\mapsto f(ugv)$ is smooth. Then the assertion follows. Q.E.D.

Let $\mathscr{H}^k_T$ be the totality of operator-valued functions

$$\boldsymbol{C}\ni s\longmapsto \mathscr{F}(s)\in \boldsymbol{B}(\mathfrak{H})$$

which satisfy the following conditions (i)~(v).

(i) $\mathscr{F}$ is an entire function.

(ii) For every non-negative integer r, there exists a constant C_r depending on $\mathscr{F}$ such that

$$\|\mathscr{F}(s)\|\le C_r(1+|s|)^{-r}e^{T|\mathrm{Re}\,s|}. \tag{6.3}$$

(iii) $(\mathscr{F}(-s)e_q,\ e_p)=\Lambda^k_{pq}(s)(\mathscr{F}(s)e_q,\ e_p) \qquad (p,\ q\in \boldsymbol{Z})$.

(iv) $(\mathscr{F}(s)e_q,\ e_p)=0 \quad$ for all $\ s\in\bigcup_{j=1,2}N^k_{pq,j}$.

(v) For every quintet of non-negative integers $a,\ b,\ c,\ r,\ M$ define $|\mathscr{F}|_{a,b,c,r,M}$ as below. Then $|\mathscr{F}|_{a,b,c,r,M}<\infty$:

$$|\mathscr{F}|_{a,b,c,r,M}=\sup_{p,q\in Z;\,j\in N}(1+|p|)^a(1+|q|)^b \tag{6.4}$$

$$\times\Big[\,|(\mathscr{F}(\cdot)e_q,\ e_p)|_{r,M}+j^c\sum_{\varepsilon=+,-}\omega^{k,\varepsilon}_{pq}(j)\Big|\Big(\mathscr{F}\Big(\varepsilon\lambda_k+j-\frac{1}{2}\Big)e_q,\ e_p\Big)\Big|\,\Big].$$

Remark 8. Conditions (i)~(iv) imply that for all $p,\ q\in\boldsymbol{Z}$, $(\mathscr{F}(\cdot)e_q,\ e_p)\in\mathscr{H}^k_{pq,T}$ for $T>0$ in (ii).

We topologize $\mathscr{H}^k_T$ by means of seminorms $|\mathscr{F}|_{a,b,c,r,M}$.

Theorem 6.3. *The linear mapping*

$$\mathscr{T}: f\longmapsto \int f(g)U^k(g,\ \cdot)dg$$

gives a topological isomorphism between $\mathscr{D}^k_T$ *and* $\mathscr{H}^k_T$.

6.2. Proof of Theorem 6.3.

Step 1. Let $f\in\mathscr{D}^k_T$ and put $\mathscr{F}=\mathscr{T}f$. Clearly $\mathscr{F}(s)\in\boldsymbol{B}(\mathfrak{H})$ and $\mathscr{F}$ is entire. Let us show (ii). Put $\mathscr{F}_r=\mathscr{T}(\Delta^r f)$. Then we have

$$\mathscr{F}_r(s)=\Big(\frac{1}{4}-s^2\Big)^r\mathscr{F}(s).$$

Once this relation is obtained, (6.3) can be proved in a completely similar way as in the proof of Theorem 4.1 by using Proposition 1.1 (3).

Since $P^k_{pq}f \in D^k_{pq,T}$ and

$$(\mathscr{F}(s)e_q, e_p) = \int P^k_{pq}f(g)u^k_{pq}(g, s)dg, \tag{6.5}$$

we have (iii) and (iv) by Theorem 5.1. To verify (v) we first note that

$$\begin{aligned}(-i)^a i^b (p+\lambda_k)^a (q+\lambda_k)^b \left(\frac{1}{4}-s^2\right)^r (\mathscr{F}(s)e_q, e_p) \\ = \int (X_0)^a (X'_0)^b \Delta^r f(g)\cdot u^k_{pq}(g, s)dg.\end{aligned} \tag{6.6}$$

Next, putting $s = \pm\lambda_k + j - \frac{1}{2}$ in (6.6) and noting (2.3) and (2.4), we have

$$\begin{aligned}(-i)^a i^b (p+\lambda_k)^a (q+\lambda_k)^b (\pm\lambda_k+j)^r (1\mp\lambda_k-j)^r \omega^{k,\pm}_{pq}(j) \times \\ \times\left(\mathscr{F}\left(\pm\lambda_k+j-\frac{1}{2}\right)e_q, e_p\right) = \int (X_0)^a (X'_0)^b \Delta^r f(g)\cdot v^{k,\pm}_{pq}(g, j)dg.\end{aligned} \tag{6.7}$$

Since $f \in \mathscr{D}^k_T$ and $|v^{k,\pm}_{pq}(g, j)| \leq 1$, we obtain (v) from (6.6) and (6.7).

In the course of the discussion above we also get the continuity of $\mathscr{T}$.

Step 2. We verify that $\mathscr{T}$ is injective. Let $f \in \mathscr{D}^k_T$ be a function such that $\mathscr{F} = \mathscr{T}f = 0$. Then we have

$$(\mathscr{F}(\cdot)e_q, e_p) = 0 \qquad \text{for all} \quad p, q \in \boldsymbol{Z}.$$

Consider $P^k_{pq}f$. Taking into account (6.5), Remark 8 and Theorem 5.1, we have $P^k_{pq}f = 0$ for all $p, q \in \boldsymbol{Z}$. By Lemma 6.2 we have $f = 0$.

Step 3. Let $\mathscr{F} \in \mathscr{H}^k_T$ be given. Define a function f on G by

$$\begin{aligned} f(g) = \\ = \frac{1}{2\pi}\sum_{p,q\in\boldsymbol{Z}}\int_0^\infty (\mathscr{F}(i\tau)e_q, e_p)\overline{u^k_{pq}(g, i\tau)}\,\tau\,\mathrm{Re\,th}\,\pi(\tau+i\lambda_k)d\tau \\ + \frac{1}{2\pi}\sum_{\varepsilon=+,-;j\in\boldsymbol{N};p,q\in\boldsymbol{Z}}\left(\varepsilon\lambda_k+j-\frac{1}{2}\right)\omega^{k,\varepsilon}_{pq}(j)\left(\mathscr{F}\left(\varepsilon\lambda_k+j-\frac{1}{2}\right)e_q, e_p\right)\overline{v^{k,\varepsilon}_{pq}(g, j)}\,.\end{aligned} \tag{6.8}$$

Condition (v) assures that the right hand side of (6.8) is absolutely convergent. To show that f is a C^∞-function, it is sufficient to verify that the differential operators X_0, $X_\pm$ are applicable indefinitely many times. This can be done in view of Proposition 2.1 (3) and the assumption $\mathscr{F} \in \mathscr{H}^k_T$. It is clear that f satisfies (6.2) (cf. Remark 7).

Step 4. We show that f in (6.8) satisfies (6.1). We can apply P^k_{pq} to the right hand side of (6.8) term by term because it is absolutely convergent. Hence we obtain

$$2\pi P^k_{pq}f(g) = \text{the right hand side of (3.11)}, \tag{6.9}$$

where we put $F^k_{pq}(s)=(\mathscr{F}(s)e_q, e_p)$ in (3.11). By Remark 8, (6.9) and Theorem 5.1, we have $P^k_{pq}f\in D^k_{pq,T}$ for $T>0$ in (ii). Since we already know that f is a C^∞-function, we have by Lemma 6.2

$$f(g)=\sum_{p,q\in Z} P^k_{pq}f(g).$$

Thus we have (6.1) because so do all $P^k_{pq}f$.

Step 5. It remains to prove the continuity of $\mathscr{T}^{-1}$. Let $\mathscr{F}_j\to 0$ in $\mathscr{H}^k_T$ and put $f_j=\mathscr{T}^{-1}\mathscr{F}_j\in\mathscr{D}^k_T$. It is sufficient to show that

$$(X_0)^a(X_+)^b(X_-)^c f_j(g)\longrightarrow 0$$

uniformly on G for every triplet of non-negative integers a, b, c, because X_0, X_1 and X_2 form a basis of $\mathfrak{g}$. This is clear by virtue of Proposition 2.1 (3).

Step 1 to Step 5 complete the proof of Theorem 6.3. Q. E. D.

6.3. We summarize here some direct consequences of Theorem 6.3.

Corollary 6.4. *$\mathscr{H}^k_T$ is a Fréchet space.*

Corollary 6.5. *The topology of $\mathscr{D}^k_T$ is also defined by another family of seminorms $|f|_{a,b,r}$ given by*

$$|f|_{a,b,r}=\sup_{g\in G}|(X_0)^a(X_0')^b\Delta^r f(g)|. \tag{6.10}$$

Proof. We denote by D^k_T the topological vector space with the same underlying space as that of $\mathscr{D}^k_T$ and with seminorms in (6.10). In the same way as in the proof of Theorem 6.3, we can prove that D^k_T is topologically isomorphic to $\mathscr{H}^k_T$. Since $\mathscr{H}^k_T$ is complete by Corollary 6.4, so is D^k_T too. By the open mapping theorem D^k_T and $\mathscr{D}^k_T$ are canonically isomorphic. Q. E. D.

Corollary 6.6. *Let $f\in\mathscr{D}^k_T$. Then we have*

$$f=\sum_{p,q\in Z} P^k_{pq}f \qquad (in\ \mathscr{D}^k_T).$$

Proof. Note that

$$(-i)^2 i^2(p+\lambda_k)^2(q+\lambda_k)^2(X_0)^a(X_0')^b\Delta^r P^k_{pq}f(g)$$
$$=\int_K\int_K \chi^k_p(u)((X_0)^{a+2}(X_0')^{b+2}\Delta^r f)(ugv)\chi^k_q(v)du\, dv.$$

Then by Corollary 6.5, $\sum_{p,q\in Z} P^k_{pq}f$ is convergent in $\mathscr{D}^k_T$. Since $\mathscr{D}^k_T$ is complete, the assertion follows from Lemma 6.2. Q. E. D.

6.4. Analogue of the Paley-Wiener theorem for $\mathscr{D}(G)$. The space $\mathscr{D}(G)$ is the inductive limit of $\mathscr{D}_T$ as $T\to\infty$, and by Lemma 6.1, $\mathscr{D}_T$ is a direct sum of $\mathscr{D}^k_T$ over $-n+1\le k\le n$. In Theorem 6.3 we have established an analogue of the Paley-Wiener theorem for the "Fourier transform" $\mathscr{T}$. Therefore extending $\mathscr{T}$ to $\mathscr{D}(G)$ naturally, we have an analogue of the Paley-Wiener theorem for $\mathscr{T}$ on $\mathscr{D}(G)$.

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