J. Math. Kyoto Univ. (JMKYAZ)
19-1 (1979) 71-90

On quasi-positive definite functions and unitary representations of groups in Pontrjagin spaces

By

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(Communicated by Prof. H. Yoshizawa, Dec. 10, 1977)

§ 0. Introduction

Let G be a topological group. The general structure of unitary representations of G in Pontrjagin spaces has been investigated by M. A. Naimark, e. g., [5]. In this paper we shall consider the relation between cyclic unitary representations of G in Pontrjagin spaces and quasi-positive definite functions on G. This is the generalization of Godement's theory in [3] concerning cyclic unitary representations of G in Hilbert spaces and positive definite functions on G. In § 1 the basic notions concerning indefinite inner product spaces are introduced, and especially the elementary properties of quasi-positive spaces and Pontrjagin spaces are stated without proofs. In § 2 we shall give the definition of quasi-positive definite functions on G, and show that every quasi-positive definite function ϕ on G is given in the form: $\phi(g)=(f, U_g f)$ $(g\in G)$, where $g\mapsto U_g$ is a unitary representation of G in a quasi-positive space $\{\mathfrak{H}, (\ ,\ )\}$ and $f\in\mathfrak{H}$. Moreover in § 3 we shall see that there exists a one to one correspondence between the space of all quasi-positive definite functions on G and the space of all isometrically equivalent classes of weakly continuous cyclic unitary representations of G in Pontrjagin spaces. In § 4 we shall generalize the results in R. Godement [3, Chapter II, A] to the case of unitary representations of G in Pontrjagin spaces. Some examples of quasi-positive definite functions are given in § 5.

Notations. Throughout this paper, G is always a topological group with the identity e whose generic elements are denoted by the small letters $g, h, \cdots$. $C(G)$ is the space of all continuous functions on G, and $P_0(G)$ is the space of all continuous positive definite functions on G. For any function f on G and $h\in G$ we denote by f_h the left translate of f by h, i. e., $f_h(g)=f(h^{-1}g)$ $(g\in G)$. $\boldsymbol{C}$ [resp. $\boldsymbol{R}$] is the complex [resp. real] number field.

§ 1. Preliminary.

1.1. Inner product spaces. We shall begin with some definitions and notions concerning inner product spaces. More detailed expositions are found in [1] and [4]. Let V be a vector space over $\boldsymbol{C}$ with an inner product $(\ ,\ )$. Here an inner product $(\ ,\ )$ is a complex valued function on $V\times V$ with the properties that $(\lambda x+\mu y, z)=\lambda(x, z)+\mu(y, z)$ and $(x, y)=\overline{(y, x)}$ for any $x, y, z\in V$ and $\lambda, \mu\in\boldsymbol{C}$. We do not require that $(\ ,\ )$ is positive definite. The pair $\{V, (\ ,\ )\}$ is called an *inner product space*. Let W be a subspace of V. W is said to be *positive* [resp. *negative*] *definite* if $(x, x)>0$ [resp. $(x, x)<0$] for all $x\in W$ except $x=0$. When $(x, x)=0$ [resp. $(x, x)\geqq 0$] for all $x\in W$, W is said to be *neutral* [resp. *non-negative*]. We now put $W^{\perp}=\{x\in V; (x, y)=0 \text{ for all } y\in W\}$ and $W^0=W\cap W^{\perp}$. Then $W^{\perp}$ is called the *orthogonal complement of* W and W^0 is the *isotropic part* of W. W is said to be *non-degenerate* if $W^0=\{0\}$, and otherwise to be *degenerate*. For any subsets A and B of V we write $A\perp B$ if $(x, y)=0$ for any $x\in A$ and $y\in B$. Let $\{W_i; 1\leqq i\leqq n\}$ be a family of subspaces of V such that $W_i\perp W_j$ for any $1\leqq i<j\leqq n$. If V is the direct sum of $\{W_i; 1\leqq i\leqq n\}$, we say that V is the *orthogonal direct sum of* $\{W_i; 1\leqq i\leqq n\}$ and denote by $V=W_1(\dot{+})W_2(\dot{+})\cdots(\dot{+})W_n$. V is said to be *fundamentally decomposable* if V is decomposed as follows:

$$V=W^-(\dot{+})W^0(\dot{+})W^+, \tag{1.1}$$

where W^-, W^0 and W^+ are negative definite, neutral and positive definite subspaces of V respectively. Any decomposition of the form (1.1) is called a *fundamental decomposition* of V. Let W be a degenerate subspace of V and π be the canonical map of W onto the quotient space W/W^0. Since $(x+u, y+v)=(x, y)$ for any $x, y\in W$ and $u, v\in W^0$, it follows that W/W^0 becomes an inner product space with the inner product given by

$$(\pi(x), \pi(y))=(x, y) \quad (\pi(x), \pi(y)\in W/W^0). \tag{1.2}$$

Let $\{V_1, (\ ,\ )_1\}$ and $\{V_2, (\ ,\ )_2\}$ be inner product spaces and consider the product space $V_1\times V_2$. For any $[x_i, y_i]\in V_1\times V_2$ $(i=1, 2)$ we put

$$([x_1, y_1], [x_2, y_2])=(x_1, x_2)_1+(y_1, y_2)_2. \tag{1.3}$$

Then $V_1\times V_2$ becomes an inner product space with the inner product given by (1.3), which is called the product space of inner product spaces V_1 and V_2.

1.2. Linear operators on inner product spaces. Let $\{V, (\ ,\ )\}$ be an inner product space. A linear operator A on V is said to be *selfadjoint* if $(Ax, y)=(x, Ay)$ for any $x, y\in V$. A linear operator U on V is *unitary* if it is bijective and satisfies $(Ux, Uy)=(x, y)$ for any $x, y\in V$. It is clear that the space of all unitary operators on V is a group under the multiplication as operators. For inner product spaces $\{V_1, (\ ,\ )_1\}$ and $\{V_2, (\ ,\ )_2\}$ a linear map T of V_1 to V_2 is said to be *isometric* if $(Tx, Ty)_2=(x, y)_1$ for any $x, y\in V_1$. We note that an isometric map of V_1 to V_2 is not necessarily injective if V_1 is degenerate. We say that $\{V_1, (\ ,\ )_1\}$ and $\{V_2, (\ ,\ )_2\}$ are *isometrically isomorphic* if there is an isometric isomorphism of V_1 onto V_2.

1.3. Quasi-positive spaces. Let $\{V, (\ ,\ )\}$ be an inner product space and W be a finite dimensional subspace of V. Then W admits a fundamental decomposition: $W=W^-(\dot{+})W^0(\dot{+})W^+$. The $\dim(W^-)$ [resp. $\dim(W^+)$] is called the negative [resp. positive] rank of W, and denoted by $r^-(W)$ [resp. $r^+(W)$]. For any $f_1, f_2, \cdots, f_n \in V$ let $H(f_1, f_2, \cdots, f_n)$ be the Hermitian matrix of n-th order whose (i, j)-element is (f_i, f_j):

$$H(f_1, f_2, \cdots, f_n)=\begin{pmatrix} (f_1, f_1)(f_1, f_2)\cdots(f_1, f_n) \\ (f_2, f_1)(f_2, f_2)\cdots(f_2, f_n) \\ \cdots \quad \cdots \\ (f_n, f_1)(f_n, f_2)\cdots(f_n, f_n) \end{pmatrix} \tag{1.4}$$

For any Hermitian matrix H we denote by $\chi^-(H)$[resp. $\chi^+(H)$] the number of negative [resp. positive] eigenvalues of H. If W is a linear span of $\{f_1, f_2, \cdots, f_n\} \subset V$, then we have

$$r^-(W)=\chi^-(H(f_1, f_2, \cdots, f_n)) \quad \text{and} \quad r^+(W)=\chi^+(H(f_1, f_2, \cdots, f_n)). \tag{1.5}$$

Using the relation (1.5), we get easily

Proposition 1.1. *Let $\{V, (\ ,\ )\}$ be an inner product space and n be a non-negative integer. Assume that V is spanned by a subset $\mathfrak{F} \subset V$. Then the following three conditions are mutually equivalent:*

(1) *V contains at least one n-dimensional negative definite subspace, and $\dim(W) \leqq n$ for any negative definite subspace W of V.*

(2) *V contains at least one finite dimensional subspace with the negative rank n, and $r^-(W) \leqq n$ for any finite dimensional subspace W of V.*

(3) *There exists a finite subset $\{f_1^0, f_2^0, \cdots, f_m^0\}$ of $\mathfrak{F}$ for which $\chi^-(H(f_1^0, f_2^0, \cdots, f_m^0))=n$, and $\chi^-(H(f_1, f_2, \cdots, f_k)) \leqq n$ for any finite subset $\{f_1, f_2, \cdots, f_k\} \subset \mathfrak{F}$.*

Definition 1.2. An inner product space $\{V, (\ ,\ )\}$ is called a quasi-positive space with negative rank n, denoted by QP_n-space, if it satisfies the equivalent conditions in Proposition 1.1. The negative rank n is written by $r^-(V)$.

1.4. Pontrjagin spaces. Let $\{\mathfrak{H}, (\ ,\ )\}$ be a non-degenerate QP_n-space ($n>0$). According to Proposition 1.1, $\mathfrak{H}$ contains an n-dimensional negative definite subspace $\mathfrak{N}$. Then $\mathfrak{P}=\mathfrak{N}^\perp$ is positive definite, and $\mathfrak{H}$ is the orthogonal direct sum of $\mathfrak{N}$ and $\mathfrak{P}$. So we have a fundamental decomposition:

$$\mathfrak{H}=\mathfrak{N}(\dot{+})\mathfrak{P} \tag{1.6}$$

and any $x \in \mathfrak{H}$ is given in the form:

$$x=x^{-}+x^{+} \quad (x^{-}\in\mathfrak{N},\ x^{+}\in\mathfrak{P}). \tag{1.7}$$

For the fundamental decomposition (1.6) there corresponds a positive definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{H}$ defined as follows:

$$\langle x, y\rangle=-(x^{-}, y^{-})+(x^{+}, y^{+}) \quad (x, y\in\mathfrak{H}). \tag{1.8}$$

We define the norm by $\|x\|=\sqrt{\langle x, x\rangle}$ $(x\in\mathfrak{H})$. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of $\mathfrak{N}$ such that $\langle e_i, e_j\rangle=-(e_i, e_j)=\delta_{ij}$ for $1\leqq i\leqq j\leqq n$. Then we have for any $x, y\in\mathfrak{H}$

$$\langle x, y\rangle=(x, y)+2\sum_{k=1}^{n}(x, e_k)(e_k, y)=(x, y)+2\sum_{k=1}^{n}\langle x, e_k\rangle\langle e_k, y\rangle. \tag{1.9}$$

Moreover we have (cf. [1, II. Lemma 11.4]).

$$|(x, y)|\leqq\|x\|\,\|y\| \quad (x, y\in\mathfrak{H}). \tag{1.10}$$

The relations (1.9) and (1.10) are essential to our discussions in §§3 and 4. It is noted in [4, Theorem 1.3] that, if $\mathfrak{H}$ becomes a Hilbert space under the inner product (1.8), then any norm topologies corresponding to fundamental decompositions of $\mathfrak{H}$ are mutually equivalent. So the follwing definition is reasonable.

Definition 1.3. A non-degenerate QP_n-space $\{\mathfrak{H}, (\ ,\ )\}$ is called a Pontrjagin space with negative rank n, denoted by Π_n-space, if $\mathfrak{H}$ becomes a Hilbert space under the inner product of the form (1.8) corresponding to a fundamental decomposition of $\mathfrak{H}$.

If $\mathfrak{P}$ in (1.6) is not complete in the norm topology and $\tilde{\mathfrak{P}}$ is the completion of $\mathfrak{P}$, then we get a Π_n-space $\tilde{\mathfrak{H}}=\mathfrak{N}(\dot{+})\tilde{\mathfrak{P}}$, which is called the completion of $\mathfrak{H}$. For a non-degenerate QP_n-space its any completions are mutually isometrically isomorphic. Any topological concepts in a Π_n-space $\mathfrak{H}$ are always defined by the Hilbert norm topology induced from the inner product of the form (1.8) corresponding to a fundamental decomposition of $\mathfrak{H}$.

1.5. Properties of QP_n- and Π_n-spaces. In order to refer in later sections, we here collect some properties concerning QP_n- and Π_n-spaces without proofs.

Lemma 1.4. *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be inner product spaces and suppose that there exists an isometric linear map of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$. Then $\mathfrak{H}_1$ is a QP_n-space if and only if so is $\mathfrak{H}_2$.*

Lemma 1.5. *(cf. [1, I. Theorem 11.7]). Any QP_n-space is fundamentally decomposable.*

Lemma 1.6. *Any subspace W of a QP_n-space is a QP_m-space for some m, $0\leqq m\leqq n$, and the quotient space W/W^0 is also a QP_m-space.*

Lemma 1.7. (*cf.* [1, IX. *Theorem* 1.4]). *Let* $\mathfrak{H}$ *be a* Π_n*-space and* $\mathfrak{H}_0$ *a dense subspace of* $\mathfrak{H}$. *Then* $\mathfrak{H}_0$ *contains an* n*-dimensional negative definite subspace.*

More generally the following is proved by the same method as in Theorem 1.4 in [1, IX].

Lemma 1.8. *Let* $\mathfrak{H}$ *be a normed space with norm* $\| \ \|$, *and* $\mathfrak{H}_0$ *be a dense subspace of* $\mathfrak{H}$. *Assume that* $\mathfrak{H}$ *admits a continuous inner product* $(\ , \ )$, *that is, it satisfies* $|(x, y)| \leqq K\|x\| \ \|y\|$ *for any* $x, y \in \mathfrak{H}$, *where* K *is a positive constant. If* $\{\mathfrak{H}, (\ , \ )\}$ *contains an* n*-dimensional negative definite subspace, then* $\mathfrak{H}_0$ *contains also an* n*-dimensional negative definite subspace. Especially* $\{\mathfrak{H}, (\ , \ )\}$ *is a* QP_n*-space if and only if so is* $\{\mathfrak{H}_0, (\ , \ )\}$.

Lemma 1.9. *Let* $\mathfrak{H}$ *be a* Π_n*-space,* $\mathfrak{K}$ *be a closed subspace of* $\mathfrak{H}$, *and set* $m = r^-(\mathfrak{K}) \leqq n$. *Then we have:*

(1) *If* $\mathfrak{K}$ *is non-degenerate, then* $\mathfrak{K}$ [*resp.* $\mathfrak{K}^\perp$] *is a* Π_m [*resp.* Π_{n-m}]*-space, and* $\mathfrak{H}$ *is the orthogonal direct sum of* $\mathfrak{K}$ *and* $\mathfrak{K}^\perp$.

(2) *If* $\mathfrak{H}$ *is the sum of* $\mathfrak{K}$ *and* $\mathfrak{K}^\perp$ *in the algebraic sense, then* $\mathfrak{K}$ *is non-degenerate.*

(3) *If* $\mathfrak{K}$ *is degenerate, then the quotient space* $\mathfrak{K}/\mathfrak{K}^0$ *becomes a* Π_m*-space and* $m < n$.

Lemma 1.10. *The product space of a* QP_l [*resp.* Π_l]*-space and a* QP_m [*resp.* Π_m]*-space is a* QP_n [*resp.* Π_n]*-space, where* $n = l + m$.

Lemma 1.11. *Let* $\mathfrak{H}_1$ *and* $\mathfrak{H}_2$ *be* Π_n*-spaces, and* U *be an isometric isomorphism of a dense subspace of* $\mathfrak{H}_1$ *onto a dense subspace of* $\mathfrak{H}_2$, *then* U *is continuous and can be extended continuously to an isometric isomorphism of* $\mathfrak{H}_1$ *onto* $\mathfrak{H}_2$.

§2. Unitary representations in QP_n-spaces and quasi-positive definite functions.

2.1. Unitary representations of groups in inner product spaces. Let $\{\mathfrak{H}, (\ , \ )\}$ be an inner product space. By a unitary representation $\boldsymbol{U} = \{U_g, \mathfrak{H}\}$ of G in $\mathfrak{H}$ we mean a homomorphism $g \mapsto U_g$ of G to the group of all unitary operators on $\mathfrak{H}$. $\{U_g, \mathfrak{H}\}$ is said to be (w)-*continuous* if the function $(x, U_g y)$ on G is in $C(G)$ for any $x, y \in \mathfrak{H}$. Let $\mathfrak{M}$ be a $\boldsymbol{U}$-invariant subspace of $\mathfrak{H}$, i.e., $U_g(\mathfrak{M}) \subseteqq \mathfrak{M}$ for all $g \in G$. Then, restricting each U_g ($g \in G$) to $\mathfrak{M}$, we get a unitary representation $\{U_g, \mathfrak{M}\}$, which is called the *partial representation* of $\{U_g, \mathfrak{H}\}$. Let $\boldsymbol{U}_i = \{U_g^{(i)}, \mathfrak{H}_i\}$ be a unitary representation of G in an inner product space $\mathfrak{H}_i$ for $i = 1, 2$. We define a unitary operator U_g ($g \in G$) on the product space $\mathfrak{H}_1 \times \mathfrak{H}_2$ as follows:

$$U_g[x, y] = [U_g^{(1)} x, U_g^{(2)} y] \quad ([x, y] \in \mathfrak{H}_1 \times \mathfrak{H}_2). \tag{2.1}$$

Then we get a unitary representation $\{U_g, \mathfrak{H}_1\times\mathfrak{H}_2\}$ of G, which is called the *product representation* of U_1 and U_2 and denoted by $U_1\times U_2$. If there exists an isometric linear map τ of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that $\tau U_g^{(1)}=U_g^{(2)}\tau$ for all $g\in G$, then U_1 is said to be *isometric* to U_2, and denoted by $U_1 \overset{\tau}{\simeq} U_2$. Moreover if τ is isomorphic, then U_1 and U_2 are said to be *isometrically equivalent*, and denoted by $U_1 \overset{\tau}{\cong} U_2$ or $U_1\cong U_2$.

2.2. Quasi-positive definite functions. Let ϕ be a function on G such that

$$\phi(g^{-1})=\overline{\phi(g)} \quad \text{for all} \quad g\in G. \tag{2.2}$$

For any $\{g_1, g_2, \cdots, g_m\}\subset G$ define an Hermitian matrix $\Phi(g_1, g_2, \cdots, g_m)$ of m-th order as follows:

$$\Phi(g_1, g_2, \cdots, g_m)=\begin{pmatrix} \phi(g_1^{-1}g_1)\,\phi(g_1^{-1}g_2)\cdots\phi(g_1^{-1}g_m) \\ \phi(g_2^{-1}g_1)\,\phi(g_2^{-1}g_2)\cdots\phi(g_2^{-1}g_m) \\ \cdots \qquad \cdots \\ \phi(g_m^{-1}g_1)\,\phi(g_m^{-1}g_2)\cdots\phi(g_m^{-1}g_m) \end{pmatrix} \tag{2.3}$$

Definition 2.1. A function ϕ on G with (2.2) is called a quasi-positive definite function with negative rank n if it has the following two properties:

$(QP)_1$ $\quad \chi^-(\Phi(g_1, g_2, \cdots, g_m))=n$ for some $\{g_1, g_2, \cdots, g_m\}\subset G$,

$(QP)_2$ $\quad \chi^-(\Phi(g_1, g_2, \cdots, g_k))\leqq n$ for any $\{g_1, g_2, \cdots, g_k\}\subset G$.

The negative rank n of ϕ is denoted by $r^-(\phi)$.

We denote by $P_n(G)$ the space of all continuous quasi-positive definite functions on G with negative rank n and set $QP(G)=\bigcup_{n=0}^{\infty}P_n(G)$. Let $L(G)$ be the linear space of all functions f on G such that $\{g\in G\,; f(g)\neq 0\}$ is finite, and let $\phi\in C(G)$ satisfy (2.2). Then on $L(G)$ we can define an inner product $(\ ,\ )_\phi$ as follows:

$$(f_1, f_2)_\phi=\sum_{g,h\in G} f_1(g)\overline{f_2(h)}\phi(g^{-1}h) \quad (f_1, f_2\in L(G)). \tag{2.4}$$

For any $g\in G$ define $\varepsilon_{(g)}\in L(G)$ by

$$\varepsilon_{(g)}(h)=\begin{cases} 1 & (h=g) \\ 0 & (h\neq g). \end{cases} \tag{2.5}$$

$L(G)$ is spanned by the family $\{\varepsilon_{(g)}\,; g\in G\}$. In the inner product space $\{L(G), (\ ,\ )_\phi\}$, using the notation (1.4), we have $\Phi(g_1, g_2, \cdots, g_m)=H(\varepsilon_{(g_1)}, \varepsilon_{(g_2)}, \cdots, \varepsilon_{(g_m)})$ for any $\{g_1, g_2, \cdots, g_m\}\subset G$. So comparing Definition 2.1 with Proposition 1.1 (3), we get immediately

Theorem 2.2. *Let $\phi\in C(G)$ satisfy* (2.2). *Then ϕ belongs to $P_n(G)$ if and only if $\{L(G), (\ ,\ )_\phi\}$ becomes a QP_n-space.*

2.3. Relation between quasi-positive definite functions and unitary representations in quasi-positive spaces.

Theorem 2.3. *For any* $\phi \in P_n(G)$ *there exists a* (w)*-continuous unitary representation* $\{U_g, \mathfrak{H}\}$ *of* G *in a* QP_n*-space* $\mathfrak{H}$ *such that* ϕ *is given in the form:*

$$\phi(g)=(f_0, U_g f_0) \quad (g\in G), \tag{2.6}$$

where $f_0 \in \mathfrak{H}$.

Proof. Let $L(G)_\phi=\{L(G), (\ ,\ )_\phi\}$. Then $L(G)_\phi$ is a QP_n-space by Theorem 2.2. For any $g\in G$ define a linear operator U_g on $L(G)$ by $U_g f=f_g$ $(f\in\mathfrak{H})$. It is easily seen that each U_g $(g\in G)$ is a unitary operator on $L(G)_\phi$, and that $\{U_g, L(G)_\phi\}$ is a unitary representation of G in the QP_n-space $L(G)_\phi$. Further for any $f_1, f_2\in L(G)$ and $g\in G$

$$\begin{aligned}(f_1, U_g f_2)_\phi &= \sum_{h,k\in G} f_1(h)\overline{f_2(g^{-1}k)}\phi(h^{-1}k) \\ &= \sum_{h,k\in G} f_1(h)\overline{f_2(k)}\phi(h^{-1}gk).\end{aligned} \tag{2.7}$$

In particular for $f_0=\varepsilon_{(e)}\in L(G)$ (cf. (2.5))

$$(f_0, U_g f_0)=\phi(g) \quad (g\in G). \tag{2.8}$$

Since ϕ is continuous, it follows from (2.7) that $\{U_g, L(G)_\phi\}$ is (w)-continuous, and from (2.8) that $\{U_g, L(G)_\phi\}$ is our desired representation of G.

q.e.d.

Conversely we have

Theorem 2.4. *Let* $\{U_g, \mathfrak{H}\}$ *be a* (w)*-continuous unitary representation of* G *in a* QP_n*-space* $\mathfrak{H}$. *For any* $f\in\mathfrak{H}$ *let* $\mathfrak{H}(f)$ *be the linear span of* $\{U_g f; g\in G\}$ *in* $\mathfrak{H}$ *and define* $\phi(g)\in C(G)$ *by* $\phi(g)=(f, U_g f)$ $(g\in G)$. *Then* ϕ *belongs to* $P_m(G)$, *where* $m=r^-(\mathfrak{H}(f))\leqq n$.

Proof. It is clear that ϕ satisfies (2.2). Let τ be a linear map of $L(G)$ onto $\mathfrak{H}(f)$ defined as follows:

$$\tau : L(G)\, x \longmapsto \tau(x)=\sum_{g\in G} x(g) U_g f \in \mathfrak{H}(f).$$

Then for any $x, y\in L(G)$

$$\begin{aligned}(\tau(x), \tau(y)) &= \sum_{g,h\in G} x(g)\overline{y(h)}(U_g f, U_h f) \\ &= \sum_{g,h\in G} x(g)\overline{y(h)}\phi(g^{-1}h)=(x, y)_\phi.\end{aligned}$$

This shows that τ is an isometric map of $\{L(G), (\ ,\ )_\phi\}$ onto the QP_m-space $\mathfrak{H}(f)$, and it follows from Lemma 1.4 that $\{L(G), (\ ,\ )_\phi\}$ becomes a QP_m-space. So we get from Theorem 2.2 that $\phi\in P_m(G)$. q.e.d.

Theorem 2.5. (1) *If* $\phi\in P_n(G)$ *and* $\lambda>0$, *then* $\lambda\phi\in P_n(G)$.

(2) *The constant function* $\phi(g)=c$ $(g\in G)$ *is in* $P_0(G)$ *if* $c\geqq 0$, *and is in* $P_1(G)$ *if* $c<0$.

(3) *If* $\phi_1\in P_{n_1}(G)$ *and* $\phi_2\in P_{n_2}(G)$, *then* $\phi=\phi_1+\phi_2\in P_m(G)$, *where* $m\leqq n_1+n_2$.

Proof. (1) and (2) are obvious. Let us see (3). By virtue of Theorem 2.3, for each $i=1, 2$ there exists a (w)-continuous unitary representation $\boldsymbol{U}_i=\{U_g^{(i)}, \mathfrak{H}_i\}$ of G in a QP_{n_i}-space $\mathfrak{H}_i$ such that ϕ_i is given in the form $\phi_i(g)=(f_i, U_g^{(i)}f_i)$ $(g\in G)$ for some $f_i\in\mathfrak{H}_i$. Consider now the product representation $\{U_g, \mathfrak{H}_1\times\mathfrak{H}_2\}$ of $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$. Then we have

$$([f_1, f_2], U_g[f_1, f_2])=(f_1, U_g^{(1)}f_1)+(f_2, U_g^{(2)}f_2)=\phi_1(g)+\phi_2(g) \quad (g\in G). \tag{2.9}$$

Since $\mathfrak{H}_1\times\mathfrak{H}_2$ is a $QP_{n_1+n_2}$-space by Lemma 1.10, it follows from (2.9) and Theorem 2.4 that $\phi\in P_m(G)$ for some $m\leqq n_1+n_2$.

§ 3. Cyclic unitary representations in Π_n-spaces and quasi-positive definite functions.

3.1. Characteristic functions of cyclic unitary representations in Π_n-spaces. Let $\boldsymbol{U}=\{U_g, \mathfrak{H}\}$ be a unitary representation of G in a Π_n-space $\{\mathfrak{H}, (\ ,\ )\}$. $f\in\mathfrak{H}$ is said to be *cyclic* with respect to $\boldsymbol{U}$ if the linear span of $\{U_g f;\ g\in G\}$ is dense in $\mathfrak{H}$. If $\boldsymbol{U}$ admits a cyclic vector $f\in\mathfrak{H}$, then $\boldsymbol{U}$ is called a *cyclic unitary representation* of G and is denoted by the triplet $\{U_g, \mathfrak{H}, f\}$. The *characteristic function* ϕ of $\{U_g, \mathfrak{H}, f\}$ is defined as follows:

$$\phi(g)=(f, U_g f) \quad (g\in G). \tag{3.1}$$

We say that two cyclic unitary representations $\boldsymbol{U}_i=\{U_g^{(i)}, \mathfrak{H}_i, f_i\}$ for $i=1, 2$ are *isometrically equivalent* if there exists an isometric isomorphism τ of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ such that

$$\tau f_1=f_2 \quad \text{and} \quad \tau U_g^{(1)}=U_g^{(2)}\tau \qquad \text{for all} \quad g\in G. \tag{3.2}$$

In this case we denote by $\boldsymbol{U}_1\overset{\tau}{\cong}\boldsymbol{U}_2$ or $\boldsymbol{U}_1\cong\boldsymbol{U}_2$.

Theorem 3.1. *For each* $i=1, 2$ *let* $\boldsymbol{U}_i=\{U_g^{(i)}, \mathfrak{H}_i, f_i\}$ *be a cyclic unitary representation of* G *in a* Π_n*-space* $\mathfrak{H}_i$ *with the characteristic function* ϕ_i. *Then* $\boldsymbol{U}_1\cong\boldsymbol{U}_2$ *if and only if* $\phi_1=\phi_2$.

Proof. Suppose that $\boldsymbol{U}_1\overset{\tau}{\cong}\boldsymbol{U}_2$. Then by (3.2) we have for any $g\in G$

$$\phi_2(g)=(f_2, U_g^{(2)}f_2)=(\tau f_1, \tau U_g^{(1)}f_1)=(f_1, U_g^{(1)}f_1)=\phi_1(g).$$

Conversely suppose that $\phi_1=\phi_2$. Let $\mathfrak{H}(f_i)$ be the linear span of $\{U_g^{(i)}f_i;\ g\in G\}$ in $\mathfrak{H}_i$ for $i=1, 2$. For any $x=\sum_{j=1}^{m}\lambda_j U_{g_j}^{(1)}f_1\in\mathfrak{H}(f_1)$, where $\lambda_j\in\boldsymbol{C}$ and $g_j\in G$ $(1\leqq j\leqq m)$, we put

$$\tau(x)=\sum_{j=1}^{m}\lambda_j U_{g_j}^{(2)}f_2\in\mathfrak{H}(f_2).$$

From the hypothesis we have for any $x, y\in\mathfrak{H}(f_1)$ and $g\in G$

$$(x, U_g^{(1)}f_1)=(\tau(x), U_g^{(2)}f_2), \tag{3.3}$$

$$(\tau(x), \tau(y))=(x, y), \tag{3.4}$$

$$\tau(f_1)=f_2 \quad\text{and}\quad \tau(U_g^{(1)}x)=U_g^{(2)}\tau(x). \tag{3.5}$$

Since $\mathfrak{H}(f_1)^{\perp}=\mathfrak{H}(f_2)^{\perp}=\{0\}$, it follows from (3.3) and (3.4) that τ is regarded as an isometric isomorphism of $\mathfrak{H}(f_1)$ onto $\mathfrak{H}(f_2)$. By Lemma 1.11 τ can be extended continuously to an isometric isomorphism of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$, and hence from (3.5) it follows that $\boldsymbol{U}_1\cong\boldsymbol{U}_2$. q.e.d.

Let $\boldsymbol{U}=\{U_g, \mathfrak{H}, f\}$ be a cyclic unitary representation of G in a Π_n-space $\mathfrak{H}$ $(n>0)$ with the characteristic function ϕ, and $\mathfrak{H}_0$ be the linear span of $\{U_g f; g\in G\}$ in $\mathfrak{H}$. Then by Lemma 1.7 there exists an n-dimensional negative definite subspace $\mathfrak{N}$ contained in $\mathfrak{H}_0$. As noted in §1.4, putting $\mathfrak{P}=\mathfrak{N}^{\perp}$, we have a fundamental decomposition of $\mathfrak{H}$:

$$\mathfrak{H}=\mathfrak{N}(\dot{+})\mathfrak{P}. \tag{3.6}$$

By $\langle\,,\rangle$ we denote the positive definite inner product on $\mathfrak{H}$ corresponding to the fundamental decomposition (3.6) (cf. ((1.8)) and by $\|\ \|$ the norm induced from $\langle\,,\rangle$. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of $\mathfrak{N}$ such that $\langle e_i, e_j\rangle=-(e_i, e_j)=\delta_{ij}$ for $1\leqq i\leqq j\leqq n$. Since U_g $(g\in G)$ is unitary, from (1.9) we have for any $x, y\in\mathfrak{H}$ and $g\in G$

$$\langle U_g x, U_g x\rangle=(x, x)+2\sum_{k=1}^{n}|(x, U_{g^{-1}}e_k)|^2 \tag{3.7}$$

$$=(x, x)+2\sum_{k=1}^{n}|\langle U_g x, e_k\rangle|^2, \tag{3.7$'$}$$

$$\langle x, U_g y\rangle=(x, U_g y)+2\sum_{k=1}^{n}(x, e_k)(U_{g^{-1}}e_k, y) \tag{3.8}$$

$$=(x, U_g y)+2\sum_{k=1}^{n}\langle x, e_k\rangle\langle e_k, U_g y\rangle. \tag{3.8$'$}$$

Let $x=\sum_{i=1}^{l}\lambda_i U_{g_i}f$ and $y=\sum_{j=1}^{m}\mu_j U_{h_j}f$ be any elements in $\mathfrak{H}_0$, where $\lambda_i, \mu_j\in\boldsymbol{C}$ and $g_i, h_j\in G$ for $1\leqq i\leqq l$ and $1\leqq j\leqq m$. Then the function $(x, U_g y)$ on G is given in terms of two sided translations of ϕ as follows:

$$(x, U_g y)=\sum_{i=1}^{l}\sum_{j=1}^{m}\lambda_i\bar{\mu}_j\phi(g_i^{-1}gh_j)\quad(g\in G). \tag{3.9}$$

Using these notations and relations (3.7)-(3.9), we prove the following two theorems.

Theorem 3.2. *Let $U=\{U_g, \mathfrak{H}, f\}$ be a cyclic unitary representation of G in a Π_n-space $\mathfrak{H}$ with the characteristic function ϕ. Then U is uniformly bounded, i.e., $K=\sup\{\|U_g\|\,;\, g\in G\}<\infty$, if and only if ϕ is bounded.*

Proof. If $K<\infty$, then it follows from (1.10) that $|\phi(g)|\leqq K\|f\|^2$ for any $g\in G$. Hence ϕ is bounded. Conversely suppose that ϕ is bounded. As $e_j, e_k \in\mathfrak{N}$, the function $(e_j, U_g e_k)$ on G is bounded by (3.9) and so is $\|U_g e_k\|^2$ by (3.7) for each $1\leqq k\leqq n$. Hence we can take a constant $M>0$ such that $\sum_{k=1}^{n}\|U_g e_k\|^2<M$ for all $g\in G$. From (1.10) and (3.7) we have $\|U_g x\|^2<(1+2M)\|x\|^2$ for any $x\in\mathfrak{H}$ and $g\in G$. Therefore U is uniformly bounded. q.e.d.

Theorem 3.3. *Let $U=\{U_g, \mathfrak{H}, f\}$ be a cyclic unitary representation of G in a Π_n-space $\mathfrak{H}$ with the characteristic function ϕ. Then the following conditions are mutually equivalent:*

(1) *U is weakly continuous, i.e., the function $\langle x, U_g y\rangle$ on G is in $C(G)$ for any $x, y\in\mathfrak{H}$.*

(2) *U is (w)-continuous.*

(3) *ϕ is continuous.*

Especially if G is a locally compact group, then the above conditions are equivalent to

(4) *U is strongly continuous, i.e., the map $G\ni g\mapsto U_g x\in\mathfrak{H}$ is continuous for any $x\in\mathfrak{H}$.*

Proof. From (3.8) and (3.8)$'$ it follows immediately that (1) and (2) are equivalent, and it is obvious that (2) implies (3). Further if G is locally compact, then it is shown in [2, Theorem 2.8] that (1) and (4) are equivalent. Thus it remains to prove that (3) implies (2). Suppose that ϕ is continuous. We consider the following subsets of $\mathfrak{H}$:

$$\mathfrak{H}_1=\{x\in\mathfrak{H}\,;\ \text{the function } (x, U_g y) \text{ on } G \text{ is in } C(G) \text{ for any } y\in\mathfrak{H}_0\},$$

$$\mathfrak{H}_2=\{x\in\mathfrak{H}\,;\ \text{the function } (x, U_g z) \text{ on } G \text{ is in } C(G) \text{ for any } z\in\mathfrak{H}\}.$$

It is obvious that $\mathfrak{H}_2\subseteqq\mathfrak{H}_1$ and $\mathfrak{H}_0\subseteqq\mathfrak{H}_1$ by (3.9). If $\mathfrak{H}_1\subseteqq\mathfrak{H}_2$, then we can conclude easily that $\mathfrak{H}=\mathfrak{H}_2$. This means that (3) implies (2). So it suffices to prove that $\mathfrak{H}_1\subseteqq\mathfrak{H}_2$. Let $x\in\mathfrak{H}_1$, $z\in\mathfrak{H}$, $g_0\in G$ and $\varepsilon>0$. In view of (3.7) the function $\|U_g x\|$ on G is continuous. Putting $M=\|U_{g_0^{-1}} x\|+1$, we take a neighborhood W_1 of g_0 such that $\|U_{g^{-1}} x\|<M$ for any $g\in W_1$. Since $\mathfrak{H}_0$ is dense in $\mathfrak{H}$, we can take $y\in\mathfrak{H}_0$ with $\|z-y\|<\varepsilon/3M$. Then the function $(x, U_g y)$ on G is continuous, and there exists a neighborhood W_2 of g_0 such that for all $g\in W_2$

$$|(x, U_{g_0} y)-(x, U_g y)|<\frac{\varepsilon}{3}.$$

So for any $g \in W_1 \cap W_2$ we have

$$|(x, U_{g_0} z) - (x, U_g z)| \leqq |(U_{g_0^{-1}} x, z-y)| + |(x, U_{g_0} y) - (x, U_g y)| \\ + |(U_{g^{-1}} x, y-z)| \\ < 2M\|z-y\| + |(x, U_{g_0} y) - (x, U_g y)| < \varepsilon.$$

Hence the function $(x, U_g z)$ on G is in $C(G)$. This shows that $x \in \mathfrak{H}_2$ and $\mathfrak{H}_1 \subseteqq \mathfrak{H}_2$. Thus the proof completes. q. e. d.

3. 2. The correspondence between quasi-positive definite functions and weakly continuous cyclic unitary representations in Π_n-spaces.

Theorem 3. 4. *Let* $\boldsymbol{U} = \{U_g, \mathfrak{H}, f\}$ *be a weakly continuous cyclic unitary representation of* G *in a* Π_n*-space* $\mathfrak{H}$. *Then the characteristic function* ϕ *of* $\boldsymbol{U}$ *is in* $P_n(G)$.

Proof. Let $\mathfrak{H}_0$ be the linear span of $\{U_g f; g \in G\}$ in $\mathfrak{H}$. Then $\mathfrak{H}_0$ is a QP_n-space by Lemma 1.7. So our assertion is an immediate consequence of Theorem 2.4. q. e. d.

Theorem 3. 5. *For any* $\phi \in P_n(G)$ *there exists a weakly continuous cyclic unitary representation of* G *in* Π_n*-space whose characteristic function is* ϕ.

Proof. Let $\mathfrak{H}_0$ be the subspace of $C(G)$ spanned by the family $\{\phi_g; g \in G\}$ and let $L(G)_\phi = \{L(G), (\ ,\ )_\phi\}$ (cf. § 2.2). Now define a linear map τ of $L(G)$ onto $\mathfrak{H}_0$ by

$$\tau : L(G) \ni \alpha \longmapsto \tau(\alpha) = \sum_{h \in G} \alpha(h) \phi_h \in \mathfrak{H}_0.$$

For any $x = \tau(\alpha)$, $y = \tau(\beta) \in \mathfrak{H}_0$, where $\alpha, \beta \in L(G)$, we put

(3. 10) $$(x, y) = (\alpha, \beta)_\phi.$$

Then using (2. 2) we have

$$(x, y) = \sum_{g, h \in G} \alpha(g) \overline{\beta(h)} \phi(g^{-1} h) = \sum_{h \in G} \overline{\beta(h)} x(h) = \sum_{g \in G} \alpha(g) \overline{y(g)}.$$

This means that (x, y) depends only on x and y. So the function (x, y) on $\mathfrak{H}_0 \times \mathfrak{H}_0$ given by (3.10) is regarded as an inner product on $\mathfrak{H}_0$, and it follows from (3.10) that τ is an isometric linear map of $L(G)_\phi$ onto $\{\mathfrak{H}_0, (\ ,\ )\}$. Since $\phi \in P_n(G)$, $L(G)_\phi$ is a QP_n-space by Theorem 2.2, and so is $\{\mathfrak{H}_0, (\ ,\ )\}$ by Lemma 1.4. The following relations are seen easily:

(3. 11) $$(x, \phi_g) = x(g) \quad \text{for any} \quad x \in \mathfrak{H}_0 \quad \text{and} \quad g \in G,$$

(3. 12) $$(\phi, \phi_g) = \phi(g) \quad \text{for any} \quad g \in G.$$

If $x \in \mathfrak{H}_0 \cap \mathfrak{H}_0^\perp$, then $x(g) = 0$ for all $g \in G$ by (3.11). This shows that $\{\mathfrak{H}_0, (\ ,\ )\}$ is non-degenerate. For any $g \in G$ define a linear operator U_g on $\mathfrak{H}_0$ by $U_g x = x_g$

$(x\in\mathfrak{H}_0)$. Then it is obvious that each U_g $(g\in G)$ is unitary, and that $\{U_g, \mathfrak{H}_0\}$ is a unitary representation of G in the non-degenerate QP_n-space $\{\mathfrak{H}_0, (\ ,\ )\}$. Let $\mathfrak{H}_\phi$ be the Π_n-space obtained from $\mathfrak{H}_0$ by the completion (cf. §1.4). Then each U_g $(g\in G)$ is extended to a unitary operator on $\mathfrak{H}_\phi$, which is also denoted by the same letter U_g. Thus we get a unitary representation $\{U_g, \mathfrak{H}_\phi\}$ of G in the Π_n-space $\mathfrak{H}_\phi$. Since $\mathfrak{H}_0$ is dense in $\mathfrak{H}_\phi$, ϕ is cyclic with respect to $\{U_g, \mathfrak{H}_\phi\}$. So from Theorem 3.3 and (3.12) it follows that $\{U_g, \mathfrak{H}_\phi, \phi\}$ is a weakly continuous cyclic unitary representation of G with the characteristic function ϕ. q. e. d.

Let $\phi\in P_n(G)$. By $\boldsymbol{U}(\phi)$ we denote always the set of all weakly continuous cyclic unitary representations of G in Π_n-spaces with the characteristic function ϕ. According to Theorem 3.1, $\boldsymbol{U}(\phi)$ is an isometrically equivalent class in the space $\boldsymbol{U}(G)$ of all weakly continuous cyclic unitary representations of G in Pontrjagin spaces. Theorems 3.4 and 3.5 show that the correspondence $\phi\mapsto\boldsymbol{U}(\phi)$ is a bijective map of $QP(G)$ onto the space of all isometrically equivalent classes in $\boldsymbol{U}(G)$. From now on we shall consider only weakly continuous representations, so that the adjective "*weakly continuous*" is omitted.

3.3. Subrepresentations and quotient representations. Let $\boldsymbol{U}=\{U_g, \mathfrak{H}\}$ [resp. $\boldsymbol{U}=\{U_g, \mathfrak{H}, f_0\}$] be a [cyclic] unitary representation of G in Π_n-space $\mathfrak{H}$, $\mathfrak{K}$ be a $\boldsymbol{U}$-invariant closed subspace of $\mathfrak{H}$ and set $m=r^-(\mathfrak{K})$. Then $\mathfrak{K}^\perp$ and $\mathfrak{K}^0$ are also $\boldsymbol{U}$-invariant. First suppose that $\mathfrak{K}$ is non-degenerate. As noted in Lemma 1.9 (1), $\mathfrak{K}$ and $\mathfrak{K}^\perp$ become Π_m- and Π_{n-m}-spaces respectively, and $\mathfrak{H}$ is the orthogonal direct sum of $\mathfrak{K}$ and $\mathfrak{K}^\perp$. Let f_1 and f_2 be the orthogonal projections of f_0 in $\mathfrak{K}$ and $\mathfrak{K}^\perp$ respectively. Then we say that the partial representations $\boldsymbol{U}_1=\{U_g, \mathfrak{K}\}$ [resp. $\boldsymbol{U}_1=\{U_g, \mathfrak{K}, f_1\}$] and $\boldsymbol{U}_2=\{U_g, \mathfrak{K}^\perp\}$ [resp. $\boldsymbol{U}_2=\{U_g, \mathfrak{K}^\perp, f_2\}$] are *subrepresentations* of $\boldsymbol{U}=\{U_g, \mathfrak{H}\}$ [resp. $\boldsymbol{U}=\{U_g, \mathfrak{H}, f_0\}$], and that $\boldsymbol{U}$ is the orthogonal direct sum of $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$. In this case we denote by $\boldsymbol{U}=\boldsymbol{U}_1(\dot{+})\boldsymbol{U}_2$. Any [cyclic] unitary representation $\boldsymbol{U}'$ of G is said to be *contained in* $\boldsymbol{U}$ if $\boldsymbol{U}$ has a subrepresentation which is isometrically equivalent to $\boldsymbol{U}'$.

Next suppose that $\mathfrak{K}$ is degenerate. Then the quotient space $\tilde{\mathfrak{K}}=\mathfrak{K}/\mathfrak{K}^0$ becomes a Π_m-space by Lemma 1.9 (3). For any $g\in G$ define a unitary operator $\tilde{U}_g$ on $\tilde{\mathfrak{K}}$ by

$$\tilde{U}_g(\pi(x))=\pi(U_g x) \quad (x\in\mathfrak{K}), \tag{3.13}$$

where π is the canonical map of $\mathfrak{K}$ onto $\tilde{\mathfrak{K}}$. Then we get a weakly continuous unitary representation $\tilde{\boldsymbol{U}}=\{\tilde{U}_g, \tilde{\mathfrak{K}}\}$ of G in the Π_m-space $\tilde{\mathfrak{K}}$, which is called *the quotient representation of* $\boldsymbol{U}$ *determined by the* $\boldsymbol{U}$*-invariant subspace* $\mathfrak{K}$. The partial representation $\{U_g, \mathfrak{K}\}$ is isometric to $\{\tilde{U}_g, \tilde{\mathfrak{K}}\}$: $\{U_g, \mathfrak{K}\}\overset{\pi}{\simeq}\{\tilde{U}_g, \tilde{\mathfrak{K}}\}$.

Under these terminology, we obtain easily

Theorem 3.6. *Let* $\boldsymbol{U}=\{U_g, \mathfrak{H}\}$ *be a unitary representation of* G *in a* Π_n*-space* $\mathfrak{H}$. *For any* $f\in\mathfrak{H}$ *let* $\mathfrak{H}_f$ *be the closed linear span of* $\{U_g f; g\in G\}$ *in* $\mathfrak{H}$

and define $\phi\in QP(G)$ *by* $\phi(g)=(f, U_g f)$ $(g\in G)$. *If* $\mathfrak{H}_f$ *is non-degenerate, then the subrepresentation* $\{U_g, \mathfrak{H}_f, f\}$ *of* $\boldsymbol{U}$ *belongs to* $\boldsymbol{U}(\phi)$. *While if* $\mathfrak{H}_f$ *is degenerate, then the quotient representation* $\{\tilde{U}_g, \tilde{\mathfrak{H}}_f, \pi(f)\}$ *of* $\boldsymbol{U}$ *determined by* $\mathfrak{H}_f$ *belongs to* $\boldsymbol{U}(\phi)$, *where* π *is the canonical map of* $\mathfrak{H}_f$ *onto* $\tilde{\mathfrak{H}}_f=\mathfrak{H}_f/\mathfrak{H}_f^0$.

§4. Product representations and normal decomposition of quasi-positive definite functions

4.1. Product representations. Let $\phi\in P_n(G)$ and $\{U_g, \mathfrak{H}, f\}\in\boldsymbol{U}(\phi)$. By $H(\phi)$ we denote the linear subspace of $C(G)$ consisting of all functions $x(g)$ given in the form: $x(g)=(x, U_g f)$ $(g\in G)$ for some $x\in\mathfrak{H}$. It is clear that $H(\phi)$ does not depend on the choice of $\{U_g, \mathfrak{H}, f\}\in\boldsymbol{U}(\phi)$. Further we can see easily

Lemma 4.1. *Let* $\phi\in P_n(G)$, $\{U_g, \mathfrak{H}, f\}\in\boldsymbol{U}(\phi)$, *and* $\{U'_g, \mathfrak{H}'\}$ *be a* (w)-*continuous unitary representation of* G *in a quasi-positive space* $\{\mathfrak{H}', (\ ,\ )'\}$. *If* $\{U'_g, \mathfrak{H}'\}\overset{\tau}{\simeq}\{U_g, \mathfrak{H}\}$ *and* $\tau(f')=f$ *for some* $f'\in\mathfrak{H}'$, *then* $H(\phi)$ *coincides with the space of all functions* $x(g)$ *given in the form:* $x(g)=(x', U'_g f')'$ $(g\in G)$ *for some* $x'\in\mathfrak{H}'$.

For $\phi\in P_n(G)$ assume that it is decomposed as follows:

$$\phi=\psi+\theta, \tag{4.1}$$

where $\psi\in P_l(G)$ and $\theta\in P_m(G)$. Then by Theorem 2.5 (3) we have $n\leqq l+m$. Throughout §§4.1 and 4.2 we use the following notations:

$\boldsymbol{U}_1=\{U_g^{(1)}, \mathfrak{H}_1, f_1\}\in\boldsymbol{U}(\psi)$, $\boldsymbol{U}_2=\{U_g^{(2)}, \mathfrak{H}_2, f_2\}\in\boldsymbol{U}(\theta)$, $H_1=H(\psi)$, $H_2=H(\theta)$,

$\mathfrak{H}=$the product Π_{l+m}-space $\mathfrak{H}_1\times\mathfrak{H}_2$, $f=[f_1, f_2]\in\mathfrak{H}_1\times\mathfrak{H}_2$,

$\{U_g, \mathfrak{H}\}=$the product representation of $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$,

$\mathfrak{K}=$the closed linear span of $\{U_g f; g\in G\}$ in $\mathfrak{H}$,

$\tilde{\mathfrak{K}}=$the quotient space $\mathfrak{K}/\mathfrak{K}^0$,

$\pi=$the canonical map of $\mathfrak{K}$ onto $\tilde{\mathfrak{K}}$, $\tilde{f}=\pi(f)$,

$\{U_g, \mathfrak{K}, f\}=$the partial representation of $\{U_g, \mathfrak{H}\}$ with the cyclic vector f,

$\{\tilde{U}_g, \tilde{\mathfrak{K}}, \tilde{f}\}=$the quotient representation of $\{U_g, \mathfrak{H}\}$ determined by $\mathfrak{K}$ with the cyclic vector $\tilde{f}$.

Then $\{\tilde{U}_g, \tilde{\mathfrak{K}}, \tilde{f}\}\in\boldsymbol{U}(\phi)$. Indeed for any $g\in G$

$$\begin{aligned}(\tilde{f}, \tilde{U}_g\tilde{f})&=(\pi(f), \pi(U_g f))=(f, U_g f)\\&=(f_1, U_g^{(1)}f_1)+(f_2, U_g^{(2)}f_2)=\psi(g)+\theta(g)=\phi(g).\end{aligned} \tag{4.2}$$

Since $\{U_g, \mathfrak{K}, f\}\overset{\pi}{\simeq}\{\tilde{U}_g, \tilde{\mathfrak{K}}, \tilde{f}\}$, it follows from Lemma 4.1 that $H(\phi)$ coincides with the space of all functions $x(g)$ given in the form:

$$x(g)=([\eta_1, \eta_2], U_g f)=(\eta_1, U_g^{(1)}f_1)+(\eta_2, U_g^{(2)}f_2)\quad(g\in G), \tag{4.3}$$

where $[\eta_1, \eta_2]\in\mathfrak{K}$. In particular, as $(\eta_1, U_g^{(i)} f_i)\in H_i$ $(i=1, 2)$, we have

$$H(\phi)\subseqq H_1+H_2. \tag{4.4}$$

In the next theorem we give a necessary and sufficient condition for any representation belonging to $\boldsymbol{U}(\phi)$ to be contained in the product representation $\{U_g, \mathfrak{H}\}$.

Theorem 4.2. *The following conditions are mutually equivalent:*

(1) $\{U_g, \mathfrak{K}\}$ *is a subrepresentation of* $\{U_g, \mathfrak{H}\}$,

(2) $H(\phi)=H_1+H_2$ *(algebraic sense in* $C(G)$*)*,

(3) $H_1\subseqq H(\phi)$ *and* $H_2\subseqq H(\phi)$.

If (1) *holds, then* $\{U_g, \mathfrak{K}, f\}\in\boldsymbol{U}(\phi)$.

Proof. Suppose that $\mathfrak{K}$ is non-degenerate. Then $\mathfrak{H}$ is the orthogonal direct sum of $\mathfrak{K}$ and $\mathfrak{K}^{\perp}$. Let $x_i(g)=(\xi_i, U_g^{(i)} f_i)\in H_i$, $\xi_i\in\mathfrak{H}_i$ for $i=1, 2$, and $[\eta_1, \eta_2]$ the orthogonal projection of $[\xi_1, \xi_2]\in\mathfrak{H}$ in $\mathfrak{K}$. Then by (4.3) we have

$$x_1(g)+x_2(g)=([\xi_1, \xi_2], U_g f)=([\eta_1, \eta_2], U_g f)\in H(\phi),$$

and $H_1+H_2\subseqq H(\phi)$. Therefore it follows from (4.4) that (1) implies (2). Conversely suppose that (2) holds. For any $[\xi_1, \xi_2]\in\mathfrak{H}$ there exists $[\eta_1, \eta_2]\in\mathfrak{K}$ such that

$$([\xi_1, \xi_2], U_g f)=(\xi_1, U_g^{(1)} f_1)+(\xi_2, U_g^{(2)} f_2)=([\eta_1, \eta_2], U_g f) \quad (g\in G).$$

So $[\xi_1, \xi_2]-[\eta_1, \eta_2]\in\mathfrak{K}^{\perp}$, and we have $\mathfrak{H}=\mathfrak{K}+\mathfrak{K}^{\perp}$. Hence $\mathfrak{K}$ is non-degenerate by Lemma 1.9 (2). Thus (2) implies (1). From (4.4) it is clear that (2) and (3) are equivalent. The last assertion follows from (4.2). q.e.d.

4.2. Normal decomposition of quasi-positive definite functions. The decomposition (4.1) of ϕ is said to be *normal* if $n=l+m$. In this case ψ and θ are called *normal components* of ϕ.

Lemma 4.3. *If* (4.1) *is a normal decomposition of* ϕ, *then* $\mathfrak{K}$ *is a* Π_n*-space, and* $\mathfrak{K}^{\perp}$ *is a Hilbert space.*

Proof. Since $\phi\in P_n(G)$ and $\{\tilde{U}_g, \tilde{\mathfrak{K}}, \tilde{f}\}\in\boldsymbol{U}(\phi)$, $\tilde{\mathfrak{K}}$ is a Π_n-space. Suppose that $\mathfrak{K}^0\neq\{0\}$. Then by Lemma 1.9 (3) $\tilde{\mathfrak{K}}$ must be a Π_k-space for some $k<l+m$. But this contradicts to the assumption $n=l+m$. Therefore $\mathfrak{K}^0=\{0\}$, $\mathfrak{K}=\tilde{\mathfrak{K}}$ is a Π_n-space, and $\mathfrak{K}^{\perp}$ is a Π_0-space, i.e., a Hilbert space. q.e.d.

The next theorem follows immediately from Theorem 4.2 and Lemma 4.3. This is a generalization of Theorem 4 in [3].

Theorem 4.4. *If* (4.1) *is a normal decomposition of* ϕ, *then we have:*

(1) *The product representation* $\{U_g, \mathfrak{H}\}$ *of* $\boldsymbol{U}_1$ *and* $\boldsymbol{U}_2$ *is the orthogonal direct sum of* $\{U_g, \mathfrak{K}\}$ *and* $\{U_g, \mathfrak{K}^{\perp}\}$, *where* $\{U_g, \mathfrak{K}, f\}\in\boldsymbol{U}(\phi)$ *and* $\{U_g, \mathfrak{K}^{\perp}\}$ *is a unitary representation in the Hilbert space* $\mathfrak{K}^{\perp}$.

(2) *Any unitary representation belonging to* $\boldsymbol{U}(\phi)$ *is contained in* $\{U_g, \mathfrak{H}\}$.

(3) $H(\phi)=H_1+H_2$, *and especially* $H_i \subseteqq H(\phi)$ *for* $i=1, 2$.

Theorem 8 in [3] holds also for our case as follows.

Theorem 4.5. *If* (4.1) *is a normal decomposition of* ϕ, *then the following three conditions are mutually equivalent:*

(1) f *is cyclic with respect to the product representation* $\{U_g, \mathfrak{H}\}$, *and* $\{U_g, \mathfrak{H}, f\} \in \boldsymbol{U}(\phi)$.

(2) $\boldsymbol{U}_1$ *or* $\boldsymbol{U}_2$ *are contained in any unitary representation belonging to* $\boldsymbol{U}(\phi)$,

(3) $H(\phi)$ *is the direct sum of* H_1 *and* H_2, *i.e.,* $H(\phi)=H_1+H_2$ *and* $H_1 \cap H_2=\{0\}$.

Proof. It is obvious that (1) implies (2). Suppose that (2) holds. Let $\boldsymbol{V}=\{V_g, \mathfrak{H}', f'\} \in \boldsymbol{U}(\phi)$ and assume that $\boldsymbol{U}_1$ is contained in $\boldsymbol{V}$. Then $\boldsymbol{V}$ has an orthogonal decomposition as follows:

$$\boldsymbol{V}=\{V_g^{(1)}, \mathfrak{H}_1', f_1'\} \,(\dot{+})\, \{V_g^{(2)}, \mathfrak{H}_2', f_2'\},$$

where $\{V_g^{(1)}, \mathfrak{H}_1', f_1'\} \cong \boldsymbol{U}_1$. Moreover we have for any $g \in G$

$$\phi(g)=(f', V_g f')=(f_1', V_g^{(1)} f_1')+(f_2', V_g^{(2)} f_2')=\psi(g)+(f_2', V_g^{(2)} f_2'),$$

and $(f_2', V_g^{(2)} f_2')=\phi(g)-\psi(g)=\theta(g)$.

So it follows from Theorem 3.1 that $\{V_g^{(2)}, \mathfrak{H}_2', f_2'\} \cong \boldsymbol{U}_2$. Thus we get

$$\{U_g, \mathfrak{H}\}=\{U_g^{(1)}, \mathfrak{H}_1\} \times \{U_g^{(2)}, \mathfrak{H}_2\} \cong \{V_g^{(1)}, \mathfrak{H}_1'\} \,(\dot{+})\, \{V_g^{(2)}, \mathfrak{H}_2'\}=\{V_g, \mathfrak{H}'\},$$

and $f=[f_1, f_2] \in \mathfrak{H}$ corresponds to $f' \in \mathfrak{H}'$ by this isometric isomorphism. Since f' is cyclic with respect to $\{V_g, \mathfrak{H}'\}$, f is cyclic with respect to $\{U_g, \mathfrak{H}\}$, and $\{U_g, \mathfrak{H}, f\} \in \boldsymbol{U}(\phi)$. Therefore (2) implies (1). By virtue of Theorem 4.3 (3), in order to prove that (1) and (3) are equivalent, it suffices to show that $\mathfrak{K}^{\perp}=\{0\}$ if and only if $H_1 \cap H_2=\{0\}$. Let $[\xi_1, -\xi_2]$ be a non-zero element in $\mathfrak{K}^{\perp}$. Then for all $g \in G$

$$([\xi_1, -\xi_2], U_g f)=(\xi_1, U_g^{(1)} f_1)-(\xi_2, U_g^{(2)} f_2)=0.$$

So $x(g)=(\xi_1, U_g^{(1)} f_1)=(\xi_2, U_g^{(2)} f_2) \in H_1 \cap H_2$. As f_1 and f_2 are cyclic, $x(g) \neq 0$ for some $g \in G$, and hence $H_1 \cap H_2 \neq \{0\}$. Conversely if $x \in H_1 \cap H_2$ and $x \neq 0$, then there exists a non-zero $[\xi_1, \xi_2] \in \mathfrak{H}$ such that $x(g)=(\xi_i, U_g^{(i)} f_i)$ for all $g \in G$ and $i=1, 2$. Hence $\mathfrak{K}^{\perp}$ contains $[\xi_1, -\xi_2] \neq 0$. Thus $\mathfrak{K}^{\perp}=\{0\}$ if and only if $H_1 \cap H_2=\{0\}$. q.e.d.

4.3. Normal components of quasi-positive definite functions. Let $\phi \in P_n(G)$ and $\boldsymbol{U}^{\phi}=\{U_g, \mathfrak{H}, f\} \in \boldsymbol{U}(\phi)$. By $N(\phi)$ we denote the set of all normal components of ϕ, that is, $\psi \in QP(G)$ is in $N(\phi)$ if and only if $\psi \in P_m(G)$ and $\phi-\psi \in P_{n-m}(G)$ for some $0 \leqq m \leqq n$. Let A be a selfadjoint operator on $\mathfrak{H}$, and set $B=I-A$ (I=the identity operator on $\mathfrak{H}$). On $\mathfrak{H}$ we define new inner products $(x, y)_A$ and $(x, y)_B$ $(x, y\in\mathfrak{H})$ as follows:

$$(x, y)_A=(Ax, y)=(x, Ay)$$

$$(x, y)_B=(Bx, y)=(x, y)-(x, y)_A.$$

Then A is called a *quasi-positive* operator with negative rank m, $0\leqq m\leqq n$, if $\{\mathfrak{H}, (\ ,\ )_A\}$ is a QP_m-space and $\{\mathfrak{H}, (\ ,\ )_B\}$ is a QP_{n-m}-space. The negative rank of A is denoted by $\chi^-(A)$. Let us denote by $\boldsymbol{A}(\boldsymbol{U}^\phi)$ the set of all quasi-positive operators A on $\mathfrak{H}$ with $\chi^-(A)\leqq n$ commuting all U_g, $g\in G$. The next is a generalization of Theorem 5 in [3].

Theorem 4.6. *For any $\phi\in P_n(G)$ and $\boldsymbol{U}^\phi=\{U_g, \mathfrak{H}, f\}\in\boldsymbol{U}(\phi)$ there exists a bijective map ρ of $\boldsymbol{A}(\boldsymbol{U}^\phi)$ onto $N(\phi)$ such that for any $A\in\boldsymbol{A}(\boldsymbol{U}^\phi)$ $\psi=\rho(A)\in N(\phi)$ is given in the form:*

$$\psi(g)=(Af, U_g f)\quad (g\in G). \tag{4.5}$$

Proof. Let $A\in\boldsymbol{A}(\boldsymbol{U}^\phi)$, $B=I-A$, $m=\chi^-(A)$, and put $l=n-m$. Then

$$(U_g x, U_g y)_A=(AU_g x, U_g y)=(U_g Ax, U_g y)=(Ax, y)=(x, y)_A,$$

$$(U_g x, U_g y)_B=(U_g x, U_g y)-(U_g x, U_g y)_A=(x, y)-(x, y)_A=(x, y)_B,$$

where $x, y\in\mathfrak{H}$ and $g\in G$. Hence each U_g $(g\in G)$ is unitary with respect to the both inner products $(\ ,\ )_A$ and $(\ ,\ )_B$. Let $\mathfrak{H}_0$ be the linear span of $\{U_g f;\, g\in G\}$ in $\mathfrak{H}$. Since $\mathfrak{H}_0$ is dense in $\mathfrak{H}$, it follows from Lemma 1.8 that $\{\mathfrak{H}_0, (\ ,\ )_A\}$ becomes a QP_m-space and $\{\mathfrak{H}_0, (\ ,\ )_B\}$ a QP_l-space. Applying Theorem 2.4 to the (w)-continuous unitary representations $\{U_g, \mathfrak{H}_0\}$ of G in $\{\mathfrak{H}_0, (\ ,\ )_A\}$ and $\{\mathfrak{H}_0, (\ ,\ )_B\}$, we have for any $g\in G$

$$\psi(g)=(f, U_g f)_A=(Af, U_g f)\in P_m(G),$$

$$\theta(g)=(f, U_g f)_B=(f, U_g f)-(f, U_g f)_A=\phi(g)-\psi(g)\in P_l(G).$$

Thus $\psi\in N(\phi)$ and we get a map ρ of $\boldsymbol{A}(\boldsymbol{U}^\phi)$ to $N(\phi)$ defined by (4.5). For $A, A'\in\boldsymbol{A}(\boldsymbol{U}^\phi)$ assume that $(Af, U_g f)=(A'f, U_g f)$ for all $g\in G$. Then for any $x=\sum_{i=1}^{k}\lambda_i U_{g_i} f\in\mathfrak{H}_0$, where $\lambda_i\in\boldsymbol{C}$ and $g_i\in G$ $(1\leqq i\leqq k)$, we have $(Ax, U_g f)=(A'x, U_g f)$ for all $g\in G$, and it follows that $A=A'$, because A and A' are continuos and $\mathfrak{H}_0$ is dense in $\mathfrak{H}$. Thus it is proved that ρ is injective. Finally we show that ρ is surjective. Let $\psi\in P_m(G)$ be in $N(\phi)$ and $\theta=\phi-\psi\in P_l(G)$, where $l=n-m$. By Theorem 4.4 (3) there exists $h\in\mathfrak{H}$ for which $\psi(g)=(h, U_g f)$ $(g\in G)$. Let us define two linear maps τ and τ' of $L(G)$ (cf. § 2.2) to $\mathfrak{H}_0$ and $\mathfrak{H}$ respectively by

$$\tau : L(G)\ni\alpha\longmapsto\tau(\alpha)=\sum_{g\in G}\alpha(g)U_g f\in\mathfrak{H}_0,$$

$$\tau' : L(G)\ni\alpha\longmapsto\tau'(\alpha)=\sum_{g\in G}\alpha(g)U_g h\in\mathfrak{H}.$$

Then we have $(\tau(\alpha), U_g h)=(\tau'(\alpha), U_g f)$ for any $\alpha\in L(G)$ and $g\in G$. If $\tau(\alpha)=0$, then $\tau'(\alpha)\in\mathfrak{H}_0^{\perp}=\{0\}$ and hence $\tau'(\alpha)=0$. So there exists uniquely a linear map A of $\mathfrak{H}_0$ onto $\tau'(L(G))$ defined by $A\tau(\alpha)=\tau'(\alpha)$ for any $\alpha\in L(G)$, that is,

$$A(\sum_{g\in G}\alpha(g)U_g f)=\sum_{g\in G}\alpha(g)U_g h \quad (\alpha\in L(G)).$$

Then we have for any $g\in G$

$$AU_g=U_g A \quad \text{on} \quad \mathfrak{H}_0, \tag{4.6}$$

$$\phi(g)=(h, U_g f)=(Af, U_g f). \tag{4.7}$$

Moreover putting $B=I-A$, we have for any $x=\tau(\alpha)$, $y=\tau(\beta)\in\mathfrak{H}_0$ $(\alpha, \beta\in L(G))$

$$(x, y)_A=(Ax, y)=(x, Ay)=\sum_{g,k\in G}\alpha(g)\overline{\beta(k)}\phi(g^{-1}k)=(\alpha, \beta)_\phi, \tag{4.8}$$

$$(x, y)_B=(Bx, y)=(x, y)-(x, y)_A=\sum_{g,k\in G}\alpha(g)\overline{\beta(k)}\theta(g^{-1}k)=(\alpha, \beta)_\theta. \tag{4.9}$$

(4.8) [resp. (4.9)] shows that τ is an isometric map of $\{L(G), (\ ,\ )_\phi\}$ [resp. $\{L(G), (\ ,\ )_\theta\}$] onto $\{\mathfrak{H}_0, (\ ,\ )_A\}$ [resp. $\{\mathfrak{H}_0, (\ ,\ )_B\}$]. Since $\phi\in P_m(G)$ and $\theta\in P_l(G)$, it follows from Theorem 2.2 and Lemma 1.4 that $\{\mathfrak{H}_0, (\ ,\ )_A\}$ is a QP_m-space and $\{\mathfrak{H}_0, (\ ,\ )_B\}$ a QP_l-space. Suppose that A is bounded on $\mathfrak{H}_0$. Then A can be extended to a selfadjoint operator of $\mathfrak{H}$, denoted also by A, and $\{\mathfrak{H}, (\ ,\ )_A\}$ and $\{\mathfrak{H}, (\ ,\ )_B\}$ become QP_m- and QP_l-spaces respectively by Lemma 1.8. Further from (4.6) and (4.7) it follows that $A\in\boldsymbol{A}(\boldsymbol{U}^\phi)$ and $\rho(A)=\phi$. So ρ is surjective. Thus it suffices to prove that A is bounded on $\mathfrak{H}_0$. If $n=m=l=0$, then $0\leqq(Ax, x)\leqq(x, x)\leqq\|x\|^2$ for any $x\in\mathfrak{H}_0$, so that A is bounded. So we may assume without loss of generality that $n\geqq m>0$. Let $\mathfrak{N}$ be an n-dimensional negative definite subspace of $\mathfrak{H}_0$ and $\mathfrak{P}$ the orthogonal complement of $\mathfrak{N}$ in $\mathfrak{H}_0$. We denote by $\langle\ ,\ \rangle$ the positive definite inner product on $\mathfrak{H}_0$ corresponding to the fundamental decomposition $\mathfrak{N}(\dot{+})\mathfrak{P}$ of $\mathfrak{H}_0$, and by $\|\ \|$ the norm induced from $\langle\ ,\ \rangle$ (cf. (1.8)). We take a basis $\{e_i; 1\leqq i\leqq n\}$ of $\mathfrak{N}$ such that $(e_i, e_j)=-\delta_{ij}$ for any $1\leqq i\leqq j\leqq n$. Then by (1.9) for any $x\in\mathfrak{H}_0$

$$\langle Ax, x\rangle=(Ax, x)+2\sum_{k=1}^{n}(x, Ae_k)(e_k, x).$$

So by (1.10) we have for any $x\in\mathfrak{H}_0$

$$|\langle Ax, x\rangle|\leqq|(Ax, x)|+K_1\|x\|^2, \tag{4.10}$$

where K_1 is a positive constant. On the other hand, since $\mathfrak{H}_0^A=\{\mathfrak{H}_0, (\ ,\ )_A\}$ is a QP_m-space and is fundamental decomposable by Lemma 1.5, $\mathfrak{H}_0^A$ is decomposed as follows:

$$\mathfrak{H}_0^A=\mathfrak{N}_A(\dot{+})_A\mathfrak{P}_A,$$

where $\mathfrak{N}_A$ is an m-dimensional negative definite subspace and $\mathfrak{P}_A$ a non-negative subspace. For any $x\in\mathfrak{H}_0$ we put $x=x^-+x^+$ $(x^-\in\mathfrak{N}_A, x^+\in\mathfrak{P}_A)$. Then for any

$x\in\mathfrak{H}_0$ $(Ax, x)=(Ax^+, x^+)+(Ax^-, x^-)$ and $(Ax^+, x^+)\geqq 0$. As $\mathfrak{N}_A$ is finite dimensional, we can find a positive constant K_2 with the properties:

$$|(Ax, x)|\leqq(Ax^+, x^+)+K_2\|x\|^2 \quad (x\in\mathfrak{H}_0), \tag{4.11}$$

$$\|x^+\|\leqq K_2\|x\| \quad (x\in\mathfrak{H}_0). \tag{4.12}$$

Similarly using the fact that $\{\mathfrak{H}_0, (\ ,\ )_B\}$ is a QP_l-space, we can take a positive constant K_3 such that for any $x\in\mathfrak{P}_A$

$$0\leqq(Ax, x)\leqq K_3\|x\|^2. \tag{4.13}$$

Combining the inequalities (4.10), (4.11), (4.12) and (4.13), we have $|\langle Ax, x\rangle|\leqq K\|x\|^2$ and hence $|\langle Ax, y\rangle|\leqq 2K\|x\|\ \|y\|$ for any $x, y\in\mathfrak{H}_0$, where K is a positive constant. Therefore A is bounded on $\mathfrak{H}_0$. This completes the proof.

q. e. d.

§5. Examples of quasi-positive definite functions

5.1. Bounded quasi-positive definite functions. Let $\psi\in C(G)$ be a positive definite, i.e., $\psi\in P_0(G)$. If the linear span $\{\psi_g\,;\,g\in G\}$ in $C(G)$ is finite dimensional, say n-dimensional, ψ is said to have positive rank n, and we put $r^+(\psi)=n$. It is obvious that $\psi\in P_0(G)$ has the positive rank n if and only if the cyclic unitary representation of G in a Hilbert space with the characteristic function ψ is n-dimensional, and that $-\psi\in P_n(G)$ for any $\psi\in P_0(G)$ with $r^+(\psi)=n$. So from Theorem 2.5 we have

Theorem 5.1. *Let $\psi\in P_0(G)$ with $n=r^+(\psi)<\infty$. Then for any $\theta\in P_0(G)$ the difference $\phi=\theta-\psi$ is a bounded quasi-positive definite function with $r^-(\phi)\leqq n$.*

The converse of Theorem 5.1 holds for amenable groups, e.g., commutative, solvable or compact groups.

Theorem 5.2. *For an amenable group G any bounded $\phi\in P_n(G)$ is given in the form: $\phi=\theta-\psi$, where $\theta, \psi\in P_0(G)$ with $r^+(\psi)=n$.*

Proof. Let $\boldsymbol{U}=\{U_g, \mathfrak{H}, f\}\in\boldsymbol{U}(\phi)$. Since $\boldsymbol{U}$ is uniformly bounded by Theorem 3.2, it follows from Theorem 1 in [6] that $\boldsymbol{U}$ is decomposed as follows: $\boldsymbol{U}=\{U_g, \mathfrak{N}, f_1\}\,(\dot{+})\,\{U_g, \mathfrak{P}, f_2\}$, where $\mathfrak{N}$ is an n-dimensional negative definite subspace and $\mathfrak{P}$ a positive definite subspace. Put $\theta(g)=(f_2, U_g f_2)$ and $-\psi(g)=(f_1, U_g f_1)$ $(g\in G)$. Then $\theta\in P_0(G)$, $-\psi\in P_n(G)$ and $\phi=\theta-\psi$. As $\mathfrak{N}$ is negative definite, the inner product space $\{\mathfrak{N}, -(\ ,\ )\}$ is an n-dimensional Hilbert space, and $\{U_g, \mathfrak{N}, f_1\}$ is regarded as a cyclic unitary representation of G in the Hilbert space $\{\mathfrak{N}, -(\ ,\ )\}$. So $\psi\in P_0(G)$ and $r^+(\psi)=n$. q.e.d.

5.2. Unbounded quasi-positive definite functions. Now consider the 2-dimensional vector space $\boldsymbol{C}^2$ with inner product $(\ ,\ )$ defined by

$$(u, v)=u_1\bar{v}_2+u_2\bar{v}_1 \quad \text{for} \quad u={}^t(u_1, u_2),\ v={}^t(v_1, v_2)\in\boldsymbol{C}^2. \tag{5.1}$$

Then $\mathfrak{H}_1^2=\{C^2, (\ ,\ )\}$ becomes a Π_1-space. Let $\chi(g)\in C(G)$ be a non-unitary character on G, that is, it satisfies the following conditions:

(5.2) $$\chi(gh^{-1})=\chi(g)/\chi(h) \qquad \text{for any} \quad g, h\in G,$$

(5.3) $$\chi(g^{-1})\neq\overline{\chi(g)} \qquad \text{for some} \quad g\in G.$$

For any $g\in G$ we define a linear operator U_g on $\mathfrak{H}_1^2$ by

$$U_g u=\begin{bmatrix} \overline{\chi(g)}\, u_1 \\ \chi(g^{-1})\, u_2 \end{bmatrix} \qquad \text{for} \quad u={}^t(u_1, u_2)\in \boldsymbol{C}^2.$$

Then it is easily seen that each U_g $(g\in G)$ is a unitary operator on $\mathfrak{H}_1^2$ and that $\boldsymbol{U}_\chi=\{U_g, \mathfrak{H}_1^2\}$ is a weakly continuous unitary representation of G. Moreover $u={}^t(u_1, u_2)\in \boldsymbol{C}^2$ is cyclic with respect to $\boldsymbol{U}_\chi$ if and only if $u_1u_2\neq 0$. Hence by Theorem 3.4 the function $\phi(g)=(u, U_g u)$ $(g\in G)$ belongs to $P_1(G)$ for any $u={}^t(u_1, u_2)$ with $u_1u_2\neq 0$. Putting $\bar{u}_1u_2=\alpha+i\beta$ $(\alpha, \beta\in \boldsymbol{R})$, we have

$$\phi(g)=(u, U_g u)=\alpha(\chi(g)+\overline{\chi(g^{-1})})+\beta i(\chi(g)-\overline{\chi(g^{-1})}) \quad (g\in G).$$

Theorem 5.3. *Let $\chi(g)$ be a non-unitary character on G. Then the following function ϕ belongs to $P_1(G)$, and is unbounded:*

$$\phi(g)=\alpha(\chi(g)+\overline{\chi(g^{-1})})+\beta i(\chi(g)-\overline{\chi(g^{-1})}) \quad (g\in G),$$

where $\alpha, \beta\in \boldsymbol{R}$ with $\alpha^2+\beta^2\neq 0$.

Now let $\chi(g)$ be a unitary character on G and $f(g)\in C(G)$ be a non-zero real character on G, that is, $f(g)$ is a non-zero real function with the following property:

(5.4) $$f(gh)=f(g)+f(h) \qquad \text{for any} \quad g, h\in G.$$

Using $\chi(g)$ and $f(g)$, we define a linear operator U_g $(g\in G)$ on $\mathfrak{H}_1^2$ by

$$U_g u=\begin{bmatrix} \overline{\chi(g)}\, u_1-i\overline{\chi(g)}f(g)\, u_2 \\ \overline{\chi(g)}\, u_2 \end{bmatrix} \qquad \text{for} \quad u={}^t(u_1, u_2)\in \boldsymbol{C}^2$$

Then U_g is unitary and $\boldsymbol{U}_{\chi,f}=\{U_g, \mathfrak{H}_1^2\}$ is a weakly continuous unitary representation of G. Moreover $u={}^t(u_1, u_2)\in \boldsymbol{C}^2$ is cyclic with respect to $\boldsymbol{U}_{\chi,f}$ if and only if $u_2\neq 0$. For any $u={}^t(u_1, u_2)$ and $g\in G$ we have

$$(u, U_g u)=(u, u)\chi(g)+u_2\bar{u}_2 i\chi(g)f(g).$$

Thus we get

Theorem 5.4. *Let $\chi(g)$ be a unitary character and $f(g)$ be a non-zero real character on G. Then the following function ϕ belongs to $P_1(G)$, and is unbounded:*

$$\phi(g)=\chi(g)(\alpha+\beta i f(g)) \quad (g\in G),$$

where $\alpha, \beta\in \boldsymbol{R}$ with $\beta>0$.

In the author's paper [7], we shall give the general form of quasi-positive definite functions on commutative groups corresponding to indecomposable cyclic unitary representations in Π_1- and Π_2-spaces.

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