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Radiation conditions and spectral theory for 2-body Schrödinger operators with "oscillating" long-range potentials, II

—Spectral representation—

By

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Introduction

The present paper is a direct continuation of our previous work (Mochizuki-Uchiyama [11]) and deals with a spectral theory for the Schrödinger operators $-\Delta+V(x)$, in exterior domain Ω of $\boldsymbol{R}^n$, with some real "oscillating" long-range potentials $V(x)$. Throughout this paper, the same notation as in [11] will be used (the list of the notation is given in §1 of [11]), and formulas, lemmas, etc. given in [11] will be quoted as (I.2.3), Lemma I.2.3, etc..

Let L be a selfadjoint realization of $-\Delta+V(x)$ in the Hilbert space $\mathcal{H}=L^2(\Omega)$, and $\mathcal{E}(\lambda)$, $\lambda\in\boldsymbol{R}$, be the spectral measure for L (the conditions on $V(x)$ required in this paper will be summarized in §1). Then, as is proved in [11], there exists a real number Λ_δ depending on the asymptotic behavior at infinity of $V(x)$ such that the operator L restricted in $\mathcal{E}((\Lambda_\delta, \infty))\mathcal{H}$ is absolutely continuous. Our purpose of the present paper is to obtain a spectral representation for this (or more restricted) part of L. Namely, we shall establish the existence of a unitary operator $\mathcal{F}_\pm$ from $\mathcal{E}((\Lambda_{\tilde\delta}, \infty))\mathcal{H}$ onto $\hat{\mathcal{H}}_{\Lambda_{\tilde\delta}}=L^2((\Lambda_{\tilde\delta}, \infty); L^2(S^{n-1}))$ (S^{n-1} being the unit sphere in $\boldsymbol{R}^n$) which diagonalizes L. In general $\Lambda_{\tilde\delta}\geqq\Lambda_\delta$ (cf., §4).

Spectral representations (or eigenfunction expansions) for the Schrödinger operators were initiated by Povzner [13], [14] and Ikebe [2]. In these few years, their results have been generalized to short-range potentials by Jäger [6], Agmon [1], Kuroda [7] and Mochizuki [10], and to "non-oscillating" long-range potentials by Ikebe [3], [4] and Saitō [15], [16] (cf., also Pinchuk [12] and Isozaki [5]). Among these works, this paper is on the same line with Jäger [6], Ikebe [3], [4] and Saitō [15], [16], and will generalize results of these works to "oscillating" long-range potentials.

In Jäger's theory, the operator $\mathcal{F}_\pm$ is obtained in the following way.

$$(0.1)\qquad (\mathcal{F}_\pm f)(\lambda, \tilde{x})=\frac{1}{\sqrt{\pi}}\lambda^{1/4}\lim_{r\to\infty} r^{(n-1)/2}e^{\mp i\sqrt{\lambda}r}(\mathcal{R}_{\lambda\pm i0}f)(r\tilde{x}),\quad \lambda>0,$$

where $u=\mathcal{R}_{\lambda\pm i0}f$ is the outgoing [incoming] solution of

$$(0.2)\qquad (-\Delta+V(x)-\lambda)u=f(x)\quad \text{in } \Omega$$

verifying the outgoing [incoming] radiation condition

$$(0.3)\qquad u\in L^2_{(-1-\varepsilon)/2}(\Omega)\ \text{ and }\ \partial_r u+\left(\mp i\sqrt{\lambda}+\frac{n-1}{2r}\right)u\in L^2_{(-1+\varepsilon)/2}(\Omega)$$

for some $\varepsilon>0$. Here $L^2_\mu(\Omega)$, $\mu\in\boldsymbol{R}$, denotes the space of functions $f(x)$ such that $(1+|x|)^\mu f(x)\in L^2(\Omega)$. Of course u is required further to satisfy a suitable (Dirichlet or Robin) boundary condition on $\partial\Omega$ if $\Omega\neq\boldsymbol{R}^n$. This result was modified by Ikebe and Saitō in the case of "non-oscillating" long-range potentials (cf., Remark 5.2). They find, roughly speaking, appropriate real modifiers $X(x, \lambda)$ and obtain $\mathcal{F}_\pm$ by (0.1) with $e^{\mp i\sqrt{\lambda}r}$ replaced by $e^{\mp i(\sqrt{\lambda}r+X(r\tilde{x},\lambda))}$. For "oscillating" long-range potentials, however, the radiation condition (0.3) does not work well, and it becomes difficult to obtain $\mathcal{F}_\pm$ by the above type of modification of $e^{\mp i\sqrt{\lambda}r}$.

In the previous paper [11] we have obtained a new formulation of the radiation condition which is applicable to a wider class of potentials. It has the form

$$(0.4)\qquad u\in L^2_{(-1-\alpha)/2}(\Omega)\ \text{ and }\ \partial_r u+k(x, \lambda\pm i0)u\in L^2_{(-1+\beta)/2}(\Omega)$$

for some α, $\beta>0$, where $k(x, \lambda\pm i0)$ solves the Riccati type equation

$$(0.5)\qquad \partial_r k+\frac{n-1}{r}k-k^2+V(x)-\lambda=O(r^{-1-\delta})\quad (\delta>0)$$

for $r=|x|$ large (the concrete form of $k(x, \lambda\pm i0)$ will be given in §1). In this paper we shall make use of this new radiation condition to modify the above mentioned results, that is, we shall show that the operator $\mathcal{F}_\pm$ can be obtained in the form

$$(0.6)\qquad (\mathcal{F}_\pm f)(\lambda, \tilde{x})=\frac{1}{\sqrt{\pi}}\lim_{r\to\infty} e^{\int^r k(s\tilde{x},\lambda\pm i0)ds}(\mathcal{R}_{\lambda\pm i0}f)(r\tilde{x})$$

for $\lambda>\Lambda_\delta$. The principle of limiting absorption established in [11] will guarantee the existence and some convenient properties of $\mathcal{R}_{\lambda\pm i0}f$.

Note that $[\Lambda_\delta, \infty)$ does not in general cover the essential spectrum of L. In this sense it remains some ambiguousness in our theory. The difficulty is caused by some bad influence of the oscillation at infinity of the potential $V(x)$. In fact, in our case, the operator L may have positive eigenvalues though $V(x)$ itself behaves like $O(r^{-1})$ at infinity (see examples of [11]). On the other hand, our theory includes a new result for "non-oscillating" long-range potentials (Corollary 5.3). This is the case that we can choose $\Lambda_{\delta}=0$.

The contents of this paper are as follows. In §1 we first summarize the main results of [11] in Proposition 1.1, and then prepare some additional propositions which easily follow from Proposition 1.1. In §§2 and 3 we construct the operator $\mathcal{F}_{\pm}$ and prove its isometry. The unitarity of $\mathcal{F}_{\pm}$ is proved in §4. In §5 we give several corollaries. Finally, in §6 we prove the unitary equivalence between L restricted in $\mathcal{E}((\Lambda_{\delta}, \infty))\mathcal{H}$ and the selfadjoint realization L_0 of $-\Delta+\Lambda_{\delta}$ in the Hilbert space $\mathcal{H}_0=L^2(\boldsymbol{R}^n)$.

§1. Assumptions and Preliminaries

Let Ω be an infinite domain in $\boldsymbol{R}^n$ with smooth compact boundary $\partial\Omega$ lying inside some sphere $S(R_0)=\{x\,;|x|=R_0\}$. We consider in Ω the Schrödinger operator $-\Delta+V(x)$, where Δ is the Laplacian and $V(x)$ is a potential function. We assume:

Assumption 1. $V(x)=V_1(x)+V_s(x)$, where $V_1(x)$ is a real-valued function belonging to a Stummel class Q_μ $(\mu>0)$, and $V_s(x)$ is a real-valued bounded measurable function in Ω. Moreover, the unique continuation property holds for both $-\Delta+V(x)$ and $-\Delta+V_1(x)$.

Assumption 2. $V_1(x)$ is an "oscillating" long-range potential such that for some $a\geqq 0$ and $1/2<\delta_j<1$ $(j=1, 2)$,

(i) $V_1(x)=O(1)$,

(ii) $\partial_r V_1(x)=O(r^{-1})$,

(iii) $\partial_r^2 V_1(x)+a V_1(x)=O(r^{-1-\delta_1})$,

(iv) $(\nabla-\tilde{x}\partial_r)V_1(x)=O(r^{-1-\delta_2})$,

(v) $(\nabla-\tilde{x}\partial_r)\partial_r V_1(x)=O(r^{-1-\delta_1})$,

(vi) $-r^{-2}A V_1(x)\equiv(\nabla-\tilde{x}\partial_r)\cdot(\nabla-\tilde{x}\partial_r)V_1(x)=O(r^{-1-2\delta_2})$

as $r=|x|\to\infty$, where $\tilde{x}=x/|x|$, $\partial_r=\partial/\partial|x|$, ∇ is the gradient and A is the minus Laplace-Beltrami operator on the unit sphere S^{n-1}. On the other hand, $V_s(x)$ is a short-range potential such that for some $0<\delta_0<1$

(vii) $V_s(x)=O(r^{-1-\delta_0})$ as $r=|x|\to\infty$.

In the following we put $\delta=\min\{\delta_1, \delta_2, \delta_0\}$. Note that the condition $\delta_j<1$ $(j=0, 1, 2)$ does not restrict the generality.

We put

$$(1.1)\qquad E(\gamma)=\limsup_{r\to\infty}\frac{1}{\gamma}\{r\partial_r V_1(x)+\gamma V_1(x)\}\qquad \text{for } \gamma>0,$$

and define Λ_σ, $\sigma>0$, as follows:

$$(1.2)\qquad \Lambda_\sigma=E(\min\{4\sigma, 2\})+a/4,$$

where $a\geqq 0$ is the constant given in (iii) of Assumption 2. As is discussed in [11; §8], Λ_{σ} is non-increasing and continuous in $\sigma>0$. Especially, we have

$$\Lambda_{\sigma}\geqq\Lambda_{1/2}=E(2)+a/4 \quad \text{for any} \quad \sigma>0. \tag{1.3}$$

Let $k(x, \lambda\pm i0)$ be a smooth function of $(x, \lambda)\in B(R_1)\times(\Lambda_{1/2}, \infty)$, $B(R_1)=\{x; |x|>R_1\}$ ($R_1>R_0$ should be chosen sufficiently large, see [11; §8]), defined by

$$k(x, \lambda\pm i0)=\mp i\sqrt{\lambda-\eta V_1(x)}+\frac{n-1}{2r}+\frac{-\eta\partial_r V_1(x)}{4\{\lambda-\eta V_1(x)\}}; \tag{1.4$_\pm$}$$

$$\eta=4\lambda/(4\lambda-a). \tag{1.5}$$

Further, for any $\mu\in\boldsymbol{R}$ and $G\subset\Omega$, let $L_{\mu}^{2}(G)$ denote the space of all functions $f(x)$ such that

$$\|f\|_{\mu,G}^{2}=\int_{G}(1+r)^{2\mu}|f(x)|^{2}dx<\infty. \tag{1.6}$$

If $\mu=0$ or $G=\Omega$, the subscript μ or G will be omitted.

Now let α, β be a pair of positive constants satisfying

$$0<\alpha+\beta\leqq 2\delta \quad \text{and} \quad 0<\alpha\leqq\beta\leqq 1. \tag{1.7}$$

Let $\lambda>\Lambda_{\beta/2}$ ($\geqq\Lambda_{\delta}$) and $f\in L_{(1+\beta)/2}^{2}(\Omega)$, and let us consider the exterior boundary-value problem

$$\begin{cases} (-\Delta+V(x)-\lambda)u=f(x) & \text{in } \Omega \\ Bu=\begin{bmatrix} u \quad \text{or} \\ \nu\cdot\nabla u+d(x)u \end{bmatrix}=0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\nu=(\nu_1, \cdots, \nu_n)$ is the outer unit normal to the boundary $\partial\Omega$ and $d(x)$ is a real-valued smooth function on $\partial\Omega$.

Definition 1.1. For solutions $u\in H_{loc}^{2}(\bar{\Omega})$ of (1.8), the outgoing (+) [or incoming (−)] radiation condition at infinity is defined by

$$u\in L_{(-1-\alpha)/2}^{2}(\Omega) \quad \text{and} \quad \partial_r u+k(x, \lambda\pm i0)u\in L_{(-1+\beta)/2}^{2}(B(R_1)), \tag{1.9$_\pm$}$$

where α, β are any constants satisfying (1.7). A solution u of (1.8) which also satisfies the radiation condition $(1.9)_+$ [or $(1.9)_-$] is called an outgoing [or incoming] solution.

Remark 1.1. If $\delta>1/2$, i.e., $\delta_0>1/2$, we can choose $\beta=1$.

The main results of our previous paper [11] can be summarized in the following proposition.

Proposition 1.1. (a) *Let K be a compact set of $(\Lambda_{\beta/2}, \infty)$. Then for any $\lambda \in K$ and $f \in L^2_{(1+\beta)/2}(\Omega)$, (1.8) has a unique outgoing [incoming] solution $u = u(x, \lambda \pm i0) = \mathcal{R}_{\lambda \pm i0} f$, which also satisfies the inequalities*

$$\|u\|_{(-1-\alpha)/2} \leqq C \|f\|_{(1+\beta)/2}, \tag{1.10}$$

$$\|\nabla u + \tilde{x} k(x, \lambda \pm i0) u\|_{(-1+\beta)/2, B(R_1)} \leqq C \|f\|_{(1+\beta)/2}, \tag{1.11}$$

where $C = C(K) > 0$ is a domain constant independent of f.

(b) *$u = \mathcal{R}_{\lambda \pm i0} f$ is continuous in $L^2_{(-1-\alpha)/2}(\Omega)$ with respect to $(\lambda, f) \in (\Lambda_{\beta/2}, \infty) \times L^2_{(1+\beta)/2}(\Omega)$.*

(c) *Let $\mathcal{R}^*_{\lambda \pm i0} : L^2_{(1+\alpha)/2}(\Omega) \to L^2_{(-1-\beta)/2}(\Omega)$ be the adjoint of $\mathcal{R}_{\lambda \pm i0}$. Then we have*

$$\mathcal{R}^*_{\lambda \pm i0} f = \mathcal{R}_{\lambda \mp i0} f \quad \text{for} \quad f \in L^2_{(1+\beta)/2}(\Omega). \tag{1.12}$$

(d) *Let L be the selfadjoint operator in the Hilbert space $\mathcal{H} = L^2(\Omega)$ defined by*

$$\begin{cases} \mathcal{D}(L) = \{u \in H^2(\Omega) ; Bu|_{\partial\Omega} = 0\} \\ Lu = -\Delta u + V(x) u \qquad \text{for} \quad u \in \mathcal{D}(L), \end{cases} \tag{1.13}$$

and let $\{\mathcal{E}(\lambda) ; \lambda \in \boldsymbol{R}\}$ be its spectral measure. Then for any Borel set $e \Subset (\Lambda_{\beta/2}, \infty)$ and $f, g \in L^2_{(1+\beta)/2}(\Omega) \subset \mathcal{H}$, we have

$$\begin{aligned} (\mathcal{E}(e) f, g) &= \frac{1}{2\pi i} \int_e (\mathcal{R}_{\lambda+i0} f - \mathcal{R}_{\lambda-i0} f, g)\, d\lambda \\ &= \pm \frac{1}{2\pi i} \int_e \{(\mathcal{R}_{\lambda \pm i0} f, g) - (f, \mathcal{R}_{\lambda \pm i0} g)\}\, d\lambda. \end{aligned} \tag{1.14}$$

Here $(\ ,\ )$ denotes the inner-product in $\mathcal{H}$, or more generally the duality between $L^2_{(-1-\alpha)/2}(\Omega)$ and $L^2_{(1+\alpha)/2}(\Omega)$:

$$(f, g) = \int_\Omega f(x) \overline{g(x)}\, dx. \tag{1.15}$$

(e) *The part of L in $\mathcal{E}((\Lambda_\delta, \infty))\mathcal{H}$ is absolutely continuous, i.e., $(\mathcal{E}(\lambda) f, f)$ for $f \in \mathcal{H}$ is absolutely continuous with respect to the Lebesgue measure on $\lambda > \Lambda_\delta$.*

Remark 1.2. In this proposition we summarized results of [11] in a slightly modified form. In [11] the pair α, β was chosen for each compact set K of (Λ_δ, ∞) so as to satisfy (1.7) and the inequality $\min\{\lambda ; \lambda \in K\} > \Lambda_{\beta/2}$. On the other hand, in this proposition we apriori gave α, β satisfying (1.7) and assumed $K \Subset (\Lambda_{\beta/2}, \infty)$. Note that there exists no essential difference between these two formulations. In fact, by the continuity and non-increasingness of Λ_σ in $\sigma > 0$, it follows that for any $N > 0$ there exists a pair $\alpha = \alpha(N)$, $\beta = \beta(N)$ satisfying (1.7) and the inequality $(\Lambda_\delta \leqq) \Lambda_{\beta/2} < \Lambda_\delta + N^{-1}$.

Remark 1.3. By (b) and Proposition I.2.1 we easily have ∇u, and hence, $(\nabla-\tilde{x}\partial_r)u$ being also continuous in $L^2_{(-1-\alpha)/2}(\Omega)$ with respect to $(\lambda, f)\in(\Lambda_{\beta/2}, \infty)\times L^2_{(1+\beta)/2}(\Omega)$.

As a corollary of the above proposition we can prove the

Proposition 1.2. *Let* $\lambda>\Lambda_{\beta/2}$, $f\in L^2_{(1+\beta)/2}(\Omega)$ *and* $u=u(x, \lambda\pm i0)=\mathcal{R}_{\lambda\pm i0}f$. *Then there exists a sequence* $r_p=r_p(\lambda, f)>R_1$ $(p=1, 2, \cdots)$ *diverging to* ∞ *as* $p\to\infty$ *such that*

$$\lim_{p\to\infty}\int_{S(r_p)}\{r^{-\alpha}|u|^2+r^{\beta}|\nabla u+\tilde{x}k(x, \lambda\pm i0)u|^2\}\,dS=0, \tag{1.16}$$

and we have for this $\{r_p\}$

$$\begin{aligned}\frac{1}{2\pi i}(\mathcal{R}_{\lambda+i0}f-\mathcal{R}_{\lambda-i0}f, f) &= \lim_{p\to\infty}\frac{1}{\pi}\int_{S(r_p)}\sqrt{\lambda-\eta V_1(x)}\,|u(x, \lambda\pm i0)|^2dS\\ &= \lim_{p\to\infty}\frac{1}{\pi}\,r_p^{n-1}\|\{\lambda-\eta V_1(r_p\cdot)\}^{1/4}u(r_p\cdot, \lambda\pm i0)\|_{\boldsymbol{h}}^2,\end{aligned}\tag{1.17}$$

where $\|\ \|_{\boldsymbol{h}}$ *is the norm in the Hilbert space* $\boldsymbol{h}=L^2(S^{n-1})$ *of all square integrable functions over the unit sphere* S^{n-1}.

Proof. The existence of the sequence $r_p=r_p(\lambda, f)$ satisfying (1.16) is obvious from (a) of Proposition 1.1. By the Green formula in $\Omega(r_p)=\{x\in\Omega;\ |x|<r_p\}$,

$$\begin{aligned}\pm\int_{\Omega(r_p)}(u\bar{f}-f\bar{u})dx &= \mp\int_{S(r_p)}\{u(\partial_r\bar{u})-(\partial_r u)\bar{u}\}\,dS\\ &= \mp\int_{S(r_p)}\{u(\partial_r\bar{u}+\overline{k(x, \lambda\pm i0)u})-(\partial_r u+k(x, \lambda\pm i0)u)\bar{u}\}\,dS\\ &\quad\mp\int_{S(r_p)}\{k(x, \lambda\pm i0)-\overline{k(x, \lambda\pm i0)}\}|u|^2dS.\end{aligned}$$

Since $\alpha\leqq\beta$ in (1.16), the first term of the right side tends to 0 as $p\to\infty$. Thus, noting

$$\pm\int_{\Omega}(u\bar{f}-f\bar{u})dx=(\mathcal{R}_{\lambda+i0}f-\mathcal{R}_{\lambda-i0}f, f)$$

and $\mp\mathrm{Im}\,k(x, \lambda\pm i0)=\sqrt{\lambda-\eta V_1(x)}$, we obtain (1.17). q.e.d.

Our spectral representation theorem will be based on the relation (1.17).

Definition 1.2. We define the smooth function $\rho=\rho(x, \lambda\pm i0)$ in $B(R_1)\times(\Lambda_{1/2}, \infty)$ as follows:

(1.18) $\rho(x, \lambda \pm i0)$

$$= \mp i\int_{R_1}^{r} \sqrt{\lambda - \eta V_1(s\tilde{x})}\, ds + \frac{n-1}{2} \log r + \frac{1}{4} \log\{\lambda - \eta V_1(x)\}.$$

Lemma 1.1. *We have*

(1.19) $$\partial_r \rho(x, \lambda \pm i0) = k(x, \lambda \pm i0),$$

(1.20) $$\rho(x, \lambda + i0) = \overline{\rho(x, \lambda - i0)},$$

(1.21) $$|e^{\rho(x, \lambda \pm i0)}| = e^{\mathrm{Re}\,\rho(x, \lambda \pm i0)} = r^{(n-1)/2}\{\lambda - \eta V_1(x)\}^{1/4},$$

(1.22) $$e^{\rho(x, \lambda \pm i0)} = r^{n-1}\{\lambda - \eta V_1(x)\}^{1/2} e^{-\overline{\rho(x, \lambda \pm i0)}},$$

(1.23) $$\partial_r^2 \rho + \frac{n-1}{r} \partial_r \rho - (\partial_r \rho)^2 + V(x) - \lambda = O(r^{-1-\delta}), \qquad \text{as } r \to \infty,$$

(1.24) $$(\nabla - \tilde{x}\partial_r)\rho = O(r^{-\delta_2}), \qquad \text{as } r \to \infty,$$

(1.25) $$-r^{-2}A\rho = (\nabla - \tilde{x}\partial_r)\cdot(\nabla - \tilde{x}\partial_r)\rho = O(r^{-2\delta_2}), \qquad \text{as } r \to \infty.$$

Proof. (1.19)$\sim$(1.22) are obvious from the definition of $\rho(x, \lambda \pm i0)$. (1.23) follows from (i), (ii), (iii) and (vii) of Assumption 2. As for the details, see Proposition I.8.2. (1.24) follows from (iv) of Assumption 2. In fact, we have noting $\delta_2 < 1$

$$(\nabla - \tilde{x}\partial_r)\rho$$

$$= \mp i r^{-1} \int_{R_1}^{r} \frac{-\eta[(\nabla - \tilde{x}\partial_r)V_1](s\tilde{x})s}{2\{\lambda - \eta V_1(s\tilde{x})\}^{1/2}} ds + \frac{-\eta(\nabla - \tilde{x}\partial_r)V_1(x)}{4\{\lambda - \eta V_1(x)\}}$$

$$= r^{-1}\int_{R_1}^{r} O(s^{-1-\delta_2})\, s\, ds + O(r^{-1-\delta_2}) = O(r^{-\delta_2}).$$

Finally, (1.25) can be proved by use of (iv) and (vi) of Assumption 2:

$$-r^{-2}A\rho$$

$$= \mp i r^{-2} \int_{R_1}^{r} \left[\frac{\eta s^{-2} A V_1(s\tilde{x})}{2\{\lambda - \eta V_1(s\tilde{x})\}^{1/2}} - \frac{\eta^2 |(\nabla - \tilde{x}\partial_r)V_1|^2(s\tilde{x})}{4\{\lambda - \eta V_1(s\tilde{x})\}^{3/2}} \right] s^2 ds$$

$$+ \frac{\eta r^{-2} A V_1(x)}{4\{\lambda - \eta V_1(x)\}} - \frac{\eta^2 |(\nabla - \tilde{x}\partial_r)V_1(x)|^2}{4\{\lambda - \eta V_1(x)\}^2}$$

$$= r^{-2}\int_{R_1}^{r} \{O(s^{-1-2\delta_2}) + O(s^{-2-2\delta_2})\}\, s^2 ds$$

$$+ O(r^{-1-2\delta_2}) + O(r^{-2-2\delta_2}) = O(r^{-2\delta_2}).$$

q.e.d.

Proposition 1.3. *Let λ, f, u and $\{r_p\}$ be as in Proposition 1.2. Then we have*

$$\lim_{p\to\infty}\frac{1}{\pi}\|e^{\rho(r_p\cdot,\lambda\pm i0)}u(r_p\cdot,\lambda\pm i0)\|_h^2 \tag{1.26}$$

$$=\frac{1}{2\pi i}(\mathcal{R}_{\lambda+i0}f-\mathcal{R}_{\lambda-i0}f,f).$$

Proof. Obvious from (1.17) and (1.21). q.e.d.

It now follows that

$$\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)=\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot,\lambda\pm i0)}u(r_p\cdot,\lambda\pm i0)\quad(p=1,2,\cdots)$$

is bounded in $\boldsymbol{h}$, and hence, contains an $\boldsymbol{h}$-weakly convergent subsequence. In the next §2 we shall consider the special case that $\delta_0>1/2$, i.e., $\delta>1/2$, and show that $\left\{\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)\right\}$ strongly converges without taking any subsequence. However, it seems difficult to show the strong convergence of this sequence for the general case $\delta_0>0$.

In the rest of this section, we shall show that if we choose $\alpha<4\delta_2-2$ in (1.7), $\left\{\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)\right\}$ itself weakly converges in $\boldsymbol{h}$, and the limit function does not depend on the choice of $\{r_p\}$ satisfying (1.16) (Proposition 1.4). These properties will play an important role in §§3 and 4 to show the existence and unitarity of the generalized Fourier transformation $\mathcal{F}_\pm$ (cf., Lemma 3.2). Note that (vi) of Assumption 2 is used only to show (1.31) of Lemma 1.2 and Proposition 1.4, and hence, in §2 we do not make use of this condition.

For $\phi=\phi(\tilde{x})\in\boldsymbol{h}$ and $\lambda>\Lambda_{1/2}$ let us put

$$v_\phi(x,\lambda\pm i0)=\begin{cases}\frac{1}{\sqrt{\pi}}e^{-\rho(x,\lambda\pm i0)}\phi(\tilde{x})\psi(r), & |x|=r>R_1\\ 0, & |x|=r\leqq R_1,\end{cases} \tag{1.27}$$

where $\psi(r)$ is a smooth function of $r>0$ such that $0\leqq\psi(r)\leqq 1$, $\psi(r)=0$ for $r<R_1+1$ and $=1$ for $r>R_1+2$.

Lemma. 1.2. *Let $\lambda>\Lambda_{1/2}$. Then we have for $\phi\in\boldsymbol{h}$*

$$v_\phi=O(r^{-(n-1)/2}),\qquad\text{as } r\to\infty, \tag{1.28}$$

$$\partial_r v_\phi+k(x,\lambda\pm i0)v_\phi=0\quad\text{in}\quad B(R_1+2), \tag{1.29}$$

$$Bv_\phi=0\quad\text{on}\quad\partial\Omega. \tag{1.30}$$

Moreover, we have for $\phi\in\mathcal{D}(A)\subset\boldsymbol{h}$

$$g_\phi\equiv(-\Delta+V(x)-\lambda)v_\phi=O(r^{-(n+1+2\tilde{\delta})/2}),\qquad\text{as } r\to\infty, \tag{1.31}$$

where $\tilde{\delta}=\min\{\delta, 2\delta_2-1\}$.

Proof. (1.28), (1.29) and (1.30) are obvious from the definition (1.27) of v_ϕ. By a simple calculation we have

$$g_\phi = g_{\phi,1} + r^{-2} A v_\phi, \tag{1.32}$$

$$g_{\phi,1} = \left\{\partial_r^2 \rho + \frac{n-1}{r}\partial_r \rho - (\partial_r \rho)^2 + V - \lambda\right\} v_\phi \tag{1.33}$$

$$-\left\{\partial_r^2 \phi + \frac{n-1}{r}\partial_r \phi - 2(\partial_r \rho)(\partial_r \phi)\right\}\frac{1}{\sqrt{\pi}} e^{-\rho}\phi.$$

$g_{\phi,1} = O(r^{-(n+1+2\delta)/2})$ by (1.23) and (1.28). On the other hand,

$$r^{-2} A v_\phi = \frac{1}{\sqrt{\pi}} e^{-\rho}\phi\{-r^{-2}(A\rho)\phi + r^{-2}A\phi$$

$$-((\nabla - \tilde{x}\partial_r)\rho)^2\phi + 2(\nabla - \tilde{x}\partial_r)\rho\cdot\nabla\phi\}.$$

Thus, by (1.21), (1.24), (1.25) and the fact $\nabla\phi = O(r^{-1})$ we have $r^{-2}Av_\phi = O(r^{-(n-1+4\delta_2)/2})$. These prove (1.31). q.e.d.

Lemma 1.3. *Let $\phi \in \mathcal{D}(A)$. Then the following relation holds for any $\lambda > \Lambda_{1/2}$ and $r_p > R_1 + 2$.*

$$\left(\frac{1}{\sqrt{\pi}} e^{\rho(r_p\cdot)} u(r_p\cdot), \phi\right)_{\boldsymbol{h}} = \int_{S(r_p)} \sqrt{\lambda - \eta V_1(x)}\, u\overline{v_\phi}\, dS \tag{1.34}$$

$$= \pm\frac{1}{2i}\left\{\int_{\Omega(r_p)} (u\overline{g_\phi} - f\overline{v_\phi})\, dx - \int_{S(r_p)} (\partial_r u + ku)\overline{v_\phi}\, dS\right\}.$$

Further, this relation can be extended to $\phi \in \mathcal{D}(A^{1/2})$ if we understand

$$\int_{\Omega(r_p)} u\overline{g_\phi}\, dx = \int_{\Omega(r_p)} \left\{u\overline{g_{\phi,1}} + (\nabla - \tilde{x}\partial_r)u\cdot\overline{(\nabla - \tilde{x}\partial_r)v_\phi}\right\} dx. \tag{1.35}$$

Proof. The first equality of (1.34) follows from the relation

$$\frac{1}{\sqrt{\pi}} e^\rho u\bar{\phi} = r^{n-1}\sqrt{\lambda - \eta V_1(x)}\, u\overline{v_\phi} \quad \text{in} \quad B(R_1+2).$$

If we note (1.29) and (1.30), the second equality is a direct consequence of the Green formula (cf., the proof of Proposition 1.2). (1.35) is obtained by means of (1.32) and an integration by parts. q.e.d.

Proposition 1.4. *Suppose that the pair $\tilde{\alpha}$, $\tilde{\beta}$ satisfies the following stronger condition:*

$$0 < \tilde{\alpha} + \tilde{\beta} < 2\delta, \quad \tilde{\alpha} < 4\delta_2 - 2 \quad \text{and} \quad 0 < \tilde{\alpha} \leqq \tilde{\beta} \leqq 1. \tag{1.36}$$

Let λ, f, u and $\{r_p\}$ be as in Proposition 1.2 *with $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$. Then $\left\{\frac{1}{\sqrt{\pi}} e^{\rho(r_p\cdot)} u(r_p\cdot)\right\}$ weakly converges in $\boldsymbol{h}$. In particular, we have for $\phi \in \mathcal{D}(A) \subset \boldsymbol{h}$*

(1.37) $$\lim_{p\to\infty}\left(\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot),\ \phi\right)_h=\pm\frac{1}{2i}\int_\Omega(u\overline{g_\phi}-f\overline{v_\phi})\,dx.$$

Thus, the limit function is independent of the choice of $\{r_p\}$.

Proof. By (1.28) and (1.31) we see that $f\overline{v_\phi}$ and $u\overline{g_\phi}$ are integrable in Ω. Moreover,

$$\lim_{p\to\infty}\int_{S(r_p)}(\partial_r u+ku)\overline{v_\phi}\,dS=0.$$

Thus, letting $p\to\infty$ in (1.34), we obtain (1.37). Since $\mathscr{D}(A)$ is dense in $\boldsymbol{h}$, the first assertion follows from (1.37) and Proposition 1.3. q.e.d.

§2. Spectral representation (isometry): Special case $\delta_0>1/2$

Throughout this section we assume $\delta_0>1/2$ in (vii) of Assumption 2. Then $\delta=\min\{\delta_1,\delta_2,\delta_0\}>1/2$, and we can choose $\beta=1>2-2\delta_2$ in (1.7). We shall show the existence of an isometry $\mathscr{F}_\pm:\mathscr{E}((\Lambda_{1/2},\infty))\mathscr{H}\to\hat{\mathscr{H}}_{\Lambda_{1/2}}=L^2((\Lambda_{1/2},\infty);\boldsymbol{h})$, which diagonalizes the operator L.

We begin by showing

Lemma 2.1. *Let* $\beta>2-2\delta_2$ *in* (1.7), *and let* λ, f, u *and* $\{r_p\}$ *be as in Proposition* 1.2 *with this* β. *Then there exists a constant* $\varepsilon(p)>0$ *independent of* $\phi\in\mathscr{D}(A^{1/2})\subset\boldsymbol{h}$ *such that* $\varepsilon(p)\to 0$ *as* $p\to\infty$ *and for* $r_q>r_p>R_1+2$

(2.1) $$\begin{aligned}&|(e^{\rho(r_q\cdot)}u(r_q\cdot)-e^{\rho(r_p\cdot)}u(r_p\cdot),\ \phi)_h|\\&\leqq\varepsilon(p)\{r_p^{-\beta/2}+r_p^{1-\delta_2-\beta/2})\|\phi\|_h+r_p^{-\beta/2}\|A^{1/2}\phi\|_h\}.\end{aligned}$$

Proof. It follows from (1.34) that

$$\begin{aligned}&\pm\frac{2i}{\sqrt{\pi}}(e^{\rho(r_q\cdot)}u(r_q\cdot)-e^{\rho(r_p\cdot)}u(r_p\cdot),\ \phi)_h\\&=\int_{B(r_p,r_q)}(u\overline{g_\phi}-f\overline{v_\phi})\,dx-\left[\int_{S(r_q)}-\int_{S(r_p)}\right](\partial_r u+ku)\overline{v_\phi}\,dS.\end{aligned}$$

By the Schwartz inequality,

$$\begin{aligned}&\left|\int_{S(r_p)}(\partial_r u+ku)\overline{v_\phi}\,dS\right|\\&\leqq\text{const}\,r_p^{-\beta/2}\left(\int_{S(r_p)}r^\beta|\partial_r u+ku|^2dS\right)^{1/2}\|\phi\|_h\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{B(r_p,r_q)}f\overline{v_\phi}\,dx\right|\\&\leqq\text{const}\left(\int_{B(r_p)}r^{1+\beta}|f|^2dx\right)^{1/2}\left(\int_{B(r_p)}r^{-n-\beta}|\phi|^2dx\right)^{1/2}\\&\leqq\text{const}\,r_p^{-\beta/2}\|f\|_{(1+\beta)/2,B(r_p)}\|\phi\|_h.\end{aligned}$$

On the other hand, if we note the inequalities

$$|g_{\phi,1}| \leqq \text{const}\, r^{-(n+1+2\delta)/2}|\phi|,$$
$$|(\nabla-\tilde{x}\partial_r)v_\phi| \leqq \text{const}\{r^{-(n-1+2\delta_2)/2}|\phi|+r^{-(n+1)/2}r|\nabla\phi|\},$$

it follows from (1.35) that

$$\left|\int_{B(r_p,r_q)} u\overline{g_\phi}\,dx\right| \leqq \int_{B(r_p,r_q)}\left|u\overline{g_{\phi,1}}+(\nabla-\tilde{x}\partial_r)u\cdot\overline{(\nabla-\tilde{x}\partial_r)v_\phi}\right|dx$$
$$\leqq \text{const}\,[\|u\|_{(-1-\alpha)/2,B(r_p)} r_p^{-\delta+\alpha/2}\|\phi\|_h$$
$$+\|\nabla u-\tilde{x}\partial_r u\|_{(-1+\beta)/2,B(r_p)}\{r_p^{1-\delta_2-\beta/2}\|\phi\|_h+r_p^{-\beta/2}\|A^{1/2}\phi\|_h\}].$$

Here $-2\delta+\alpha\leqq-\beta$ by (1.7) and $|\nabla u-\tilde{x}\partial_r u|\leqq|\nabla u+\tilde{x}ku|$. Thus, choosing

$$\varepsilon(p)=\text{const}\left[\sup_{l\geqq p}\left(\int_{S(r_l)}\{r^{-\alpha}|u|^2+r^\beta|\nabla u+\tilde{x}ku|^2\}\,dS\right)^{1/2}\right.$$
$$\left.+\|f\|_{(1+\beta)/2,B(r_p)}+\|u\|_{(-1-\alpha)/2,B(r_p)}+\|\nabla u+\tilde{x}ku\|_{(-1+\beta)/2,B(r_p)}\right],$$

we obtain (2.1). q. e. d.

Lemma 2.2. *Let β, λ, f, u and $\{r_p\}$ be as in the above lemma. Then $\left\{\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)\right\}$ weakly converges in* $\boldsymbol{h}$. *In particular, we have for $\phi\in\mathcal{D}(A^{1/2})$*

$$\lim_{p\to\infty}\left(\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot),\ \phi\right)_h \tag{2.2}$$
$$=\pm\frac{1}{2i}\int_\Omega\{u\overline{g_{\phi,1}}+(\nabla-\tilde{x}\partial_r)u\cdot\overline{(\nabla-\tilde{x}\partial_r)v_\phi}-f\overline{v_\phi}\}\,dx.$$

Thus, the limit function is independent of the choice of $\{r_p\}$.

Proof. Obvious from Lemma 2.1, Proposition 1.3 and Lemma 1.3.

q. e. d.

Lemma 2.3. *We have for $u=\mathcal{R}_{\lambda\pm i0}f$*

$$e^{\rho(r\cdot)}u(r\cdot)\in\mathcal{D}(A^{1/2})\quad(r>R_1), \tag{2.3}$$

$$\|e^{\rho(r_p\cdot)}u(r_p\cdot)\|_h\leqq M<\infty\quad(p=1,2,\cdots), \tag{2.4}$$

$$\|A^{1/2}e^{\rho(r_p\cdot)}u(r_p\cdot)\|_h\leqq\varepsilon(p)(r_p^{1-\beta/2}+r_p^{1-\delta_2+\alpha/2}). \tag{2.5}$$

Proof. (2.3) is obvious since $\rho(x)$ is smooth and $u\in H^2_{\text{loc}}(\bar{\Omega})$. (2.4) follows from Proposition 1.3. Finally, (2.5) is proved as follows:

$$\|A^{1/2}e^{\rho(r_p\cdot)}u(r_p\cdot)\|_{\boldsymbol{h}}^2=\int_{S(r_p)}r_p^{-n+1}|A^{1/2}e^{\rho(r_p\tilde{x})}u(r_p\tilde{x})|^2dS$$

$$=\int_{S(r_p)}r^{-n+3}|(\nabla-\tilde{x}\partial_r)e^{\rho}u|^2dS\leqq\int_{S(r_p)}r^{-n+3}|\nabla(e^{\rho}u)|^2dS$$

$$\leqq\text{const}\int_{S(r_p)}r^2|\nabla u+\tilde{x}ku+(\nabla\rho-\tilde{x}\partial_r\rho)u|^2dS$$

$$\leqq\varepsilon(p)^2(r_p^{2-\beta}+r_p^{2-2\delta_2+\alpha}).$$

q. e. d.

Proposition 2.1. *Let* $\lambda>\Lambda_{1/2}$, $f\in L_1^2(\Omega)$ *and* $u=u(\cdot,\lambda\pm i0)=\mathcal{R}_{\lambda\pm i0}f$. *Then* $\left\{\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)\right\}$ *strongly converges in* $\boldsymbol{h}$, *where* $\{r_p\}$ *is any sequence specified in Proposition* 1.2 *with* $\beta=1$.

Proof. Let

$$h=\text{weak}\lim_{p\to\infty}\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)\quad\text{in}\quad\boldsymbol{h}. \tag{2.6}$$

Then it follows from Lemma 2.1 (by letting $q\to\infty$) that

$$\left|\left(h-\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot),\ \phi\right)_{\boldsymbol{h}}\right|\leqq\varepsilon(p)(\|\phi\|_{\boldsymbol{h}}+r_p^{-\beta/2}\|A^{1/2}\phi\|_{\boldsymbol{h}})$$

for $\phi\in\mathcal{D}(A^{1/2})$. Here we put $\phi=\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)$. Then by virtue of (2.4) and (2.5), we have

$$\left|\left(h-\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot),\ \frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)\right)_{\boldsymbol{h}}\right|$$

$$\leqq\varepsilon(p)\{M+\varepsilon(p)(r_p^{1-\beta}+r_p^{1-\delta_2+\alpha/2-\beta/2})\}.$$

Since $1-\beta=0$ and $1-\delta_2+\alpha/2-\beta/2=(1-2\delta_2+\alpha)/2<0$, letting $p\to\infty$, we have

$$\lim_{p\to\infty}\left(h-\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot),\ \frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)\right)_{\boldsymbol{h}}=0,$$

that is,

$$\lim_{p\to\infty}\left\|\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)\right\|_{\boldsymbol{h}}=\|h\|_{\boldsymbol{h}}.$$

Hence, we see that $\left\{\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u(r_p\cdot)\right\}$ strongly converges in $\boldsymbol{h}$. q. e. d.

Definition 2.1. For $\lambda>\Lambda_{1/2}$ let $\mathcal{F}_{\pm}(\lambda):L_1^2(\Omega)\to\boldsymbol{h}$ be defined by

$$\mathcal{F}_{\pm}(\lambda)f=\text{strong}\lim_{p\to\infty}\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot,\lambda\pm i0)}(\mathcal{R}_{\lambda\pm i0}f)(r_p\cdot), \tag{2.7}$$

where $\{r_p\}$ is any sequence specified in Proposition 1.2 with $\beta=1$.

Remark 2.1. $\mathcal{F}_{\pm}(\lambda)$ is independent of the choice of $\{r_p\}$ since the weak limit (2.6) does not depend on the choice of $\{r_p\}$ (see Lemma 2.2).

By Proposition 1.3 we have for $f\in L_1^2(\Omega)$ and $\lambda>\Lambda_{1/2}$

$$\|\mathcal{F}_{\pm}(\lambda)f\|_{\boldsymbol{h}}^2=\frac{1}{2\pi i}(\mathcal{R}_{\lambda+i0}f-\mathcal{R}_{\lambda-i0}f,\ f). \tag{2.8}$$

Thus, for any λ in a compact set $K\subset(\Lambda_{1/2},\infty)$ we have from (a) of Proposition 1.1

$$\|\mathcal{F}_{\pm}(\lambda)f\|_{\boldsymbol{h}}^2\leqq C(K)\|f\|_1^2. \tag{2.9}$$

This implies that $\mathcal{F}_{\pm}(\lambda)$, $\lambda>\Lambda_{1/2}$, is a bounded linear operator from $L_1^2(\Omega)$ into $\boldsymbol{h}$: $\mathcal{F}_{\pm}(\lambda)\in B(L_1^2(\Omega),\boldsymbol{h})$.

Moreover, we have the

Lemma 2.4. *$\mathcal{F}_{\pm}(\lambda)f\in\boldsymbol{h}$ depends continuously on $(\lambda, f)\in(\Lambda_{1/2},\infty)\times L_1^2(\Omega)$.*

Proof. By (2.8) and (b) of Proposition 1.1 we see that $\|\mathcal{F}_{\pm}(\lambda)f\|_{\boldsymbol{h}}$ is continuous in $(\lambda, f)\in(\Lambda_{1/2},\infty)\times L_1^2(\Omega)$. Thus, we have only to show that for any smooth $\phi\in\boldsymbol{h}$, $(\mathcal{F}_{\pm}(\lambda)f,\phi)_{\boldsymbol{h}}$ is continuous in the same domain. We return to the relation (2.2) of Lemma 2.2. Since $g_{\phi,1}\in L_1^2(\Omega)$, $(\nabla-\tilde{x}\partial_r)v_\phi\in L^2(\Omega)$ and $v_\phi\in L_{-1}^2(\Omega)$ are continuous with respect to $\lambda>\Lambda_{1/2}$, we then have the desired continuity of $(\mathcal{F}_{\pm}(\lambda)f,\phi)_{\boldsymbol{h}}$ also by (b) of Proposition 1.1 (cf., Remark 1.3).

q. e. d.

In virtue of this lemma, (2.8) and (d) of Proposition 1.1 we see $\mathcal{F}_{\pm}(\lambda)f\in\hat{\mathcal{H}}_{\Lambda_{1/2}}$ and

$$\|\mathcal{E}((\Lambda_{1/2},\infty))f\|^2=\|\mathcal{F}_{\pm}(\cdot)f\|_{\hat{\mathcal{H}}_{\Lambda_{1/2}}}^2=\int_{\Lambda_{1/2}}^{\infty}\|\mathcal{F}_{\pm}(\lambda)f\|_{\boldsymbol{h}}^2\,d\lambda. \tag{2.10}$$

Hence, we have the

Lemma 2.5. *Let $\mathcal{F}_{\pm}: L_1^2(\Omega)\to\hat{\mathcal{H}}_{\Lambda_{1/2}}$ be defined by*

$$(\mathcal{F}_{\pm}f)(\lambda)=\mathcal{F}_{\pm}(\lambda)f \quad \text{for } \lambda>\Lambda_{1/2}. \tag{2.11}$$

Then $\mathcal{F}_{\pm}$ can be extended by continuity to a partial isometric operator from $\mathcal{H}$ into $\hat{\mathcal{H}}_{\Lambda_{1/2}}$ with initial set $\mathcal{E}((\Lambda_{1/2},\infty))\mathcal{H}$, which will be denoted by $\mathcal{F}_{\pm}$ also. Moreover, we have for any Borel set $e\subset(\Lambda_{1/2},\infty)$ and $f, g\in\mathcal{H}$

$$(\mathcal{E}(e)f,\ g)=\int_e((\mathcal{F}_{\pm}f)(\lambda),\ (\mathcal{F}_{\pm}g)(\lambda))_{\boldsymbol{h}}\,d\lambda. \tag{2.12}$$

We are now ready to prove the following spectral representation theorem associated with the Schrödinger operator L.

Theorem 2.1. *Let $\delta_0>1/2$ in* (vii) *of Assumption* 2, *and let $\mathcal{F}_\pm : \mathcal{H} \to \hat{\mathcal{H}}_{\Lambda_{1/2}}$ be as given in the above lemma.*

(a) (*Diagonal representation of L*) *For any bounded Borel function $\alpha(\lambda)$ on $\boldsymbol{R}$ and $f \in \mathcal{H}$ we have*

$$(\mathcal{F}_\pm \alpha(L) f)(\lambda) = \alpha(\lambda)(\mathcal{F}_\pm f)(\lambda) \qquad \text{for a.e. } \lambda > \Lambda_{1/2}. \tag{2.13}$$

(b) (*Inversion formula*) *Let $\mathcal{F}_\pm^* : \hat{\mathcal{H}}_{\Lambda_{1/2}} \to \mathcal{H}$ be the adjoint operator of $\mathcal{F}_\pm$. Then we have for $\hat{f} \in \hat{\mathcal{H}}_{\Lambda_{1/2}}$*

$$\mathcal{F}_\pm^* \hat{f} = \text{strong} \lim_{N\to\infty} \int_{\Lambda_{1/2}+1/N}^{N} \mathcal{F}_\pm(\lambda)^* \hat{f}(\lambda)\, d\lambda \qquad \text{in } \mathcal{H}, \tag{2.14}$$

where $\mathcal{F}_\pm(\lambda)^* : \boldsymbol{h} \to L^2_{-1}(\Omega)$ is the adjoint of $\mathcal{F}_\pm(\lambda)$. In particular, the following inversion formula holds for $f \in \mathcal{H}$.

$$\mathcal{E}((\Lambda_{1/2}, \infty)) f = \text{strong} \lim_{N\to\infty} \int_{\Lambda_{1/2}+1/N}^{N} \mathcal{F}_\pm(\lambda)^* (\mathcal{F}_\pm f)(\lambda)\, d\lambda. \tag{2.15}$$

(c) (*Eigenoperator*) *$\mathcal{F}_\pm(\lambda)^*$ is an eigenoperator of L with eigenvalue $\lambda(>\Lambda_{1/2})$ in the sense that*

$$(\mathcal{F}_\pm(\lambda)^* \phi, (L-\lambda) u) = 0 \tag{2.16}$$

for any $\phi \in \boldsymbol{h}$ and $u \in C_0^\infty(\bar{\Omega})$ satisfying the boundary condition $Bu|_{\partial\Omega} = 0$.

Proof. (a) We have only to show the assertion for $\alpha(\lambda) = \chi_e(\lambda)$, the characteristic function of any Borel set e in $(\Lambda_{1/2}, \infty)$. We note $\chi_e(L) = \mathcal{E}(e)$. By (2.12) we have for any $e \subset (\Lambda_{1/2}, \infty)$ and $f \in \mathcal{H}$

$$\|\mathcal{E}(e) f\|^2 = \int_e \|(\mathcal{F}_\pm f)(\lambda)\|_{\boldsymbol{h}}^2\, d\lambda.$$

It then follows that

$$0 = \|\mathcal{E}(e)(\mathcal{E}(e)-1) f\|^2 = \int_e \|(\mathcal{F}_\pm(\mathcal{E}(e)-1) f)(\lambda)\|_{\boldsymbol{h}}^2\, d\lambda$$

$$= \int_e \|(\mathcal{F}_\pm \mathcal{E}(e) f)(\lambda) - (\mathcal{F}_\pm f)(\lambda)\|_{\boldsymbol{h}}^2\, d\lambda$$

and

$$0 = \|\mathcal{E}(e')\mathcal{E}(e) f\|^2 = \int_{e'} \|(\mathcal{F}_\pm \mathcal{E}(e) f)(\lambda)\|_{\boldsymbol{h}}^2\, d\lambda,$$

where $e' = (\Lambda_{1/2}, \infty) \backslash e$. These relations show that

$$(\mathcal{F}_\pm \mathcal{E}(e) f)(\lambda) = \begin{cases} (\mathcal{F}_\pm f)(\lambda) & \text{for a.e. } \lambda \in e \\ 0 & \text{for a.e. } \lambda \in e', \end{cases}$$

which was to be proved.

(b) To show (2.14) we first note that the integral of the right side makes sense since $\mathcal{F}_{\pm}(\lambda)^*$ is measurable (cf., Lemma 2.4). Let $e_N=(\Lambda_{1/2}+1/N, N)$. Then for every $\hat{f}\in\hat{\mathcal{H}}_{\Lambda_{1/2}}$ and $g\in L_1^2(\Omega)$

$$\int_{e_N}(\mathcal{F}_{\pm}(\lambda)^*\hat{f}(\lambda),\ g)\,d\lambda=\int_{e_N}(\hat{f}(\lambda),\ \mathcal{F}_{\pm}(\lambda)g)_h\,d\lambda$$

$$=\int_{e_N}(\hat{f}(\lambda),\ (\mathcal{F}_{\pm}g)(\lambda))_h\,d\lambda.$$

As is proved in (a), $\chi_{e_N}\mathcal{F}_{\pm}g=\mathcal{F}_{\pm}\mathcal{E}(e_N)g$. Hence,

$$\int_{e_N}(\mathcal{F}_{\pm}(\lambda)^*\hat{f}(\lambda),\ g)\,d\lambda=(\hat{f},\ \mathcal{F}_{\pm}\mathcal{E}(e_N)g)_{\hat{\mathcal{H}}_{\Lambda_{1/2}}}=(\mathcal{E}(e_N)\mathcal{F}_{\pm}^*\hat{f},\ g).$$

Since $L_1^2(\Omega)$ is dense in $\mathcal{H}$, this shows that

$$\mathcal{E}(e_N)\mathcal{F}_{\pm}^*\hat{f}=\int_{e_N}\mathcal{F}_{\pm}(\lambda)^*\hat{f}(\lambda)\,d\lambda.$$

Thus, letting $N\to\infty$, we obtain (2.14) since the final set of $\mathcal{F}_{\pm}^*$ is $\mathcal{E}((\Lambda_{1/2},\infty))\mathcal{H}$. (2.15) directly follows from (2.14) if we note $\mathcal{F}_{\pm}^*\mathcal{F}_{\pm}f=\mathcal{E}((\Lambda_{1/2},\infty))f$ by (2.12).

(c) Let $f=(L-\lambda)u$. Then $f\in L_1^2(\Omega)$, and hence,

$$(\mathcal{F}_{\pm}(\lambda)^*\phi,\ (L-\lambda)u)=(\phi,\ \mathcal{F}_{\pm}(\lambda)f)_h.$$

Moreover, we have $u=\mathcal{R}_{\lambda+i0}f$ $(=\mathcal{R}_{\lambda-i0}f)$ since $u\in C_0^\infty(\bar{\Omega})$ satisfies the outgoing (as well as incoming) radiation condition. Thus, in view of the definition of $\mathcal{F}_{\pm}(\lambda)$ and the fact $u\in C_0^\infty(\Omega)$, we have $\mathcal{F}_{+}(\lambda)f$ $(=\mathcal{F}_{-}(\lambda)f)=0$, which implies (2.16).

The proof of the theorem is now complete. q. e. d.

§ 3. Spectral representation (isometry): General case $\delta_0>0$

In this section we return to the general case $\delta_0>0$. We shall construct the operator $\mathcal{F}_{\pm}$ by use of a perturbation method (cf., Ikebe [4]), and prove a spectral representation theorem for the operator L restricted in $\mathcal{E}((\Lambda_\delta,\infty))\mathcal{H}$. Note that in general $\Lambda_\delta\geqq\Lambda_{1/2}$.

Let $L_1=-\Delta+V_1(x)$ be the selfadjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(L_1)=\mathcal{D}(L)$, and let $\{\mathcal{E}_1(\lambda);\lambda\in\boldsymbol{R}\}$ be its spectral measure. Since $V_s(x)$ is assumed to be bounded in Ω, L is then a perturbed operator of L_1:

$$L=L_1+V_s. \tag{3.1}$$

Let $\mathcal{R}_{1,\lambda\pm i0}$ and $\mathcal{F}_{1,\pm}(\lambda)$ denote, respectively, the operator $\mathcal{R}_{\lambda\pm i0}$ and $\mathcal{F}_{\pm}(\lambda)$ with $V(x)=V_1(x)$ (since $V_s(x)=0$ in this case, we can admit $\delta_0>1/2$, and $\mathcal{F}_{1,\pm}(\lambda)$, $\lambda>\Lambda_{1/2}$, is well defined by (2.7) with $\mathcal{R}_{\lambda\pm i0}$ replaced by $\mathcal{R}_{1,\lambda\pm i0}$).

Lemma 3.1. *For any $N>0$ there exists a pair $\tilde{\alpha}=\tilde{\alpha}(N)$, $\tilde{\beta}=\tilde{\beta}(N)$ satisfying the conditions*

$$0<\tilde{\alpha}+\tilde{\beta}<2\delta,\ \tilde{\alpha}<4\delta_2-2,\quad 0<\tilde{\alpha}\leqq\tilde{\beta}\leqq 1 \tag{3.2}$$
$$\text{and}\quad \Lambda_\delta\leqq\Lambda_{\tilde{\beta}/2}<\Lambda_\delta+N^{-1}.$$

Proof. The assertion follows from the continuity and non-increasingness of Λ_σ in $\sigma>0$. q. e. d.

Now let $\lambda>\Lambda_\delta$, and $\tilde{\alpha}=\tilde{\alpha}(N)$, $\tilde{\beta}=\tilde{\beta}(N)$ be the pair given in the above lemma, where N should be chosen large so that $\lambda>\Lambda_\delta+N^{-1}$. Then for $f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$, $u_1=\mathcal{R}_{1,\lambda\pm i0}f$ is the unique outgoing [incoming] solution of (1.8) with $V=V_1$:

$$\begin{cases} (-\Delta+V_1(x)-\lambda)u_1=f & \text{in } \Omega \\ Bu_1=0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

$$u_1\in L^2_{(-1-\tilde{\alpha})/2}(\Omega)\quad\text{and}\quad \partial_r u_1+k(x,\lambda\pm i0)u_1\in L^2_{(-1+\tilde{\beta})/2}(B(R_1)). \tag{3.4$_\pm$}$$

Lemma 3.2. *If $\lambda>\Lambda_\delta$, the operator $\mathcal{F}_{1,\pm}(\lambda): L^2_1(\Omega)\to \boldsymbol{h}$ can be extended to a bounded linear operator from $L^2_{(1+\tilde{\beta})/2}(\Omega)$ to $\boldsymbol{h}$. Denoting the extended operator by $\mathcal{F}_{1,\pm}(\lambda)$ again, we have for $f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$ and $\phi\in\boldsymbol{h}$,*

$$\|\mathcal{F}_{1,\pm}(\lambda)f\|^2_{\boldsymbol{h}}=\frac{1}{2\pi i}(\mathcal{R}_{1,\lambda+i0}f-\mathcal{R}_{1,\lambda-i0}f,\ f), \tag{3.5}$$

$$(\mathcal{F}_{1,\pm}(\lambda)f,\ \phi)_{\boldsymbol{h}}=\lim_{p\to\infty}\left(\frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}u_1(r_p\cdot),\ \phi\right)_{\boldsymbol{h}}, \tag{3.6}$$

where $u_1=\mathcal{R}_{1,\lambda\pm i0}f$ and $\{r_p\}$ is any sequence diverging to ∞ such that

$$\lim_{p\to\infty}\int_{S(r_p)}\{r^{-\tilde{\alpha}}|u_1|^2+r^{\tilde{\beta}}|\nabla u_1+\tilde{x}k(x,\lambda\pm i0)u_1|^2\}\,dS=0. \tag{3.7}$$

Proof. The extensibility of $\mathcal{F}_{1,\pm}(\lambda)$ and the relation (3.5) easily follow from (2.8) and Proposition 1.1 (a). (3.6) follows from (1.37) of Proposition 1.4 since we have for $f\in L^2_1(\Omega)$ and $\phi\in\mathcal{D}(A)$ (which is dense in $\boldsymbol{h}$)

$$(\mathcal{F}_{1,\pm}(\lambda)f,\ \phi)_{\boldsymbol{h}}=\pm\frac{1}{2i}\int_\Omega(u_1\overline{g'_\phi}-f\overline{v_\phi})\,dx, \tag{3.8}$$

g'_ϕ being the function defined by (1.31) with $V=V_1$, and the both sides are continuous in $f\in L^2_{(1+\tilde{\beta})/2}(\Omega)$. q. e. d.

Lemma 3.3. *$\mathcal{F}_{1,\pm}(\lambda)f\in\boldsymbol{h}$ depends continuously on $(\lambda, f)\in(\Lambda_{\tilde{\beta}/2},\infty)\times L^2_{(1+\tilde{\beta})/2}(\Omega)$.*

Proof. The assertion is obvious since the right hand sides of (3.5) and (3.8) are both continuous in $(\lambda, f)\in(\Lambda_{\tilde{\beta}/2},\infty)\times L^2_{(1+\tilde{\beta})/2}(\Omega)$ (cf., the proof of Lemma 2.4 and (1.31)). q. e. d.

Lemma 3.4. (a) *$V_s \in B(L^2_{(-1-\tilde{\alpha})/2}(\Omega), L^2_{(1+\tilde{\beta})/2}(\Omega))$ and $V_s^* f = V_s f$ for $f \in L^2_{(-1-\tilde{\alpha})/2}(\Omega)$, where $V_s^* \in B(L^2_{(-1-\tilde{\beta})/2}(\Omega), L^2_{(1+\tilde{\alpha})/2}(\Omega))$.*

(b) *$V_s \mathcal{R}_{\lambda \pm i0} \in B(L^2_{(1+\tilde{\beta})/2}(\Omega), L^2_{(1+\tilde{\beta})/2}(\Omega))$ for any $\lambda > \Lambda_\delta + N^{-1}$.*

Proof. Obvious from (vii) of Assumption 2 and the condition (3.2) on $\tilde{\alpha}$, $\tilde{\beta}$.
q. e. d.

Now, for $\lambda > \Lambda_\delta$ and $f \in L^2_{(1+2)\delta/2}(\Omega)$ $(\subset L^2_{(1+\tilde{\beta})/2}(\Omega))$ let $u = \mathcal{R}_{\lambda \pm i0} f$. Then we have from (1.8)

$$(3.9) \qquad \begin{cases} (-\Delta + V_1(x) - \lambda) u = (1 - V_s \mathcal{R}_{\lambda \pm i0}) f & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

$(1 - V_s \mathcal{R}_{\lambda \pm i0}) f \in L^2_{(1+\tilde{\beta})/2}(\Omega)$ by Lemma 3.4, and u satisfies the outgoing [incoming] radiation condition $(3.4)_\pm$. Thus,

$$(3.10) \qquad u = \mathcal{R}_{\lambda \pm i0} f = \mathcal{R}_{1, \lambda \pm i0} (1 - V_s \mathcal{R}_{\lambda \pm i0}) f.$$

Definition 3.1. For $\lambda > \Lambda_\delta$ let $\mathcal{F}_\pm(\lambda) \in B(L^2_{(1+2\delta)/2}(\Omega), \boldsymbol{h})$ be defined by

$$(3.11) \qquad \mathcal{F}_\pm(\lambda) = \mathcal{F}_{1, \pm}(\lambda)(1 - V_s \mathcal{R}_{\lambda \pm i0})$$

Remark 3.1. If $\delta_0 > 1/2$, $\mathcal{F}_\pm(\lambda)$ defined above coincides with the operator given by Definition 2.1. In fact, for $\lambda > \Lambda_{1/2}$ and $f \in L^2_1(\Omega)$ we have from (3.10)

$$(3.12) \qquad \frac{1}{\sqrt{\pi}} \operatorname*{strong\,lim}_{p \to \infty} e^{\rho(r_p \cdot)} (\mathcal{R}_{\lambda \pm i0} f)(r_p \cdot) = \mathcal{F}_{1, \pm}(\lambda)((1 - V_s \mathcal{R}_{\lambda \pm i0}) f)$$

since $(1 - V_s \mathcal{R}_{\lambda \pm i0}) f \in L^2_1(\Omega)$ in this case.

Lemma 3.5. *For any Borel set $e \Subset (\Lambda_\delta, \infty)$ and $f, g \in L^2_{(1+2\delta)/2}(\Omega)$ we have*

$$(3.13) \qquad (\mathcal{E}(e) f, g) = \int_e (\mathcal{F}_\pm(\lambda) f, \mathcal{F}_\pm(\lambda) g)_{\boldsymbol{h}}\, d\lambda.$$

Proof. By Proposition 1.1 (c), Lemma 3.4 (a) and (3.10),

$$\begin{aligned}
(\mathcal{R}_{\lambda + i0} f - \mathcal{R}_{\lambda - i0} f, g) &= \pm \{(\mathcal{R}_{\lambda \pm i0} f, g) - (f, \mathcal{R}_{\lambda \pm i0} g)\} \\
&= \pm [(\mathcal{R}_{1, \lambda \pm i0}\{1 - V_s \mathcal{R}_{\lambda \pm i0}\} f, g) - (f, \mathcal{R}_{1, \lambda \pm i0}\{1 - V_s \mathcal{R}_{\lambda \pm i0}\} g)] \\
&= \pm [(\mathcal{R}_{1, \lambda \pm i0}\{1 - V_s \mathcal{R}_{\lambda \pm i0}\} f, \{1 - V_s \mathcal{R}_{\lambda \pm i0}\} g) \\
&\quad - (\{1 - V_s \mathcal{R}_{\lambda \pm i0}\} f, \mathcal{R}_{1, \lambda \pm i0}\{1 - V_s \mathcal{R}_{\lambda \pm i0}\} g)] \\
&= (\{\mathcal{R}_{1, \lambda + i0} - \mathcal{R}_{1, \lambda - i0}\}\{1 - V_s \mathcal{R}_{\lambda \pm i0}\} f, \{1 - V_s \mathcal{R}_{\lambda \pm i0}\} g).
\end{aligned}$$

Here $\{1 - V_s \mathcal{R}_{\lambda \pm i0}\} f$, $\{1 - V_s \mathcal{R}_{\lambda \pm i0}\} g \in L^2_{(1+\tilde{\beta})/2}(\Omega)$ by Lemma 3.4. Thus, it follows from Lemma 3.2 that

$$\frac{1}{2\pi i}(\{\mathcal{R}_{\lambda+i0}-\mathcal{R}_{\lambda-i0}\}f,\ g)$$
$$=(\mathcal{F}_{1,\pm}(\lambda)\{1-V_s\mathcal{R}_{\lambda\pm i0}\}f,\ \mathcal{F}_{1,\pm}(\lambda)\{1-V_s\mathcal{R}_{\lambda\pm i0}\}g)_{\boldsymbol{h}}$$
$$=(\mathcal{F}_\pm(\lambda)f,\ \mathcal{F}_\pm(\lambda)g)_{\boldsymbol{h}}.$$

Integrate both sides over e with respect to λ (the measurability of $\mathcal{F}_\pm(\lambda)$ in $\lambda>\Lambda_\delta$ is guaranteed by Lemma 3.3 and Proposition 1.1 (b)). Then by means of Proposition 1.1 (d) we obtain (3.13). q. e. d.

With the aid of Lemma 3.5 we can now prove the following spectral representation theorem for L.

Theorem 3.1. (a) *(Diagonal representation of L) Let* $\mathcal{F}_\pm : L^2_{(1+2\delta)/2}(\Omega)\to\hat{\mathcal{H}}_{\Lambda_\delta}$ *be defined by*

$$(3.14)\qquad (\mathcal{F}_\pm f)(\lambda)=\mathcal{F}_\pm(\lambda)f\quad (\lambda>\Lambda_\delta \text{ and } f\in L^2_{(1+2\delta)/2}(\Omega)).$$

Then $\mathcal{F}_\pm$ *can be extended by continuity to a partial isometric operator from* $\mathcal{H}$ *into* $\hat{\mathcal{H}}_{\Lambda_\delta}$ *with the initial set* $\mathcal{E}((\Lambda_\delta,\infty))\mathcal{H}$. *Further, for any bounded Borel function* $\alpha(\lambda)$ *on* $\boldsymbol{R}$ *and* $f\in\mathcal{H}$ *we have*

$$(3.15)\qquad (\mathcal{F}_\pm\alpha(L)f)(\lambda)=\alpha(\lambda)(\mathcal{F}_\pm f)(\lambda)\quad \text{for a.e. } \lambda>\Lambda_\delta.$$

(b) *(Inversion formula) Let* $\mathcal{F}_\pm^* : \hat{\mathcal{H}}_{\Lambda_\delta}\to\mathcal{H}$ *be the adjoint operator of* $\mathcal{F}_\pm$. *Then we have for* $\hat{f}\in\hat{\mathcal{H}}_{\Lambda_\delta}$

$$(3.16)\qquad \mathcal{F}_\pm^*\hat{f}=\text{strong}\lim_{N\to\infty}\int_{\Lambda_\delta+1/N}^{N}\mathcal{F}_\pm(\lambda)^*\hat{f}(\lambda)\,d\lambda\quad \text{in } \mathcal{H},$$

where $\mathcal{F}_\pm(\lambda)^* : \boldsymbol{h}\to L^2_{(-1-2\delta)/2}(\Omega)$ *is the adjoint of* $\mathcal{F}_\pm(\lambda)$. *In particular, the following inversion formula holds for* $f\in\mathcal{H}$.

$$(3.17)\qquad \mathcal{E}((\Lambda_\delta,\infty))f=\text{strong}\lim_{N\to\infty}\int_{\Lambda_\delta+1/N}^{N}\mathcal{F}_\pm(\lambda)^*(\mathcal{F}_\pm f)(\lambda)\,d\lambda.$$

(c) *(Eigenoperator)* $\mathcal{F}_\pm(\lambda)^*$ *is an eigenoperator of L with eigenvalue* λ ($>\Lambda_\delta$) *in the sense that*

$$(3.18)\qquad (\mathcal{F}_\pm(\lambda)^*\phi,\ (L-\lambda)u)=0$$

for any $\phi\in\boldsymbol{h}$ *and* $u\in C_0^\infty(\bar{\Omega})$ *satisfying the boundary condition* $Bu|_{\partial\Omega}=0$.

Proof. As we see in the proof of Theorem 2.1, the assertions (a) and (b) are easily proved by use of the relation (3.13). To show (c), let $f=(L-\lambda)u$. Then f is of compact support, and hence $\in L^2_{(1+2\delta)/2}(\Omega)$. Thus,

$$(\mathcal{F}_\pm(\lambda)^*\phi,\ (L-\lambda)u)=(\mathcal{F}_{1,\pm}(\lambda)^*\phi,\ \{1-V_s\mathcal{R}_{\lambda\pm i0}\}f)$$
$$=(\mathcal{F}_{1,\pm}(\lambda)^*\phi,\ (L_1-\lambda)u)=0$$

by (3.11), (3.9) and Theorem 2.1 (c). q. e. d.

§4. Unitarity of $\mathcal{F}_{\pm}$

We want to show that the ranges $\mathcal{R}(\mathcal{F}_+)$ of $\mathcal{F}_+$ and $\mathcal{R}(\mathcal{F}_-)$ of $\mathcal{F}_-$ coincide and are equal to $\hat{\mathcal{H}}_{\Lambda_\delta}$. In this section we shall give a partial answer to this problem. Our results will be satisfactory to some restricted cases (cf., the corollaries of the next section).

We can prove the following

Theorem 4.1. (a) *We have*

$$\mathcal{R}(\mathcal{F}_{\pm})\supset\{\hat{f}\in\hat{\mathcal{H}}_{\Lambda_\delta}\,;\hat{f}(\lambda)=0 \quad for\ a.e.\quad \lambda\in(\Lambda_\delta,\ \Lambda_{\tilde{\delta}})\} \tag{4.1}$$

or equivalently

$$\begin{aligned}\mathcal{N}(\mathcal{F}_{\pm}^{*})&\equiv\{\hat{f}\in\hat{\mathcal{H}}_{\Lambda_\delta}\,;\ \mathcal{F}_{\pm}^{*}f=0\}\\ &\subset\{\hat{f}\in\hat{\mathcal{H}}_{\Lambda_\delta}\,;\hat{f}(\lambda)=0 \quad for\ a.e.\quad \lambda\in(\Lambda_{\tilde{\delta}},\ \infty)\},\end{aligned} \tag{4.2}$$

where $\tilde{\delta}=\min\{\delta,\ 2\delta_2-1\}$.

(b) *Both* $\mathcal{F}_+$ *and* $\mathcal{F}_-$ *map* $\mathcal{E}((\Lambda_{\tilde{\delta}},\ \infty))\mathcal{H}$ *onto* $\hat{\mathcal{H}}_{\Lambda_{\tilde{\delta}}}=L^2((\Lambda_{\tilde{\delta}},\ \infty)\,;\boldsymbol{h})$, *that is,* $\mathcal{F}_{\pm}$ *restricted in* $\mathcal{E}((\Lambda_{\tilde{\delta}},\ \infty))\mathcal{H}$ *are unitary operators.*

Remark 4.1. If $\delta_2\geqq(\delta+1)/2$, we have $\tilde{\delta}=\delta$.

Lemma 4.1. *For* $\phi\in\mathcal{D}(A)\subset\boldsymbol{h}$ *and* $\lambda>\Lambda_{1/2}$ *let* v_ϕ *be the function given by* (1.27), *and let* $g_\phi'=(-\Delta+V_1(x)-\lambda)v_\phi$. *Then* $g_\phi'\in L^2_{(1+\tilde{\beta})/2}(\Omega)$ *for any* $0<\tilde{\beta}<2\tilde{\delta}$. *Moreover, if* $\lambda>\Lambda_{\tilde{\beta}/2}$, v_ϕ *coincides with the outgoing* [*incoming*] *solution of* (3.3) *with* $f=g_\phi'$, *that is,* $v_\phi=\mathcal{R}_{1,\lambda\pm i0}g_\phi'$.

Proof. Obvious from Lemma 1.2. q.e.d.

Lemma 4.2. *For any* $N>0$ *there exists a pair* $\tilde{\alpha}=\tilde{\alpha}(N)$, $\tilde{\beta}=\tilde{\beta}(N)$ *satisfying the conditions*

$$\begin{aligned}&0<\tilde{\alpha}+\tilde{\beta}<2\tilde{\delta},\quad \tilde{\alpha}<4\delta_2-2,\quad 0<\tilde{\alpha}\leqq\tilde{\beta}\leqq1\\ &and\quad (\Lambda_\delta\leqq)\,\Lambda_{\tilde{\delta}}\leqq\Lambda_{\tilde{\beta}/2}<\Lambda_{\tilde{\delta}}+N^{-1}.\end{aligned} \tag{4.3}$$

Proof. The assertion is proved by the same argument as in the proof of Lemma 3.1. q.e.d.

Proof of Theorem 4.1. The assertion (b) is easily proved by (a) of this theorem and (a) of Theorem 3.1. Thus, we have only to prove (a).

Let $\hat{f}\in\mathcal{N}(\mathcal{F}_{\pm}^{*})$, namely, let f be orthogonal to $\mathcal{R}(\mathcal{F}_{\pm})$. By Theorem 3.1 (a) we have for any Borel set $e\Subset(\Lambda_\delta,\ \infty)$ and $g\in\mathcal{H}$,

$$(\mathcal{F}_{\pm}^{*}\chi_e\hat{f},\ g)=(\hat{f},\ \chi_e\mathcal{F}_{\pm}g)_{\hat{\mathcal{H}}_{\Lambda_\delta}}=(\hat{f},\ \mathcal{F}_{\pm}\mathcal{E}(e)g)_{\hat{\mathcal{H}}_{\Lambda_\delta}}=0,$$

which implies $\mathcal{F}_{\pm}^{*}\chi_e\hat{f}=0$, i.e., $\chi_e\hat{f}\in\mathcal{N}(\mathcal{F}_{\pm}^{*})$. In view of Theorem 3.1 (b), we then have

$$\mathcal{F}_{\pm}^{*}\chi_e\hat{f}=\int_e \mathcal{F}_{\pm}(\lambda)^*\hat{f}(\lambda)\,d\lambda=0 \qquad \text{for any} \quad e\Subset(\Lambda_{\tilde{\delta}},\infty).$$

Thus, it follows that $\mathcal{F}_{\pm}(\lambda)^*\hat{f}(\lambda)=0$ for a.e. $\lambda>\Lambda_{\tilde{\delta}}$.

By Definition 3.1 and the "resolvent" equation

$$\mathcal{R}_{\lambda\pm i0}-\mathcal{R}_{1,\lambda\pm i0}=-\mathcal{R}_{1,\lambda\pm i0}V_s\mathcal{R}_{\lambda\pm i0}=-\mathcal{R}_{\lambda\pm i0}V_s\mathcal{R}_{1,\lambda\pm i0},$$

we have $\mathcal{F}_{1,\pm}(\lambda)=\mathcal{F}_{\pm}(\lambda)(1+V_s\mathcal{R}_{1,\lambda\pm i0})$, and hence,

$$\mathcal{F}_{1,\pm}(\lambda)^*=(1+\mathcal{R}_{1,\lambda\pm i0}^{*}V_s^{*})\mathcal{F}_{\pm}(\lambda)^*.$$

Therefore, to complete the proof, we have only to show that $\mathcal{F}_{1,\pm}(\lambda)^*\hat{f}(\lambda)=0$ for a.e. $\lambda>\Lambda_{\tilde{\delta}}$ implies $\hat{f}(\lambda)=0$ for a.e. $\lambda>\Lambda_{\tilde{\delta}}$.

For $\lambda>\Lambda_{\tilde{\delta}}+N^{-1}$ $(N>0)$ let $\tilde{\alpha}$, $\tilde{\beta}$ be as in Lemma 3.1, and let $g\in L^2_{(1+\tilde{\beta})/2}(\Omega)$ and $v=\mathcal{R}_{1,\lambda\pm i0}\,g$. Then by Lemma 3.2 we have

$$(4.4) \qquad 0=(\mathcal{F}_{1,\pm}(\lambda)^*\hat{f}(\lambda),\ g)=(\hat{f}(\lambda),\ \mathcal{F}_{1,\pm}(\lambda)g)_h$$

$$=\lim_{p\to\infty}(\hat{f}(\lambda),\ \frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}v(r_p\cdot))_h \qquad \text{for a.e.} \quad \lambda>\Lambda_{\tilde{\delta}}+N^{-1}.$$

Here, we restrict ourselves to the case $\lambda>\Lambda_{\tilde{\delta}}+N^{-1}$ and choose $\tilde{\alpha}$, $\tilde{\beta}$ as in Lemma 4.2. Then by Lemma 4.1, $g'_\phi\in L^2_{(1+\tilde{\beta})/2}(\Omega)$ and $v_\phi=\mathcal{R}_{1,\lambda\pm i0}\,g'_\phi$. Putting $g=g'_\phi$ in (4.4), we have

$$0=\lim_{p\to\infty}(\hat{f}(\lambda),\ \frac{1}{\sqrt{\pi}}e^{\rho(r_p\cdot)}v_\phi(r_p\cdot))_h$$

$$=\lim_{p\to\infty}(\hat{f}(\lambda),\ \frac{1}{\pi}e^{\rho(r_p\cdot)}e^{-\rho(r_p\cdot)}\phi)_h=\frac{1}{\pi}(\hat{f}(\lambda),\ \phi)_h$$

for a.e. $\lambda>\Lambda_{\tilde{\delta}}+N^{-1}$. $\phi\in\mathcal{D}(A)$ being arbitrary, this implies $\hat{f}(\lambda)=0$ for a.e. $\lambda>\Lambda_{\tilde{\delta}}+N^{-1}$. Letting $N\to\infty$, we have the assertion (a) of the theorem. q.e.d.

§ 5. Some corollaries of the above results

Corollary 5.1. *If $\delta_0\geqq 1/2$ and $\delta_2\geqq 3/4$ in Assumption 2, $\mathcal{F}_{\pm}$ is a unitary operator from $\mathcal{E}((\Lambda_{1/2},\infty))\mathcal{H}$ onto $\hat{\mathcal{H}}_{\Lambda_{1/2}}$.*

Proof. Since $\tilde{\delta}=\min\{\delta,2\delta_2-1\}\geqq 1/2$ by condition, the assertion is obvious from Theorems 3.1 and 4.1. q.e.d.

Corollary 5.2. *If $V_1(x)$ satisfies, in place of* (ii) *of Assumption 2, the following stronger condition:*

(ii)′ $$\partial_r V_1(x)=o(r^{-1}) \qquad as \quad r\to\infty,$$

then $\mathcal{F}_{\pm}$ is a unitary operator from $\mathcal{E}((\Lambda,\infty))\mathcal{H}$ onto $\hat{\mathcal{H}}_{\Lambda}$, where $\Lambda=\limsup_{r\to\infty}V_1(x)+a/4$.

Proof. In this case, $E(\gamma)=\limsup_{r\to\infty} V_1(x)$ for any $\gamma>0$ in (1.1). Thus, we see $\Lambda_\sigma=\limsup_{r\to\infty} V_1(x)+a/4$ for any $\sigma>0$, and the assertion follows from Theorems 3.1 and 4.1. q.e.d.

Remark 5.1. As an example which satisfies the condition stated in the above corollary, we have

$$V(x)=\sin(\log(\log r))+O(r^{-1-\delta_0}) \qquad \text{as} \quad r\to\infty.$$

The potential $\sin(\log(\log r))$ satisfies the conditions (i), (ii)′ and (iii) with $a=0$. Thus, in this case we have $\Lambda=\limsup_{r\to\infty}[\sin(\log(\log r))]=1$.

Corollary 5.3. *If* $V_1(x)$ *satisfies, in place of* (i), (ii) *and* (iii) *of Assumption* 2, *the following stronger conditions:*

(i)′ $$V_1(x)=o(1),$$

(ii)′ $$\partial_r V_1(x)=o(r^{-1}),$$

(iii)′ $$\partial_r^2 V_1(x)=O(r^{-1-\delta_1}) \quad (\delta_1>1/2)$$

as $r\to\infty$, *then* $\mathcal{F}_\pm$ *becomes a unitary operator from the absolutely continuous subspace* $\mathcal{H}_{ac}=\mathcal{E}((0,\infty))\mathcal{H}$ *for* L *onto* $\hat{\mathcal{H}}_0=L^2((0,\infty);\boldsymbol{h})$.

Proof. Note that, in this case, the essential spectrum of L consists of the non-negative real axis $[0,\infty)$. On the other hand, by (1.2), $\Lambda_\sigma=0$ for any $\sigma>0$. These show that $\mathcal{E}((\Lambda_{\tilde\delta},\infty))\mathcal{H}$ coincides with the absolutely continuous subspace $\mathcal{H}_{ac}=\mathcal{E}((0,\infty))\mathcal{H}$ for L. q.e.d.

Remark 5.2. The above result is a partial generalization of results of Ikebe [3], [4] and Saitō [15], [16]. Corollary 5.3 includes the following type of potentials:

$$V_1(x)=\frac{\text{const}}{\log r} \quad \text{or} \quad V_1(x)=\frac{\text{const}}{\log(\log r)}.$$

However, these potentials are not covered by the results of Ikebe and Saitō. Roughly speaking, their assumptions on $V_1(x)$ are as follows:

$$\nabla^\nu V_1(x)=O(r^{-\nu-\delta_1}) \quad (\delta_1>0), \quad \nu=0, 1, \cdots, m,$$

where $m=2$ if $\delta_1>1/2$ and $m>2/\delta_1$ if $1/2\geqq\delta_1>0$.

§6. Unitary equivalence between $\mathcal{E}((\Lambda_{\tilde\delta},\infty))L$ and $L_0=-\Delta+\Lambda_{\tilde\delta}$

Let $L_0=-\Delta+\Lambda_{\tilde\delta}$ be the selfadjoint operator in the Hilbert space $\mathcal{H}_0=L^2(\boldsymbol{R}^n)$ with domain $\mathcal{D}(L_0)=H^2(\boldsymbol{R}^n)$, and let $\{\mathcal{E}_0(\lambda);\lambda\in\boldsymbol{R}\}$ be its spectral measure. In this section we shall show the existence of unitary operators $U_\pm$ from $\mathcal{H}_0$ to $\mathcal{E}((\Lambda_{\tilde\delta},\infty))\mathcal{H}$ which intertwine the operators L_0 and L.

We denote by $\mathcal{F}_0(\lambda)$ the operator $\mathcal{F}_+(\lambda)$ corresponding to $L=L_0$. Namely, for $\lambda>\Lambda_{\bar{\delta}}$ and $f\in L_1^2(\boldsymbol{R}^n)$ let

$$\mathcal{F}_0(\lambda)f=\text{strong}\lim_{p\to\infty}\frac{1}{\sqrt{\pi}}e^{\rho_0(r_p,\lambda+i0)}(\mathcal{R}_{0,\lambda+i0}f)(r_p\cdot)\in\boldsymbol{h}; \tag{6.1}$$

$$\rho_0(r,\lambda+i0)=-i(\sqrt{\lambda-\Lambda_{\bar{\delta}}})r+\frac{n-1}{2}\log r+\frac{1}{4}\log(\lambda-\Lambda_{\bar{\delta}}), \tag{6.2}$$

where $u_0=\mathcal{R}_{0,\lambda+i0}f$ is the outgoing solution of $(-\Delta+\Lambda_{\bar{\delta}}-\lambda)u_0=f$ in $\boldsymbol{R}^n$, and $\{r_p\}$ is any sequence diverging to ∞ as $p\to\infty$ such that

$$\lim_{p\to\infty}\int_{S(r_p)}\left\{r^{-\alpha}|u_0|^2+r\left|\nabla u_0+\tilde{x}\left(-i\sqrt{\lambda-\Lambda_{\bar{\delta}}}+\frac{n-1}{2r}\right)u_0\right|^2\right\}dS=0 \tag{6.3}$$

for some $0<\alpha<1$. Let $\mathcal{F}_0 : L_1^2(\boldsymbol{R}^n)\to\hat{\mathcal{H}}_{\Lambda_{\bar{\delta}}}=L^2((\Lambda_{\bar{\delta}},\infty);\boldsymbol{h})$ be defined by

$$(\mathcal{F}_0f)(\lambda)=\mathcal{F}_0(\lambda)f. \tag{6.4}$$

Then, as is proved, $\mathcal{F}_0$ can be uniquely extended to a unitary operator from $\mathcal{E}_0((\Lambda_{\bar{\delta}},\infty))\mathcal{H}_0=\mathcal{H}_0$ to $\hat{\mathcal{H}}_{\Lambda_{\bar{\delta}}}$, which diagonalizes L_0. We denote the extended operator by $\mathcal{F}_0$ again.

Definition 6.1. Let $U_\pm\in B(\mathcal{H}_0,\mathcal{E}((\Lambda_{\bar{\delta}},\infty))\mathcal{H})$ be defined by

$$U_\pm=\mathcal{F}_\pm^*\mathcal{F}_0. \tag{6.5}$$

Theorem 6.1. *$U_\pm : \mathcal{H}_0\to\mathcal{E}((\Lambda_{\bar{\delta}},\infty))\mathcal{H}$ are unitary operators which intertwine L_0 and $\mathcal{E}((\Lambda_{\bar{\delta}},\infty))L$:*

$$\mathcal{E}((\Lambda_{\bar{\delta}},\infty))L=U_\pm L_0U_\pm^*; \tag{6.6}$$

$$L_0=U_\pm^*\mathcal{E}((\Lambda_{\bar{\delta}},\infty))LU_\pm. \tag{6.7}$$

Proof. It follows from Theorems 3.1 and 4.1 that for any Borel set $e\subset(\Lambda_{\bar{\delta}},\infty)$

$$\mathcal{F}_0^*\chi_e\mathcal{F}_0=\mathcal{E}_0(e) \text{ in } \mathcal{H}_0,\quad \mathcal{F}_0\mathcal{E}_0(e)\mathcal{F}_0^*=\chi_e \text{ in } \hat{\mathcal{H}}_{\Lambda_{\bar{\delta}}},$$

$$\mathcal{F}^*\chi_e\mathcal{F}=\mathcal{E}(e) \text{ in } \mathcal{H}, \text{ and } \mathcal{F}\mathcal{E}(e)\mathcal{F}^*=\chi_e \text{ in } \hat{\mathcal{H}}_{\Lambda_{\bar{\delta}}}.$$

The above assertions are obvious from these relations if we note $\mathcal{E}_0((\Lambda_{\bar{\delta}},\infty))=I$ (the identity in $\mathcal{H}_0$). q. e. d.

Remark 6.1. $U_\pm$ may be said the stationary wave operators (cf., Pinchuk [12] and Isozaki [5]).

Remark 6.2. The operator $\mathcal{F}_0$ is essentially the Fourier transformation, that is, for any $f\in C_0^\infty(\boldsymbol{R}^n)$ we have

(6.8) $$[\mathcal{F}_0(\lambda)f](\tilde{x}) = C(n)\frac{1}{\sqrt{2}}(\lambda-\Lambda_{\tilde{\delta}})^{(n-2)/4}(2\pi)^{-n/2}\int_{R^n} e^{-i\sqrt{\lambda-\Lambda_{\tilde{\delta}}}\,\tilde{x}\cdot y} f(y)\,dy$$

for $\lambda>\Lambda_{\tilde{\delta}}$, where $C(n)=e^{-i\pi(n-3)/4}$.

(6.8) can be verified in the following way (for details, cf., e.g., Matsumura [8] and Mochizuki [9]). By the Fourier inversion formula,

$$[\mathcal{R}_{0,\lambda+i0}f](x)=(2\pi)^{-n/2}\lim_{\varepsilon\downarrow 0}\int_{R^n}\frac{e^{ix\cdot\xi}\hat{f}(\xi)}{|\xi|^2+\Lambda_{\tilde{\delta}}-(\lambda+i\varepsilon)}\,d\xi$$

in $L^2_{(-1-\alpha)/2}(\boldsymbol{R}^n)$, where

$$\hat{f}(\xi)=(2\pi)^{-n/2}\int_{R^n} e^{-i\xi\cdot y}f(y)\,dy.$$

Applying the asymptotic form

$$\int_{S^{n-1}} e^{i|\xi|x\cdot\omega}\hat{f}(|\xi|\omega)\,dS_\omega$$

$$=\hat{f}(|\xi|\tilde{x})\Big(\frac{2\pi}{r|\xi|}\Big)^{(n-1)/2} e^{i(r|\xi|-\pi(n-1)/4)}$$

$$+\hat{f}(-|\xi|\tilde{x})\Big(\frac{2\pi}{r|\xi|}\Big)^{(n-1)/2} e^{-i(r|\xi|-\pi(n-1)/4)}+q(|\xi|\tilde{x});$$

$$\nabla^\nu q(x)=O(r^{-(n+1)/2})\quad(\nu=0,1,\cdots)\qquad\text{as}\quad r\to\infty,$$

we then have

$$[\mathcal{R}_{0,\lambda+i0}f](r\tilde{x})=(2\pi)^{-1/2}e^{-i\pi(n-1)/4}$$

$$\times r^{-(n-1)/2}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\frac{|\xi|^{(n-1)/2}\hat{f}(|\xi|\tilde{x})\,e^{ir|\xi|}}{|\xi|^2+\Lambda_{\tilde{\delta}}-(\lambda+i\varepsilon)}\,d|\xi|+O(r^{-n/2}).$$

Hence, by (6.1) and (6.2),

$$[\mathcal{F}_0(\lambda)f](\tilde{x})=C(n)\sqrt{2}(\lambda-\Lambda_{\tilde{\delta}})^{1/4}(2\pi i)^{-1}$$

$$\times\lim_{p\to\infty}\Big[e^{-i\sqrt{\lambda-\Lambda_{\tilde{\delta}}}\,r_p}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\frac{|\xi|^{(n-1)/2}\hat{f}(|\xi|\tilde{x})\,e^{ir_p|\xi|}}{|\xi|^2-(\lambda-\Lambda_{\tilde{\delta}}+i\varepsilon)}\,d|\xi|\Big]$$

$$=C(n)\sqrt{2}(\lambda-\Lambda_{\tilde{\delta}})^{1/4}(\lambda-\Lambda_{\tilde{\delta}})^{(n-1)/4}\frac{1}{2\sqrt{\lambda-\Lambda_{\tilde{\delta}}}}\hat{f}(\sqrt{\lambda-\Lambda_{\tilde{\delta}}}\,\tilde{x}).$$

This is what is to be shown.

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