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A note on Rees algebras of two dimensional local domains

By

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§ 0. Introduction

If R is a Cohen-Macaulay ring (C-M ring for short) and $\{a_1, a_2, \cdots, a_r\}$ is a regular sequence, then the Rees algebra $R[a_1^n t, a_1^{n-1}a_2 t, \cdots, a_r^n t]$ is a C-M ring for any positive integer n [4]. But even if R is not a C-M ring, the Rees algebra is sometimes a C-M ring (See [5] for example).

Now, our aim of this paper is to give some conditions for $R[at, bt]$ to be a C-M ring in case that $(R, \mathfrak{m})$ is a two dimensional local domain and $\{a, b\}$ is a system of parameters of R.

At first, in § 1 we will give some conditions for the kernel of the natural epimorphism $R[X, Y] \to R[at, bt]$ to have a linear base.

In § 2, using the results in § 1, we will prove the following:

Theorem: *Let* $(R, \mathfrak{m})$ *be a two dimensional local domain and* $\{a, b\}$ *be a system of parameters of* R. *Then* $R[at, bt]$ *is a C-M ring if and only if* $(aR: b) \cap (bR: a) = aR \cap bR$. *Moreover,* $R[at, bt]$ *is a C-M ring for every system of parameters* $\{a, b\}$ *of* R *if and only if* R *is a Buchsbaum ring.*

In § 3, the case of Gorenstein ring will be treated and we will show that $R[at, bt]$ is a Gorenstein ring for every (or equivalently some) system of parameters $\{a, b\}$ of R if and only if R is a Gorenstein ring.

Throughout this paper, we always denote by R or $(R, \mathfrak{m})$ a two dimensional local domain with maximal ideal $\mathfrak{m}$, by T the subring $R[at, bt]$ of a polynomial ring $R[t]$ where $\{a, b\}$ is a system of parameters of R and by K the kernel of the ring epimorphism $\varphi: R[X, Y] \to T$ given by $\varphi(X) = at$, $\varphi(Y) = bt$, where X, Y are indeterminates.

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§ 1. Linear base

Proposition (1.1). $K = \sqrt{(aY - bX)R[X, Y]}$.

Proof. $\sqrt{(aY-bX)R[X, Y]}\subseteq K$ is obvious, since K is a prime ideal and $aY-$ $-bX\in K$. To prove the opposite inclusion, we have to show that any minimal prime ideal P of $(aY-bX)R[X, Y]$ contains K. If both a and b are contained in P, $\mathfrak{m}R[X, Y]$ is contained in P and hence ht $(P)\geqq 2$, which is impossible. Therefore we may assume $a\notin P$. Let f be any element of K. As the map φ is homogeneous, we may assume that f is a form of degree n. Then, since we have $a^n f\in(aY-bX)R[X, Y]$ $\subseteq P$, f is contained in P. Thus we have $K\subseteq P$.

Definition (1.2). We say that K *has a linear base* if K is generated by linear polynomials. Namely $K=BR[X, Y]$, where $B=\{dX-cY;\ c, d\in R,\ da=cb\}$. In this case, if we take $(bR:a)=(d_1, d_2, \cdots, d_k)R$, $(aR:b)=(c_1, c_2, \cdots, c_m)R$, obviously $K=B_1$, where $B_1=\{d_iX-c_jY;\ 1\leqq i\leqq k,\ 1\leqq j\leqq m\}\cap B$.

At first we consider some equivalent conditions that K has a linear base.

Proposition (1.3). *For a system of parameters* $\{a, b\}$ *of* R, *the following conditions are equivalent*:

(i) *K has a linear base*

(ii) $b^{n+1}R\cap a((a, b)R)^n\subseteq ab^nR$ *for all* $n\geqq 0$.

(iii) $a^{n+1}R\cap b((a, b)R)^n\subseteq a^nbR$ *for all* $n\geqq 0$.

Proof. (i)$\Rightarrow$(ii): Let r be any element of $b^{n+1}R\cap a((a, b)R)^n$. Then we have

$$r=-r_0b^{n+1}=r_{n+1}a^{n+1}+r_na^nb+\cdots+r_1ab^n \qquad (r_i\in R)$$

Put $f(X, Y)=r_{n+1}X^{n+1}+r_nX^nY+\cdots+r_1XY^n+r_0Y^{n+1}$ and we have $f(X, Y)\in K$. Hence, by the assumption

$$f(X, Y)=\sum_{i,j}(d_iX-c_jY)g_{i,j}(X, Y), \qquad (d_iX-c_jY)\in B_1$$

Therefore, r_0 is contained in $(c_1, c_2, \cdots, c_k)=(aR:b)$ and $r=-r_0b^{n+1}\in ab^nR$.

(ii)$\Rightarrow$(i): Let $f(X, Y)$ be any element of K. We will show that $f(X, Y)$ is contained in $BR[X, Y]$. We may assume that $f(X, Y)$ is a form of degree $n+1$. We will show $f(X, Y)\in BR[X, Y]$ by induction on n. When $n=0$, $f(X, Y)=rX+sY\in K$ implies $ra+sb=0$ and we have $f(X, Y)=rX-(-s)Y\in BR[X, Y]$. Now, let $n\geqq 1$ and put $f(X, Y)=r_0Y^{n+1}+r_1Y^nX+\cdots+r_{n+1}X^{n+1}$ and we have $r_0b^{n+1}+r_1b^na+\cdots$ $+r_{n+1}a^{n+1}=0$, hence $r_0b^{n+1}\in b^{n+1}R\cap a((a, b)R)^n\subseteq ab^nR$. Therefore $r_0b^{n+1}=r_0'ab^n$ for some $r_0'\in R$ and

$$r_0Y^{n+1}\equiv r_0'XY^n \qquad (\text{mod } BR[X, Y]).$$

Now, let $g(X, Y)=(r_1+r_0')Y^n+r_2XY^{n-1}+\cdots+r_{n+1}X^n$ and we have $f(X, Y)\equiv$ $Xg(X, Y)$ (mod $BR[X, Y]$). Since K is a prime ideal and $X\notin K$, $g(X, Y)\in K$. As $g(X, Y)$ is a form of degree n, by induction hypothesis, we have $g(X, Y)\in BR[X, Y]$. Thus $f(X, Y)\in BR[X, Y]$.

The equivalence of (i) and (iii) is proved similarly.

Now, we consider the following several conditions for a system of parameters $\{a, b\}$ of R:

(I) $(aR: b)\cap(bR: a)=aR\cap bR$

(II) 1) $(aR: b^n)=(aR: b)$ for every $n>0$

2) $(aR: b^n)=(aR: b)$ for some $n>1$

3) $(aR: b^2)=(aR: b)$

(III) 1) $(b^{n+1}R: a)\subseteqq b^nR$ for every $n>0$

2) $(b^{n+1}R: a)\subseteqq b^nR$ for some $n>0$

3) $(b^2R: a)\subseteqq bR$

Then, we have

Proposition (1.4).

(i) *Three conditions in* (II) *are equivalent.*

(ii) *Three condtions in* (III) *are equivalent.*

(iii) *We have the following hierarchy*:

(I)$\Rightarrow$(II)$\Rightarrow$(III)$\Rightarrow$*K has a linear base.*

(iv) *Three statements that the condition* (I), *respectively* (II) *and* (III), *holds for every system of parameters* $\{a, b\}$ *of R are equivalent.*

Proof. (i) It suffices to prove 2)$\Rightarrow$1). It is obvious for $m<n$, since $(aR: b)\subseteqq(aR: b^m)\subseteqq(aR: b^n)$. For $m>n$, it is obtained by induction, since $(aR: b^m)=((aR: b^{m-1}): b)$.

(ii) It suffices to prove 2)$\Rightarrow$1). For $m<n$, it follows directly from $(b^{n+1}R: a)\supseteqq(b^{m+1}R: a)b^{n-m}$. For $m>n$, let $x\in(b^{m+1}R: a)$ and $xa=b^{m+1}r$ $(r\in R)$. Since $x\in(b^{m+1}R: a)\subseteqq(b^{n+1}R: a)\subseteqq b^nR$, we have $x=b^nx'$ for some $x'\in R$ and $x'a=b^{m-n+1}r$. Therefore we have $x'\in b^{m-n}R$ by induction and $x=b^nx'\in b^mR$.

(iii) (I)$\Rightarrow$(II): Take $x\in(aR: b^2)$, $xb^2=ar$ and we have $xb\in(aR: b)\cap bR\subseteqq(aR: b)\cap(bR: a)=aR\cap bR\subseteqq aR$. Thus $x\in(aR: b)$.

(II)$\Rightarrow$(III): Take $x\in(b^2R: a)$, $xa=b^2r$ $(r\in R)$ and we have $r\in(aR: b^2)=(aR: b)$. Thus $br=as$ for some $s\in R$ and $x=bs\in bR$.

(III)$\Rightarrow$$K$ has a linear base: This follows from Proposition (1.3), since we have $b^{n+1}R\cap a((a, b)R)^n\subseteqq b^{n+1}R\cap aR=a(b^{n+1}R: a)\subseteqq ab^nR$.

(iv) We have only to prove that (I) holds if (III) holds for every system of parameters $\{a, b\}$ of R. Let $x\in(aR: b)\cap(bR: a)$ and $xb=ar$, $xa=bs$ for some r, $s\in R$. Then we have $a^2r=b^2s$, hence $r\in(b^2R: a^2)$. Since $\{a^2, b\}$ is also a system of parameters of R, by the assumption we have $r\in(b^2R: a^2)\subseteqq bR$ and hence x is contained in aR. Quite similarly we have $x\in bR$ since $s\in(a^2R: b^2)$ and $\{b^2, a\}$ is also a system of parameters of R.

§ 2. Cohen-Macaulayness of $T=R[at, bt]$

Lemma (2.1). ht $(a, bt)T=2$.

Proof. It is well known $\mathrm{ht}(a, bt)T \leqq 2$, since T is Noetherian. Let $\mathfrak{P}$ be a minimal prime ideal of $(a, bt)T$. As $b(at)=a(bt) \in \mathfrak{P}$, we have $b \in \mathfrak{P}$ or $at \in \mathfrak{P}$, hence $(m, bt)T \subseteqq \mathfrak{P}$ or $(a, at, bt)T \subseteqq \mathfrak{P}$. Since $\mathfrak{m}R[X, Y]$ is a prime ideal of $R[X, Y]$ and $K \subseteqq R[X, Y]$ by Proposition (1.1), $\mathfrak{m}T$ is a prime ideal of T. Hence, if $(\mathfrak{m}, bt)T \subseteqq \mathfrak{P}$, we have $0 \subsetneqq \mathfrak{m}T \subsetneqq \mathfrak{P}$. Similarly $(at, bt)T$ is also a prime ideal of T, since $K \subseteqq (X, Y)R[X, Y]$ by Proposition (1.1), and we have $0 \subsetneqq (at, bt)T \subsetneqq \mathfrak{P}$ if $(a, at, bt)T \subseteqq \mathfrak{P}$. Therefore in each cases we have $\mathrm{ht}(\mathfrak{P}) \geqq 2$ and $\mathrm{ht}(a, bt)T=2$ is proved.

Lemma (2.2). *For a system of parameters $\{a, b\}$ of R, $\{a, bt\}$ is a regular sequence in T if and only if $(bR: a^2)=(bR: a)$.*

Proof. Let $(bR: a) \subsetneqq (bR: a^2)$ and $ra^2=sb$, $ra \notin bR$ for some $r, s \in R$. Then we have $(rat)a \in (bt)T$ and $rat \notin btT$, which implies $\{bt, a\}$ is not a regular sequence in T and hence $\{a, bt\}$ is not a regular sequence.

Conversely, assume $(bR: a)=(bR: a^2)$. Then K has a linear base by Proposition (1.4). Therefore φ induces an isomorphism $R[X]/(bR: a)XR[X] \cong T/(bt)T$. To prove that $\{a, bt\}$ is a regular sequence in T, it suffices to show that a is nonzero-divisor on $R[X]/(bR: a)XR[X]$. If $af(X) \in (bR: a)R[X] \cdot X$, the coefficients of $f(X)$ are contained in $(bR: a^2)=(bR: a)$ and we have $f(X) \in (bR: a)XR[X]$.

Corollary (2.3). *For a system of parameters $\{a, b\}$ of R, if T is a C-M ring, then $\{a, bt\}$ is a regular sequence in T and K has a linear base.*

Proof. Since T is C-M and $\mathrm{ht}(a, bt)T=2$ by Lemma (2.1), we have $\mathrm{grade}(a, bt)T=2$ and hence $\{a, bt\}$ is a regular sequence. And from this, it follows that K has a linear base by Lemma (2.2) and Proposition (1.4).

Now, we give a characterization for T to be a C-M ring.

Theorem (2.4). *For a system of parameters $\{a, b\}$ of R, $T=R[at, bt]$ is a Cohen-Macaulay ring if and only if $(aR: b) \cap (bR: a)=aR \cap bR$.*

Proof. In both cases when T is a C-M ring and when $(aR: b) \cap (bR: a)=aR \cap bR$ holds, we have $\{a, bt\}$ is a regular sequence and K has a linear base by Corollary (2.3), Proposition (1.4) and Lemma (2.2). Therefore φ induces an isomorphism

$$R[X]/\mathrm{A} \cong T/(a, bt, at+b)T$$

where $\mathrm{A}=aR[X]+(bR: a)XR[X]+(X+b)R[X]$. It is easily seen that the formar is a ring of dimension 0. Thus, since $\dim T=3$, T is a C-M ring if and only if $\{a, bt, at+b\}$ is a regular sequence in T, that is, $at+b$ is a nonzero-divisor on $T/(a, bt)T$, which is equivalent to that $X+b$ is a nonzero-divisor on $R[X]/aR[X]+(bR: a)XR[X]$.

Now we assume that T is a *C-M* ring. And let $r \in (aR: b) \cap (bR: a)$. Then $r(X+b) \in aR[X]+(bR: a)XR[X]$ and hence we have $r \in (aR[X]+(bR: a)XR[X]) \cap R = aR$. Since we can proceed all arguments on a and b replaced, r is also contained in bR. Thus $(aR: b)\cap(bR: a)=aR\cap bR$.

Conversely, we assume $(aR: b)\cap(bR: a)=aR\cap bR$. We have only to prove that $(X+b)$ is a nonzero-divisor on $R[X]/(aR[X]+(bR: a)XR[X])$. Take $f(X)=r_0+r_1X+\cdots+r_nX^n \in R[X]$ such that $(X+b)f(X) \in aR[X]+(bR: a)XR[X]$, and we have

$$r_n,\ r_i+r_{i+1}b \in aR+(bR: a) \qquad (0\leqq i\leqq n-1)$$
$$r_0 \in (aR: b)$$

From this we get

$$r_i \in aR+(bR: a) \qquad \text{for } i=0, 1, 2, \cdots, n$$

and hence

$$r_iX^i \in aR[X]+(bR: a)XR[X] \qquad \text{for } i=1, 2, \cdots, n$$

Furthermore

$$\begin{aligned} r_0 \in (aR+(bR: a))\cap(aR: b) &= aR+((bR: a)\cap(aR: b)) \\ &= aR+(aR\cap bR)=aR. \end{aligned}$$

Thus $f(X) \in aR[X]+(bR: a)XR[X]$.

We call $\{a, b\}$ a *weakly regular sequence* if $\mathfrak{m}(aR: b)\subseteqq aR$ and $(R, \mathfrak{m})$ is called a *Buchsbaum ring* (or *I*-ring) if each system of parameters of R forms a weakly regular sequence, [2 or 3]. We have

Theorem (2.5). *$T=R[at, bt]$ is a Cohen-Macaulay ring for every system of parameters $\{a, b\}$ of R if and only if R is a Buchsbaum ring.*

Proof. If R is a Buchsbaum ring, each system of parameters $\{a, b\}$ of R satisfies the condition (II) by [2. Theorem 5]. Theorefore T is a *C-M* ring for each system of parameters by Proposition (1.4) and Theorem (2.4).

Cnoversely, let T be C-M for every system of parameters $\{a, b\}$ of R. By [2. Theorem 5] it suffices to prove that $(aR: b)=(aR: b_1)$ for any system of parameters $\{a, b\}$ and $b_1 \in R$ such that $\mathrm{ht}\,(a, b_1)R=2$. There exists integer n such that $b_1^n \in (a, b)R$, for $(a, b)R$ is an $\mathfrak{m}$-primary ideal. Then for each $r \in (aR: b)$ $rb_1^n \in aR$, that is $r \in (aR: b)_1^n$. Thus we have $(aR: b)\subseteqq(aR: b_1^n)$. Since $\{a, b_1\}$ is also a system of parameters, $T_1=R[at, b_1t]$ is a *C-M* ring by the assumption and we have $(aR: b_1^n)=(aR: b_1)$ by Theorem (2.4) and Proposition (1.4). Thus we have $(aR: b)\subseteqq(aR: b_1)$. The opposite inclusion follows quite similarly.

§ 3. Gorensteinness of $T=R[at, bt]$

On Gorensteinness of T, we have the following.

Theorem (3.1). *The following conditions are equivalent:*

(i) *R is a Gorenstein ring*

(ii) *$T=R[at, bt]$ is a Gorenstein ring for every system of parameters $\{a, b\}$ of R.*

(iii) *$T=R[at, bt]$ is a Gorenstein ring for some system of parameters $\{a, b\}$ of R.*

Proof. (i)⇒(ii): If R is Gorenstein, R is *C-M* and we have $(aR: b)=aR$ and $(bR: a)=bR$. Therefore by Proposition (1.4) and Definition (1.2), T is isomorphic to $R[X, Y]/(bX-aY)R[X, Y]$, which is obviously Gorenstein.

(ii)⇒(iii): trivial.

(iii)⇒(i): Let $\mathrm{A}=aR[X]+(bR: a)XR[X]+(X+b)R[X]$. Then we have an isomorphism

$$R[X]/\mathrm{A}\cong T/(a, bt, at+b)T=T_{\mathfrak{N}}/(a, bt, at+b)T_{\mathfrak{N}}$$

where $\mathfrak{N}$ is the irrelevant maximal ideal $(\mathfrak{m}, at, bt)T$ of T, and $\{a, bt, at+b\}$ is a regular sequence in T, as we saw in the proof of Theorem (2.4). Therefore $R[X]/\mathrm{A}$ is a 0-dimensional local Gorenstein ring and hence A is an irreducible ideal. Now, if we can prove that $\{a, b\}$ is a regular sequence in R, then R is *C-M* and we have $R/(a, b^2)R\cong R[X]/\mathrm{A}$, which is Gorenstein. This implies R is Gorenstein and the proof will be completed.

In the first place, we note:

$$\mathrm{A}\cap R=aR+b(bR: a)$$

and the conditions in Proposition (1.4) hold by Theorem (2.4).

Now, put $\mathrm{A}_1=(aR: b)R[X]+\mathrm{A}$, $\mathrm{A}_2=(bR: a)R[X]+\mathrm{A}$. We claim $\mathrm{A}=\mathrm{A}_1\cap \mathrm{A}_2$. Because, for any $f\in \mathrm{A}_1\cap \mathrm{A}_2$, we can take $c_0\in (aR: b)$, $d_0\in (bR: a)$ such that $f\equiv c_0\equiv d_0$ (mod A), since $(aR: b)X\equiv(aR: b)(-b)\equiv 0$ (mod A) and $(bR: a)X\equiv 0$ (mod A). Then $c_0-d_0\in \mathrm{A}\cap R=aR+b(bR: a)$ and we have $c_0-d_0=ar+bd'$, for some $r\in R$, $d'\in (bR: a)$. So $c_0-ar=d_0+bd'$ is contained in $(aR: b)\cap(bR: a)=aR\cap bR$. Thus we get $c_0\in aR$ and $f\equiv c_0\equiv 0$ (mod A).

Since A is an irreducible ideal, $\mathrm{A}=\mathrm{A}_1$ or $\mathrm{A}=\mathrm{A}_2$. If $\mathrm{A}=\mathrm{A}_1$, $(aR: b)\subseteq \mathrm{A}\cap R=aR+b(bR: a)$. Hence we have $(aR: b)=(aR: b)\cap(aR+b(bR: a))\subseteq aR+((aR: b)\cap(bR: a))=aR+(aR\cap bR)=aR$. Thus $(aR: b)=aR$ is obtained. If $\mathrm{A}=\mathrm{A}_2$, $(bR: a)\subseteq A\cap R=aR+b(bR: a)$. Hence we have $(bR: a)=(bR: a)\cap(aR+b(bR: a))=((bR: a)\cap aR)+((bR: a)\cap b(bR: a))\subseteq((bR: a)\cap(aR: b))+bR=(aR\cap bR)+bR=bR$. Thus we have $(bR: a)=bR$.

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