J. Math. Kyoto Univ. (JMKYAZ)
19-2 (1979) 371-374

On the Gorensteinness of determinantal loci

By

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(Communicated by Prof. M. Nagata, May 9, 1978)

1. Introduction

Let k be a field and let $0<t<n$ be integers. Let $X=[X_{ij}]$ be an n by n symmetric matrix of indeterminates over the field k. We put $S=k[X]$, the polynomial ring generated by $\{X_{ij}\}_{1\leq i\leq j\leq n}$ over k. Let $\mathrm{I}_t(X)$ denote the ideal of S generated by the $t+1$ by $t+1$ minors of the matrix X. Finally we put $R=S/\mathrm{I}_t(X)$. In this situation Kutz [**6**] gave the following nice result: *R is a Cohen-Macaulay domain with* $\dim R=nt-t(t-1)/2$ (c.f. Theorem 1). The aim of our paper is to study farther when R is a Gorenstein ring.

Our result is

Theorem (1.1). *R is a Gorenstein ring if and only if $n\not\equiv t$* (2).

Recently the author [**3**] has showed that R is a normal ring with $\mathrm{Cl}(R)=\boldsymbol{Z}/2\boldsymbol{Z}$ (Here $\mathrm{Cl}(R)$ denotes the divisor class group of R.), and this fact will play a key role in the present paper.

In case the field k has characteristic 0, the ring R appears as the ring of invariants. More explicitly, let U be a t by n matrix of indeterminates over k. We put $A=k[U]$ and $G=O(t, k)$, the orthogonal group. Let the group G act on the ring A as k-automorphisms by taking U onto MU for every $M\in G$. Then, as is known classically, the ring A^G of invariants is generated by the entries of the n by n symmetric matrix $Y={}^tU\cdot U$ and the ideal of relations on Y is generated by the $t+1$ by $t+1$ minors and the symmetry conditions (c.f. [**9**], Ch. II, 15 and 17). Thus from (1.1) we deduce at once the following invariant-theoretic result.

Corollary (1.2). *For A and G as above, A^G is a Gorenstein ring if and only if $n\not\equiv t$* (2).

2. Proof of Theorem (1.1)

We put $x_{ij}=X_{ij} \bmod \mathrm{I}_t(X)$ and denote by $\mathfrak{p}$ the ideal of R generated by the t by t minors of the t by n matrix $[x_{ij}]_{1\leq i\leq t,1\leq j\leq n}$. Note that $\mathfrak{p}$ is a prime ideal of height 1 (c.f. [**6**], Theorem 1). Moreover the class $\mathrm{cl}(\mathfrak{p})$ of $\mathfrak{p}$ generates the group

Cl(R) and has order 2 (c.f. [3], Proof of Theorem).

We begin with the following

Lemma (2.1). *$\mathfrak{p}$ is minimally generated by $\binom{n}{t}$ elements.*

Proof. Let $\mathscr{F}$ be the set of all the subsets of $\{1, 2, \cdots, n\}$ with $\sharp J=t$. For $J \in \mathscr{F}$ we put $x_J=\det(x_{\alpha j_\beta})_{1\leqq\alpha,\beta\leqq t}$ and $X_J=\det(X_{\alpha j_\beta})_{1\leqq\alpha,\beta\leqq t}$, where $J=\{j_1, j_2, \cdots, j_t\}$ with $j_1<j_2<\cdots<j_t$. (Hence $\mathfrak{p}=(x_J/J \in \mathscr{F})$ and $x_J=X_J \bmod \mathrm{I}_t(X)$.) Of course, in order to prove the assertion, it suffices to show that $\{x_J\}_{J\in\mathscr{F}}$ is a minimal system of generators of the graded ideal $\mathfrak{p}$ of R.

Assume the contrary and choose an identity $x_J=\sum_{K\neq J} c_K x_K$ for some $J \in \mathscr{F}$ and for some family $\{c_K\}_{K\in\mathscr{F}-\{J\}}$ of elements of k. Then, as $X_J\equiv\sum_{K\neq J} c_K X_K \bmod \mathrm{I}_t(X)$, we have actually $X_J=\sum_{K\neq J} c_K X_K$. (Note that each X_K is a homogeneous element of degree t and that the ideal $\mathrm{I}_t(X)$ is generated by homogeneous elements of degree $t+1$.) Now put $T=k[\{X_{ij}\}_{1\leqq i\leqq t, 1\leqq j\leqq n, \text{ and } i\leqq j}]$ and let φ: $T\to S$ be the k-algebra map such that

$$\varphi(X_{ij})=\begin{cases}\delta_{ij}X_{ij} & (j \in J \text{ and } j\leqq t)\\ X_{ij} & (j \in J \text{ and } t<j)\\ 0 & (j \notin J)\end{cases}$$

for $1\leqq i\leqq t$, $1\leqq j\leqq n$, and $i\leqq j$. Then clearly $\varphi(X_J)\neq 0$, and $\varphi(X_K)=0$ for every $K \in \mathscr{F}-\{J\}$. But this is impossible, since $\varphi(X_J)=\sum_{K\neq J} c_K\varphi(X_K)$. Thus we conclude that $\mathfrak{p}$ is minimally generated by $\binom{n}{t}$ elements.

We put $K_R=\mathrm{Ext}_S^g(R, S)$ ($g\equiv\dim S-\dim R=n(n+1)/2-nt+t(t-1)/2$) and call it the canonical module of R. The fundamental properties of K_R are discussed by the author and Watanabe [5], some of which we shall need to prove (1.1). So we will summarize them with a sketch of proof.

Proposition (2.2). (a) *K_R is contained in R as a divisorial ideal.*

(b) *R is a Gorenstein ring if and only if K_R is principal.*

Proof. (a) Because K_R is a Cohen-Macaulay R-module with $\dim K_R=\dim R$, this assertion follows from the facts that R is a Noetherian integrally closed domain and that $\mathrm{Q}(R)\otimes_R K_R\cong\mathrm{Q}(R)$ (Here $\mathrm{Q}(R)$ denotes the quotient field of R.).

(b) See [5], (2.1.3).

Proof of Theorem (1.1). First we will discuss in case $t=1$. Let $P=k[X_1, X_2, \cdots, X_n]$ be a polynomial ring and φ: $S\to P$ the k-algebra map which takes X_{ij} onto X_iX_j ($1\leqq i\leqq j\leqq n$). Then $\mathrm{Ker}\,\varphi=\mathrm{I}_1(X)$ and $\mathrm{Im}\,\varphi=k[\{X_iX_j\}_{1\leqq i\leqq j\leqq n}]$. As the latter ring is a so-called Veronesean subring of P and as the Veronesean subring of order 2 is a Gorenstein ring if and only if $n\equiv 0$ (2) (c.f. [2] or [7]), we have the assertion.

Now consider the general case. Suppose that R is a Gorenstein ring and we will show that $n \not\equiv t$ (2). Assume that this assertion holds for $t-1$ ($t \geqq 2$). We put $\tilde{S}=S[X_{11}^{-1}]$, $\tilde{k}=k[X_{11}^{-1}, \{X_{1j}\}_{1 \leq j \leq n}]$, and $Y_{ij}=X_{ij}-X_{i1}X_{1j}/X_{11}$ ($2 \leqq i, j \leqq n$). Then $\tilde{S}=\tilde{k}[\{Y_{ij}\}_{2 \leq i \leq j \leq n}]$, and $\{Y_{ij}\}_{2 \leq i \leq j \leq n}$ are algebraically independent over $\tilde{k}$. Moreover in this situation the ideal $\tilde{I}=\mathrm{I}_t(X)\tilde{S}$ coincides with the ideal generated by the t by t minors of the new symmetric $n-1$ by $n-1$ matrix $Y=[Y_{ij}]$ (of indeterminates over $\tilde{k}$). Hence we see that $n-1 \not\equiv t-1(2)$ by the hypothesis of induction on t, because $k' \otimes_{\tilde{k}} \tilde{S}/\tilde{I} \cong k'[Y]/I_{t-1}(Y)$ where k' denotes the quotient field of $\tilde{k}$ and because $k' \otimes_{\tilde{k}} \tilde{S}/\tilde{I}=k' \otimes_{\tilde{k}} R[x_{11}^{-1}]$ is a Gorenstein ring by the standard assumption. Thus we have $n \not\equiv t$ (2).

Conversely suppose that $n \not\equiv t$ (2). If R were not a Gorenstein ring, we would choose t so that it is minimal among such counterexamples. Of course $t \geqq 2$. We regard $K=K_R$ as a divisorial ideal of R (c.f. (a) of (2.2)). Then $\mathrm{cl}(K) \neq 0$ by (b) of (2.2), and so $\mathrm{cl}(K)=\mathrm{cl}(\mathfrak{p})$ because the group $\mathrm{Cl}(R)$ has order 2 and $\mathrm{cl}(\mathfrak{p}) \neq 0$ (c.f. [3], Proof of Theorem). Let $\tilde{S}, \tilde{k}, k', \tilde{I}$, and Y be as above and put $\tilde{R}=k' \otimes_{\tilde{k}} \tilde{S}/\tilde{I}$ ($=k'[Y]/I_{t-1}(Y)$). Then the minimality of t guarantees that $\tilde{R}$ is a Gorenstein ring, since $n-1 \not\equiv t-1$ (2) by the standard assumption. On the other hand, we have

$$\begin{aligned} K_{\tilde{R}} &= \mathrm{Ext}^{g}_{k' \otimes_{\tilde{k}} \tilde{S}}\left(\tilde{R}, k' \otimes_{\tilde{k}} \tilde{S}\right) \\ &\cong \mathrm{Ext}^{g}_{S}(R, S) \otimes_{S} \left(k' \otimes_{\tilde{k}} \tilde{S}\right) \\ &= K\tilde{R}. \end{aligned}$$

(Notice that $g \equiv \dim S-\dim R=\dim k' \otimes_{\tilde{k}} \tilde{S}-\dim \tilde{R}=n(n+1)/2-nt+t(t-1)/2$.) Hence $K\tilde{R}$ is principal as so is $K_{\tilde{R}}$ (See (b) of (2.2).). Therefore we conclude that $\mathrm{cl}(\mathfrak{p}\tilde{R})=0$, as $\mathrm{cl}(K\tilde{R})=\mathrm{cl}(\mathfrak{p}\tilde{R})$ in $\mathrm{Cl}(\tilde{R})$. But this is impossible. (In fact, $\mathfrak{p}\tilde{R}$ coincides with the ideal generated by the $t-1$ by $t-1$ minors of the $t-1$ by $n-1$ matrix $[Y_{ij}]_{2 \leq i \leq t, 2 \leq j \leq n}$ mod $\mathrm{I}_{t-1}(Y)$ and the latter ideal is minimally generated by $\binom{n-1}{t-1}$ ($\geqq 2$) elements. See (2.1).) Thus we have that R is a Gorenstein ring and this completes the proof of Theorem (1.1).

Let $\mathfrak{m}=(x_{ij}/1 \leqq i \leqq j \leqq n)$ be the irrelevant maximal ideal of R. We put $\mathrm{r}(R)=\dim_k \mathrm{Ext}^d_R(k, R)$ ($k=R/\mathfrak{m}$ and $d=\dim R$) and call it the type of R. This invariant $\mathrm{r}(R)$ measures how the ring R differs from Gorenstein rings. R is a Gorenstein ring if and only if $\mathrm{r}(R)=1$.

Corollary (2.3). $\mathrm{r}(R)=\binom{n}{t}$ *if* $n \equiv t$ (2).

Proof. Suppose that $n \equiv t$ (2). Then R is not a Gorenstein ring by (1.1). Hence $K_R \cong \mathfrak{p}$, as $\mathrm{cl}(K_R)=\mathrm{cl}(\mathfrak{p})$. On the other hand, the cardinality of a minimal system of generators of K_R is equal to the number $\mathrm{r}(R)$ (c.f. [5], (2.1.8)). Thus we have $\mathrm{r}(R)=\binom{n}{t}$ by (2.1).

Remark (2.4). (1.1) and (2.3) have been known by the author and Tachibana [**4**], in case $t=n-2$ and the field k has characteristic not equal to 2.

Remark (2.5). Let k be an arbitrary Noetherian ring and let $0<t<n$ be integers. Let X be an n by n symmetric matrix of indeterminates over k. We put $S=k[X]$ and denote by $I_t(X)$ the ideal of S generated by the $t+1$ by $t+1$ minors of the matrix X. Then, as $S/I_t(X)$ is k-free, we can deduce easily from (1.1) that *$S/I_t(X)$ is a Gorenstein ring if and only if k is a Gorenstein ring and* $n\not\equiv t$ (2).

Remark (2.6). Let k be a Noetherian ring and r, s, t integers such that $0<t<\min\{r, s\}$. Let X be an r by s matrix of indeterminates over k. We put $S=k[X]$ and denote by $I_t(X)$ the ideal of S generated by the $t+1$ by $t+1$ minors of the matrix X. Then we can prove similarly as the symmetric case that *$S/I_t(X)$ is a Gorenstein ring if and only if k is a Gorenstein ring and $r=s$.* Of course this result has been already known by Svanes [**8**] (The *only if* part is due to the author [**1**].), and so we omit the detail.

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