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Semi-homogeneous vector bundles on an abelian variety

By

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Introduction

In his paper [2], Atiyah completely classified the vector bundles on an elliptic curve (see also Oda [12]). His results have been extended to the curves of higher genus in various ways. Another direction in which we must try to extend it is the study of vector bundles on an abelian variety X of higher dimension. In contrast with the case $g=\dim X=1$, there are "too" many vector bundles on X when $g>1$. The main object of our study in this article is the vector bundle of the following type:

Definition. A vector bundle E on X is semi-homogeneous if for every $x \in X$, there exists a line bundle L on X such that

$$T_x^*(E) \cong E \otimes L,$$

where T_x is the translation of X by x (see also Definition 5.2).

By this definition, the difference between the case $g=1$ and the case $g>1$ can be explained by the following two facts:

(1) Every indecomposable vector bundle on an elliptic curve is semi-homogeneous.

(2) Semi-homogeneous vector bundles on X are very special when $g>1$.

(1) will be easily seen by the following result of Atiyah ([2] Theorem 10): If two indecomposable vector bundles E and E' on an elliptic curve have the same rank and the same degree, then $E' \cong E \otimes L$ for some line bundle L on X.

To show (2) is one of our main aims of this article.

Abelian varieties have a lot of automorphisms, that is, translations. As for the vector bundles which are invariant under all translations (such vector bundles are said to be homogeneous), their structure has been considerably clarified by Matsushima [6], Morimoto [9] and Miyanishi [7]. In § 4, we shall complete their results by i) determining the category of homogeneous vector bundles (Theorem 4.19) and ii) computing their cohomologies (Theorem 4.12). (In the case $g=1$,

i) was implicitly done by Oda [12]. The essential tool for ii) is that of Mumford which was used in the construction of the dual abelian variety ([10] § 13).) These results are inevitable for our study of semi-homogeneous vector bundles.

In contrast with rational homogeneous spaces, there exist many line bundles on abelian varieties. In fact, the connected component $\mathrm{Pic}^{\circ}(X)$ of $\mathrm{Pic}(X)$ has the same dimension as X. It is this fact that makes our definition of semi-homogeneous vector bundles meaningful.

In § 5, semi-homogeneous vector bundles will be characterized by various properties under the condition that they are simple (Theorem 5.8). Among those the property $\dim_k H^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E))=g$ is the simplest one. The equivalence (1) and (4) in Theorem 5.8 is the generalization of the characterization given by Morikawa [8] and Oda [12], [13] to the case where the characteristic p of the base field is arbitrary. The keys of our proof are the pro-representability and the relative representability of the moduli of simple vector bundles (they are summarized in § 1) and nice properties of the group scheme $\Sigma(E)$. (See § 2. For the basic idea, we owe to Takemoto [15].) These enable us to overcome the difficulty which arises when $p>0$ (see Remark 3.18 and the proof of Proposition 2.6).

§ 6 is devoted to the study of semi-homogeneous vector bundles which may not be simple. The similar results to homogeneous vector bundles will be obtained for a certain category $\boldsymbol{S}_\delta$ consisting of semi-homogeneous vector bundles (Theorem 6.19). This almost determines the category $\boldsymbol{S}_\delta$ and reduces the study of semi-homogeneous vector bundles to the case they are simple.

In the final section, we shall again consider the simple semi-homogeneous vector bundles. Various group schemes attached to them will be explicitly determined and it will be shown that their ranks and other numerical invariants are much restricted when $g>1$ (Remark 7.13).

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Notation. Throughout this article k denotes a fixed algebraically closed field of characteristic $p\geq 0$. By a scheme we understand a scheme of finite type over k. For a scheme X, $\underline{X}$ is the contravariant functor on (Sch) to (Sets) associated to X. For $\mathcal{O}_X$-modules F and G, $\mathscr{H}om_{\mathcal{O}_X}(F, G)$ denotes the sheaf of $\mathcal{O}_X$-homomorphisms, while $\mathrm{Hom}_{\mathcal{O}_X}(F, G)$ is the set of global $\mathcal{O}_X$-homomorphisms. $\mathscr{E}nd_{\mathcal{O}_X}(F)$ and $\mathrm{End}_{\mathcal{O}_X}(F)$ are $\mathscr{H}om_{\mathcal{O}_X}(F, F)$ and $\mathrm{Hom}_{\mathcal{O}_X}(F, F)$, respectively. We use the terms "vector bundles" (resp. "line bundles") and "locally free sheaves" (resp. "invertible sheaves") interchangeably. For a vector bundle E on X, $r(E)$ is the rank of E. $E^{\vee}=\mathscr{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ denotes the dual vector bundle of E (especially, for a vector space V, $V^{\vee}$ is its dual vector space). For a positive integer t, $\overset{t}{\wedge} E$ is the t-th exterior power of E. In the case $t=r(E)$ it is a line bundle which we denote by $\det(E)$.

For a scheme X, $\mathrm{Pic}(X)$ is the set of isomorphic classes of line bundles on X and $\mathrm{Pic}^{\circ}(X)$ is the subset of $\mathrm{Pic}(X)$ consisting of elements represented by line bundles which are algebraically equivalent to zero. $\mathrm{Pic}(X)$ is an abelian group and $\mathrm{Pic}^{\circ}(X)$

is its subgroup with respect to the tensor product $\otimes$. We denote the quotient $\mathrm{Pic}(X)/\mathrm{Pic}^{\circ}(X)$ by $\mathrm{NS}(X)$.

§ 1. Simple vector bundles

In this section we shall summarize the facts about families of simple vector bundles. (A vector bundle E on a complete variety X is said to be simple if $\mathrm{End}_{\mathcal{O}_X}(E) \cong k$.) They seems to be well-known in the case of line bundles and the proof is similar to the case (see [10] § 10). Thus we only sketch them for the convenience of readers.

Let $f: V \to S$ be a proper, flat, integral morphism. Let F and G be locally free sheaves on V. There exist a coherent $\mathcal{O}_S$-module A and an isomorphism of functors on quasi-coherent $\mathcal{O}_S$-modules M:

$$f_*(\mathcal{H}om_{\mathcal{O}_V}(G, F) \otimes_{\mathcal{O}_S} M) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_S}(A, M) \tag{1.1}$$

(**EGA** III (7.7.6)). For an affine morphism, $\alpha: T \to S$, this gives rise to isomorphisms

$$(f_T)_*(\mathcal{H}om_{\mathcal{O}_{V_T}}(G_T, F_T)) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_S}(A, \mathcal{O}_T) \tag{1.2}$$

$$\mathrm{Hom}_{\mathcal{O}_{V_T}}(G_T, F_T) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(A, \mathcal{O}_T). \tag{1.3}$$

Let Z be the closed subscheme of S defined by the ideal $\mathcal{I} = \mathrm{Ann}(A)$.

Lemma 1.4. *Let $\alpha: T \to S$ be a morphism. If $F_T \cong G_T \otimes_{\mathcal{O}_T} N$ for some invertible sheaf N on T, then α factors through $Z \hookrightarrow S$.*

Proof. We may assume that α is affine. Since $F_T \cong G_T \otimes_{\mathcal{O}_T} N$,

$$\mathcal{H}om_{\mathcal{O}_{V_T}}(G_T, F_T) \cong \mathcal{E}nd_{\mathcal{O}_{V_T}}(G_T) \otimes_{\mathcal{O}_T} N.$$

Hence there exists an injective homomorphism

$$(f_T)^*(N) \longrightarrow \mathcal{H}om_{\mathcal{O}_{V_T}}(G_T, F_T). \tag{1.4.1}$$

By the assumption on f, we have

$$(f_T)_*(\mathcal{O}_{V_T}) \cong \mathcal{O}_T$$

Hence, taking the direct image of both sides of (1.4.1), we have an injective homomorphism

$$N \longrightarrow \mathcal{H}om_{\mathcal{O}_S}(A, \mathcal{O}_T).$$

Therefore, it follows from the definition of $\mathcal{I}$ that $\mathcal{I} \cdot N = 0$. Since N is an invertible $\mathcal{O}_T$-module, $\mathcal{I} \cdot \mathcal{O}_T = 0$, which completes our proof. q.e.d.

Obviously the underlying set of Z is

$$\mathrm{Supp}(A) = \{s \in S \mid \text{there exists a non-zero homomorphism } \varphi: G_s \longrightarrow F_s\},$$

and hence the set $W=\{s\in S|G_s\cong F_s\}$ is contained in it.

Proposition 1.5. *W is a constructible set.*

Proof. The functor $T\rightsquigarrow\mathrm{Isom}_{\mathcal{O}_{V_T}}(G_T, F_T)$ on (Sch/S) is representable by an open subscheme Y of $V(A)$ (see [11] Lemma (II.11) and Remark (II.14)). Since $W=\beta(Y)$, it is a constructible set, where $\beta\colon Y\to S$ is the structure morphism. q.e.d.

To obtain further results we have to assume that G is S-simple, *i.e.*, the natural homomorphism $\mathcal{O}_S\to f_*(\mathcal{E}nd_{\mathcal{O}_V}(G))$ is an isomorphism. From this assumption we have

Lemma 1.6. *For every $w\in W$, there exists an open neighbourhood U of w in Z such that $A|_U\cong\mathcal{O}_Z|_U$.*

Proof. By (1.3),

$$\mathrm{Hom}_{\mathcal{O}_{V_s}}(G_s, F_s)\cong\mathrm{Hom}_{k(s)}(A\otimes k(s), k(s))\cong[A\otimes k(s)]^\vee$$

for all $s\in S$. Since G is S-simple, we have $\dim_{k(w)}(A\otimes k(w))=1$ for all $w\in W$. By Nakayama's lemma, there exist an open neighbourhood $\tilde{U}$ of w and an ideal $\mathcal{J}$ of $\mathcal{O}_{\tilde{U}}$ such that $A|_{\tilde{U}}\cong\mathcal{O}_{\tilde{U}}/\mathcal{J}$. Obviously $\mathcal{J}=\mathcal{I}|_{\tilde{U}}$, whence $A|_U\cong\mathcal{O}_Z|_U$ for $U=\tilde{U}\cap Z$. q.e.d.

Proposition 1.7. *W is an open subset of Z.*

Proof. By (1.3) and Lemma 1.6, the map

$$\mathrm{Hom}_{\mathcal{O}_{V_U}}(G_U, F_U)\longrightarrow\mathrm{Hom}_{\mathcal{O}_{V_w}}(G_w, F_w)$$

is surjective for a U in Lemma 1.5. Hence there exists a homomorphism $\varphi\colon G_U\to F_U$ such that $\varphi\otimes k(w)$ is an isomorphism. The rest is routine. q.e.d.

By virtue of the above proposition, W can be regarded as an open subscheme of Z.

Theorem 1.8. *W represents the following functor of* (Sch) *to* (Sets);

$T\rightsquigarrow\{\alpha\colon T\to S|F_T\cong G_T\otimes_{\mathcal{O}_T}M$ *for some invertible sheaf M on T*$\}$.

Proof. Put $L=(f_W)_*(\mathcal{H}om_{\mathcal{O}_{V_W}}(G_W, F_W))$. This is an invertible sheaf on W by (1.2) and Lemma 1.6. We see easily that the natural homomorphism

$$\lambda\colon G_W\otimes_{\mathcal{O}_W}L\longrightarrow F_W$$

is an isomorphism. This and Lemma 1.4 prove our theorem. q.e.d.

Under the assumption that F or G is S-simple, we shall refer to the above subscheme W of S as the maximal subscheme over which F and G are isomorphic to each other.

Let E be a simple vector bundle on a complete variety X. Let $\boldsymbol{C}$ be the category of artin local rings over a field k and $\mathcal{D}$ the functor on $\boldsymbol{C}$ defined by

$\mathcal{D}(A)=\{\tilde{E}|\tilde{E}$ is a vector bundle on X_A such that $(\tilde{E})_0=\tilde{E}\otimes_A(A/\mathfrak{m})$ is

isomorphic to $E\}$/isom.
For this functor, we have the following.

Proposition 1.9. *The functor $\mathscr{D}$ is pro-representable by a pro-couple (R, ξ) and, moreover, the Zariski tangent space t_R of R is canonically isomorphic to $H^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E))$. We shall call this R the local moduli of E.*

For the terminology and the proof, see Schllesinger [14]. If one notes the following lemma, he would see that the proof of the pro-representability of Picard functor in [14] works in our case without any modifications.

Lemma 1.10. *If $\tilde{E} \in \mathscr{D}(A)$, then* $\mathrm{End}_{\mathcal{O}_{X_A}}(\tilde{E}) \cong A$.
The proof is easy and we omit it.

§2. The group scheme $\Sigma(E)$

From now on, we shall fix an abelian variety X of dimension g over k. In this section we shall give a condition for a vector bundle E on X to be isomorphic to the direct image $\pi_*(E')$ for some isogeny π: $Y \to X$ and vector bundle E' on Y (Proposition 2.6).

In general we regard X as a variety. Hence, by a point of X, we mean a k-rational point of X. But, in some places, regarding X as a scheme, we shall consider subschemes of X which may not be reduced or $X \times S$ for an arbitrary scheme S.

We denote by $\hat{X}$ the dual abelian variety of X. For a point $\hat{x}$ of $\hat{X}$, we denote by $P_{\hat{x}}$ the line bundle in $\mathrm{Pic}^\circ(X)$ corresponding to $\hat{x}$. Moreover, for an arbitrary scheme S and an S-valued point f of $\hat{X}$, we denote by P_f the line bundle $(1_X \times f)^*(\mathscr{P})$ on $X_S = X \times S$, where $\mathscr{P}$ is the normalized Poincaré bundle on $X \times \hat{X}$.

Let E be a vector bundle of rank r on X.

Definition 2.1. $\Sigma^\circ(E) = \{\hat{x} \in \hat{X} | E \otimes P_{\hat{x}} \cong E\}$.

If $E \cong P_{\hat{x}} \otimes E$, then $\det(E) \cong P_{\hat{x}}^{\otimes r} \otimes \det(E)$, whence $r\hat{x} = 0$. Hence $\Sigma^\circ(E)$ is contained in the r-torsion of $\hat{X}$.

Let p_1 (resp. p_2) be the projection of $X \times \hat{X}$ to the first (resp. second) factor. We shall apply the results in § 1 to the morphism p_2 and the couple of vector bundles $F = (p_1)^*(E) \otimes \mathscr{P}$ and $G = (p_1)^*(E)$. By virtue of (1.1), there exist a coherent $\mathcal{O}_{\hat{X}}$-module A and an isomorphism of functors on quasi-coherent $\mathcal{O}_{\hat{X}}$-modules M:

$$(2.2) \qquad (p_2)_*((p_1)^*(\mathscr{E}nd_{\mathcal{O}_X}(E)) \otimes \mathscr{P} \otimes_{\mathcal{O}_{\hat{X}}} M) \cong \mathscr{H}om_{\mathcal{O}_{\hat{X}}}(A, M).$$

Let us assume that E is a simple vector bundle. It is easily seen that G is an $\hat{X}$-simple vector bundle.

Definition 2.3. $\Sigma(E)$ is the maximal subscheme of $\hat{X}$ over which F and G are isomorphic to each other.

By virtue of Theorem 1.8, $\Sigma(E)$ represents the following functor of (Sch) to

(Sets);

(2.3.1) $S \rightsquigarrow \{f \in \underline{\hat{X}}(S) | E_S \otimes P_f \cong E_S \otimes_{\mathcal{O}_S} N$ for some invertible sheaf N on $S\}$.
Obviously this is a subgroup functor of $\underline{\hat{X}}$. Hence $\Sigma(E)$ is a subgroup scheme of $\hat{X}$. As in the case of $\Sigma^{\circ}(E)$, we see that $\Sigma(E)$ is contained in the scheme-theoretic kernel $(\hat{X})_r$ of the multiplication $r_{\hat{X}}: \hat{X} \to \hat{X}$ by r. Especially, $\Sigma(E)$ is a finite commutative group scheme. Since every invertible sheaf on $\Sigma(E)$ is trivial, there exists an isomorphism

$$(2.3.2) \qquad (p_1)^*(E) \otimes (\mathscr{P}|_{X \times \Sigma(E)}) \cong (p_1)^*(E),$$

where $p_1: X \times \Sigma(E) \to X$ is the projection.

The relation of A and $\Sigma(E)$ is as follows.

Lemma 2.4. *$A \cong \mathcal{O}_{\Sigma(E)} \oplus A'$ for some coherent $\mathcal{O}_{\hat{X}}$-module A' and moreover,* $\mathrm{Supp}(A') \cap \Sigma(E) = \emptyset$.

Proof. By Proposition 1.7, $\Sigma(E)$ is open in Z. Since $\Sigma(E)$ is finite, we have, by Lemma 1.6,

$$A|_{\Sigma(E)} \cong \mathcal{O}_Z|_{\Sigma(E)} \cong \mathcal{O}_{\Sigma(E)}$$

Since $\Sigma(E)$ is a union of some connected components of Z, $A|_{\Sigma(E)} \cong \mathcal{O}_{\Sigma(E)}$ is a direct summand of A. Hence we have $A \cong \mathcal{O}_{\Sigma(E)} \oplus A'$ for some coherent $\mathcal{O}_{\hat{X}}$-module A'. Let $\hat{x} \in \Sigma(E)$. Since E is simple, we have

$$\dim_k (A \otimes k(\hat{x})) = 1 = \dim_k (\mathcal{O}_{\Sigma(E)} \otimes k(\hat{x})).$$

Therefore $A' \otimes k(\hat{x}) = 0$, which proves the second assertion. q.e.d.

For a vector bundle E on X, let $\mathscr{pgl}(E)$ be the cokernel of the natural injection i of $\mathcal{O}_X$ to $\mathscr{End}_{\mathcal{O}_X}(E)$. Let $Tr: \mathscr{End}_{\mathcal{O}_X}(E) \to \mathcal{O}_X$ be the trace map. We see easily that $Tr \circ i = r \cdot id_E$, where $r = r(E)$. Hence if $(r(E), p) = 1$, then $\mathscr{End}_{\mathcal{O}_X}(E) \cong \mathcal{O}_X \oplus \mathscr{pgl}(E)$ and especially $H^1(i)$ is injective. In general this is not true.

Lemma 2.5. *If E is a simple vector bundle on X, then*

$$(1) \qquad \mathrm{Lie}^{(p)}(\Sigma(E)) \cong \mathrm{Ker}[H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathscr{End}_{\mathcal{O}_X}(E))],$$

$$(2) \qquad H^0(X, \mathscr{pgl}(E)) \cong \mathrm{Lie}^{(p)}(\Sigma(E))$$

Proof. There are natural identifications

$$(2.5.1) \qquad H^1(X, \mathcal{O}_X) \cong \mathrm{Lie}^{(p)}(\hat{X})$$

(see [10] § 15) and

$$(2.5.2) \qquad H^1(X, \mathscr{End}_{\mathcal{O}_X}(E)) \cong t_R$$

(see Proposition 1.9), where R is the local moduli of E. Hence (1) follows from the definition of $\Sigma(E)$. (2) is derived from (1) and the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{End}_{\mathcal{O}_X}(E) \longrightarrow \mathscr{pgl}(E) \longrightarrow 0.$$

q.e.d.

$\Sigma(E)$ has the following nice property.

Proposition 2.6. *Let E be a simple vector bundle on X and suppose that $\Sigma(E)\neq 0$. Then there are a non-trivial isogeny π: $Y\to X$ and a simple vector bundle E_1 on Y such that $\pi_*(E_1)\cong E$.*

Proof. Our proof consists of several steps.

Step I. Let G be a simple subgroup scheme of $\Sigma(E)$. Let p: $\hat{X}\to\hat{X}/G$ be the natural isogeny and π: $Y\to X$ be the dual isogeny of p, where Y is the dual abelian variety of $\hat{X}/G$. $\mathrm{Ker}(\pi)$ is canonically isomorphic to the dual $\hat{G}$ of G ([10] § 15). We shall construct a vector bundle E_1 on Y so that $\pi_*(E_1)\cong E$.

Step II. Restricting (2.3.2) to $X\times G$ we have

$$(p_1)^*(E)\otimes\mathscr{P}|_{X\times G}\cong(p_1)^*(E),$$

where p_1 is the projection of $X\times G$ to X. Taking the direct image by p_1 we have

$$E\otimes(p_1)_*(\mathscr{P}|_{X\times G})\cong E\otimes_k\Gamma(G, \mathcal{O}_G).$$

Since $(p_1)_*(\mathscr{P}|_{X\times G})\cong\pi_*(\mathcal{O}_Y)$ ([12] Corollary 1.7), we have

$$\pi_*(\pi^*(E))\cong E\otimes\pi_*(\mathcal{O}_Y)\cong E^{\oplus l}, \tag{2.6.1}$$

where l is the order of G. This and the assumption that E is simple imply that

$$\dim_k \mathrm{End}_{\mathcal{O}_Y}(\pi^*(E))=\dim_k \mathrm{Hom}_{\mathcal{O}_X}(E, \pi_*\pi^*(E))=l. \tag{2.6.2}$$

Step III. Since G is simple, l is a prime number. When $(l, p)=1$, $G\cong\boldsymbol{Z}/l\boldsymbol{Z}$ and when $l=p$, G is isomorphic to $\boldsymbol{Z}/p\boldsymbol{Z}$, μ_p or α_p. The structure of the k-algebra $A=\mathrm{End}_{\mathcal{O}_Y}(\pi^*(E))$ is one of the following:

Case 1. $A\cong k[T]/(T^l-1)$ if $G\cong\boldsymbol{Z}/l\boldsymbol{Z}$ and $l\neq p$.

Case 2. $A\cong k[T]/(T^p)$ if $G\cong\boldsymbol{Z}/p\boldsymbol{Z}$.

Case 3. $A\cong k[T]/(T^p-T)$ if $G\cong\mu_p$.

Case 4. $A\cong k[T]/(T^p)$ if $G\cong\alpha_p$.

We shall prove the above only in the case 4. The others are similar (cf. [15] Lemma (1.12) for the case 1).

The embedding $G\hookrightarrow\hat{X}$ determines a non-trivial cohomology class $\alpha\in H^1(X, \mathcal{O}_X)$ (cf. (2.5.1)) uniquely up to the constant multiplications. Since $G\cong\alpha_p$,

$$\alpha^{(p)}=0 \tag{2.6.3}$$

By the definition of the isogeny π, we have

$$\pi^*(\alpha)=0 \tag{2.6.4}$$

in $H^1(Y, \mathcal{O}_Y)$. Let $\{A_{ij}\}\in Z^1(X, \boldsymbol{GL}_r(\mathcal{O}_X))$ be a 1-cocycle which defines E for some affine open covering $\{U_i\}$ of X. For the covering $\{U_i\}$, α is represented by a

1-cocycle $\{a_{ij}\}$, $a_{ij}\in \Gamma(U_i\cap U_j, \mathcal{O}_X)$. By (2.6.3), there exists a 0-cochain $\{f_i\}$, $f_i\in\Gamma(U_i, \mathcal{O}_X)$, such that

(2.6.3′) $$a_{ij}^p=f_i-f_j \quad \text{for all } i \text{ and } j.$$

By (2.6.4), there exists $\{t_i\}$, $t_i\in\Gamma(U_i, \pi_*(\mathcal{O}_Y))$, such that

(2.6.4′) $$a_{ij}=t_i-t_j \quad \text{for all } i \text{ and } j.$$

By Lemma 2.5, (2), there exists $\{B_i\}$, $B_i\in\Gamma(U_i, \boldsymbol{M}_r(\mathcal{O}_X))$, such that

(2.6.5) $$a_{ij}A_{ij}=B_iA_{ij}-A_{ij}B_j,$$

where $\boldsymbol{M}_r(\mathcal{O}_X)$ is the sheaf consisting of all the (r, r)-matrices whose coefficients are in $\mathcal{O}_X$. By (2.6.3′) and (2.6.4′), we can find a constant c in k such that $t_i^p-f_i=c$ for all i. Replacing f_i by f_i-c we may assume that $t_i^p=f_i$ for all i. By (2.6.4′) and (2.6.5), we have

$$(B_i-t_iI)A_{ij}=A_{ij}(B_j-t_jI),$$

where I denotes the unit matrix. Hence $\{B_i-t_iI\}$ determines an endomorphism φ of $\pi^*(E)$. Let $K(X)$ and $K(Y)$ be the function fields of X and Y, respectively. Since $t_i\not\in K(X)$ and $[K(Y)\colon K(X)]=p$, the endomorphisms $id, \varphi,\dots, \varphi^{p-1}$ are linearly independent in $\mathrm{End}_{\mathcal{O}_Y}(\pi^*(E))$. Moreover, since $(B_i-t_iI)^p=B_i^p-f_iI$, φ^p is induced by an endomorphism of E. Since E is simple, φ^p is a scalar multiplication. By this and (2.6.2), we see that $A\cong k[T]/(T^p)$.

Step IV. In the case 1 and 3, the semi-simple endomorphism forces $\pi^*(E)$ to decompose into a direct sum of its eigenspaces:

$$\pi^*(E)\cong E_1\oplus\cdots\oplus E_l.$$

Hence we have

$$\pi_*(\pi^*(E))\cong\pi_*(E_1)\oplus\cdots\oplus\pi_*(E_l).$$

By the Krull-Schmidt theorem of locally free sheaves (see [2]) and (2.6.1), we have $\pi_*(E_i)\cong E$ for all $i=1, 2,\dots, l$.

In the case 2 and 4, there exists an endomorphism φ such that $\varphi^p=0$ and $\varphi^{p-1}\neq 0$. We have two filtrations of $\pi^*(E)$ by its subsheaves:

$$\begin{array}{ccccccc} 0\neq \mathrm{Ker}(\varphi) & \subset & \mathrm{Ker}(\varphi^2) & \subset\cdots\subset & \mathrm{Ker}(\varphi^{p-1}) & \subset & \pi^*(E) \\ \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \| \\ 0\neq \mathrm{Im}(\varphi^{p-1}) & \subset & \mathrm{Im}(\varphi^{p-2}) & \subset\cdots\subset & \mathrm{Im}(\varphi) & \subset & \pi^*(E). \end{array}$$

Since π_* is an exact functor, we have

$$\begin{array}{ccccccc} 0\neq \mathrm{Ker}(\chi) & \subset & \mathrm{Ker}(\chi^2) & \subset\cdots\subset & \mathrm{Ker}(\chi^{p-1}) & \subset & \pi_*(\pi^*(E)) \\ \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \| \\ 0\neq \mathrm{Im}(\chi^{p-1}) & \subset & \mathrm{Im}(\chi^{p-2}) & \subset\cdots\subset & \mathrm{Im}(\chi) & \subset & \pi_*(\pi^*(E)), \end{array}$$

where $\chi=\pi_*(\varphi)$. Since χ is an endomorphism of $\pi_*(\pi^*(E))\cong E^{\oplus p}$ and E is simple, we see that

$$\mathrm{Ker}(\chi^i)=\mathrm{Im}(\chi^{p-i})\cong E^{\oplus i} \qquad \text{for all} \quad i=1, 2,\dots, p.$$

This yields that $\mathrm{Ker}(\varphi^i)=\mathrm{Im}(\varphi^{p-i})$ and that $\pi_*(E_i)\cong E^{\oplus i}$ for all $i=1, 2,\dots, p$, where $E_i=\mathrm{Ker}(\varphi^i)$. Moreover, we have $\pi_*(E_i/E_{i-1})\cong E$ for all $i=1, 2,\dots, p$.

It suffices, therefore, to show that all the E_i are locally free $\mathcal{O}_Y$-modules. This follows from the following lemma. (Apply the lemma to the case where $f=\varphi^i$ and $g=\varphi^{p-i}$.)

Lemma 2.7. *Let R be a regular local ring and let M and N be free R-modules with finite ranks. Let $f: M\to N$ and $g: N\to M$ be homomorphisms such that $\mathrm{Im}(f)=\mathrm{Ker}(g)$ and that $\mathrm{Im}(g)=\mathrm{Ker}(f)$. Then $\mathrm{Ker}(f)$ and $\mathrm{Ker}(g)$ are free R-modules.*

Proof. The infinite complex

$$0 \longleftarrow \mathrm{Im}(g) \longleftarrow N \xleftarrow{f} M \xleftarrow{g} N \xleftarrow{f} M \longleftarrow \cdots\cdots$$

is considered as a free resolution of the R-module $\mathrm{Im}(g)$. Since the cohomological dimension of R is finite, both $\mathrm{Ker}(f)$ and $\mathrm{Ker}(g)$ are free. q.e.d.

For an isogeny $\pi: Y\to X$ and a vector bundle E_1 on Y, we see that $r(\pi_*(E_1))=(\deg\pi)\cdot r(E_1)$. This and Proposition 2.6 show the following.

Corollary 2.8. *For every simple vector bundle E on X, there exist an isogeny $\pi: Y\to X$ and a simple vector bundle E' on Y with $\Sigma(E')=0$, such that $\pi_*(E')\cong E$.*

§3. The group scheme $\Phi(E)$

For a line bundle L on X, the homomorphism $\phi_L: X\to\hat{X}$ can be defined. ϕ_L is the unique morphism such that

$$(1_X\times\phi_L)^*(\mathscr{P})\cong m^*(L)\otimes(p_1)^*(L)^{-1}\otimes(p_2)^*(L)^{-1},$$

where p_1 and p_2 are projections of $X\times X$ to X and $m: X\times X\to X$ is the group law of X. As a map $\phi_L(x)=T_x^*(L)\otimes L^{-1}$ for every point $x\in X$, where $T_x: X\to X$ is the translation of X by x. For details, see Mumford [10] § 13.

For a vector bundle E on X, we define an analogue corresponding to ϕ_L in the case of line bundles.

Definition 3.1. For a vector bundle E on X,

$$\Phi^\circ(E)=\{(x, \hat{x})\in X\times\hat{X}\,|\,T_x^*(E)\cong E\otimes P_{\hat{x}}\}.$$

Proposition 3.2. *$\Phi^\circ(E)$ is a closed subgroup of $X\times\hat{X}$.*

Proof. Let p_{12}, p_{13}, p_1, etc. be the various projections of $X\times X\times\hat{X}$. We shall apply the results in § 1 to the morphism $p_{23}: X\times X\times\hat{X}\to X\times\hat{X}$ and the couple of vector bundles $F=(p_{12})^*(m^*(E))\otimes(p_{13})^*(\mathscr{P})^{-1}$ and $G=(p_1)^*(E)$. Obviously $\Phi^\circ(E)=\{(x, \hat{x})\in X\times\hat{X}\,|\, F|_{X\times\{x\}\times\{\hat{x}\}}\cong G|_{X\times\{x\}\times\{\hat{x}\}}\}$. Thus $\Phi^\circ(E)$ is a constructible set by Proposition 1.5. On the other hand, $\Phi^\circ(E)$ is closed under the multiplication.

Hence our proposition is proved. q.e.d.

Let p°: $\Phi^\circ(E) \to X$ be the restriction of the projection of $X \times \hat{X}$ to X. Set-theoretically the kernel of p° is isomorphic to $\Sigma^\circ(E)$. Hence we have

Proposition 3.3. *For a vector bundle E on X, we have* $\dim \Phi^\circ(E) \le g$. *p° is surjective if and only if* $\dim \Phi^\circ(E) = g$.

If E is simple, we have more information on $\Phi^\circ(E)$. First we have

Proposition 3.4. *If E is a simple vector bundle, then $\Phi^\circ(E)$ is a union of some connected components of the closed set $\Psi^\circ(E) = \{(x, \hat{x}) \in X \times \hat{X} |$ there exists a non-zero homomorphism f: $T_x^*(E) \to E \otimes P_{\hat{x}}\}$.*

Proof. On one hand, Proposition 3.2 implies that $\Phi^\circ(E)$ is closed in $\Psi^\circ(E)$. On the other hand, it is open in $\Psi^\circ(E)$ by Proposition 1.7. Thus we have proved our proposition. q.e.d.

Secondly, we have

Definition 3.5. For a simple vector bundle E on X, $\Phi(E)$ is the maximal subscheme of $X \times \hat{X}$ over which F and G are isomorphic to each other, where F and G are the same as in the proof of Proposition 3.2.

By virtue of Theorem 1.8, $\Phi(E)$ represents the following functor on (Sch);

(3.5.1) $S \rightsquigarrow \{(h, f) \in \underline{X}(S) \times \underline{\hat{X}}(S) | T_h^*(E_S) \cong E_S \otimes P_f \otimes_{\mathcal{O}_S} N$ for some invertible sheaf N on $S\}$,

where $T_h = (m \circ (1 \times h), p_2)$: $X \times S \to X \times S$.

Clearly this functor is a subgroup functor of $X \times \hat{X}$. Hence $\Phi(E)$ is a subgroup scheme of $X \times \hat{X}$.

Let p (resp. q) be the restriction of the projection p_1: $X \times \hat{X} \to X$ (resp. p_2: $X \times \hat{X} \to \hat{X}$) to $\Phi(E)$.

Lemma 3.6. *$p^*(E) \cong L^{\oplus r}$ for some line bundle L on $\Phi(E)$, where $r = r(E)$.*

Proof. If h: $S \to X$ factors the morphism p: $\Phi(E) \to X$, then by (3.5.1), we have $T_h^*(E_S) \cong E_S \otimes L$ for some line bundle L on X_S. Restricting this isomorphism to $\{0\} \times S$, we have that $h^*(E) \cong (\mathcal{O}_S^{\oplus r}) \otimes L$. Putting $h = p$, we have our lemma. q.e.d.

By (2.3.1) and (3.5.1), we have

Lemma 3.7. *For a simple vector bundle E on X, the (scheme-theoretic) kernel of p is isomorphic to $\Sigma(E)$.*

For the morphism q, we can manage a similar business to p.

Definition 3.8. For a simple vector bundle E on X, $K(E)$ is the maximal subscheme of X over which the vector bundles $m^*(E)$ and $(p_1)^*(E)$ on $X \times X$ are isomorphic to each other, where $X \times X$ is regarded as an X-scheme via the projection $p_2: X \times X \to X$.

Under this definition we have

Proposition 3.9. *Let E be a simple vector bundle on X.*

(1) *$K(E)$ represents the following functor on* (Sch);

$$S \rightsquigarrow \{h \in \underline{X}(S) \mid T_h^*(E_S) \cong E_S \otimes_{\mathcal{O}_S} N \text{ for some invertible sheaf } N \text{ on } S\}. \tag{3.9.1}$$

(2) *There exist a line bundle L on $K(E)$ and an isomorphism*

$$\mu^*(E) \cong (p_1)^*(E) \otimes (p_2)^*(L) \tag{3.9.2}$$

on $X \times K(E)$, where μ is the restriction of $m: X \times X \to X$ to $X \times K(E)$.

(3) *The scheme-theoretic kernel of q is isomorphic to $K(E)$.*

We shall consider the relation between $\Phi^\circ(E)$ and $\Phi^\circ(F)$ in the case where E and F are related with each other.

Definition 3.10. For a vector bundle E on X, $\Phi^{\circ\circ}(E)$ is the neutral component of $\Phi^\circ(E)$.

Proposition 3.11. *Let F be a vector bundle on X.*

(1) *If E is a direct summand of F, then $\Phi^{\circ\circ}(F) \subseteq \Phi^{\circ\circ}(E)$.*

(2) *If F has a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

such that $F_i/F_{i-1} \cong E$ $(i = 1, 2, \ldots, n)$ for a simple vector bundle E, then $\Phi^{\circ\circ}(F) \subseteq \Phi^{\circ\circ}(E)$.

Proof. (1) Assume that $\Phi^{\circ\circ}(F) \not\subseteq \Phi^{\circ\circ}(E)$. We can choose an infinite sequence $a_1, a_2, \ldots$ of points in $\Phi^{\circ\circ}(F)$ so that a_i and a_j do not belong to the same coset of $\Phi^{\circ\circ}(E)$ whenever $i \neq j$. Put $a_i = (x_i, \hat{x}_i) \in X \times \hat{X}$. From the choice of a_i's, we see that $E_i = T_{x_i}^*(E) \otimes P_{\hat{x}_i}^{-1}$ is a direct summand of F for every i and that $E_i \not\cong E_j$ whenever $i \neq j$, which contradicts to the Krull-Schmidt theorem of locally free sheaves (see [2]).

(2) If $T_x^*(F) \cong F \otimes P_{\hat{x}}$, then there exists a non-zero homomorphism $f: T_x^*(E) \to E \otimes P_{\hat{x}}$. Hence $\Phi^\circ(F) \subseteq \Psi^\circ(E)$. Since E is simple, our assertion follows from Proposition 3.4. q.e.d.

Proposition 3.12. *Let $\pi: Y \to X$ be an isogeny and F a vector bundle on Y. For $E = \pi_*(F)$, we have*

$$(\pi \times 1)(\Phi^\circ(F) \times_{\hat{Y}} \hat{X}) \subseteq \Phi^\circ(E).$$

Proof. A point of $\Phi^\circ(F) \times_{\hat{Y}} \hat{X}$ corresponds to a point $(y, \hat{x}) \in Y \times \hat{X}$ such that $(y, \hat{\pi}(\hat{x})) \in \Phi^\circ(F)$. Hence $T_y^*(F) \cong F \otimes P_{\hat{\pi}(\hat{x})} \cong F \otimes \pi^*(P_{\hat{x}})$. Taking the direct image by π, we have

$$(T_{\pi(y)})^*(\pi_*(F)) \cong \pi_*(T_y^*(F)) \cong \pi_*(F) \otimes P_{\hat{x}}.$$

This proves our proposition. q.e.d.

Let us refine the above proposition in the case where both E and F are simple.

Lemma 3.13. *Let F be a simple vector bundle on Y. If $E=\pi_*(F)$ is simple, then*

$$K(F) \cap \mathrm{Ker}(\pi)=0,$$

where $\cap$ means the scheme-theoretic intersection.

Proof. The followings are easily verified;

$$\mathrm{End}_{\mathcal{O}_X}(E) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\pi^*\pi_*(F), F)$$

and
$$\pi^*\pi_*(F) \cong (p_1)_*(\nu^*(F)),$$

where ν is the restriction of $m\colon X \times X \to X$ to $X \times \mathrm{Ker}(\pi)$. Since $H=K(F) \cap \mathrm{Ker}(\pi)$ is finite, we have

$$\nu^*(F)|_{X \times H} \cong (p_1)^*(F)$$

by (3.9.2). Hence there exists a surjection $\pi^*\pi_*(F) \to F^{\oplus h}$, where h is the order of H. Therefore $h \le \dim_k \mathrm{Hom}_{\mathcal{O}_Y}(\pi^*\pi_*(F), F)=\dim_k \mathrm{End}_{\mathcal{O}_X}(E)$. By the assumption, $h=1$, which proves our lemma. q.e.d.

In the above lemma, the converse is true if F is a line bundle ([12] Theorem 1.2).

Proposition 3.14. *Let $\pi\colon Y \to X$ be an isogeny and F a simple vector bundle on Y. Assume that $E=\pi_*(F)$ is simple. Then the restriction α of $\pi \times 1_{\hat{X}}$ to $\Phi(F) \times_{\hat{Y}} \hat{X}$ factors through $\Phi(E)$. Moreover, α is a closed immersion.*

$$\begin{array}{ccc} Y \times \hat{X} & \xrightarrow{\pi \times 1} & X \times \hat{X} \\ \cup & & \cup \\ \Phi(F) \times_{\hat{Y}} \hat{X} & \hookrightarrow & \Phi(E) \end{array}$$

Proof. An S-valued point of $\Phi(F) \times_{\hat{Y}} \hat{X}$ corresponds to a pair $(h, f) \in \underline{Y}(S) \times \underline{\hat{X}}(S)$ such that $(h, \hat{\pi}(f)) \in \underline{\Phi(F)}(S)$. Hence

$$T_h^*(F_S) \cong F \otimes P_{\hat{\pi}(f)} \otimes_{\mathcal{O}_S} N$$

for some invertible sheaf N on S. Since $P_{\hat{\pi}(f)} \cong (\pi_S)^*(P_f)$, taking the direct image by π_S we have

$$T_{\pi(h)}{}^*(\pi_*(F)_S) \cong (\pi_S)_*(T_h^*(F_S)) \cong (\pi_*(F))_S \otimes P_f \otimes_{\mathcal{O}_S} N.$$

Hence $(\pi(h), f) \in \Phi(E)(S)$, which proves the first assertion. We see that

$$\mathrm{Ker}(\alpha) \cong (\Phi(F) \times_{\hat{Y}} \hat{X}) \cap (\mathrm{Ker}(\pi) \times \{\hat{0}\}) \cong K(F) \cap \mathrm{Ker}(\pi).$$

Hence by Lemma 3.13, we have our second assertion. q.e.d.

Corollary 3.15. *Under the situation of Proposition 3.14, we have the following.*

(1) *The projection* $\Phi(F)\times_{\hat{Y}}\hat{X}\to\hat{X}$ *induces a closed immersion*

$$((\mathrm{Ker}(\pi)\times\hat{Y})\cap\Phi(F))\times_{\hat{Y}}\hat{X}\hookrightarrow\Sigma(E).$$

Especially, when F is a line bundle, the projection of $\mathrm{Ker}(\pi)\times_{\hat{Y}}\hat{X}$ *to* $\hat{X}$ *induces a closed immersion*

$$\mathrm{Ker}(\pi)\times_{\hat{Y}}\hat{X}\hookrightarrow\Sigma(E), \tag{3.15.1}$$

where $\mathrm{Ker}(\pi)$ *is regarded as a* $\hat{Y}$*-scheme via* $\phi_F|_{\mathrm{Ker}(\pi)}\colon \mathrm{Ker}(\pi)\hookrightarrow\hat{Y}$.

(2) $\pi\colon Y\to X$ *induces a closed immersion*

$$K(F)\hookrightarrow K(E) \tag{3.15.2}$$

Proof. Restricting the morphism α in Proposition 3.14 to $\mathrm{Ker}(\pi)\times\hat{X}$ (resp. $Y\times\{\hat{0}\}$), we have the first assertion of (1) (resp. the assertion (2)). The second assertion of (1) is easily derived from the fact that for a line bundle F, $\Phi(F)$ is the graph of ϕ_L. q.e.d.

In the case where E is simple, $\dim\Phi(E)$ is closely related with the dimension of the local moduli of E. In fact,

Proposition 3.16. *Let E be a simple vector bundle on X and R its loacl moduli. Then we have*

$$g\le 2g-\dim\Phi(E)\le\dim R\le\dim_k H^1(X,\mathscr{E}nd_{\mathcal{O}_X}(E)).$$

Proof. The first inequality follows from Proposition 3.3 and the third inequality from Proposition 1.9. It suffices to show the second inequality. By the Poincaré's complete reducibility theorem, there exists an abelian subvariety Y of $X\times\hat{X}$ such that

$$Y\cap\Phi(E)\ \text{is finite,} \tag{3.16.1}$$

$$Y\ \text{and}\ \Phi(E)\ \text{generate}\ X\times\hat{X}. \tag{3.16.2}$$

Let us consider the family $F'=F|_{X\times Y}$ of vector bundles on X, where F is the same as in the proof of Proposition 3.2. Since $F'|_{X\times\{0\}}\cong E$, there exists a homomorphism $R\to\hat{\mathcal{O}}_{Y,0}$ of complete local rings. The dimension of the fibre $\hat{\mathcal{O}}_{Y,0}\otimes_R(R/\mathfrak{m})$ should be 0. For otherwise there exists an artin local k-algebra $A_n=\hat{\mathcal{O}}_{Y,0}/\mathfrak{a}_n$ of length n for an arbitrarily large n such that $F'|_{X\times\mathrm{Spec}(A_n)}$ is isomorphic to $E\otimes_k A_n$, which contradicts to (3.16.1). Hence we have that $\dim Y\le\dim R$. Since $\dim Y=2g-\dim\Phi(E)$, we have our proposition. q.e.d.

Corollary 3.17. *Under the situation of Proposition* 3.16, *we have*

(1) $\dim_k H^1(X,\mathscr{E}nd_{\mathcal{O}_X}(E))\ge g$

(2) *if* $\dim_k H^1(X,\mathscr{E}nd_{\mathcal{O}_X}(E))=g$, *then* $\dim\Phi(E)=g$ *and the ring R is a regular local ring.*

Remark 3.18. If $(r(E),p)=1$, then $\mathscr{E}nd_{\mathcal{O}_X}(E)$ contains $\mathcal{O}_X$ as a direct summand. Hence (1) of Corollary 3.17 is trivial in this case.

In the proof of Proposition 3.16, if $p=0$, then $Y \cap \Phi(E)$ is a discrete finite group. Hence the natural map

$$T_{Y,0} \longrightarrow H^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E))$$

is injective, where $T_{Y,0}$ is the tangent space of Y at 0. Thus, in this case, we can prove the inequality

$$2g-\dim \Phi(E) \leq \dim_k H^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E))$$

without aid of Proposition 1.9.

§4. Homogeneous vector bundles

For a scheme V, we denote by $\boldsymbol{C}_V$ the category of coherent $\mathcal{O}_V$-modules. Let us consider the functor $\mathscr{S}$ on $\boldsymbol{C}_{\hat{X}}$ to $\boldsymbol{C}_X$ defined by

$$\mathscr{S}(M)=(p_1)_*(\mathscr{P}\otimes_{\mathcal{O}_{\hat{X}}} M), \tag{4.1}$$

for every $M \in \boldsymbol{C}_{\hat{X}}$, where p_1 (resp. p_2) is the projection of $X \times \hat{X}$ to X (resp. $\hat{X}$).

Proposition 4.2. *$\mathscr{S}$ is a left exact functor. Moreover, $\mathscr{S}$ has a left adjoint functor $\mathscr{T}$ and for every $N \in \boldsymbol{C}_X$ and $M \in \boldsymbol{C}_{\hat{X}}$, there exists a functorial isomorphism*

$$(p_2)_*\mathscr{H}om_{\mathcal{O}_{X\times\hat{X}}}((p_1)^*(N), \mathscr{P}\otimes_{\mathcal{O}_{\hat{X}}}M) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_{\hat{X}}}(\mathscr{T}(N), M). \tag{4.2.1}$$

Proof. The functor $\mathscr{S}$ is a composition of the functor $\mathscr{S}'$ on $\boldsymbol{C}_{\hat{X}}$ to $\boldsymbol{C}_{X\times\hat{X}}$ and the functor $(p_1)_*$, where $\mathscr{S}'(M)=\mathscr{P}\otimes_{\mathcal{O}_X} M$ for every $M \in \boldsymbol{C}_{\hat{X}}$. Since $\mathscr{P}$ is an $\mathcal{O}_{\hat{X}}$-flat module, $\mathscr{S}'$ is an exact functor. Since p_2 is a proper morphism, $\mathscr{S}'$ has a left adjoint functor $\mathscr{T}'$ and for every $L \in \boldsymbol{C}_{X\times\hat{X}}$ and $M \in \boldsymbol{C}_{\hat{X}}$, there exists a functorial isomorphism

$$(p_2)_*\mathscr{H}om_{\mathcal{O}_{X\times\hat{X}}}(L, \mathscr{P}\otimes_{\mathcal{O}_{\hat{X}}}M) \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_{\hat{X}}}(\mathscr{T}'(L), M), \tag{4.2.2}$$

(see ***EGA*** III (7.7.2)). As is well known, $(p_1)_*$ is left exact and has a left adjoint $(p_1)^*$. Therefore $\mathscr{S}=((p_1)_*)\circ\mathscr{S}'$ is left exact and $\mathscr{T}=\mathscr{T}'\circ((p_1)^*)$ is the left adjoint of $\mathscr{S}$. Putting $L=(p_1)^*(N)$ in (4.2.2), we have (4.2.1). q.e.d.

Lemma 4.3. *Let $M \in \boldsymbol{C}_{\hat{X}}$ and $N \in \boldsymbol{C}_X$. Then we have the following isomorphisms:*

$$\mathscr{S}(M\otimes P_x)\cong T_x^*(\mathscr{S}(M)),\ \mathscr{S}(T_{\hat{x}}^*(M))\cong \mathscr{S}(M)\otimes P_{\hat{x}}^{-1}, \tag{4.3.1}$$

$$\mathscr{T}(N\otimes P_{\hat{x}})\cong T_{-\hat{x}}^*(\mathscr{T}(N)),\ \mathscr{T}(T_x^*(N))\cong \mathscr{T}(N)\otimes P_x, \tag{4.3.2}$$

where P_x is the line bundle in $\mathrm{Pic}^\circ(\hat{X})$ *corresponding to $x \in X \cong (\hat{\hat{X}})$.*

Proof. (4.3.1) is easily verified by the following properties of the Poincaré bundle $\mathscr{P}$;

$$\mathscr{P}\otimes_{\mathscr{O}_{\hat{X}}} P_x \cong (T_{(x,0)})^*(\mathscr{P})$$

and $$(T_{(0,\hat{x})})^*(\mathscr{P}) \cong \mathscr{P}\otimes_{\mathscr{O}_X} P_{\hat{x}}.$$

The isomorphisms (4.3.2) are derived by (4.3.1) and the adjointness property of $\mathscr{S}$ and $\mathscr{T}$. q.e.d.

Other properties of $\mathscr{S}$ and $\mathscr{T}$ will be pursued in a forthcoming paper. In this section we shall consider them in a special case to obtain some results related with homogeneous vector bundles.

Definition 4.4. A vector bundle F on X is homogeneous if $T_x^*(F)\cong F$ for every point x of X. We denote by $\boldsymbol{H}_X$ the full subcategory of $\boldsymbol{C}_X$ consisting of the zero sheaf and all homogeneous vector bundles on X.

Definition 4.5. A vector bundle F on X is unipotent if F has a filtration

$$0=F_0\subset F_1\subset\cdots\subset F_{n-1}\subset F_n=F$$

such that $F_i/F_{i-1}\cong\mathscr{O}_X$ for all $i=1,\dots, n$. We denote by $\boldsymbol{U}_X$ the full subcategory of $\boldsymbol{C}_X$ consisting of the zero sheaf and all unipotent vector bundles on X.

Definition 4.6. $\boldsymbol{C}_{\hat{X}}^f$ is the full subcategory of $\boldsymbol{C}_{\hat{X}}$ consisting of all coherent $\mathscr{O}_{\hat{X}}$-modules supported on a finite set.

Definition 4.7. We put $B=\mathscr{O}_{\hat{X},\hat{0}}$. $\mathrm{Mod}_f(B)$ is the category of B-modules of finite length. We identify $\mathrm{Mod}_f(B)$ with the full subcategory of $\boldsymbol{C}_{\hat{X}}^f$ consisting of all coherent $\mathscr{O}_{\hat{X}}$-modules with $\mathrm{Supp}(M)=\{\hat{0}\}$.

In the rest of this section we shall show that the functors $\mathscr{S}$ and $\mathscr{T}$ give us close relations among the above categories.

Lemma 4.8. *If $M\in\boldsymbol{C}_{\hat{X}}^f$, then $\mathscr{S}(M)\in\boldsymbol{H}_X$ and $r(\mathscr{S}(M))=\mathrm{length}(M)$. If $M\in\mathrm{Mod}_f(B)$, then $\mathscr{S}(M)\in\boldsymbol{U}_X$.*

Proof. If $M\in\boldsymbol{C}_{\hat{X}}^f$, then M has a composition series by $k(\hat{x})$, $\hat{x}\in\hat{X}$. Since the restriction of $\mathscr{S}$ to $\boldsymbol{C}_{\hat{X}}^f$ is exact, $\mathscr{S}(M)$ has a composition series by $\mathscr{S}(k(\hat{x}))=P_{\hat{x}}$. Hence $r(\mathscr{S}(M))=\mathrm{length}(M)$ and if $\mathrm{Supp}(M)=\{\hat{0}\}$, then $\mathscr{S}(M)$ is unipotent. The homogenity of $\mathscr{S}(M)$ follows from (4.3.1). q.e.d.

Lemma 4.9. *If $U\in\boldsymbol{U}_X$ and $\hat{x}\neq\hat{0}$, then $H^i(X, U\otimes P_{\hat{x}})=0$ for all i.*

Proof. It suffices to show the lemma when $U\cong\mathscr{O}_X$. But in that case it is well-known ([10] § 8).

Lemma 4.10. *If $U\in\boldsymbol{U}_X$, then we have*

$$\mathrm{length}\, R^g(p_2)_*((p_1)^*(U)\otimes\mathscr{P})=r(U).$$

Proof. There exists an exact sequence

$$0\longrightarrow V\longrightarrow U\longrightarrow\mathscr{O}_X\longrightarrow 0$$

for some $V\in \boldsymbol{U}_X$. Since $R^g(p_2)_*(\mathscr{P})\cong k(\hat{0})$ and $R^{g-1}(p_2)_*(\mathscr{P})=0$ ([10] § 13), we have

$$\operatorname{length} R^g(p_2)_*((p_1)^*(U)\otimes\mathscr{P})=\operatorname{length} R^g(p_2)_*((p_1)^*(V)\otimes\mathscr{P})+1.$$

Hence an induction on $r(U)$ completes our proof. q.e.d.

Proposition 4.11. *If $U\in \boldsymbol{U}_X$, then $\mathscr{T}(U)$ has a support at the origin $\hat{0}$.*

Proof. Putting $N=U$ and $M=k(\hat{x})$ in (4.2.1), we have

$$H^0(X,\ U^\vee\otimes P_{\hat{x}})\cong \operatorname{Hom}_{\mathcal{O}_{\hat{X}}}(\mathscr{T}(U),\ k(\hat{x})).$$

Hence, by Lemma 4.9, if $\hat{x}\neq 0$, then $\operatorname{Hom}_{\mathcal{O}_{\hat{X}}}(T(U),\ k(\hat{x}))=0$. Since $\operatorname{Hom}_{\mathcal{O}_{\hat{X}}}(\mathscr{T}(U), k(\hat{x}))\cong[\mathscr{T}(U)\otimes k(\hat{x})]^\vee$, the stalk $\mathscr{T}(U)_{\hat{x}}$ of $\mathscr{T}(U)$ at $\hat{x}$ is zero whenever $\hat{x}\neq\hat{0}$. Hence $\mathscr{T}(U)$ is supported at the origin $\hat{0}$. q.e.d.

By Lemma 4.8 and Proposition 4.11, $\mathscr{S}$ (resp. $\mathscr{T}$) defines a functor on $\operatorname{Mod}_f(B)$ (resp. $\boldsymbol{U}_X$) to $\boldsymbol{U}_X$ (resp. $\operatorname{Mod}_f(B)$), which we also denote by $\mathscr{S}$ (resp. $\mathscr{T}$). Then we have the following theorem.

Theorem 4.12.

(1) *$\mathscr{S}$ and $\mathscr{T}$ give an equivalence of categories between $\operatorname{Mod}_f(B)$ and $\boldsymbol{U}_X$.*

(2) *For $M, N\in\operatorname{Mod}_f(B)$, let $U=\mathscr{S}(M)$ and $V=\mathscr{S}(N)$. Then we have the following correspondences:*

i) $r(U)=\operatorname{length}(M)$.

ii) *the dual vector bundle $U^\vee$ of U corresponds to $(-1)_B^*\mathscr{D}(M)$, where $\mathscr{D}$ is the dualizing functor of $\operatorname{Mod}_f(B)$ and $(-1)_B$: $B\to B$ is the isomorphism induced by $(-1)_{\hat{X}}$: $\hat{X}\to\hat{X}$.*

iii) *$U\otimes V$ corresponds to $M*N$, where $M*N$ is $M\underset{k}{\otimes}N$ regarded as a B-module via the co-multiplication μ: $\hat{B}\to\hat{B}\otimes\hat{B}$ of the formal group $\hat{B}$.*

iv) *$H^i(X,\ U)\cong\operatorname{Ext}^i_B(k,\ M)$ for all i, where $k=B/\mathfrak{m}$.*

Remark 4.13. By the above theorem, the category $\boldsymbol{U}_X$ is determined by the local ring B, whence essentially by $g=\dim X$. But the operations $^\vee$ and $\otimes$ depend on the formal group scheme structure of B (especially in the case $p>0$).

Let $U\in\boldsymbol{U}_X$. By Lemma 4.6, $R^i(p_2)_*((p_1)^*(U^\vee)\otimes\mathscr{P})$ are concentrated at the origin. There exists a complex

$$K^\cdot:0\longrightarrow K^0\longrightarrow K^1\longrightarrow\cdots\longrightarrow K^{g-1}\longrightarrow K^g\longrightarrow 0$$

of locally free B-modules of finite rank which gives the direct images of $(p_1)^*(U^\vee)\otimes\mathscr{P}$ universally, that is, there exists an isomorphism of functors

$$R^i(p_2)_*((p_1)^*(U^\vee)\otimes\mathscr{P}\otimes_{\mathcal{O}_{\hat{X}}}M)\cong H^i(K^\cdot\otimes_B M) \tag{4.14.1}$$

on the category of B-modules M ([10] § 5, see also § 13). Since B is a regular local ring and $H^i(K^\cdot)$ are artinian modules, $H^i(K^\cdot)=0$ for $0\le i<g$. (See [10] § 13

Lemma. The regularity of B is not necessary. One sees easily that it is sufficient that B is Cohen-Macaulay.) Put $Q=R^g(p_2)_*((p_1)^*(U^\vee)\otimes\mathscr{P})=H^g(K^\cdot)$. Then the sequence

$$0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots \longrightarrow K^{g-1} \longrightarrow K^g \longrightarrow Q \longrightarrow 0.$$

is exact. Let

$$(K^\cdot)^\vee : 0 \longrightarrow (K^g)^\vee \longrightarrow (K^{g-1})^\vee \longrightarrow \cdots \longrightarrow (K^1)^\vee \longrightarrow (K^0)^\vee \longrightarrow 0$$

be the dual complex of $K^\cdot$. Then we also have that $H^i((K^\cdot)^\vee)=0$ for $0\leq i<g$. Putting $R=H^g((K^\cdot)^\vee)=\mathrm{Ext}^g_B(Q, B)$, we have the exact sequence

(4.14.2)
$$0 \longrightarrow (K^g)^\vee \longrightarrow (K^{g-1})^\vee \longrightarrow \cdots \longrightarrow (K^1)^\vee \longrightarrow (K^0)^\vee \longrightarrow R \longrightarrow 0.$$

Since B is Gorenstein, $\mathrm{Ext}^g_B(\cdot, B)$ is the dualizing functor $\mathscr{D}$ of B and we have

(4.14.3) $$\mathrm{length}(R)=\mathrm{length}(Q).$$

Lemma 4.15. $R\cong\mathscr{T}(U)$.

Proof. $\mathrm{Hom}_B(R, M)\cong \mathrm{Ker}[\mathrm{Hom}_B((K^0)^\vee, M) \longrightarrow \mathrm{Hom}_B((K^1)^\vee, M)]$

$$\cong \mathrm{Ker}[K^0\otimes_B M \longrightarrow K^1\otimes_B M]\cong H^0(K^\cdot\otimes_B M).$$

Hence by (4.14.1), we have

$$\mathrm{Hom}_B(R, M)\cong(p_2)_*((p_1)^*(U^\vee)\otimes\mathscr{P}\otimes_{\mathcal{O}_{\hat{X}}} M).$$

By this, (4.2.1) and Proposition 4.11, we have our lemma. q.e.d.

Proof of Theorem 4.12

(1) Since $\mathscr{T}$ is a left adjoint functor of $\mathscr{S}$, there exist morphisms of functors φ: $id_{\boldsymbol{U}_X}\to\mathscr{S}\circ\mathscr{T}$ and ψ: $\mathscr{T}\circ\mathscr{S}\to id_{\mathrm{Mod}_f(B)}$. It suffices to show that φ and ψ are isomorphisms. First we note that $\varphi(U)$ and $\psi(M)$ are not zero for every $U\neq 0$ in $\boldsymbol{U}_X$ and $M\neq 0$ in $\mathrm{Mod}_f(B)$. By Lemma 4.10, (4.14.3) and Lemma 4.15, we have

(4.16.1) $$\mathrm{length}\,\mathscr{T}(U)=r(U),$$

for every $U\in\boldsymbol{U}_X$. Hence by Lemma 4.8, we have

(4.16.2) $$r(\mathscr{S}(\mathscr{T}(U)))=r(U),$$

(4.16.3) $$\mathrm{length}(\mathscr{T}(\mathscr{S}(M)))=\mathrm{length}(M),$$

for every $U\in\boldsymbol{U}_X$ and $M\in\mathrm{Mod}_f(B)$. Since $r(\mathscr{S}(\mathscr{T}(\mathcal{O}_X)))=1$, $\mathscr{S}(\mathscr{T}(\mathcal{O}_X))\cong\mathcal{O}_X$ and $\varphi(\mathcal{O}_X)$ is an isomorphism. Let

$$0 \longrightarrow V \longrightarrow U \longrightarrow \mathcal{O}_X \longrightarrow 0$$

be an exact sequence in $\boldsymbol{U}_X$. Then we have the sequence

(4.16.4) $$0 \longrightarrow \mathscr{S}(\mathscr{T}(V)) \longrightarrow \mathscr{S}(\mathscr{T}(U)) \longrightarrow \mathscr{S}(\mathscr{T}(\mathcal{O}_X)) \longrightarrow 0$$

in $\boldsymbol{U}_X$. Since $\mathscr{T}$ is right exact and $\mathscr{S}$ is exact, (4.16.4) is exact in the middle and the right terms. By (4.16.2), we have

$$r(\mathscr{S}(\mathscr{T}(U)))=r(\mathscr{S}(\mathscr{T}(V)))+r(\mathscr{S}(\mathscr{T}(\mathcal{O}_X))).$$

Hence the sequence (4.16.4) in $\boldsymbol{U}_X$ is exact. Therefore we have the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "\varphi(V)"] & U \arrow[r] \arrow[d, "\varphi(U)"] & \mathcal{O}_X \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathscr{S}(\mathscr{T}(V)) \arrow[r] & \mathscr{S}(\mathscr{T}(U)) \arrow[r] & \mathscr{S}(\mathscr{T}(\mathcal{O}_X)) \arrow[r] & 0.
\end{tikzcd}$$

By the 5-lemma, if $\varphi(V)$ is an isomorphism, then $\varphi(U)$ is also an isomorphism. Hence by induction on $r(U)$, we see that $\varphi(U)$ is an isomorphism for every $U\in\boldsymbol{U}_X$. For the morphism ψ it is similarly proved.

(2) i) has been already proved in Lemma 4.8. We omit the proof of ii) and iii) since we shall not use them later. By Lemma 4.15, we have to compute $\mathrm{Ext}^i_B(k, R)$ to prove iv). Since B is Gorenstein, we have

(4.16.5) $$\begin{aligned}\mathrm{Ext}^i_B(k, B)&=0 \quad \text{for} \quad i\neq g,\\ &\cong k \quad \text{for} \quad i=g.\end{aligned}$$

Hence for every free B-module L, we have

(4.16.6) $$\begin{aligned}\mathrm{Ext}^i_B(k, L)&=0 \quad \text{for} \quad i\neq g,\\ &\cong L\otimes_B k \quad \text{for} \quad i=g.\end{aligned}$$

Hence by the exact sequence (4.14.2), we have

$$\mathrm{Ext}^i_B(k, R)\cong H^i((K^\cdot)^\vee\otimes_B k)\simeq H^i((K^\cdot\otimes_B k)^\vee)\cong[H^{g-i}(K^\cdot\otimes_B k)]^\vee.$$

By (4.14.1) and the duality theorem, we have

$$\mathrm{Ext}^i_B(k, R)\cong[H^{g-i}(X, U^\vee)]^\vee\cong H^i(X, U). \qquad \text{q.e.d.}$$

The following theorem was proved in Matsushima [6] and Morimoto [9], in the case $p=0$.

Theorem 4.17. *Let F be a vector bundle on X. Then the following conditions are equivalent.*

i) *F is homogeneous,*

ii) *there exist line bundles P_i in $\mathrm{Pic}^\circ(X)$ and $U_i\in\boldsymbol{U}_X$ such that $F\cong\bigoplus_i(P_i\otimes U_i)$.*

Proof. i)⇒ii): This was proved in Miyanishi [7]. ii)⇒i): Since $P_i\in\mathrm{Pic}^\circ(X)$, $T^*_x(P_i)\cong P_i$ for every $x\in X$. Hence it suffices to show that every unipotent vector bundle U is homogeneous. By virtue of Theorem 4.12, (1), there exists $M\in\mathrm{Mod}_f(B)$ such that $U\cong\mathscr{S}(M)$. Hence by Lemma 4.8, U is homogeneous.

q.e.d.

As a corollary to the above theorem, we have the following proposition which will play an important role later.

Proposition 4.18. *Let F be a homogeneous vector bundle.*

(1) *There exists a line bundle M in* $\mathrm{Pic}^{\circ}(X)$ *such that $H^0(X, F\otimes M)\neq 0$.*

(2) *The following conditions are equivalent.*

i) $H^0(X, F)\neq 0$,

ii) $H^0(X, F^{\vee})\neq 0$,

iii) $H^1(X, F)\neq 0$,

iv) *F contains a unipotent vector bundle $U\neq 0$ as a direct summand.*

Proof. By Lemma 4.9 and Theorem 4.17, i) and ii) are equivalent to iv), and iii) implies iv). If $U\neq 0$, then $\mathscr{T}(U)\neq 0$. Since $\mathscr{T}(U)$ is artinian, $\mathrm{Ext}^i_B(k, \mathscr{T}(U))\neq 0$ for $0\leq i\leq g$. Therefore iv) implies iii). q.e.d.

The category $\boldsymbol{H}_X$ is determined as follows.

Theorem 4.19. *$\mathscr{S}$ and $\mathscr{T}$ give an equivalence of categories between $\boldsymbol{C}_X^f$ and $\boldsymbol{H}_X$.*

Proof. As in the proof of (1) of Theorem 4.12, we have only to show that φ: $id_{\boldsymbol{H}_X}\to\mathscr{S}\circ\mathscr{T}$ and ψ: $\mathscr{T}\circ\mathscr{S}\to id_{\boldsymbol{C}_{\hat{X}}^f}$ are isomorphisms. Let $F\in\boldsymbol{H}_X$. We have to show that $\varphi(F)$ is an isomorphism. Since $\mathscr{S}$ and $\mathscr{T}$ are additive functors, we may assume that F is indecomposable. Then, by virtue of Theorem 4.12, $F\cong P_{\hat{x}}\otimes U$ for some $\hat{x}\in\hat{X}$ and $U\in\boldsymbol{U}_X$. By Lemma 4.3, we have only to show the assertion in the case $F\cong U$. In this case φ is an isomorphism by Theorem 4.12. For the morphism ψ, it is similar. q.e.d.

Let π: $Y\to X$ be an isogeny. The following is obvious.

Lemma 4.20. *If $F\in\boldsymbol{H}_Y$ and $G\in\boldsymbol{H}_X$, then $\pi_*(F)\in\boldsymbol{H}_X$ and $\pi^*(G)\in\boldsymbol{H}_Y$.*

Hence π_* (resp. π^*) defines a functor of $\boldsymbol{H}_Y$ (resp. $\boldsymbol{H}_X$) to $\boldsymbol{H}_X$ (resp. $\boldsymbol{H}_Y$). As a special case of Lemma 4.20, $\pi_*(\mathcal{O}_Y)$ is a homogeneous vector bundle on X. In fact, we have

Proposition 4.21. *$\pi_*(\mathcal{O}_Y)\cong(p_1)_*(\mathscr{P}|_{X\times G})$, that is, $\pi_*(\mathcal{O}_X)\cong\mathscr{S}(\mathcal{O}_G)$, where $G=\mathrm{Ker}(\hat{\pi})$.*

For the proof see [12]. As a corollary to the above we have,

Corollary 4.22. *If $\hat{\pi}$ is separable, then $\pi_*(\mathcal{O}_Y)\cong\bigoplus_{\hat{x}\in\mathrm{Ker}(\hat{\pi})}P_{\hat{x}}$. If $\hat{\pi}$ is purely inseparable, then $\pi_*(\mathcal{O}_Y)$ is an indecomposable unipotent vector bundle on X.*

Proposition 4.21 is a special case of the following theorem.

Theorem 4.23. *Let π: $Y\to X$ be an isogeny and $\hat{\pi}$: $\hat{X}\to\hat{Y}$ the dual isogeny of π. Then the following diagrams are commutative.*

$$\begin{array}{ccc} \boldsymbol{C}^f_{\hat{X}} & \xrightarrow[\sim]{\mathscr{S}} & \boldsymbol{H}_X \\ {\scriptstyle (\hat{\pi})^*}\uparrow & & \uparrow{\scriptstyle \pi_*} \\ \boldsymbol{C}^f_{\hat{Y}} & \xrightarrow[\sim]{\mathscr{S}} & \boldsymbol{H}_Y \end{array}$$

and

$$\begin{array}{ccc} \boldsymbol{C}^f_{\hat{X}} & \xrightarrow[\sim]{\mathscr{S}} & \boldsymbol{H}_X \\ {\scriptstyle (\hat{\pi})_*}\downarrow & & \downarrow{\scriptstyle \pi^*} \\ \boldsymbol{C}^f_{\hat{Y}} & \xrightarrow[\sim]{\mathscr{S}} & \boldsymbol{H}_Y \end{array}$$

The proof is not difficult. We omit it since we shall not use it later.

§5. Semi-homogeneous vector bundles

Every vector bundle E on an abelian variety X has two natural deformations. The one is the family $\{T_y^*(E) | x \in X\}$ and the other is the family $\{E \otimes P_{\hat{x}} | \hat{x} \in \hat{X}\}$. The algebraic group $\Phi^\circ(E)$ considered in §3 represents a relation between these two deformations. The object studied in the rest of this paper is the vector bundle E such that the former family is included in the latter.

Proposition 5.1. *Let E be a vector bundle on X. Then the following conditions are equivalent:*

(1) *for every $x \in X$, there exists a line bundle L on X such that $T_x^*(E) \cong E \otimes L$,*

(2) *$p^\circ : \Phi^\circ(E) \longrightarrow X$ is surjective,*

(3) $\dim \Phi^\circ(E) = g$,

(4) *for every $x \in X$, there exists an isomorphism φ_x of $\boldsymbol{P}(E)$ which covers the translation T_x, where $\boldsymbol{P}(E)$ is the projective bundle associated with E.*

Proof. The equivalence of (1) and (4) follows from the well-known fact that $\boldsymbol{P}(E) \cong \boldsymbol{P}(F)$ as X-schemes if and only if $E \cong F \otimes L$ with some line bundle L on X. If $T_x^*(E) \cong E \otimes L$, then we have $T_x^*(\det(E)) \cong \det(E) \otimes L^{\otimes r}$, where $r = r(E)$. Hence $L^{\otimes r} \in \mathrm{Pic}^\circ(X)$. Since $\mathrm{NS}(X)$ is torsion free, L itself is contained in $\mathrm{Pic}^\circ(X)$. Hence (1) implies (2). The converse is trivial. The equivalence of (2) and (3) was shown in Proposition 3.3. q.e.d.

Definition 5.2. A vector bundle E on X is semi-homogeneous if E satisfies the equivalent conditions of Proposition 5.1.

Our aim of this section is to characterize semi-homogeneous vector bundles. In the first place, we have

Proposition 5.3. *Let F be a semi-homogeneous vector bundle on X.*

(1) *Every direct summand of F is semi-homogeneous.*

(2) *If F has a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

such that $F_i/F_{i-1} \cong E$ $(i=1, 2,\dots, n)$ *for a simple vector bundle* E, *then* E *is also semi-homogeneous.*

Proof. This is an immediate consequence of Proposition 3.11 and Proposition 5.1. q.e.d.

Semi-homogeneous vector bundles behave nicely under isogenies.

Proposition 5.4. *Let* π: $Y \to X$ *and* τ: $X \to Z$ *be isogenies of abelian varieties. Let* E *be a vector bundle on* X.

(1) *If* E *is semi-homogeneous, then so are* $\pi_*(E)$ *and* $\tau^*(E)$.

(2) *Assume that* E *is simple or* $\hat{\pi}$ *(resp.* τ*) is separable. If* $\pi^*(E)$ *(resp.* $\tau_*(E)$*) is semi-homogeneous, then so is* E.

Proof. (1) Since π is surjective, it is easily seen that $\pi^*(E)$ is semi-homogeneous. It easily follows from Proposition 3.12 that $\tau_*(E)$ is semi-homogeneous.

(2) If $\pi^*(E)$ is semi-homogeneous then $\pi_*\pi^*(E) \cong E \otimes \pi_*(\mathcal{O}_Y)$ is semi-homogeneous by (1). If $\hat{\pi}$ is separable, then, by Corollary 4.22, $\pi_*(\mathcal{O}_Y)$ decomposes into the direct sum $\bigoplus_{\hat{x} \in \mathrm{Ker}(\hat{\pi})} P_{\hat{x}}$. Hence E is contained in $\pi_*\pi^*(E)$ as a direct summand. Therefore, by (1) of Proposition 5.3, E is semi-homogeneous. Suppose that E is simple. By the above result, we may assume that $\hat{\pi}$ is purely inseparable. Then, by Corollary 4.22, $\pi_*(\mathcal{O}_Y)$ is unipotent. Hence our assertion follows from (2) of Proposition 5.3.

If $\tau_*(E)$ is semi-homogeneous, then $\tau^*\tau_*(E) = (p_1)_*\nu^*(E)$ is semi-homogeneous, where ν: $X \times \mathrm{Ker}(\tau) \to X$ is the restriction of the multiplication m: $X \times X \to X$. If τ is separable, then E is a direct summand of $\tau^*\tau_*(E)$. Hence E is semi-homogeneous. Therefore, we may assume that τ is purely inseparable and $\deg(\tau) = p$. Set $G = \mathrm{Ker}(\tau)$. Since $\mathcal{O}_G$ is an artin local ring and $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$, $\mathcal{O}_G \cong k[T]/(T^p)$ (which also follows from the structure theorem of local group schemes). Hence the filtration

$$0 = \mathfrak{m}^p \subset \mathfrak{m}^{p-1} \subset \cdots \subset \mathfrak{m} \subset \mathcal{O}_G$$

induces a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{p-1} \subset F_p = \tau^*\tau_*(E)$$

such that $F_i/F_{i-1} \cong E$ $(1 \le i \le p)$. Therefore, as before, we see that if E is simple, then it is semi-homogeneous. q.e.d.

Remark 5.5. In Proposition 5.3, (2) and 5.4, (2), the assumption of simpleness or separability of isogeny is superfluous. We shall show it in a forthcoming paper.

As a special case of (1) of Proposition 5.4, we have that $E = \pi_*(L)$ is semi-homogeneous for every line bundle L on Y.

Proposition 5.6. *It is necessary and sufficient for* E *to be simple that* $\mathrm{Ker}(\pi)$

$\cap K(L)=(0)$.

Proof. We have seen the necessity in Lemma 3.13. For the sufficiency, see [12] Theorem 1.2. q.e.d.

Lemma 5.7. *Let E be a simple semi-homogeneous vector bundle on X. If $\Sigma(E)=0$, then E is a line bundle.*

Proof. The assumption implies that $p\colon \Phi(E)\to X$ is an isomorphism. In view of Lemma 3.6, $r(E)$ must be one. q.e.d.

We are now ready to prove the main theorem in this section.

Theorem 5.8. *Let E be a simple vector bundle on X. Then the following conditions are equivalent to one another*:

(1) $\dim_k H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(E))=g$,

(1′) $\dim_k H^j(X, \mathcal{E}nd_{\mathcal{O}_X}(E))=\binom{g}{j}$ *for all* $j=1, 2, \dots, g$,

(2) *E is semi-homogeneous,*

(3) *$\mathcal{E}nd_{\mathcal{O}_X}(E)$ is a homogeneous vector bundle,*

(4) *there exist an isogeny $\pi\colon Y\to X$ and a line bundle L on Y such that $E\cong\pi_*(L)$.*

Proof. We shall prove the theorem following the diagram below:

$$\begin{array}{ccccc} (1) & \Longrightarrow & (2) & \Longleftrightarrow & (4) \\ \Uparrow & & \Downarrow & & \\ (1') & \Longleftarrow & (3) & & \end{array}$$

The implication (1)⇒(2) is an immediate consequence of Corollary 3.17, (2) and Proposition 5.1. The implication (2)⇒(3) is obvious.

(3)⇒(1′): Let A be as in (2.2). Then $A=\mathcal{T}(\mathcal{E}nd_{\mathcal{O}_X}(E))$ by (4.2.1). Since E is simple, we have, by Lemma 2.4, that $A\cong\mathcal{O}_{\Sigma(E)}\oplus A'$ for some A' and that $\mathrm{Supp}(A')\cap\Sigma(E)=(0)$. Since $\mathcal{E}nd_{\mathcal{O}_X}(E)$ is homogeneous, $A\in \boldsymbol{C}^f_{\hat{X}}$. Therefore, A' is also contained in $\boldsymbol{C}^f_{\hat{X}}$. Since $\hat{0}\notin\mathrm{Supp}(A')$, $H^i(X, \mathcal{S}(A'))=0$ for all i. Thus we have

(5.8.1) $$H^i(X, \mathcal{E}nd_{\mathcal{O}_X}(E))\cong H^i(X, \mathcal{S}(\mathcal{O}_{\Sigma(E)})\oplus\mathcal{S}(A'))\cong H^i(X, \mathcal{S}(\mathcal{O}_{\Sigma(E)}))$$

for all i.

Let G be the connected component of identity in $\Sigma(E)$. By the similar reason, we have

(5.8.2) $$H^i(X, \mathcal{S}(\mathcal{O}_{\Sigma(E)}))\cong H^i(X, \mathcal{S}(\mathcal{O}_G)) \quad \text{for all } i.$$

By virtue of Theorem 4.12, we have

(5.8.3) $$\dim_k H^i(X, \mathcal{S}(\mathcal{O}_G))=\mathrm{length}_B\,\mathrm{Ext}^i_B(k, \mathcal{O}_G),$$

where $B=\mathcal{O}_{\hat{X},\hat{0}}$ and $k=k(\hat{0})=B/\mathfrak{m}$. By virtue of Cartier's theorem of local group schemes ([3] III, §3, 6.3), we have that $\mathcal{O}_G\cong B/\sum_{i=1}^{g} t_i^{p^{\alpha_i}}B$, where $\{t_i\}$ is a regular system of parameters of B and α_i's are some non-negative integers. Using the Koszul complex we see easily that

$$\text{(5.8.4)} \qquad \mathrm{length}_B \mathrm{Ext}^i_B(k, \mathcal{O}_G) = \binom{g}{i} \qquad \text{for all} \quad i=0, 1,\dots, g.$$

By (5.8.1), (5.8.2), (5.8.3) and (5.8.4), we have (1′).

The implication (1′)⇒(1) is obvious and the implication (4)⇒(2) is a special case of Proposition 5.4 as we have seen before Lemma 5.6.

(2)⇒(4): By Corollary 2.8, there exist an isogeny $\pi: Y \to X$ and a vector bundle E' on Y with $\Sigma(E')=0$ such that $\pi_*(E') \cong E$. By (2) of Proposition 5.4, E' is also a simple semi-homogeneous vector bundle. Therefore, E' is a line bundle by Lemma 5.7, which implies (4). q.e.d.

The natural injection i of $\mathcal{O}_X$ to $\mathcal{E}nd_{\mathcal{O}_X}(E)$ induces a homomorphism $H^1(i)$ of $H^1(X, \mathcal{O}_X)$ to $H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(E))$. As is stated before Lemma 2.5, if $(r(E), p) = 1$, then $H^1(i)$ is injective. Surprisingly, the converse is true if E is simple and semi-homogeneous. In fact,

Proposition 5.9. *Let E be a simple vector bundle on X. Then the following conditions are equivalent*:

(1) *$H^1(i)$: $H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(E))$ is an isomorphism,*

(1′) *$H^j(i)$: $H^j(X, \mathcal{O}_X) \xrightarrow{\sim} H^j(X, \mathcal{E}nd_{\mathcal{O}_X}(E))$ is an isomorphism for all $j=1, 2,\dots, g$,*

(2) *E is semi-homogeneous and $(r(E), p)=1$,*

(3) *E is semi-homogeneous and $\Sigma(E)$ is reduced.*

Proof. (1)⇒(1′): Since $\dim_k H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(E)) = \dim_k H^1(X, \mathcal{O}_X) = g$, $\mathcal{E}nd_{\mathcal{O}_X}(E)$ is homogeneous by virtue of Theorem 5.8, whence so is $\mathcal{P}gl(E)$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{i} \mathcal{E}nd_{\mathcal{O}_X}(E) \longrightarrow \mathcal{P}gl(E) \longrightarrow 0.$$

Since $H^1(i)$ is injective and E is simple, we see that $H^0(X, \mathcal{P}gl(E))=0$. By virtue of Theorem 4.17 and Lemma 4.9, we have that $H^j(X, \mathcal{P}gl(E))=0$ for all j. By the above exact sequence, we have (1′).

(1′)⇒(2): Consider the trace map

$$Tr: \mathcal{E}nd_{\mathcal{O}_X}(E) \longrightarrow \mathcal{O}_X.$$

By the canonical isomorphism $[\mathcal{E}nd_{\mathcal{O}_X}(E)]^\vee \rightleftarrows \mathcal{E}nd_{\mathcal{O}_X}(E)$, Tr is identified with the dual homomorphism $i^\vee$ of i. Therefore, by the duality theorem, $H^1(Tr)$ is identified with

$$H^{g-1}(i)^\vee : [H^{g-1}(X, \mathcal{E}nd_{\mathcal{O}_X}(E))]^\vee \longrightarrow [H^{g-1}(X, \mathcal{O}_X)]^\vee.$$

Hence, by the assumption, both $H^1(i)$ and $H^1(Tr)$ are isomorphisms. It follows that $H^1(Tr \circ i)$ is an isomorphism. Since $H^1(Tr \circ i)(a) = r \cdot a$ for every $a \in H^1(X, \mathcal{O}_X)$, we have $(r, p)=1$.

The implication (2)⇒(3) follows from the fact that $\Sigma(E)$ is a closed subscheme of $(\hat{X})_r$. The implication (3)⇒(1) is obvious if one takes Lemma 2.5 into account.

q.e.d.

As an application of the above, we have

Proposition 5.10. *Assume that $p>0$ and that the p-rank of X is maximal. If E is a simple semi-homogeneous vector bundle on X, then there exist a separable isogeny $\pi\colon Y\to X$ and a line bundle L on Y such that $E\cong\pi_*(L)$.*

Proof. If $(r(E), p)=1$, then the assertion is clear. Assume that p divides $r(E)$. Then $\Sigma(E)$ is not reduced by virtue of Proposition 5.9. Since the p-rank of X is maximal, $\Sigma(E)$ contains a subgroup scheme G isomorphic to μ_p ([10] § 15). By *Step* I in the proof of Proposition 2.6, there exist an isogeny $\pi'\colon X'\to X$ and a vector bundle E' on X' such that a) $(\pi')_*(E')\cong E$ and b) $\mathrm{Ker}(\pi')\cong\hat{G}\cong\boldsymbol{Z}/p\boldsymbol{Z}$. b) means that π' is separable. By virtue of (2) of Proposition 5.4, E' is also simple and semi-homogeneous. Hence repeating the above argument we have our proposition.

q.e.d.

§ 6. The category $\mathbf{S}_\delta$

Semi-homogeneous vector bundles are characterized in the preceding section when they are simple. In this section we shall study semi-homogeneous vector bundles which may not be simple. Let us begin with a definition.

Definition 6.1. Let E be a vector bundle on X. A vector bundle F on X is said to be E-potent if F has a filtration

$$0=F_0\subset F_1\subset\cdots\subset F_{n-1}\subset F_n=F$$

such that $F_i/F_{i-1}\cong E$ for all $i=1, 2,\dots, n$. We denote by $\boldsymbol{U}_{X,E}$ the full subcategory of $\boldsymbol{C}_X$ consisting of all E-potent vector bundles and the zero sheaf on X.

It is easily seen that if E is a simple vector bundle, then $\boldsymbol{U}_{X,E}$ is an abelian subcategory of $\boldsymbol{C}_X$ and E is a unique "simple" object in $\boldsymbol{U}_{X,E}$.

For a vector bundle E on X, a natural functor

$$\alpha_E\colon \boldsymbol{U}_X\longrightarrow\boldsymbol{U}_{X,E}$$

is defined by $\alpha_E(U)=U\otimes E$ and $\alpha_E(\varphi)=\varphi\otimes 1$ for each U, $U'\in\boldsymbol{U}_X$ and $\varphi\in\mathrm{Hom}_{\mathcal{O}_X}(U, U')$.

Proposition 6.2. *If E is a simple semi-homogeneous vector bundle and if $(r(E), p)=1$, then α_E is an equivalence of categories.*

Proof. Consider the natural injection

$$\lambda\colon \mathcal{H}om_{\mathcal{O}_X}(V, U)\longrightarrow\mathcal{H}om_{\mathcal{O}_X}(V\otimes E, U\otimes E)$$

for U, $V\in\boldsymbol{U}_X$. By the canonical isomorphism $\mathcal{H}om_{\mathcal{O}_X}(V\otimes E, U\otimes E)\cong\mathcal{E}nd_{\mathcal{O}_X}(E)$

$\otimes \mathcal{H}om_{\mathcal{O}_X}(V, U)$, λ is identified with $i \otimes \mathcal{H}om_{\mathcal{O}_X}(V, U)$, where $i: \mathcal{O}_X \to \mathcal{E}nd_{\mathcal{O}_X}(E)$ is the natural injection. By Proposition 5.9, $H^j(i)$ is an isomorphism for all j. Hence by the 5-lemma argument, we see that $H^j(i \otimes W)$ is an isomorphism for every $W \in \boldsymbol{U}_X$. Thus we have two natural isomorphisms

(6.2.1) $\quad H^0(\lambda): \mathrm{Hom}_{\mathcal{O}_X}(V, U) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(V \otimes E, U \otimes E)$,

(6.2.2) $\quad H^1(\lambda): H^1(X, \mathcal{H}om_{\mathcal{O}_X}(V, U)) \xrightarrow{\sim} H^1(X, \mathcal{H}om_{\mathcal{O}_X}(V \otimes E, U \otimes E))$.

By (6.2.1), α_E is a full embedding. Hence it suffices to show the following lemma.

Lemma. For every $F \in \boldsymbol{U}_{X,E}$, there exists a $U \in \boldsymbol{U}_X$ such that $F \cong U \otimes E$.

Proof. We prove our lemma by induction on $r(F)$. If $r(F) \leq r(E)$, then the assertion is trivial. If $r(F) > r(E)$, then there is an exact sequence $0 \to F' \to F \to F'' \to 0$ in $\boldsymbol{U}_{X,E}$ such that both F' and F'' are non-zero. By induction hypothesis, there exist U and V in $\boldsymbol{U}_X$ such that $F' \cong U \otimes E$ and $F'' \cong V \otimes E$. The isomorphism (6.2.2) means that every extension of F'' by F' is derived by tensoring E from some extension of V by U. Hence there exists a $W \in \boldsymbol{U}_X$ such that $F \cong W \otimes E$, which completes our proof. q.e.d.

Let $\pi: Y \to X$ be an isogeny and L a line bundle on Y. For $E = \pi_*(L)$, a functor

$$\beta = \beta_{\pi,L}: \boldsymbol{U}_Y \longrightarrow \boldsymbol{U}_{X,E}$$

is defined by $\beta(U) = \pi_*(U \otimes L)$ and $\beta(\varphi) = \pi_*(\varphi \otimes 1)$ for each $U, U' \in \boldsymbol{U}_Y$ and $\varphi \in \mathrm{Hom}_{\mathcal{O}_Y}(U, U')$. In this situation we have

Proposition 6.3. *If π is separable and E is simple, then β is an equivalence of categories.*

Proof. Let $U, V \in \boldsymbol{U}_Y$ and $\varphi \in \mathrm{Hom}_{\mathcal{O}_Y}(V, U)$. Then φ induces a homomorphism $\varphi \otimes 1$ of $V \otimes L$ to $U \otimes L$. $\beta(\varphi)$ is, by definition, the homomorphism $\pi_*(\varphi \otimes 1)$ of $\pi_*(V \otimes L)$ to $\pi_*(U \otimes L)$. Since π^* is a left adjoint of π_*, there exists an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\pi_*(V \otimes L), \pi_*(U \otimes L)) \xleftarrow[\sim]{\theta} \mathrm{Hom}_{\mathcal{O}_Y}(\pi^*\pi_*(V \otimes L), U \otimes L),$$

of functors on U and V. It is easily seen that $\theta^{-1}(\beta(\varphi))$ is the composition $\pi^*\pi_*(V \otimes L) \xrightarrow{\varepsilon} V \otimes L \xrightarrow{\varphi \otimes 1} U \otimes L$, where ε is the canonical homomorphism. Similarly, the natural homomorphism

(6.3.1)
$$\begin{aligned} \pi_*(\mathcal{H}om_{\mathcal{O}_Y}(V \otimes L, U \otimes L)) &\longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\pi_*(V \otimes L), \pi_*(U \otimes L)) \\ &\cong \pi_*(\mathcal{H}om_{\mathcal{O}_Y}(\pi^*\pi_*(V \otimes L), U \otimes L)) \end{aligned}$$

equals to $\pi_*(\mu)$, where

$$\mu: \mathcal{H}om_{\mathcal{O}_Y}(V \otimes L, U \otimes L) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\pi^*\pi_*(V \otimes L), U \otimes L)$$

is the homomorphism induced by the natural surjection ε. On the other hand, we see that

(6.3.2) $$\pi^*\pi_*(V\otimes L)\cong \bigoplus_{a\in \mathrm{Ker}(\pi)} T_a^*(V\otimes L)\cong V\otimes L\otimes(\bigoplus_{a\in\mathrm{Ker}(\pi)} P_{\phi_L(a)})$$

because π is separable and V is homogeneous (Theorem 4.17). Since E is simple, we have, by Lemma 3.13, that $\phi_L(a)\neq 0$ for every $a\neq 0$ in $\mathrm{Ker}(\pi)$. Hence the restriction $\varepsilon(a)$ of ε to $V\otimes L\otimes P_{\phi_L(a)}$ is zero for every $a\neq 0$ in $\mathrm{Ker}(\pi)$ by Lemma 4.9. On the other hand, since ε is surjective, $\varepsilon(0)$ is an isomorphism. Hence μ is an isomorphism onto the direct summand $\mathscr{H}om_{\mathcal{O}_Y}(V\otimes L, U\otimes L)$ of $\mathscr{H}om_{\mathcal{O}_Y}(\pi^*\pi_*(V\otimes L), U\otimes L)$. For the other direct summands $H_a=\mathscr{H}om_{\mathcal{O}_Y}(V\otimes L\otimes P_{\phi_L(a)}, U\otimes L)$, we see that $H^i(X, H_a)=0$ for all i, where $a\neq 0$. Therefore, $H^i(\mu)$ is an isomorphism for each i. Hence, by (6.3.1), we have isomorphisms

(6.3.3) $\mathrm{Hom}_{\mathcal{O}_Y}(V\otimes L, U\otimes L)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{O}_X}(\pi_*(V\otimes L), \pi_*(U\otimes L))$

(6.3.4) $H^1(Y, \mathscr{H}om_{\mathcal{O}_Y}(V\otimes L, U\otimes L))\xrightarrow{\sim} H^1(X, \mathscr{H}om_{\mathcal{O}_X}(\pi_*(V\otimes L), \pi_*(U\otimes L)))$

It is easily seen that the homomorphism (6.3.3) is β and that the homomorphism (6.3.4) sends each extension class $0\to U\to W\to V\to 0$ in $\boldsymbol{U}_Y$ to the extension class $0\to\beta(U)\to\beta(W)\to\beta(V)\to 0$ in $\boldsymbol{U}_{X,E}$. Hence the rest is similar to the latter part of the proof of Proposition 6.2. q.e.d.

As a corollary to the above two proposition, we have

Proposition 6.4. *Let E be a simple semi-homogeneous vector bundle on X. Assume that one of the following conditions holds*:

(1) $(r(E), p)=1$,

(2) *the p-rank of X is maximal.*

Then every E-potent vector bundle is semi-homogeneous.

Proof. By Theorem 4.17, every unipotent vector bundle is homogeneous. Hence, if (1) holds, our assertion follows from Proposition 6.3 and if (2) holds, our assertion follows from Proposition 5.4 and 6.4.

Remark 6.5. Proposition 6.4 is true without the assumption (1) or (2). We shall prove it in a forthcoming paper.

Let E be a vector bundle on X. We denote by $\delta(E)$ the equivalence class of $\frac{\det(E)}{r(E)}$ in $\mathrm{NS}(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$. The following equalities are easily verified:

$$\delta(E\otimes F)=\delta(E)+\delta(F)$$

and $$\delta(E^\vee)=-\delta(E).$$

Definition 6.6. For a $\delta\in\mathrm{NS}(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, $\boldsymbol{S}_\delta$ is the full subcategory of $\boldsymbol{C}_X$ consisting of all semi-homogeneous vector bundles E with $\delta(E)=\delta$ and the zero sheaf.

If $F\in\boldsymbol{U}_{X,E}$, then it is clear that $\delta(F)=\delta(E)$. Thus Proposition 6.4 implies that if E is a simple vector bundle in $\boldsymbol{S}_\delta$, then $\boldsymbol{S}_\delta$ contains $\boldsymbol{U}_{X,E}$ under the assumption (1) or (2) in the proposition. We shall show that $\boldsymbol{S}_\delta$ is covered by such $\boldsymbol{U}_{X,E}$'s

(Proposition 6.18).

Lemma 6.7. *Let E be a semi-homogeneous vector bundle on X. Then for all $x \in X$, we have*

$$T_{rx}^{*}(E) \cong E \otimes P_{\phi_D(x)},$$

where $r = r(E)$ and $D = \det(E)$.

Proof. Since E is semi-homogeneous,

$$T_x^*(E) \cong E \otimes M \tag{6.7.1}$$

for some line bundle M in $\mathrm{Pic}^\circ(X)$. Hence we have

$$T_{rx}^*(E) \cong E \otimes M^{\otimes r} \tag{6.7.2}$$

On the other hand, taking determinants of both sides of (6.7.1), we have $T_x^*(\det(E)) \cong \det(E) \otimes M^{\otimes r}$. This and (6.7.2) prove our lemma. q.e.d.

Lemma 6.8. *Let F and G be semi-homogeneous vector bundles. Then $\delta(F) = \delta(G)$ if and only if $\Phi^{\circ\circ}(F) = \Phi^{\circ\circ}(G)$.*

Proof. By Lemma 6.8, $\Phi^\circ(F)$ contains $Y_F = \{(rx, \phi_D(x)) \in X \times \hat{X} | x \in X\}$, where $r = r(F)$ and $D = \det(F)$. Since $\dim \Phi^\circ(F) = \dim Y_F$ $(=g)$ and Y_F is irreducible, we have $\Phi^{\circ\circ}(F) = Y_F$. It is easily seen that $Y_F = Y_G$ if and only $\delta(F) = \delta(G)$. Hence we have our lemma. q.e.d.

Proposition 6.9. *Let F and G be vector bundles on X. Let $\delta \in \mathrm{NS}(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$.*

(1) *$F \in \boldsymbol{S}_0$ if and only if F is homogeneous.*

(2) *$F \oplus G \in \boldsymbol{S}_\delta$ if and only if both F and G are contained in $\boldsymbol{S}_\delta$.*

Proof. If F is homogeneous, then F is semi-homogeneous and $\det(F)$ is also homogeneous, which implies that $F \in \boldsymbol{S}_0$. The converse is obvious by Lemma 6.7. (2) is an immediate consequence of Lemma 6.8 and (1) of Proposition 3.11. q.e.d.

Let F and G be vector bundles in $\boldsymbol{S}_\delta$. Then $\mathscr{H}om_{\mathcal{O}_X}(F, G)$ is a homogeneous vector bundle by (1) of the above proposition. Applying Proposition 4.18 to this vector bundle, we have

Proposition 6.10. *Let F and G be vector bundles in $\boldsymbol{S}_\delta$.*

(1) *There exists a non-zero homomorphism $\varphi: F \to G \otimes M$ for some line bundle M in* $\mathrm{Pic}^\circ(X)$.

(2) *The following conditions are equivalent:*

i) *there exists a non-zero homomorphism $f: F \to G$,*

ii) *there exists a non-zero homomorphism $g: G \to F$,*

iii) $H^1(X, \mathscr{H}om_{\mathcal{O}_X}(F, G)) \neq 0$.

Let E be a semi-homogeneous vector bundle. We put $r = r(E)$. For every line bundle L on X and integer l, $(l_X)^*(L) \sim L^{\otimes l^2}$, where $\sim$ means the algebraic equivalence ([10] § 8). Since

$$\det((r_X)^*(E)) \cong (r_X)^*(\det(E)) \sim \det(E)^{\otimes r^2},$$

we have $\det((r_X)^*(E) \otimes \det(E)^{\otimes -r}) \sim \mathcal{O}_X$. Hence, by (1) of Proposition 6.9, we have

Lemma 6.11. *$(r_X)^*(E) \cong \det(E)^{\otimes r} \otimes H$, where H is a homogeneous vector bundle.*

By Theorem 4.17, for every line bundle L we have $\chi(L \otimes H) = r \cdot \chi(L)$. Hence

$$\begin{aligned} r^{2g} \cdot \chi(E) &= \chi((r_X)^*(E)) = \chi(\det(E)^{\otimes r} \otimes H) \quad \text{(Lemma 6.11)} \\ &= r \cdot \chi(\det(E)^{\otimes r}) = r^{g+1} \cdot \chi(\det(E)). \end{aligned}$$

Thus we have proved

Proposition 6.12. *If E is a semi-homogeneous vector bundle, then $\chi(E) = \dfrac{\chi(\det(E))}{r(E)^{g-1}}$.*

We shall fix an ample line bundle $\mathcal{O}(1)$ on X and use the terms "stable" and "semi-stable" in the sense of Gieseker [4]. For a non-torsion coherent $\mathcal{O}_X$-module E we denote by P_E the polynomial such that $P_E(m) = \dfrac{\chi(E(m))}{r(E)}$ for all integers m, where $r(E)$ is the rank of E at the generic point of X.

Proposition 6.13. *Every semi-homogeneous vector bundle is semi-stable.*

Proof. Obviously, every unipotent vector bundle is semi-stable. Therefore, by Theorem 4.17, every homogeneous vector bundle is semi-stable. Let F be a semi-homogeneous vector bundle. By Lemma 6.11, we see that $F' = (r_X)^*(F)$ is semi-stable, where $r = r(F)$. If G is an $\mathcal{O}_X$-submodule of F, then $G' = (r_X)^*(G)$ is an $\mathcal{O}_X$-submodule of F'. Hence $P_{G'}(m) \leq P_{F'}(m)$ for $m \gg 0$. Therefore, by the following lemma, $P_G(m) \leq P_F(m)$ for $m \gg 0$, which completes our proof.

Lemma 6.14. *Let F be a coherent $\mathcal{O}_X$-module. Let $F' = (r_X)^*(F)$. Then $P_{F'}(r^2 m) = r^{2g} P_F(m)$ for all integers m, where $r = r(F)$.*

Proof. Obviously, $r(F) = r(F')$. On the other hand,

$$\chi(F'(r^2 m)) = \chi((r_X)^*(F) \otimes \mathcal{O}(r^2 m)) = \chi((r_X)^*(F(m)))$$

because $\mathcal{O}(r^2)$ is algebraically equivalent to $(r_X)^*(\mathcal{O}(1))$. Hence $\chi(F'(r^2 m)) = (\deg(r_X)) \cdot \chi(F(m)) = r^{2g} \cdot \chi(F(m))$. Thus our lemma is proved. q.e.d.

By Lemma 6.13, every semi-homogeneous vector bundle has a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

such that $E_i = F_i / F_{i-1}$ is stable and $P_{E_i} = P_F$ for all $i = 1, 2, \ldots, n$. Moreover, $E_1, \ldots, E_n$ are determined uniquely up to permutations (see [4]). We fix an index i. If

$(x, \hat{x}) \in \Phi^\circ(F)$, then $T_x^*(F) \otimes P_{\hat{x}}^{-1} \cong F$. Hence, for each $(x, \hat{x}) \in \Phi^\circ(F)$, there exists an index j such that $T_x^*(E_i) \otimes P_x^{-1} \cong E_j$. In other words, $\Phi^\circ(F) \subseteq \bigcup_{1 \leq j \leq n} \Phi_{ij}$, where $\Phi_{ij} = \{(x, \hat{x}) \in X \times \hat{X} \mid T_x^*(E_i) \cong E_j \otimes P_{\hat{x}}\}$. Since E_i is stable, Φ_{ij} is a closed subset of $X \times \hat{X}$ (Langton [5]. Though the concept of stability in it differs from what we are using in this section, the proof works in our case without any modifications). Moreover, Φ_{ii} is a closed subgroup and Φ_{ij} is a coset of Φ_{ii} if $\Phi_{ij} \neq \phi$. Hence we have that $\Phi^{\circ\circ}(F) \subseteq \Phi_{ii}$. Thus E_i satisfies the condition (1) of Proposition 5.1. Therefore, E_i is a semi-homogeneous vector bundle (the fact that E_i is locally free is deduced immediately from that condition) and $\delta(E_i) = \delta(F)$ by Lemma 6.8. Thus we have proved

Proposition 6.15. *Every vector bundle F in $\boldsymbol{S}_\delta$ has a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

such that $E_i = F_i / F_{i-1}$ is a stable vector bundle in $\boldsymbol{S}_\delta$ for all $i = 1, 2, \ldots, n$.

Proposition 6.16. *Every simple semi-homogeneous vector bundle is stable.*

Proof. Let E be a simple vector bundle in $\boldsymbol{S}_\delta$. Assume that E is not stable. By Proposition 6.15, there exists a proper subbundle E' of E which is stable and belongs to $\boldsymbol{S}_\delta$. Then by (2) of Proposition 6.10, there exists a non-zero homomorphism $g: E \to E' \subset E$, which contradicts to our assumption. q.e.d.

By Proposition 6.12, we see that if $E, E' \in \boldsymbol{S}_\delta$, then $P_E = P_{E'}$. Hence by the above proposition, we see that for every simple vector bundles E and E' in $\boldsymbol{S}_\delta$, every non-zero homomorphism $f: E \to E'$ is an isomorphism (see [4]).

Proposition 6.17. *Let E and E' be simple vector bundles in $\boldsymbol{S}_\delta$.*

(1) *There exists a line bundle M in* $\mathrm{Pic}^\circ(X)$ *such that $E' \cong E \otimes M$.*

(2) *Let F (resp. F') be an E (resp. E')-potent vector bundle. If $E \not\cong E'$, then* $\mathrm{Hom}_{\mathcal{O}_X}(F, F') = 0$ *and* $H^1(X, \mathcal{H}om_{\mathcal{O}_X}(F, F')) = 0$.

Proof. (1) follows from the above result and (1) of Proposition 6.10. If $E \not\cong E'$, then $\mathrm{Hom}_{\mathcal{O}_X}(E, E') = 0$, whence $H^1(X, \mathcal{H}om_{\mathcal{O}_X}(E, E')) = 0$ by (2) of Proposition 6.10. Put $H = \mathcal{H}om_{\mathcal{O}_X}(E, E')$. Since $\mathcal{H}om_{\mathcal{O}_X}(F, F')$ is H-potent, $H^0(X, H) = 0$ and $H^1(X, H) = 0$, we have (2) by cohomology exact sequence. q.e.d.

The followings are main results in this section.

Proposition 6.18. *If $F \in \boldsymbol{S}_\delta$, then there exist simple vector bundles $E_1, \ldots, E_n$ in $\boldsymbol{S}_\delta$ such that $F \cong \bigoplus_{i=1}^{n} F_i$, where $F_i \in \boldsymbol{U}_{X, E_i}$ for all $i = 1, 2, \ldots, n$.*

Proof. We prove our proposition by induction on $r(F)$. By Proposition 6.15, there exists an exact sequence $0 \to E \to F \xrightarrow{\varphi} G \to 0$, where $E, G \in \boldsymbol{S}_\delta$ and E is simple. If $G \in \boldsymbol{U}_{X,E}$, then so is F. If $G \notin \boldsymbol{U}_{X,E}$, then by induction hypothesis, there exists a simple vector bundle E' which is not isomorphic to E such that $G \cong G' \oplus G''$, for some $G' \in \boldsymbol{U}_{X,E'}$. We see that $F = \varphi^{-1}(G') + \varphi^{-1}(G'')$ and $\varphi^{-1}(G') \cap \varphi^{-1}(G'') = E$. By (2) of Proposition 6.17, the exact sequence $0 \to E \to \varphi^{-1}(G') \to G' \to 0$ splits. Hence $F \cong G' \oplus \varphi^{-1}(G'')$, which completes our proof because $G', \varphi^{-1}(G'') \in \boldsymbol{S}_\delta$ by (2) of Proposition 6.9. q.e.d.

Let $\boldsymbol{C}_i$ ($i \in I$) be a family of abelian categories. Let $\boldsymbol{C} = \bigoplus_{i \in I} \boldsymbol{C}_i$ be the following category:

(1) An object a of $\boldsymbol{C}$ is a sequence $(a_i)_{i \in I}$ such that $a_i \in \boldsymbol{C}_i$ for every $i \in I$ and $a_i = 0$ for all but a finite number of i.

(2) A morphism $f: (a_i) \to (b_i)$ is a collection of morphisms $f_i: a_i \to b_i$ in $\boldsymbol{C}_i$ for every $i \in I$.

Theorem 6.19. *Let $\delta \in \mathrm{NS}(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$.*

(1) *$\boldsymbol{S}_\delta$ is a full subcategory of $\bigoplus_E \boldsymbol{U}_{X,E}$, where E runs over the isomorphic classes of all simple vector bundles in $\boldsymbol{S}_\delta$.*

(2) *Assume that one of the following conditions holds.*

i) *For some (any) simple vector bundle E in $\boldsymbol{S}_\delta$, $(r(E), p) = 1$.*

ii) *The p-rank of X is maximal.*

Then the category $\boldsymbol{S}_\delta$ is equivalent to the category $\bigoplus_E \boldsymbol{U}_{X,E}$.

Proof. (1) is an immediate consequence of Proposition 6.18 and (2) of Proposition 6.17. (2) follows from (1) and Proposition 6.4. q.e.d.

Remark 6.20. As was stated in Remark 6.5, the condition i) or ii) of (2) in the above theorem is superfluous.

Here we have to show that the category $\boldsymbol{S}_\delta$ is not trivial.

Lemma 6.21. *Let $\pi: Y \to X$ be an isogeny and E a vector bundle on Y. For $F = \pi_*(E)$, we have $\pi^*(\det(F)) \sim (\det(E))^{\otimes d}$, where $d = \deg(\pi)$ and $\sim$ means the algebraic equivalence.*

Proof. Assume $\pi = \pi_1 \circ \pi_2$ for some isogenies $\pi_1: Z \to X$ and $\pi_2: Y \to Z$. It is easily seen that if our lemma holds for π_1 and π_2, then it does for π. Hence we may assume that $\deg(\pi)$ is a prime number. If π is separable, then $\pi^*(\pi_*(E)) \cong \bigoplus_{x \in \mathrm{Ker}(\pi)} T_x^*(E)$. Since $\det(T_x^*(E)) \sim \det(E)$, we have $\pi^*(\det(F)) \sim \det(E)^{\otimes d}$. Assume that π is purely inseparable and $d = p$. As we have seen in the proof of Proposition 5.4, $\pi^*(\pi_*(E))$ has a filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{p-1} \subset G_p = \pi^*(F)$$

such that $G_i/G_{i-1} \cong E$ for all $i = 1, 2, \ldots, p$. Hence $\det(\pi^*(F)) \cong \det(E)^{\otimes d}$. q.e.d.

Proposition 6.22. *For every $\delta \in \mathrm{NS}(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, there exists a vector bundle in $\boldsymbol{S}_\delta$.*

Proof. Let $\delta = \dfrac{[L]}{l}$, where $[L]$ is the equivalence class of L in $\mathrm{NS}(X)$ and l is a positive integer. Put $F = (l_X)_*(L^{\otimes l})$. By Proposition 5.4, F is semi-homogeneous. By the above lemma, we have

$$\det(F)^{\otimes l^2} \sim (l_X)^*(\det(F)) \sim L^{\otimes l^{2g+1}}$$

Since $r(F)=l^{2g}$, we have $\delta(F)=\frac{l^{2g-1}[L]}{l^{2g}}=\frac{[L]}{l}=\delta$. Therefore, F is contained in $\boldsymbol{S}_\delta$. q.e.d.

By Proposition 6.15, we have

Corollary 6.23. *For every $\delta \in \mathrm{NS}(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, there exists a simple semi-homogeneous vector bundle E with $\delta(E)=\delta$.*

§7. Simple semi-homogeneous vector bundles

In the preceding section we have seen that for every $\delta \in \mathrm{NS}(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, there exists a simple vector bundle in $\boldsymbol{S}_\delta$ (Corollary 6.23). Proposition 6.17 tells us that simple vector bundles in $\boldsymbol{S}_\delta$ are unique up to tensoring line bundles in $\mathrm{Pic}^\circ(X)$. Furthermore, Theorem 6.19 shows that simple vector bundles play a key role in $\boldsymbol{S}_\delta$. Hence our task is to study more closely simple vector bundles in $\boldsymbol{S}_\delta$. In this section we shall investigate the various group schemes associated with those vector bundles.

Proposition 7.1. *Let E be a simple semi-homogeneous vector bundle on X. Then $\mathrm{ord}\,\Sigma(E)=r(E)^2$ and $\mathcal{E}nd_{\mathcal{O}_X}(E)\cong\mathcal{S}(\mathcal{O}_{\Sigma(E)})$.*

Since E is stable and $\mathcal{E}nd_{\mathcal{O}_X}(E)$ is homogeneous, our proposition follows from Theorem 4.19 and the following lemma.

Lemma 7.2. *For every stable vector bundle E, $\mathcal{O}_{\Sigma(E)}\cong\mathcal{T}(\mathcal{E}nd_{\mathcal{O}_X}(E))$.*

Proof. Let A be as in (2.2). Since E is stable, for every line bundle L in $\mathrm{Pic}^\circ(X)$, every non-zero homomorphism $f\colon E\to E\otimes L$ is an isomorphism. Hence $\mathrm{Supp}(A)=\mathrm{Supp}(\Sigma(E))$. Therefore, our lemma follows from Lemma 2.4. q.e.d.

Here we add one property which characterizes simple semi-homogeneous vector bundles.

Proposition 7.3. *For a simple vector bundle E on X, the following conditions are equivalent*:

i) *E is semi-homogeneous,*

ii) *There exist an isogeny $\pi\colon Y\to X$ and a line bundle M on Y such that $\pi^*(E)\cong L^{\oplus r}$, where $r=r(E)$.*

Proof. Put $R=\Phi^{\circ\circ}(E)$. Since E is simple, R is considered as the reduced scheme associated with the neutral component of $\Phi(E)$. Let p' be the restriction to R of the projection $p_1\colon X\times\hat{X}\to X$. If E is semi-homogeneous, then $p'\colon R\to X$ is an isogeny. Hence the implication i)$\Rightarrow$ii) is obtained from Lemma 3.6. The converse is clear by (2) of Proposition 5.4. q.e.d.

We shall investigate such an isogeny as in ii) of the above proposition. Let $\rho\colon Z\to X$ be the dual of the natural isogeny $\hat{X}\to\hat{X}/\Sigma(E)$, where Z is the dual abelian variety of $\hat{X}/\Sigma(E)$.

Lemma 7.4. *Let E be a stable vector bundle. For every isogeny π: $Y \to X$, we have*

$$\dim_k \mathrm{End}_{\mathcal{O}_Y}(\pi^*(E)) = \mathrm{ord}\,(\Sigma(E) \cap \mathrm{Ker}\,(\hat{\pi})).$$

Proof. The following isomorphisms of k-vector spaces are easily verified:

$$\mathrm{End}_{\mathcal{O}_Y}(\pi^*(E)) \cong \mathrm{Hom}_{\mathcal{O}_X}(E, \pi_*\pi^*(E)) \cong \mathrm{Hom}_{\mathcal{O}_X}(E, E \otimes \pi_*(\mathcal{O}_Y))$$
$$\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}nd_{\mathcal{O}_X}(E), \pi_*(\mathcal{O}_Y)).$$

By Proposition 4.21. $\pi_*(\mathcal{O}_Y) \cong \mathcal{S}(\mathcal{O}_{\mathrm{Ker}(\hat{\pi})})$. Hence, by the definition of $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}nd_{\mathcal{O}_X}(E), \pi_*(\mathcal{O}_Y)) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}(\mathcal{E}nd_{\mathcal{O}_X}(E)), \mathcal{O}_{\mathrm{Ker}(\hat{\pi})})$. This and Lemma 7.2 complete our proof. q.e.d.

Lemma 7.5. *Let E be a simple semi-homogeneous vector bundle. Let π: $Y \to X$ be an isogeny of abelian varieties. If $\pi^*(E) \cong M^{\oplus r}$ for some line bundle M on Y, then there exists an isogeny π': $Y \to Z$ such that $\pi = \rho \circ \pi'$.*

Proof. If $\pi^*(E) \cong M^{\oplus r}$, then $\dim_k \mathrm{End}_{\mathcal{O}_Y}(\pi^*(E)) = r^2$. Hence by Proposition 7.1 and Lemma 7.4, $\Sigma(E) \subseteq \mathrm{Ker}\,(\hat{\pi})$. There exists an isogeny f: $\hat{X}/\Sigma(E) \to \hat{Y}$ such that $\hat{\pi} = f \circ \hat{\rho}$. Hence we have $\pi = \rho \circ \pi'$ for $\pi' = \hat{f}$. q.e.d.

Proposition 7.6. *Let E be a simple semi-homogeneous vector bundle and p the restriction to $\Phi(E)$ of the projection p_1: $X \times \hat{X} \to X$.*

(1) *There exists an isomorphism φ: $\Phi(E) \xrightarrow{\sim} Z$ such that $\rho \circ \varphi = p$.*

(2) *For an isogeny π: $Y \to X$, the following conditions are equivalent:*

i) *$\pi^*(E) \cong M^{\oplus r}$ for some line bundle M on Y,*

ii) *There exists an isogeny π': $Y \to \Phi(E)$ such that $\pi = \mathrm{p} \circ \pi'$.*

Proof. Let p': $R \to X$ be as in the proof of Proposition 7.3. By Lemma 7.5, there is an isogeny φ: $R \to Z$ such that $p' = \rho \circ \varphi$. By Lemma 3.7, $\mathrm{Ker}\,(p) \cong \Sigma(E)$. Hence $\deg(p') \leqq r^2$. On the other hand, $\deg(\rho) = r^2$. Hence $\deg(\varphi) = 1$ and $\deg(p') = r^2$, which implies that φ is an isomorphism and $R \cong \Phi(E)$. Thus we have proved (1). (2) is an immediate consequence of (1) and Lemma 7.5. q.e.d.

Among others the above proposition implies that $\Phi(E)$ is an abelian variety for every simple semi-homogeneous vector bundle E. Therefore, by Lemma 6.7, we have

Proposition 7.7. *If E is simple and semi-homogeneous, then $\Phi(E) = \mathrm{Im}$ $[X \xrightarrow{(r_X, \phi_D)} X \times \hat{X}]$, where $r = r(E)$ and $D = \det(E)$.*

Corollary 7.8. *Let E be a simple semi-homogeneous vector bundle on X. Then there are exact sequences of group schemes*

$$(7.8.1) \qquad 0 \longrightarrow X_r \cap K(D) \longrightarrow X_r \xrightarrow{\phi_D} \Sigma(E) \longrightarrow 0$$

$$(7.8.2) \qquad 0 \longrightarrow X_r \cap K(D) \longrightarrow K(D) \xrightarrow{r_X} K(E) \longrightarrow 0$$

Proof. By the proposition

$$0 \longrightarrow X_r \cap K(D) \longrightarrow X \xrightarrow{\alpha} \Phi(E) \longrightarrow 0$$

is an exact sequence, where $\alpha=(r_X, \phi_D)$. Since $\Sigma(E)=\mathrm{Ker}(p)$, $\alpha^{-1}(\Sigma(E))=\mathrm{Ker}(\alpha\circ p)=X_r$. Similarly $\alpha^{-1}(K(E))=K(D)$. By these isomorphisms, our corollary is clear. q.e.d.

Corollary 7.9. *For a simple semi-homogeneous vector bundle E, we have*
(1) $\dim K(E)=\dim K(D)$,
(2) *if* $\chi(E)\neq 0$, *then* $\mathrm{ord}\, K(E)=\chi(E)^2$.

Proof. (1) is clear by the exact sequence (7.8.2). If $\chi(E)\neq 0$, then $\chi(D)\neq 0$ by Proposition 6.12. Hence $\mathrm{ord}\, K(D)=\chi(D)^2$ (see [10] § 16). By the exact sequences (7.8.1) and (7.8.2), we have

$$\frac{\mathrm{ord}\, X_r}{\mathrm{ord}\, \Sigma(E)}=\frac{\mathrm{ord}\, K(D)}{\mathrm{ord}\, K(E)}\,.$$

Since $\mathrm{ord}\, \Sigma(E)=r^2$ and $\chi(D)^2=r^{2g-2}\cdot\chi(E)^2$ (Proposition 6.12), (2) is easily derived by direct computation. q.e.d.

Remark 7.10. In general, we can prove the inequalities $\mathrm{ord}\, \Sigma(E)\leq r(E)^2$ and in the case $\chi(E)\neq 0$, $\mathrm{ord}\, K(E)\leq\chi(E)^2$ for every simple vector bundle E on X.

We summarize the results in this section in the following theorem.

Theorem 7.11. *Let* $\delta=\frac{[L]}{l}$, *where* $[L]$ *is the equivalence class in* $\mathrm{NS}(X)$ *of a line bundle L and l is a positive integer.*
(1) *There exists a simple vector bundle* $E=E_\delta$ *in* $\boldsymbol{S}_\delta$.
(2) *Every simple vector bundle in* $\boldsymbol{S}_\delta$ *is isomorphic to* $E\otimes M$ *for some line bundle M in* $\mathrm{Pic}^\circ(X)$.
(3) $\Phi(E)=\mathrm{Im}\,[X\xrightarrow{(l_X,\phi_L)} X\times\hat{X}]$.
(4) *There are exact sequences of group schemes*

$$0 \longrightarrow X_l\cap K(L) \longrightarrow X_l \xrightarrow{\phi_L} \sum(E) \longrightarrow 0$$

and

$$0 \longrightarrow X_l\cap K(L) \longrightarrow K(L) \xrightarrow{l_X} K(E) \longrightarrow 0.$$

(5) $\mathrm{ord}(X_l\cap K(L))=u^2$ *for some positive integer u. For this u, we have* $r(E)=\frac{l^g}{u}$ *and* $\chi(E)=\frac{\chi(L)}{u}$.

Proof. (1) and (2) has been proved in §6. If $\frac{[L]}{l}=\frac{[L']}{l'}$ in $\mathrm{NS}(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, then $\mathrm{Im}\,[X\xrightarrow{(l_X,\phi_L)} X\times\hat{X}]=\mathrm{Im}\,[X\xrightarrow{(l_{X'},\phi_{L'})} X\times\hat{X}]$. Hence (3) and (4) follow from Proposition 7.7 and Corollary 7.8. (5) is an immediate consequence of (4). q.e.d.

Corollary 7.12. *Assume that l is minimal among the positive integers l' such that* $\delta=\frac{[L']}{l'}$ *for some line bundle L'. Then*

(1) *l divides $r(E)$ and $r(E)$ divides l^g. Especially, $l \leq r(E) \leq l^g$.*

(2) *If $(l, \chi(L))=1$, then $r(E)=l^g$ and $\chi(E)=\chi(L)$. Moreover, $\Sigma(E)=X_l$ and $K(E)=K(L)$.*

Remark 7.13. Let (X, L) be a principally polarized abelian variety. By the above theorem, for each pair of integers (n, m) with $n>0$ and $(n, m)=1$, there exists a simple semi-homogeneous vector bundle $E_{n,m}$ with $r(E_{n,m})=n^g$, $\det(E) = L^{\otimes mn^{g-1}}$ and $\chi(E_{n,m})=m^g$. In addition, if $\mathrm{NS}(X) \cong \boldsymbol{Z}$, then they are all the simple semi-homogeneous vector bundles on X modulo tensoring line bundles in $\mathrm{Pic}^\circ(X)$.

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