J. Math. Kyoto Univ. (JMKYAZ)
18-2 (1978) 229-238

On stochastic control of a Wiener process

By

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(Communicated by Prof. S. Watanabe, May 4, 1977)

Introduction

In this paper we shall discuss the problem of linear stochastic control.

In the stochastic control, contrary to the deterministic one, there are few examples in which the optimal controls can be simply calculated even in the linear case. R. S. Liptser [4] showed an interesting example of partially observable stochastic control in which an optimal control exists and furthermore it is written in an explicit form. In his paper he treated a simple one-dimensional linear system with bounded measurable control functions and simple cost function.

We shall extend his result to the multidimensional case and obtain an optimal trajectory corresponding to the optimal control. The method we use relies on the theory of filtering due to Fujisaki-Kallianpur-Kunita [1] and on a comparison theorem for solutions of stochastic differential equations due to Ikeda-Watanabe [3]. Since the latter is essentially one-dimensional, we must necessarily restrict the control system to be of a simple form. Nevertheless our result has remarkable features such that an optimal control exists in very wide class in which the control function is bounded measurable but not necessarily continuous (indeed, in our case an optimal control is not continuous) and the cost function is not continuous, while in the majority of the results which are already obtained both functions are smooth (cf. W. M. Wonham [6]). We shall mention this point concretely in the section 2.

The author expresses his hearty thanks to Prof. S. Watanabe for his valuable advices and to Prof. T. Komatsu and Prof. H. Kunita for informative discussions.

§1. Formulations

Let T be a positive number. Let $W=(W_t)$ and $w=(w_t)$, $0\leq t\leq T$, be m-dimensional Wiener process with $W_0=0$ and n-dimensional one with $w_0=0$ respectively. Let $\theta=(\theta_t)$ and $\zeta=(\zeta_t)$, $0\leq t\leq T$, be m-dimensional stochastic process and n-dimensional one respectively, which are given by the following linear stochastic differential equation with initial value $(\theta_0, 0)$;

$$(1.1)\qquad \begin{cases} d\theta_t = u_t dt + B_t dW_t \\ d\zeta_t = a_t\theta_t + b_t dw_t, \qquad \zeta_0 = 0, \end{cases}$$

where the coefficients B_t, b_t, and a_t are all non-random matrix-valued functions of t, which are bounded measurable. B_t and b_t are an (m, m)-orthogonal matrix and an (n, n)-orthogonal matrix respectively, and a_t is an (n, m)-matrix such that $a_t^* a_t = c_1 I_m$ where c_1 is positive constant and I_m is (m, m)-identity matrix, a_t^* represents the transposed matrix of a_t and from now on it is assumed that every vector is column one and transposed matrix or vector is written with $*$. $u = (u_t)$ is the *control* which depends upon the apriori distribution of θ_0 and the informations being obtained by $\{\zeta_s, s \leq t\}$. So we call m-vector (θ_t) the *state of channel*, n-vector (ζ_t) the *output*.

Now we consider the following problem.

Problem. Find a control u in suitable class, which minimizes the following cost function $J(u)$;

$$(1.2)\qquad J(u) = E^u\left[\int_0^T \omega(\theta_t)\, dt\right],$$

where (θ_t) is the solution of the stochastic equation (1.1), and function $\omega(x)$, $x \in R^m$, is defined by

$$(1.3)\qquad \omega(x) = \begin{cases} 0 & \text{for } \|x\| \leq H \\ 1 & \text{for } \|x\| > H, \end{cases}$$

where $\|x\| = (\sum_{i=1}^m x_i^2)^{1/2}$ for $x = (x_1, x_2, \ldots, x_m)$ and H is a given positive constant. Since the control does not depend upon (θ_t) the problem is called partially observable one.

Now we shall formulate the problem precisely. Let $\boldsymbol{C}^n$ be the Banach space of all $\boldsymbol{R}^n$-valued continuous functions over $[0, T]$ with uniform norm and for each t, $0 \leq t \leq T$, denote by $\mathscr{B}_t^n$ the sub-σ-field of the topological Borel field $\mathscr{B}^n$ on $\boldsymbol{C}^n$, which is generated by the cylinder sets up to the time t. Let $\boldsymbol{\Psi}$ be the class of $\mathbf{R}^m$-valued functions $\psi(t, w)$ over $[0, T] \times \boldsymbol{C}^n$ which satisfy the following three conditions;

($\boldsymbol{\Psi}$.1) $\psi(t, w)$ is measurable in (t, w),

($\boldsymbol{\Psi}$.2) for each t, $w \to \psi(t, w)$ is measurable with respect to $\mathscr{B}_t^n$,

($\boldsymbol{\Psi}$.3) $\|\psi(t, w)\| \leq k$, where $\|\cdot\|$ is the Euclidian norm in $\boldsymbol{R}^m$ and k is a given positive constant.

The notion of a solution of the equation (1.1) with $u_t = \psi(t, \zeta)$, $0 \leq t \leq T$, $\psi \in \boldsymbol{\Psi}$, is defined in a usual way as follows.

Definition 1.1. (θ, ζ) is a solution of the equation (1.1) with the initial distribution μ corresponding to $\psi \in \boldsymbol{\Psi}$ if it is a family of stochastic processes (θ, ζ),

θ being m-dimensional and ζ being n-dimensional with $\zeta_0=0$, defined on a probability space on which there exist mutually independent m-dimensional Wiener process W with $W_0=0$ and n-dimensional one w with $w_0=0$ such that

(1) $\sigma\{W_t-W_{t'}, w_t-w_{t'}; t, t'\geq s\}$ is independent of $\sigma\{W_\tau, w_\tau, \theta_\tau, \zeta_\tau; \tau\leq s\}$ and (θ, ζ) satisfies

(2) $$\theta_t=\theta_0+\int_0^t u_s ds+\int_0^t B_s dW_s$$

$$\zeta_s=\int_0^t a_s\theta_s ds+\int_0^t b_s dw_s,$$

(3) the distribution of θ_0 is μ.

By Girsanov's theorem, we have the following proposition which assures us that there always exists a solution.

Proposition 1.1. *For any $\psi\in\boldsymbol{\Psi}$, and any probability law μ on $\boldsymbol{R}^m$, there exists a unique (in the sense of law) solution of the equation* (1.1) *corresponding to ψ and with the initial distribution μ.*

Definition 1.2. A solution (θ, ζ) of the equation (1.1) corresponding to some $\psi\in\boldsymbol{\Psi}$ and with some probability measure μ on $\boldsymbol{R}^m$ as its initial distribution is called an *admissible system*. Moreover we call $u_t=\psi(t, \zeta)$ an *admissible control*.

Let $\mathscr{U}$ be the class of all admissible controls determined as above from all possible admissible systems. Therefore our problem should be to obtain u_0 in $\mathscr{U}$ such that

$$J(u_0)=\min_{u\in\mathscr{U}} J(u). \tag{1.4}$$

Definition 1.3. Control $u_0\in\mathscr{U}$ is called *optimal* if (1.4) holds.

For a given $\psi\in\boldsymbol{\Psi}$ we consider the stochastic differential equation (1.1) with $u_t=\psi(t, \zeta)$. A solution (θ, ζ) of the equation is called a *strong solution* if it is expressed as a causal function of (θ_0, W, w). As is well known the existence of a strong solution is clearly related to the pathwise uniqueness of solutions (see Yamada-Watanabe [7]). Here are two examples of the cases where strong solutions exist.

Example 1.1. (W. M. Wonham [6]) $\psi(t, w)$ satisfies the conditions ($\boldsymbol{\Psi}$.1), ($\boldsymbol{\Psi}$.2) and it is uniformly Lipshitz continuous w.r.t. the variable w in the sense of the norm of Banach space. Moreover it is Hölder continuous in t for each w and it takes its value in a compact convex subset of $\boldsymbol{R}^m$.

Example 1.2. (Fujisaki-Kallianpur-Kunita [1]) $\psi(t, w)$ satisfies the conditions ($\boldsymbol{\Psi}$.1), ($\boldsymbol{\Psi}$.2) and the following conditions;

$$\|\psi(t, w)-\psi(t, w')\|_m^2\leq\int_0^t\|w(s)-w'(s)\|_n^2 d\Gamma(s), \qquad 0\leq t\leq T, \tag{1.5}$$

(1.6) $$\|\psi(t, w)\|_m^2 \leq K(1+\int_0^t \|w(s)\|^2 d\Gamma(s)), \qquad 0 \leq t \leq T,$$

where $\Gamma(dt)$ is a bounded positive measure on $[0, T]$, K is a positive constant, $\| \ \|_m$ and $\| \ \|_n$ are Euclidian norms in $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$ respectively.

§2. Completely observable problem

Let $(\Omega, \mathscr{F}, P, W, w, \theta, \zeta, u)$ be a solution corresponding to $\psi \in \boldsymbol{\Psi}$. By $\mathscr{F}_t$ we mean the σ-field generated by $\{\zeta_s, s \leq t\}$ and put $\mathscr{F} = \bigvee_{0 \leq t \leq T} \mathscr{F}_t$. m-vector m_t and (m, m)-symmetric matrix σ_t^2 are defined by the following formulas;

(2.1) $$m_t^i = E[\theta_t^i | \mathscr{F}_t], \qquad 1 \leq i \leq m,$$

(2.2) $$(\sigma_t^2)^{ij} = E[(\theta_t^i - m_t^i)(\theta_t^j - m_t^j) | \mathscr{F}_t], \qquad 1 \leq i, j \leq m.$$

Then A. N. Shirjaev and others (see [5]) proved the following lemma.

Lemma 2.1. *If the initial distribution μ of θ_0 is* m-dim. *normal one $N(m, \sigma^2)$, where m is a vector in $\boldsymbol{R}^m$ and $\sigma^2 = c_2 I_m$ (c_2 is positive constant and I_m is (m, m)-identity matrix), then for each t $\mathscr{F}_t$-conditional distribution $P(\theta_t \in \cdot | \mathscr{F}_t)$ of θ_t is also normal one $N(m_t, \sigma_t^2)$. Furthermore m_t and σ_t^2 satisfy the following equations*;

(2.3) $$dm_t = u_t dt + \sigma_t^2 a_t^* dv_t, \qquad m_0 = m,$$

(2.4) $$\frac{d\sigma_t^2}{dt} = I_m - c_1(\sigma_t^2)^2, \qquad \sigma_0^2 = \sigma^2,$$

where $\boldsymbol{R}^n$-valued stochastic process $v = (v_t)$, $0 \leq t \leq T$, is given by

(2.5) $$dv_t = d\zeta_t - a_t m_t dt, \qquad v_0 = 0.$$

Remark 2.1. It is well known (see [1]) that v of (2.5) is $(\mathscr{F}_t)$-adapted n-dim. standard Wiener process such that for $t > s$ $v_t - v_s$ and $\mathscr{F}_s$ are independent. v is often called the innovation process.

Remark 2.2. Ordinary differential equation (2.4) is Ricatti's equation and then it has an unique solution for given initial condition. Since $\sigma_0^2 = c_2 I_m$, we can easily calculate the solution of (2.4). It is shown that

(2.6) $$\sigma_t^2 = c_3(t) I_m,$$

(2.7) $$c_3(t) = \frac{(1 + c_1^{1/2} c_2) e^{2c_1^{1/2} t} - (1 - c_1^{1/2} c_2)}{(1 + c_1^{1/2} c_2) e^{2c_1^{1/2} t} + (1 - c_1^{1/2} c_2)}.$$

Therefore the conditional variance σ_t^2 defined by (2.2) actually does not depend upon (u_t) and is indeed non-random. Since all c_i $(i = 1, 2)$ are positive constants, $c_3(t)$ is also positive and bounded function of t.

By using these results we have the following.

Lemma 2.2. *The cost function* $J(u)$ *of* (1.2) *is represented as follows;*

$$J(u)=E^u\left[\int_0^T f(t,\ \|m_t\|)dt\right], \tag{2.8}$$

where f is a bounded positive function over $[0,\ T]\times \boldsymbol{R}^1$ *and it is monotonely increasing in* $\boldsymbol{R}^1$ *for any fixed* $t\in[0,\ T]$ *and* $\|m_t\|=\{\sum_{i=1}^m (m_t^i)^2\}^{1/2}$.

Proof. By the formula (1.2),

$$J(u)=E^u\left[\int_0^T \omega(\theta_t)dt\right]=E^u\left[\int_0^T E^u[\omega(\theta_t)|\mathscr{F}_t]dt\right]$$

$$=E^u\left[\int_0^T dt\int_{\mathbf{R}^m}\omega(x)\,p(t,\ x)\,dx_1dx_2\cdots dx_m\right],$$

where $p(t,\ x)$ is m-dim. density of the normal distribution with mean m_t and variance σ_t^2 which are defined by (2.1) and (2.2) respectively. Since $\omega(x)=0$ for $\|x\|\le H$ and $\omega(x)=1$ for $\|x\|>H$, we have

$$J(u)=E^u\left[\int_0^T dt\int_{\|x\|>H}(2\pi c_3(t))^{-m/2}e^{-(x-m_t)^*(x-m_t)/2c_3(t)}dx\right]$$

$$=E^u\left[\int_0^T dt\int_{\|x+m_t\|>H}(2\pi c_3(t))^{-m/2}e^{-(x_1^2+x_2^2+\cdots+x_m^2)/2c_3(t)}\right.$$

$$\left.\times\, dx_1dx_2\cdots dx_m\right],$$

where $c_3(t)$ is given by (2.7) and $x=(x_1,\ x_2,\dots,\ x_m)$.
But the integral $\int_{\|x+m_t\|>H}(2\pi c_3(t))^{-m/2}e^{-(x_1^2+\cdots+x_m^2)/2c_3(t)}dx$ is clearly bounded and positive function over $[0,\ T]\times\mathbf{R}^1$ and it increases monotonely as $\|m_t\|$ increases because this integral is rotation invariant w.r.t. $\|m_t\|$. Q. E. D.

By Lemma 2.1 and Lemma 2.2 our first problem is transferred into the following problem, which is completely observable. We call this the *second problem.* Let P_t be an $(m,\ n)$-matrix such that $P_tP_t^*=c_tI_m$, where c_t is a bounded and measurable function of t. Let (β_t) be n-dim. standard Wiener process with $\beta_0=0$. Moreover suppose that (u_t) is a process taking its values in $\mathbf{R}^m$ such that $\sigma\{u_s;\ s\le t\}\vee\sigma\{\beta_s;\ s\le t\}$ is independent of $\sigma\{\beta_{t''}-\beta_{t'};\ t\le t'<t''\le T\}$ and $\|u_t\|\le k$.

Now we consider the following m-dim. state equation and cost function;

$$dx_t=u_tdt+P_td\beta_t,\qquad x_0=x, \tag{2.9}$$

$$\hat{J}(u)=E^u\left[\int_0^T g(t,\ \|x_t\|)dt\right], \tag{2.10}$$

where x is a constant m-vector and g is a function over $[0,\ T]\times\boldsymbol{R}^1$ having the same properties as f of Lemma 2.2. Then the second problem is to find a process u

(called control) which minimizes $\hat{J}(u)$. To this problem next theorem is well known.

Theorem 2.3. (Ikeda-Watanabe [3]) *To the second problem there exist an optimal* control (u_t) *and* n-dim. *standard Wiener process* (β_t) *such that* (u_t) *minimizes* $\hat{J}(u)$, *and moreover* u_t *is written as follows*;

$$u_t = U(x_t), \qquad 0 \le t \le T, \tag{2.11}$$

$$U(x) = \begin{cases} -k\frac{x}{\|x\|} = -k\left(\frac{x_1}{\|x\|}, \dots, \frac{x_m}{\|x\|}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases} \tag{2.12}$$

where (x_t) *is a unique solution* (*in the sense of law*) *of* (2.13).

$$dx_t = U(x_t)dt + P_t d\beta_t, \qquad x_0 = x. \tag{2.13}$$

By this fact it is natural to expect that $u_t = U(m_t)$, where m_t is given by (2.3), is optimal in the first problem. From now on we shall show that this function is actually optimal in our sense; namely we shall show that a solution $(\Omega, \mathscr{F}, P, W, w, \theta, \zeta, u)$ of the equation (1.1) corresponding to some $\psi \in \Psi$ exists such that $u_t = \psi(t, \zeta)$ coincides with $U(m_t)$ and this u is optimal. This is discussed in the next section and the rest of this section is devoted to a preparation for it.

Let (β_t) be n-dim. standard Wiener process and let $U(x)$ be given in (2.12). Assume that Q_t is a non-negative definite (m, m)-matrix and R_t is an (m, n)-matrix such that $R_t R_t^* > 0$. Both Q_t and R_t depend on t. Let's consider the following m-dim. stochastic differential equation with initial value x (x is a given constant);

$$dx_t = U(x_t)dt - Q_t x_t dt + R_t d\beta_t, \qquad x_0 = x. \tag{2.14}$$

Then we obtain the lemma which assures the existence of a strong solution in the sense of the section 1.

Lemma 2.4. *Stochastic differential equation* (2.14) *has a unique strong solution.*

Proof. The existence of a solution in the sense of law is easily proved by the routine method of Girsanov. Then it is sufficient to show that any two solutions of the equation (2.14) corresponding to the same Wiener process and the same initial value are equal with probability 1. So let X_t and Y_t be any two solutions having common Wiener process and initial value. Since both X_t and Y_t satisfy the equation (2.14), by taking the difference of them,

$$X_t - Y_t = \int_0^t \{U(X_s) - U(Y_s)\}ds - \int_0^t Q_s(X_s - Y_s)ds. \tag{2.15}$$

Put $\Delta_t = X_t - Y_t$ or $\Delta_t^i = X_t^i - Y_t^i$, $1 \le i \le m$, and let's show $\Delta_t^i = 0$ for any i and t. By differentiating both sides of (2.15) we have for any i,

$$\frac{d\Delta_t^i}{dt} = U^i(X_t) - U^i(Y_t) - \sum_{j=1}^m Q_t^{ij}\Delta_t^j, \qquad \Delta_0^i = 0,$$

where $U^i(x)=-k\frac{x_i}{\|x\|}$ $(x\neq 0)$, $U^i(x)=0$ $(x=0)$. Multiply Δ_t^i to both sides and sum up from 1 to m with respect to i, then

$$\sum_{i=1}^m \Delta_t^i \frac{d\Delta_t^i}{dt}=\sum_{i=1}^m \Delta_t^i\{U^i(X_t)-U^i(Y_t)\}-\sum_{i=1}^m Q_t^{ij}\Delta_t^i\Delta_t^j.$$

Since matrix Q_t is non-negative definite by the assumption, the second term of the right side is negative. Next it is shown that the first term is also negative. In fact,

$$\sum_{i=1}^m \Delta_t^i\{U^i(X_t)-U^i(Y_t)\}=k\sum_{i=1}^m \Delta_t^i\left\{-\frac{X_t^i}{\|X_t\|}+\frac{Y_t^i}{\|Y_t\|}\right\}$$

$$=k\sum_{i=1}^m\left\{-\frac{(X_t^i)^2}{\|X\|}+\frac{X_t^iY_t^i}{\|X_t\|}+\frac{X_t^iY_t^i}{\|Y_t\|}-\frac{(Y_t^i)^2}{\|Y_t\|}\right\}$$

$$=k\left\{-\|X_t\|-\|Y_t\|+\left(\frac{1}{\|X_t\|}+\frac{1}{\|Y_t\|}\right)\sum_{i=1}^m X_t^iY_t^i\right\}.$$

But by Schwarz's inequality it is shown that

$$\left|\sum_{i=1}^m X_t^iY_t^i\right|\leq\{\sum_{i=1}^m (X_t^i)^2\}^{1/2}\{\sum_{i=1}^m (Y_t^i)^2\}^{1/2}=\|X_t\|\cdot\|Y_t\|,$$

therefore we have

$$\sum_{i=1}^m \Delta_t^i\{U^i(X_t)-U^i(Y_t)\}\leq 0. \qquad \therefore \quad \sum_{i=1}^m \Delta_t^i\frac{d\Delta_t^i}{dt}\leq 0.$$

Clearly this is equal to say that $\frac{d}{dt}\{\sum_{i=1}^m(\Delta_t^i)^2\}\leq 0$. Since $\sum_{i=1}^m(\Delta_t^i)^2\geq 0$ and $\Delta_o^i=0$ for all i, thus we obtain $\Delta_t^i=0$ for all i and t. Q.E.D.

Remark 2.3. By this theorem it is clear that the stochastic differential equation (2.13) has a unique solution in the sense of pathwise.

§3. Main theorem

Theorem 3.1. *If the initial distribution of θ_0 is normal $N(m, \sigma^2)$ then there exist some $\psi\in\Psi$ and a solution (θ, ζ) of the equation* (1.1) *corresponding to ψ such that $u_t=\psi(t, \zeta)=U(m_t)$ is optimal, where $U(x)$, $x\in \boldsymbol{R}^m$, is given by* (2.12) *and*

$$m_t=E^u[\theta_t|\mathscr{F}_t], \tag{3.1}$$

for each t, $\mathscr{F}_t$ is σ-field generated by $\{\zeta_s, s\leq t\}$.

Remark 3.1. Most essential difference between W. M. Wonham and us is in the admissible class of the control variables. In fact he treated functions which are continuous in w and Hölder continuous in t as the element of Ψ (see Remark 2.1). Then it is possible to say that for any Wiener process he obtained an optimal control minimizing the given (smooth) cost function, while we find it in the widest class whose element is bounded measurable and the cost function is not continuous. In our case naturally Wiener process depends on the control u.

Proof. Since the proof of theorem is long, we shall divide it into four parts.

1. Let's consider the following m-dim. stochastic differential equation with initial condition $x_0=m$;

$$dx_t=U(x_t)dt-c_1c_3(t)x_tdt+c_3(t)a_t^*d\zeta_t, \qquad x_0=m, \tag{3.2}$$

where (ζ_t) is n-dim. Wiener process, $c_3(t)$ is given by (2.7) and $U(x)$ is the same as (2.12). Since $c_1c_3(t)>0$ and $R_t=c_3(t)a_t^*$ satisfies $R_tR_t^*=c_tI_m$ by the assumption (c_t is a positive constant depending on t), then by Lemma 2.4 the equation (3.2) has a unique strong solution. That is to say, there exists a unique $\boldsymbol{R}^m$-valued function $F(t, w)$ over $[0, T]\times \boldsymbol{C}^n$ which satisfies the following three conditions;
($F.1$) it is measurable w.r.t. (t, w),
($F.2$) for fixed t $(0\leq t\leq T)$, $w\to F(t, w)$ is $\mathscr{B}_t^n$-measurable,
($F.3$) for n-dim. Wiener process ζ, if we set $x_t=F(t, \zeta)$ then this is a unique solution of (3.2).

2. Let $(\Omega, \mathscr{F}, Q)$ be a probability space on which n-dim. Wiener process (ζ_t) is defined. Let (θ_t) be m-dim. Wiener process which is independent of (ζ_t), and we can assume that θ_0 has a given normal distribution $N(m, \sigma^2)$. Let $\mathscr{G}_t$ be the σ-field such that $\{(\theta, \zeta), \mathscr{G}_t, Q\}$ is $(m+n)$-dim. Wiener process. Now if we put

$$A_t=\begin{pmatrix} B_t & 0 \\ 0 & b_t \end{pmatrix} \tag{3.3}$$

then A_t is an $(m+n, m+n)$-matrix such that $A_tA_t^*=I_{m+n}$. Here let's define $(m+n)$-vectors η_t, S_t, and $\tilde{\gamma}_t$ as follows.

$$\eta_t=\begin{pmatrix} \theta_t \\ \zeta_t \end{pmatrix}, \quad S_t=\begin{pmatrix} U(x_t) \\ a_t\theta_t \end{pmatrix}, \quad \tilde{\gamma}_t=\begin{pmatrix} \tilde{W}_t \\ \tilde{w}_t \end{pmatrix},$$

where $\tilde{W}_t$ and $\tilde{w}_t$ are given by the formula

$$\begin{aligned} d\tilde{W}_t&=d\theta_t-U(x_t)dt, \qquad \tilde{W}_0=0, \\ d\tilde{w}_t&=d\zeta_t-a_t\theta_tdt, \qquad \tilde{w}_0=0, \end{aligned} \tag{3.4}$$

or equivalently

$$d\tilde{\gamma}_t=d\eta_t-S_tdt, \qquad \tilde{\gamma}_0=0. \tag{3.5}$$

Since (η_t) is $(m+n)$-dim. Wiener process w.r.t. $(\mathscr{G}_t, Q)$ and both a_t and $U(x)$ are bounded in t, we can show by using Girsanov's theorem that $(\tilde{\gamma}_t)$ of (3.5) is also $(m+n)$-dim. Wiener process with $\tilde{\gamma}_0=0$ w.r.t. probability measure $dP=\varphi_t|_{\mathscr{G}_t}dQ$, where φ_t is uniformly integrable martingale $(\mathscr{G}_t, Q)$ defined by

$$\varphi_t = \exp\left\{\int_0^t S_s^* d\eta_s - \frac{1}{2}\int_0^t \|S_s\|^2 ds\right\}. \tag{3.6}$$

Let $\gamma_t = \begin{pmatrix} W_t \\ w_t \end{pmatrix}$ be defined by

$$d\gamma_t = A_t^* d\tilde{\gamma}_t, \qquad \gamma_0 = 0. \tag{3.7}$$

Then (γ_t) is also $(m+n)$-dim. Wiener process $(\mathscr{G}_t, P)$ because A_t is an orthogonal matrix,and this is equivalent to say that (W_t) is m-dim. Wiener process with $W_0=0$ and (w_t) is n-dim. one with $w_0=0$. As it holds that $d\tilde{\gamma}_t = A_t d\gamma_t$, (3.5) is

$$A_t d\gamma_t = d\eta_t - S_t dt, \qquad \gamma_0 = 0, \tag{3.8}$$

or in the component wise,

$$\begin{cases} d\theta_t = U(x_t)dt + B_t dW_t \\ d\zeta_t = a_t\theta_t dt + b_t dw_t, \end{cases} \tag{3.9}$$

where the distribution of θ_0 is normal $N(m, \sigma^2)$ and $\zeta_0 = 0$.

3. Next by the filtering equation we obtain the following equation of (m_t) which is defined in (3.1);

$$dm_t = U(x_t)dt + c_3(t)a_t^* dv_t, \qquad m_0 = m, \tag{3.10}$$

where (v_t) is given by (2.5). Then (3.10) is

$$dm_t = U(x_t)dt + c_3(t)a_t^* d\zeta_t - c_1c_3(t)m_t dt, \qquad m_0 = m. \tag{3.11}$$

But (x_t) is a unique solution of the equation (3.2) for the same (ζ_t) and m as those in (3.11), then by taking the difference between x_t and m_t,

$$x_t - m_t = -\int_0^t c_1c_3(s)(x_s - m_s)ds.$$

Since $x_0 = m_0 = m$ and $c_1c_3(t)$ is bounded, it is easy to show that $x_t = m_t$ for all t with probability 1. Therefore we can write (3.9) as follows;

$$\begin{aligned} d\theta_t &= U(m_t)dt + B_t dW_t \\ d\zeta_t &= a_t\theta_t dt + b_t dw_t. \end{aligned} \tag{3.12}$$

4. Now it is easy to show that $u_t = U(m_t)$ is an admissible optimal control and (θ, ζ) given by (3.12) is an optimal solution in our sense. In fact, first of all $u \in \mathscr{U}$ is clear because $u_t = U(m_t) = U(x_t) = U(F(t, \zeta))$ w.p.1 and $\psi(t, w) = U(F(t, w))$ belongs to the class $\boldsymbol{\Psi}$ as a function over $[0, T] \times \boldsymbol{C}^n$. Next let $(\psi', \theta', \zeta', u', m')$ be arbitrary admissible system. Then by Lemma 2.2,

$$E^u\left[\int_0^T \omega(\theta_t)dt\right] = E^u\left[\int_0^T f(t, \|m_t\|)dt\right]$$

$$E^{u'}\left[\int_0^T \omega(\theta_t')dt\right]=E^{u'}\left[\int_0^T f(t, \|m_t'\|)dt\right].$$

But by Theorem 2.3 we have

$$E^{u}\left[\int_0^T f(t, \|m_t\|)dt\right]\leq E^{u'}\left[\int_0^T f(t, \|m_t'\|)dt\right],$$

therefore $E^{u}\left[\int_0^T \omega(\theta_t)dt\right]\leq E^{u'}\left[\int_0^T \omega(\theta_t')dt\right]$. Q. E. D.

Corollary 3.2. Whenever there exists a function $F(t, w)$ such that $m_t=F(t, \zeta)$, where $F(t, w)$ satisfies $(F.1)\sim(F.3)$ in the proof of Theorem 3.1, $u_t=U(m_t)$ is optimal.

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