J. Math. Kyoto Univ. (JMKYAZ)
18-2 (1978) 209-219

Tauberian theorems of exponential type

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(Received May 9, 1977)

§0. Introduction

The notion of regularly varying functions, which was introduced by Karamata, extended greatly the Hardy-Littlewood Tauberian theorem and simplified its proof. According to Karamata's Tauberian theorem, a nondecreasing function $a(t)$, $t \geqq 0$ varies regularly at 0, if and only if its Laplace transform $F(\lambda)$ varies regularly at ∞ (see [2] or [9]). However, his method provides us with little information in a case where $a(t)$ or $F(\lambda)$ varies in an exponential order (cf. [3]). Such a case is of interest in some problems in probability theory and studied by Varadhan [10] and by Fukushima [3] *etc.* Similar problems have been studied by many authors. L. Davies [1] and Nagai [7] (or [4]) studied the relation between the asymptotic behaviour of $a(t)$ as $t \to \infty$ and that of $F(\lambda)$ as $\lambda \to -\infty$. Davies [1] and Kôno [5] treated the case where the Laplace transform is replaced by the moments.

The aim of this paper is to give a Tauberian theorem in a most general form. In section 1 the main theorem is stated with its proof. In section 2, we apply it to various cases and see that the Tauberian theorems mentioned above are obtained as special cases of our theorem.

§1. Main theorem

Throughout this section we assume α to be a fixed positive number and $f(x)$ ($\not\equiv$ *const.*) to be a real valued nondecreasing function defined on the interval $(0, \infty)$ such that $f(\xi^\beta)$ is concave for some $\beta (>\alpha)$. Note that $f(\xi^\alpha)$ is also concave. Therefore without difficulty we see that

$$g(x) = \sup_{\xi > 0} \{f(\xi^\alpha) + x\xi\}, \qquad x < 0,$$

is a nondecreasing convex function and that $g(x) > f(0+)$. In fact $g(x)$ is strictly increasing in $x \in (-\infty, 0)$. For convenience we define $g(0) = f(+\infty)$ and $g(-\infty) = f(0+)$. So $g(x)$ $(-\infty \leqq x \leqq 0)$ is a continuous function with values in $[-\infty, \infty]$. Notice that for each $A \in (-\infty, 0)$, there exists a positive solution of $f(\xi^\alpha) + A\xi = g(A)$ and that this solution is unique. Indeed the first assertion is clear because

$$\lim_{\xi\to\infty} f(\xi^\alpha)+x\xi=-\infty,$$

and
$$\lim_{\xi\downarrow 0} f(\xi^\alpha)+x\xi=f(0+)<g(A).$$

So we prove the uniqueness. Assume there exist two solutions $\lambda_1<\lambda_2$. Then since $f(\xi^\alpha)+A\xi$ is concave, $f(\xi^\alpha)+A\xi=g(A)$ holds in the interval $[\lambda_1, \lambda_2]$. But this contradicts the concavity of $f(\xi^\beta)$. Using a similar argument, we see that $f(\xi^\alpha)+A\xi=B$ has two positive solutions for each $A\in(-\infty, 0)$, $B\in(f(0+), g(A))$.

Now we state our main theorem:

Theorem 1. *Suppose $\mu(dx)$ be a finite Borel measure on $(0, \infty)$ and $L(x)$ be a slowly varying function. Set*

$$F(\lambda)=\int_0^\infty \exp\{\lambda f(x/\phi(\lambda))\}\mu(dx)$$

where $\phi(\lambda)=\lambda^\alpha L(\lambda)$.

Then;

(i) $$-\infty\leqq A_1\leqq \varliminf_{x\to\infty}\frac{1}{x}\log\mu(\phi(x), \infty)$$

$$\leqq \varlimsup_{x\to\infty}\frac{1}{x}\log\mu(\phi(x), \infty)\leqq A_2\leqq 0$$

implies

$$(*)\qquad g(A_1)\leqq \varliminf_{\lambda\to\infty}\frac{1}{\lambda}\log F(\lambda)\leqq \varlimsup_{\lambda\to\infty}\frac{1}{\lambda}\log F(\lambda)\leqq g(A_2).$$

(ii) *Conversely, if $A_2\neq 0$, then $(*)$ implies*

$$\frac{\lambda_2}{\lambda_1}A_2\leqq \varliminf_{x\to\infty}\frac{1}{x}\log\mu(\phi(x), \infty)$$

$$\leqq \varlimsup_{x\to\infty}\frac{1}{x}\log\mu(\phi(x), \infty)\leqq A_2$$

where $\lambda_1[\lambda_2]$ is the least [largest] solution of

$$f(\xi^\alpha)+A_2\xi=g(A_1).$$

$\left(\frac{\lambda_2}{\lambda_1}A_2\right.$ *is to be read* $\left.-\infty \text{ if } A_1=-\infty\right)$.

(iii) *If $f(+\infty)<\infty$ and if $\varliminf_{\lambda\to\infty}\frac{1}{\lambda}\log F(\lambda)\geqq B>f(0+)$, then,*

$$\varliminf_{x\to\infty}\frac{1}{x}\log\mu(\phi(x), \infty)\geqq (B-f(+\infty))/\lambda_3$$

where $\lambda_3=\sup\{\lambda: f(\lambda^\alpha)<B\}$.

Remark. The constant $\frac{\lambda_2}{\lambda_1}A_2$ which appeared in (ii) depends not only on

A_1 but also on A_2. We easily see that $\lambda_1=\lambda_2$ holds if and only if $A_1=A_2$. Furthermore if $A_1>-\infty$,

$$\lim_{A_2\downarrow A_1}\uparrow \frac{\lambda_2}{\lambda_1}A_2=A_1,$$

$$\lim_{A_2\uparrow 0}\downarrow \frac{\lambda_2}{\lambda_1}A_2=(g(A_1)-f(+\infty))/\lambda_3.$$

So we can regard (iii) as an extreme case of (ii).

Corollary.

(i) $\overline{\lim_{x\to\infty}}\frac{1}{x}\log\mu(\phi(x),\infty)=A \qquad (-\infty\leqq A\leqq 0)$
if and only if

$$\overline{\lim_{\lambda\to\infty}}\frac{1}{\lambda}\log F(\lambda)=g(A).$$

(ii) $\lim_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\infty)=A\ (<0)$
if and only if

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log F(\lambda)=g(A).$$

In case $f(+\infty)<\infty$, the assumption $A<0$ can be removed.

Proof of Corollary. Since $g(x)$ is strictly increasing, (i) follows from (i) and (ii) of Theorem 1. For the proof of (ii), we have only to bear in mind that $\lambda_1=\lambda_2$ if $A_1=A_2$. In the case where $A=0$ and $f(+\infty)<\infty$, we can make use of (iii) of Theorem 1. Q.E.D.

For the proof of Theorem 1, we prepare some lemmas.

Lemma 1.

(i) $\overline{\lim_{\lambda\to\infty}}\frac{1}{\lambda}\log F(\lambda)\geqq g\Big(\overline{\lim_{x\to\infty}}\frac{1}{x}\log\mu(\phi(x),\infty)\Big)$,

(ii) $\underline{\lim_{\lambda\to\infty}}\frac{1}{\lambda}\log F(\lambda)\geqq g\Big(\underline{\lim_{x\to\infty}}\frac{1}{x}\log\mu(\phi(x),\infty)\Big)$.

Proof. We need nothing but Chebyshef's inequality. Let $A=\overline{\lim_{x\to\infty}}\frac{1}{x}\log\mu(\phi(x),\infty)$. In case $A=-\infty$, (i) is trivial because $g(-\infty)=f(0+)$. So we assume $A\neq-\infty$. Then for each $\xi>0$,

$$F(\lambda)\geqq\int_{\phi(\xi\lambda)}^{\infty}e^{\lambda f(x/\phi(\lambda))}\mu(dx)$$
$$\geqq e^{\lambda f(\phi(\xi\lambda)/\phi(\lambda))}\mu(\phi(\xi\lambda),\infty).$$

Hence

$$\overline{\lim_{\lambda\to\infty}}\frac{1}{\lambda}\log F(\lambda)\geqq f(\xi^{\alpha})+A\xi,$$

which proves (i). Similarly we have (ii). Q.E.D.

The following lemma plays a key role in this paper.

Lemma 2. *If* $\overline{\lim_{x\to\infty}}\frac{1}{x}\log\mu(\phi(x),\infty)\leqq A$ $(-\infty<A<0)$, *then,*

(i) $\overline{\lim_{\lambda\to\infty}}\frac{1}{\lambda}\log\int_{\phi(\mu\lambda)}^{\infty}e^{\lambda f(x/\phi(\lambda))}\mu(dx)\leqq f(\mu^{\alpha})+A\mu$ *for each* $\mu>\lambda_0$,

(ii) $\overline{\lim_{\lambda\to\infty}}\frac{1}{\lambda}\log\int_{0}^{\phi(\mu\lambda)}e^{\lambda f(x/\phi(\lambda))}\mu(dx)\leqq f(\mu^{\alpha})+A\mu$ *for each* $0<\mu<\lambda_0$,

where $\lambda_0(>0)$ *is the unique solution of*

$$f(\lambda^{\alpha})+A\lambda=g(A). \tag{1.1}$$

Proof. Set

$h_i(x,\delta)=f((1+\delta)x^{\alpha(1+(-1)^i\delta)})+A(1-\delta)x$. Then, clearly, $h_i(x,\delta)$, $i=1, 2$, are continuous in $\delta\in[0,1)$ and are concave in x provided $0\leqq\delta\leqq\left(\frac{\beta}{\alpha}-1\right)\wedge 1$. Since $h_i(\mu,0)<h_i(\lambda_0,0)$ $(\mu\neq\lambda_0)$, there exist positive constants c and $\delta_0\left(<\left(\frac{\beta}{\alpha}-1\right)\wedge 1\right)$ such that

$$\{h_i(\mu,\delta)-h_i(\lambda_0,\delta)\}/(\mu-\lambda_0)\geqq c>0 \qquad \text{for}\quad \delta\in(0,\delta_0),\qquad i=1, 2.$$

On the other hand, the concavity of h_i provides us with

$$h_i(x,\delta)\leqq h_i(\mu,\delta)+\frac{h_i(\mu,\delta)-h_i(\lambda_0,\delta)}{\mu-\lambda_0}(x-\mu),\qquad 0<x<\mu.$$

Hence, if we set $h(x,\delta)=\max\{h_i(x,\delta),\ i=1, 2\}$, then

$$h(x,\delta)\leqq h(\mu,\delta)+c(x-\mu),\qquad 0<x<\mu,\ 0<\delta<\delta_0.$$

Next we remark that for each $\delta>0$, there exists a positive constant N_δ such that

$$\frac{\phi(y)}{\phi(x)}\leqq\begin{cases}(1+\delta)(y/x)^{\alpha(1+\delta)} & \text{for}\quad y\geqq x\geqq N_\delta\\ (1+\delta)(y/x)^{\alpha(1-\delta)} & \text{for}\quad x\geqq y\geqq N_\delta\end{cases}$$

and

$$\mu(\phi(x),\infty)\leqq e^{A(1-\delta)x} \qquad \text{for}\quad x\geqq N_\delta.$$

The first inequality can be verified if we make use of the canonical representation of ϕ (cf. [2], p. 282). Now fix a positive number ε and set

$$\mu_k=\mu-\varepsilon k,\qquad k=1, 2,\dots.$$

Then if $\mu_{k+1}\xi$, $\xi\geqq N_\delta$,

$$\int_{\phi(\mu_{k+1}\xi)}^{\phi(\mu_k\xi)} e^{\xi f(x/\phi(\xi))}\mu(dx)$$
$$\leqq \exp\{\xi f(\phi(\mu_k\xi)/\phi(\xi))\}\mu(\phi(\mu_{k+1}\xi),\ \infty)$$
$$\leqq \exp\xi\{f((1+\delta)\mu_k^{\alpha(1\pm\delta)})+A(1-\delta)\mu_{k+1}\}$$
$$\leqq \exp\xi\{h(\mu_k,\ \delta)-A(1-\delta)\varepsilon\}$$
$$\leqq \exp\xi\{h(\mu,\ \delta)-k\varepsilon c-A(1-\delta)\varepsilon\}.$$

Therefore, if $\xi \geqq N_\delta$,

$$\int_{\phi(N_\delta+\varepsilon)}^{\phi(\mu\xi)} e^{\xi f(x/\phi(\xi))}\mu(dx)$$
$$\leqq \sum_{k;\mu_{k+1}\xi\geqq N_\delta}\int_{\phi(\mu^{k+1}\xi)}^{\phi(\mu_k\xi)}$$
$$\leqq (\exp\xi\{h(\mu,\ \delta)-A(1-\delta)\varepsilon\})/(1-e^{-c\varepsilon\xi})$$

which implies

$$\overline{\lim_{\xi\to\infty}}\frac{1}{\xi}\log\int_{\phi(N_\delta+\varepsilon)}^{\phi(\mu\xi)} e^{\xi f(x/\phi(\xi))}\mu(dx)$$
$$\leqq h(\mu,\ \delta)-A(1-\delta)\varepsilon.$$

Hence,

$$\overline{\lim_{\xi\to\infty}}\frac{1}{\xi}\log\int_0^{\phi(\mu\xi)} e^{\xi f(x/\phi(\xi))}\mu(dx)$$
$$\leqq \max\{f(0+),\ h(\mu,\ \delta)-A(1-\delta)\varepsilon\}.$$

Letting $\varepsilon\downarrow 0$, $\delta\downarrow 0$, we obtain (ii). Similarly we can prove (i). Q.E.D.

Lemma 3. *If* $\overline{\lim_{x\to\infty}}\frac{1}{x}\log\mu(\phi(x),\ \infty)\leqq A$ $(-\infty<A<0)$, *then,*

$$\overline{\lim_{\lambda\to\infty}}\frac{1}{\lambda}\log F(\lambda)\leqq g(A).$$

Proof. Let λ_0 be as in Lemma 2. We choose λ_1 and λ_2 so that $0<\lambda_1<\lambda_0<\lambda_2<\infty$. Then,

$$\overline{\lim_{\xi\to\infty}}\frac{1}{\xi}\log\int_{\phi(\lambda_1\xi)}^{\phi(\lambda_2\xi)} e^{\xi f(x/\phi(\xi))}\mu(dx)$$
$$\leqq \overline{\lim_{\xi\to\infty}}\frac{1}{\xi}\log\{e^{\xi f(\phi(\lambda_2\xi)/\phi(\xi))}\mu(\phi(\lambda_1\xi),\ \infty)\}$$
$$\leqq f(\lambda_2^\alpha)+A\lambda_1.$$

Therefore, by Lemma 2,

$$\overline{\lim_{\lambda\to\infty}}\frac{1}{\lambda}\log F(\lambda)\leqq \max\{f(\lambda_1^\alpha)+A\lambda_1,\ f(\lambda_2^\alpha)+A\lambda_1,\ f(\lambda_2^\alpha)+A\lambda_2\}.$$

Letting $\lambda_1 \uparrow \lambda_0$, $\lambda_2 \downarrow \lambda_0$, we see

$$\overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log F(\lambda) \leqq f(\lambda_0^\alpha) + A\lambda_0 = g(A).$$

Q.E.D.

Lemma 4. *Suppose that*

$$\overline{\lim_{x\to\infty}} \frac{1}{x} \log \mu(\phi(x), \infty) \leqq A \qquad (-\infty < A < 0) \tag{1.2}$$

and that

$$\underline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log F(\lambda) \geqq B > f(0+). \tag{1.3}$$

Then,

$$\underline{\lim_{x\to\infty}} \frac{1}{x} \log \mu(\phi(x), \infty) \geqq \frac{\lambda_2}{\lambda_1} A$$

where $\lambda_1 \leqq \lambda_2$ are the solutions of

$$f(\lambda^\alpha) + A\lambda = B, \qquad \lambda > 0. \tag{1.4}$$

Proof. Since Lemma 3, (1.2) and (1.3) imply $B \leqq g(A)$, we see that (1.4) has two solutions which coincide if and only if $B = g(A)$. Now choose η_1 and η_2 so that $0 < \eta_1 < \lambda_1 \leqq \lambda_2 < \eta_2 < \infty$. Then by Lemma 2,

$$\overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log \int_0^{\phi(\eta_1\lambda)} e^{\lambda f(x/\phi(\lambda))} \mu(dx) \leqq f(\eta_1^\alpha) + A\eta_1 \tag{1.5}$$
$$< f(\lambda_1^\alpha) + A\lambda_1 = B,$$

$$\overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log \int_{\phi(\eta_2\lambda)}^{\infty} e^{\lambda f(x/\phi(\lambda))} \mu(dx) \leqq f(\eta_2^\alpha) + A\eta_2 \tag{1.6}$$
$$< f(\lambda_2^\alpha) + A\lambda_2 = B.$$

(1.3), (1.5) and (1.6) imply

$$\underline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log \int_{\phi(\eta_1\lambda)}^{\phi(\eta_2\lambda)} e^{\lambda f(x/\phi(\lambda))} \mu(dx) \geqq B.$$

On the other hand we have

$$\underline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log \int_{\phi(\eta_1\lambda)}^{\phi(\eta_2\lambda)} e^{\lambda f(x/\phi(\lambda))} \mu(dx)$$

$$\leqq f(\eta_2^\alpha) + \eta_1 \underline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log \mu(\phi(\lambda), \infty), \quad \text{(see the proof of Lemma 3).}$$

Thus we see

$$\underline{\lim}_{\lambda\to\infty}\frac{1}{\lambda}\log\mu(\phi(\lambda),\ \infty)\geqq\frac{1}{\eta_1}(B-f(\eta_2^\alpha)).$$

Hence, letting $\eta_1\uparrow\lambda_1$, and $\eta_2\downarrow\lambda_2$, we obtain the assertion since

$$B-f(\lambda_2^\alpha)=\lambda_2 A.$$

Q.E.D.

Lemma 5. *If* $-\infty\leqq A\leqq 0$, *then each of the conditions*

$$\overline{\lim}_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\ \infty)\leqq A \tag{1.7}$$

and

$$\overline{\lim}_{\lambda\to\infty}\frac{1}{\lambda}\log F(\lambda)\leqq g(A) \tag{1.8}$$

implies the other.

Proof. Since $g(x)$ is monotone, it is easy to see that (1.8) implies (1.7) by Lemma 1. So we prove the converse. Since $g(0)=f(\infty)$, (1.8) is trivial if $A=0$. In case $-\infty<A<0$, the assertion is proved in Lemma 3. Therefore, if $A=-\infty$, (1.8) is also valid replacing A by any $A'(>A)$. Hence

$$\overline{\lim}_{\lambda\to\infty}\frac{1}{\lambda}\log F(\lambda)\leqq\inf_{A'}g(A')=g(A)$$

which completes our proof.

Now it is easy to prove Theorem 1. (i) follows from Lemmas 1 and 5, while (ii) is an easy consequence of Lemmas 5 and 4. To prove (iii), we have only to notice an inequality

$$F(\xi)\leqq e^{\xi f(\phi(\lambda\xi)/\phi(\xi))}\mu(0,\ \phi(\lambda\xi))+e^{\xi f(\infty)}\mu(\phi(\lambda\xi),\ \infty),\qquad \xi,\ \lambda>0. \tag{1.9}$$

(see, for instance, p. 448 of [10])
Indeed (1.9) shows that

$$\lambda\,\underline{\lim}_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\ \infty)\geqq -f(\infty)+B$$

provided $f(\lambda^\alpha)<B$.

§2. Applications

Set $f(x)=x$ and $0<\alpha<1$. Then the assumptions in section 1 are clearly satisfied, and we see $g(x)=(1-\alpha)(\alpha/-x)^{\alpha/(1-\alpha)}$. Let $\phi(x)$ be a positive function varying regularly at ∞ with exponent α and $\psi(x)$ be the asymptotic inverse of $x/\phi(x)$ (cf. Seneta [9]). Apparently $\psi(x)$ varies regularly at ∞ with exponent $1/(1-\alpha)$. Now we have the following as a special case of Theorem 1.

Theorem 2.

(i) $-\infty \leqq -A_1 \leqq \varliminf_{x\to\infty} \frac{1}{x} \log \mu(\phi(x), \infty) \leqq \varlimsup_{x\to\infty} \frac{1}{x} \log \mu(\phi(x), \infty) \leqq -A_2 \leqq 0$

implies

$$(2.1)\qquad (1-\alpha)(\alpha/A_1)^{\alpha/(1-\alpha)} \leqq \varliminf_{\lambda\to\infty} \frac{1}{\psi(\lambda)} \log \int_0^\infty e^{\lambda x}\mu(dx)$$

$$\leqq \varlimsup_{x\to\infty} \frac{1}{\psi(\lambda)} \log \int_0^\infty e^{\lambda x}\mu(dx) \leqq (1-\alpha)(\alpha/A_2)^{\alpha/(1-\alpha)}.$$

(ii) *Conversely if* (2.1) *holds with* $0<A_2\leqq A_1<\infty$, *then*

$$-\frac{\lambda_2}{\lambda_1}A_2 \leqq \varliminf_{x\to\infty} \frac{1}{x} \log \mu(\phi(x), \infty) \leqq \varlimsup_{x\to\infty} \frac{1}{x} \log \mu(\phi(x), \infty) \leqq -A_2$$

where λ_1 $[\lambda_2]$ *is the least* [*largest*] *solution of*

$$\xi^\alpha - \xi = (1-\alpha)(\alpha A_2/A_1)^{\alpha/(1-\alpha)}.[1]$$

The latter half of this theorem is a generalization of the result of Davies [1], and the following corollary includes Nagai's Tauberian theorem which was derived from Minlos-Povzner's theorem (cf. [7], [6]).

Corollary 1.

(i) $\lim_{x\to\infty} \frac{1}{x} \log \mu(\phi(x), \infty) = -A < 0$ *holds if and only if*

$$\lim_{\lambda\to\infty} \frac{1}{\psi(\lambda)} \log \int_0^\infty e^{\lambda x}\mu(dx) = (1-\alpha)(\alpha/A)^{\alpha/(1-\alpha)}.$$

(ii) $\varlimsup_{x\to\infty} \frac{1}{x} \log \mu(\phi(x), \infty) = -A$ $(0 \leqq A \leqq \infty)$ *holds if and only if*

$$\varlimsup_{\lambda\to\infty} \frac{1}{\psi(\lambda)} \log \int_0^\infty e^{\lambda x}\mu(dx) = (1-\alpha)(\alpha/A)^{\alpha/(1-\alpha)}.$$

As an easy consequence of the preceding corollary, we also have the following;

Corollary 2. *Let* $v_i(dx)$, $i=1, 2$, *be two Radon measures on the line such that* $\int_{-\infty}^\infty e^{\lambda x} v_i(dx) < \infty$ *for all sufficiently large* λ. *Suppose*

$$\log \int_{-\infty}^\infty e^{\lambda x} v_1(dx) = \log \int_{-\infty}^\infty e^{\lambda x} v_2(dx) + O(\lambda), \quad as \quad \lambda \uparrow \infty.$$

Then, for each slowly varying $L(x)$ *and constant* $\rho>1$, *we have the following*;

(i) $\varlimsup_{x\to\infty} \frac{1}{x^\rho L(x)} \log v_1(x, \infty) = \varlimsup_{x\to\infty} \frac{1}{x^\rho L(x)} \log v_2(x, \infty)$.

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[1] It is easy to see that the ratio of the two solutions of this equation equals that of

$$\xi^\alpha - A_2\xi = (1-\alpha)(\alpha/A_1)^{\alpha/(1-\alpha)}.$$

(ii) $\lim_{x\to\infty} \frac{1}{x^{\rho}L(x)} \log v_1(x, \infty) = A \qquad (-\infty \leqq A < 0)$

if and only if

$$\lim_{x\to\infty} \frac{1}{x^{\rho}L(x)} \log v_2(x, \infty) = A.$$

For an application of this corollary, see [4], for instance.

We next show that our theorem includes Fukushima's Tauberian theorem in [3]. Let $a(x)$, $x \geqq 0$, be a nondecreasing right-continuous function with $a(0)=0$ such that $\int_0^\infty e^{-\lambda x} da(x)$ is finite for sufficiently large λ. Assume x_0 be a continuity point of $a(x)$. Then $b(x) = a(1/x_0) - a(1/x+0)$ for $x > x_0$ and $=0$ for $0 \leqq x \leqq x_0$, defines a finite Stieltjes measure $db(x)$. Now set $f(x) = -1/x$, $\alpha > 0$ in Theorem 1. Then we see $g(x) = -(1+\alpha)(-x/\alpha)^{\alpha/(\alpha+1)}$. Hence we obtain, for example, $\overline{\lim}_{x\to\infty} x^{-1/\alpha} \log(b(\infty) - b(x)) = A$ $(-\infty \leqq A \leqq 0)$ is equivalent to

$$\overline{\lim_{\lambda\to\infty}} \lambda^{-1/(1+\alpha)} \log \int_0^\infty e^{-\lambda/x} db(x) = -(1+\alpha)(-A/\alpha)^{\alpha/(1+\alpha)}.$$

After a change of notation we see that $\overline{\lim}_{x\downarrow 0} x^{1/\alpha} \log a(x) = A$ is equivalent to

$$\overline{\lim_{\lambda\to\infty}} \lambda^{-1/(1+\alpha)} \log \int_0^\infty e^{-\lambda x} da(x) = -(1+\alpha)(-A/\alpha)^{\alpha/(1+\alpha)}.$$

Thus, similarly, we obtain the following;

Theorem 3.

(i) $-\infty \leqq -A_1 \leqq \underline{\lim}_{x\downarrow 0} x^{1/\alpha} \log a(x) \leqq \underline{\lim}_{x\downarrow 0} x^{1/\alpha} \log a(x) \leqq -A_2 \leqq 0 \quad (\alpha > 0)$

implies

(**) $$-(1+\alpha)(A_1/\alpha)^{\alpha/(1+\alpha)} \leqq \underline{\lim}_{\lambda\to\infty} \lambda^{-1/(1+\alpha)} \log \int_0^\infty e^{-\lambda x} da(x)$$

$$\leqq \overline{\lim_{\lambda\to\infty}} \lambda^{-1/(1+\alpha)} \log \int_0^\infty e^{-\lambda x} da(x) \leqq -(1+\alpha)(A_2/\alpha)^{\alpha/(1+\alpha)}.$$

(ii) *Conversely, if* $A_2 \neq 0$, *then* (**) *implies*

$$-\frac{\lambda_2}{\lambda_1} A_2 \leqq \underline{\lim}_{x\downarrow 0} x^{1/\alpha} \log a(x) \leqq \overline{\lim_{x\downarrow 0}} x^{1/\alpha} \log a(x) \leqq -A_2$$

where $\lambda_1[\lambda_2]$ *is the least [largest] solution of*

$$x^{-\alpha} + x = (1+\alpha)(\alpha A_2/A_1)^{-\alpha/(1+\alpha)}.$$

(iii) $\underline{\lim}_{\lambda\to\infty} \lambda^{-1/(1+\alpha)} \log \int_0^\infty e^{-\lambda x} da(x) \geqq -B \qquad (0 \leqq B < \infty)$

implies $\underline{\lim}_{x\downarrow 0} x^{1/\alpha} \log a(x) \geqq -B^{(1+\alpha)/\alpha}$.

Finally we give an application which is of interest in the probabilistic point of view. Set $f(x)=\log x$, $\alpha>0$, and $\phi(x)=x^{\alpha}L(x)$. Then $g(-x)=\alpha\log(\alpha/ex)$. Let us denote by a_n the n-th moment of $\mu(dx)$. Remark that $\overline{\lim}_{n\to\infty} \sqrt[n]{a_n}/\phi(n)=e^{-A}$ is equivalent to

$$\overline{\lim}_{\lambda\to\infty}\frac{1}{\lambda}\log\int_0^{\infty}e^{\lambda f(x/\phi(\lambda))}\mu(dx)=-A, \quad etc.$$

Thus, from Theorem 1, we obtain the following;

Theorem 4.

(i) $$-\infty\leqq -A_1\leqq \underline{\lim}_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\infty)$$
$$\leqq \overline{\lim}_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\infty)\leqq -A_2\leqq 0$$

implies

(2.2) $$(\alpha/eA_1)^{\alpha}\leqq \underline{\lim}_{n\to\infty}\sqrt[n]{a_n}/\phi(n)\leqq \overline{\lim}_{n\to\infty}\sqrt[n]{a_n}/\phi(n)\leqq(\alpha/eA_2)^{\alpha}.$$

(ii) *Conversely, if* $A_2\neq 0$, *then* (2.2) *implies*

$$-\frac{\lambda_2}{\lambda_1}A_2\leqq \underline{\lim}_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\infty)$$
$$\leqq \overline{\lim}_{x\to\infty}\frac{1}{x}\log\mu(\phi(x),\infty)\leqq -A_2$$

where λ_1 [λ_2] *is the least* [*largest*] *solution of*

$$\log\eta-\eta=\log\frac{A_2}{A_1}-1.$$

Using Stirling's formula, we easily see that Theorem 4 includes Corollary of Davies [1] and Theorem 2 of Kôno [5].

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