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On semi-C-reducibility, T-tensor $=0$ and $S4$-likeness of Finsler spaces

By

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The purpose of the present paper is to introduce the concepts called semi-C-reducibility and $S4$-likeness of Finsler space and to consider relations between these concepts and other important ones which are familiar to us.

The concept "semi-C-reducibility" is a generalization of the well-known C-reducibility and a certain restriction of the quasi-C-reducibility. The concept "$S4$-likeness" is introduced based on the fact that the v-curvature tensor S_{hijk} of any four-dimensional Finsler space is of a special form, similarly to the case of the concept "S3-likeness".

The notations and terminology are used the ones of the monograph [17] without comment. The introduction of two new concepts are done by the first author and the contents of the final section is due to the work of the second author only.

§ 1. Semi-C-reducibility

Throughout the paper we denote by F^n an n-dimensional Finsler space with a fundamental function $L(x, y)$ ($y^i=\dot{x}^i$), the fundamental tensor g_{ij} and the angular metric tensor $h_{ij}=L(\dot{\partial}_i\dot{\partial}_j L)=g_{ij}-l_i l_j$ ($l_i=\dot{\partial}_i L$).

We are concerned with special forms of the $(h)hv$-torsion tensor $C_{ijk}=(\dot{\partial}_k g_{ij})/2$. First of all, it should be noted that C_{ijk} of any F^2 is written in the form

$$LC_{ijk}=Im_i\, m_j m_k,$$

where I is the main scalar and we refer to the Berwald frame (l^i, m^i) (VI § 6 of [24], § 28 of [17]). If the torsion vector $C_i=C_{ijk}g^{jk}$ has the non-zero length C, then we have $m_i=\pm C_i/C$ and C_{ijk} is written in the form

$$3C_{ijk}=h_{ij}C_k+h_{jk}C_i+h_{ki}C_j. \tag{1.1}$$

Next, consider an F^3 with non-zero C ([19]). We can refer to the Moór frame (l^i, m^i, n^i) ($m^i=C^i/C$) ([12], § 29 of [17]) and C_{ijk} is written in the form

(1.2) $$LC_{ijk}=Hm_im_jm_k-\mathfrak{S}^{1)}_{(ijk)}\{Jm_im_jn_k-Im_in_jn_k\}+Jn_in_jn_k,$$

where H, I, J are main scalars.

Now, one of the authors proposed a special form of C_{ijk} ([11]):

(1.3) $$C_{ijk}=A_{ij}B_k+A_{jk}B_i+A_{ki}B_j,$$

where A_{ij} is a symmetric tensor and B_i a covariant vector. The equations $A_{i0}=0$ and $B_0=0$ were shown. The angular metric tensor h_{ij} has these properties of A_{ij} and it was also shown that $A_{ij}=h_{ij}$ implies $B_i=C_i/(n+1)$. Thus we are led to the special form

(1.4) $$(n+1)C_{ijk}=h_{ij}C_k+h_{jk}C_i+h_{ki}C_j.$$

It follows from (1.1) that (1.4) imposes no restriction on any F^2. A non-Riemannian F^n $(n\geqq 3)$ with C_{ijk} of the form (1.4) is called *C-reducible*. It has been, however, concluded in a recent paper ([18]) that the metric of *any* C-reducible F^n is only of the Randers type or the Kropina type.

On the other hand, one of the authors proposed another special case of (1.3) such that B_i is equal to the torsion vector C_i ([15]):

(1.5) $$C_{ijk}=A_{ij}C_k+A_{jk}C_i+A_{ki}C_j,$$

and non-Riemannian F^n $(n\geqq 3)$ with C_{ijk} of the form (1.5) was called *quasi-C-reducible*. It was shown that any non-Riemannian F^n $(n\geqq 3)$ with the so-called (α,β)-metric is quasi-C-reducible and F^3 is quasi-C-reducible iff $J=0$ in (1.2) ([12]).

Really speaking, A_{ij} of any (α,β)-metric is of a special form $A_{ij}=\lambda h_{ij}+\mu C_iC_j$ with some scalars λ and μ. It is also verified easily that any quasi-C-reducible F^3 $(J=0)$ has this form. Further we should recall that one of the authors has already treated A_{ij} of this form in a previous paper ([20]). In the case of this A_{ij} we have

(1.6) $$C_{ijk}=\lambda(h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)+3\mu C_iC_jC_k.$$

We deal with C_{ijk} of the form (1.6). Contraction of (1.6) by g^{jk} gives $(n+1)\lambda+3C^2\mu=1$; two scalars λ and μ are not independent. In case of $\lambda=0$ we have to take accont into Brickell's theorem ([4]), because $C_{ijk}=3\mu C_iC_jC_k$ causes immediately vanishing of the v-curvature tensor $S_{hijk}=C_h{}^r{}_kC_{rij}-C_h{}^r{}_jC_{rik}$.

Next, in case of $C=0$, Deicke's theorem ([5]) must be taken into account. Further, in case of $\mu=0$, (1.6) is solely reduced to (1.4). Paying attention to these circumstances we are naturally led to the following definition:

Definition. A Finsler space F^n $(n\geqq 3)$ with the non-zero length C of the torsion vector C^i is called *semi-C-reducible*, if the $(h)hv$-torsion tensor C_{ijk} is of the form

(1.7) $$C_{ijk}=[p/(n+1)](h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)+(q/C^2)C_iC_jC_k,$$

1) $\mathfrak{S}_{(ijk)}$ means cyclic permutation of indices i,j,k and summation.

where p and $q\ (=1-p)$ do not vanish. p is called the *characteristic scalar* of the F^n.

It is easily seen that the v-curvature tensor S_{hijk} of a semi-C-reducible F^n is written in the form

$$L^2 S_{hijk} = h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk}, \tag{1.8}$$

where the symmetric tensor M_{ij} is defined by

$$M_{ij}/L^2 = -[(pC)^2/2(n+1)^2]h_{ij} - [p^2/(n+1)^2 + pq/(n+1)]C_iC_j. \tag{1.9}$$

§ 2. T-tensor $=0$

In 1972 one of the authors ([10]) and H. Kawaguchi ([7]) independently found an important tensor

$$T_{hijk} = LC_{hij}|_k + l_hC_{ijk} + l_iC_{hjk} + l_jC_{hik} + l_kC_{hij}, \tag{2.1}$$

where $C_{hij}|_k$ is the v-covariant derivative of C_{hij}. This is called the *T-tensor*. Finsler spaces with the vanishing T-tensor constitute an important and interesting class (§ 28 of [17]). We denote *such a Finsler space by* TF^n in the following. Thus

$$LC_{hij}|_k = -l_hC_{ijk} - l_iC_{hjk} - l_jC_{hik} - l_kC_{hij} \tag{2.2}$$

is the system of partial differential equations which is the characteristic of the fundamental function L of TF^n.

From (2.2) we immediately obtain

$$LC_i|_j = -l_iC_j - l_jC_i, \tag{2.3}$$

$$C^i|_i = 0. \tag{2.4}$$

As to the length C of the torsion vector C^i, (2.3) yields

$$LC^2|_i = -2C^2 l_i, \tag{2.5}$$

which further implies

$$C^2|_i\, C^i = 0. \tag{2.6}$$

Consider the normalized torsion vector $A^i = LC^i$, which is (0)p-homogeneous. The length of A^i is equal to LC in our notations. It is remarkable that (2.5) is written as $\dot{\partial}_i C^2/C^2 + \dot{\partial}_i L^2/L^2 = 0$, so that we obtain

Theorem 1. *The length LC of the normalized torsion vector $A^i = LC^i$ of any TF^n is constant in every tangent space of TF^n.*

Pay attention to the integrability condition

$$C_{hij}|_k|_l - C_{hij}|_l|_k = -\mathfrak{S}_{(hij)}\{C_{hir}S_j{}^r{}_{kl}\},$$

of (2.2), which is one of the Ricci identities (§ 17 of [17]). Applying this condition to (2.2), we have

$$(2.7)\qquad \mathfrak{S}_{(hij)}\{h_{hk}C_{ijl}-h_{lh}C_{ijk}-L^2C_h{}^r{}_iS_{rjkl}\}=0.$$

It is noted that (2.7) is a system of *algebraic* equations satisfied by C_{ijk} of TF^n.

We shall derive various equations from (2.7) for the later use. First, applying $\mathfrak{S}_{(jkl)}$ to (2.7), we obtain

$$(2.8)\qquad \mathfrak{S}_{(jkl)}\{C_i{}^r{}_jS_{rhkl}+C_h{}^r{}_jS_{rikl}\}=0.$$

Contraction of (2.8) by g^{hl} gives

$$(2.9)\qquad \mathfrak{A}_{(jk)}{}^{2)}\{C_i{}^r{}_jS_{rk}+C_j{}^s{}_rS_i{}^r{}_{sk}\}-C_rS_i{}^r{}_{jk}=0,$$

where $S_{rk}=S_r{}^i{}_{ki}$ is the so-called *v-Ricci tensor*.

Secondly we contract (2.7) by g^{hl} and moreover by g^{ij} to obtain

$$(2.10)\qquad nC_{ijk}=h_{ki}C_j+h_{kj}C_i-L^2(C_i{}^s{}_rS_j{}^r{}_{sk}+C_j{}^s{}_rS_i{}^r{}_{sk}+C_i{}^r{}_jS_{rk}),$$

$$(2.11)\qquad (n-2)C_k+L^2C^rS_{rk}=0.$$

Thirdly we contract (2.7) by g^{hi} to obtain

$$(2.12)\qquad h_{lj}C_k-h_{kj}C_l+L^2C^rS_{rjkl}=0.$$

The equation (2.10) is rather interesting. In fact, while C_{ijk} is symmetric in all indices, the right-hand side of (2.10) is seemingly not symmetric in j and k. The symmetric form of C_{ijk} is easily derived from (2.10) by applying $\mathfrak{S}_{(ijk)}$:

$$(2.13)\qquad 3nC_{ijk}=\mathfrak{S}_{(ijk)}\{2h_{ij}C_k-2L^2C_i{}^s{}_rS_j{}^r{}_{sk}-L^2C_i{}^r{}_jS_{rk}\}.$$

Further, we apply the Christoffel method (§ 5 of [17]) to (2.10) to get

$$(2.14)\qquad nC_{ijk}=2(h_{ik}C_j-L^2C_j{}^s{}_rS_k{}^r{}_{si})-L^2(C_i{}^r{}_jS_{rk}+C_j{}^r{}_kS_{ri}-C_k{}^r{}_iS_{rj}).$$

§ 3. $S4$-likeness

It is well-known ([9], § 29 of [17]) that the v-curvature tensor S_{hijk} of any F^3 is of the form

$$(3.1)\qquad L^2S_{hijk}=S(h_{hj}h_{ik}-h_{hk}h_{ij}),$$

where the scalar S is called the *v-curvature*. One of the authors introduced the concept of $S3$-likeness ([9]): F^n ($n\geqq 4$) is called $S3$-like if S_{hijk} is written in the form (3.1). It is known that the v-curvature S of any $S3$-like F^n is a function of position alone. Recently appear various papers concerned with $S3$-like Finsler spaces ([1],

2) $\mathfrak{A}_{(jk)}$ means interchange of indices j,k and subtraction.

[2], [3], [22] and [23]). The v-curvature S of any TF^3 is equal to -1 ([13], § 29 of [17]). The following is a generalization of this fact:

Theorem 2. *The v-curvature S of any non-Riemannian $S3$-like TF^n is equal to -1.*

Proof. Substitution from (3.1) into (2.10) and (2.11) yields

$$(S+1)(nC_{ijk}-C_ih_{jk}-C_jh_{ik})=0, \qquad (S+1)(n-2)C_k=0.$$

Thus $S+1\neq 0$ causes $C_{ijk}=0$ and the proof is completed.

Remark. Theorem 2 asserts that the indicatrix of any non-Riemannian $S3$-like TF^n is flat (§ 31 of [17]), very strange circumstances.

Next it is known ([16], § 31 of [17]) that the v-curvature tensor S_{hijk} of any F^4 is written in the form

$$L^2S_{hijk}=h_{hj}M_{ik}+h_{ik}M_{hj}-h_{hk}M_{ij}-h_{ij}M_{hk}, \tag{3.2}$$

where M_{ij} is a symmetric tensor and satisfies $M_{0j}=0$. Thus, similarly to the case of the $S3$-likeness, we are led to the following definition:

Definition. A non-Riemannian F^n $(n\geqq 5)$ is called *$S4$-like* if the v-curvature tensor S_{hijk} is written in the form (3.2) where M_{ij} is a symmetric and indicatory tensor (§ 31 of [17]).

From (1.8) we immediately have

Theorem 3. *Any semi-C-reducible F^n $(n\geqq 5)$ is $S4$-like.*

The reverse of Theorem 3 is true on the assumption of T-tensor $=0$. Precisely speaking, we have

Theorem 4. *Consider a TF^n $(n\geqq 4)$ with the non-zero length C of the torsion vector C^i and suppose that the TF^n is not $S3$-like.*

(1) *The space TF^4 is semi-C-reducible and the scalar $M=g^{ij}M_{ij}$ is not equal to $-3/2$.*

(2) *If the TF^n $(n\geqq 5)$ is $S4$-like, it is semi-C-reducible and the M is not equal to $(1-n)/2$.*

Proof. From (3.2) the v-Ricci tensor S_{ij} is written as

$$L^2S_{ij}=(n-3)M_{ij}+Mh_{ij}. \tag{3.3}$$

Substitution from (3.3) into (2.11) yields

$$C_i+C^rM_{ri}=\phi C_i, \qquad \phi=-(M+1)/(n-3). \tag{3.4}$$

By means of (3.2) and (3.4), the equation (2.12) gives

$$M_{ij}=[(2\phi-1)/C^2]C_iC_j-\phi h_{ij}. \tag{3.5}$$

From (3.2), (3.3) and (3.5) the equation (2.9) is written as

$$(2\phi-1)\mathfrak{A}_{(jk)}\{h_{ij}(B_k-C_k)+(n-2)B_{ij}C_k\}=0, \tag{3.6}$$

where we put $B_{ij}=C_i{}^r{}_jC_r/C^2$ and $B_i=B_{ij}C^j$. If $2\phi-1=0$, then M_{ij} is proportional to h_{ij} from (3.5), so that the space is essentially $S3$-like. Therefore we get $2\phi-1\neq 0$, i.e., $2M\neq 1-n$.

We contract (3.6) by C^k to obtain

$$C^2\psi h_{ij}+C_iC_j+(n-2)C^2B_{ij}=(n-2)B_iC_j+B_jC_i,$$

where $\psi=(B_i-C_i)C^i/C^2$. Because the left-hand side of the above is symmetric in indices, we get $B_i=(\psi+1)C_i$ easily from $n\geqq 4$. Thus the above is written as

$$B_{ij}=-\{\psi/(n-2)\}h_{ij}+\{1+(n-1)\psi/(n-2)\}C_iC_j/C^2. \tag{3.7}$$

It follows from (3.5) that (3.2) is written in the form

$$\begin{aligned}L^2S_{hijk}=&-2\phi(h_{hj}h_{ik}-h_{hk}h_{ij})\\&+[(2\phi-1)/C^2]\mathfrak{A}_{(jk)}\{h_{hj}C_iC_k+h_{ik}C_hC_j\}.\end{aligned} \tag{3.8}$$

Finally, substituting from (3.8) into (2.14) and making use of (3.7), we arrive at the form (1.7) of C_{ijk}, where $p=-(n+1)\psi/(n-2)$.

§ 4. The T-tensor of semi-C-reducible Finsler spaces

It is known ([14], § 30 of [17]) that the T-tensor of any C- reducible Finsler space is of an elegant form. The purpose of the present section is to consider the T-tensor of semi-C-reducible Finsler spaces. We shall use following notations:

$$p_i=\dot{\partial}_i p,\quad p_c=p_iC^i/C^2,\quad \alpha=C^i|_i/C^2,\quad \beta=C^2|_i\, C^i/C^4. \tag{4.1}$$

Now from (1.7) we obtain

$$\begin{aligned}C_{ijk}|_h=&\mathfrak{S}_{(ijk)}\{h_{ij}(pC_k|_h+C_kp_h)/(n+1)\\&-[p/(n+1)L]h_{hi}(C_jl_k+C_kl_j)+qC_iC_jC_k|_h/C^2\}\\&-(p_h+qC^2|_h/C^2)C_iC_jC_k/C^2.\end{aligned} \tag{4.2}$$

While $C_{ijk}|_h$ is symmetric in all indices without any assumption, the right-hand side of (4.2) is seemingly not symmetric in k, h. From this point of view, it is conjectured that the symmetry property of the right-hand side of (4.2) may impose some restriction on the characteristic scalar p. We shall examine this in the following.

Pay attention first to the fact that $C_{ijk}|_hg^{kh}=C_{khi}|_j\, g^{kh}=C_i|_j$. It then follows from (4.2) that

$$(4.3)\quad \begin{aligned}(n+1-2p)C_i|_j=&(p\alpha+p_c)C^2h_{ij}+C_ip_j+C_jp_i-(n-1)p(C_il_j+C_jl_i)/L\\ &+(n+1)q(C_iC^2|_j+C_jC^2|_i)/2C^2+(n+1)\{q(\alpha-\beta)-p_c\}C_iC_j.\end{aligned}$$

By applying the contraction by C^i to (4.3), we obtain

$$(4.4)\quad C^2|_j/C^2=UC_j+[2/(n-1)p]p_j-2l_j/L,$$

where the coefficient U is given by

$$(4.5)\quad (n-1)pU=2(nq+1)\alpha-(n+1)q\beta-2(n-1)p_c.$$

Substitution from (4.4) into (4.3) yields the following form of $C_i|_j$:

$$(4.6)\quad \begin{aligned}(n+1-2p)C_i|_j=&(p\alpha+p_c)C^2h_{ij}+(n+1-2p)[(C_ip_j+C_jp_i)/(n-1)p\\ &-(C_il_j+C_jl_i)/L+VC_iC_j],\end{aligned}$$

where V is given by

$$(4.7)\quad (n+1-2p)V=(n+1)\{(\alpha-\beta+U)q-p_c\}.$$

We consider four scalars α, β, U and V appearing in the above. These satisfy (4.5) and (4.7). We obtain further two equations arising from the definition (4.1) of α and β. That is, the contraction of (4.6) by g^{ij} gives

$$(4.8)\quad (n-1)(n+1-2p)pV=(n+1)[(n-1)pq\alpha-\{2+(n-3)p\}p_c].$$

Contraction of (4.4) by C^j does

$$(4.9)\quad (n-1)p(\beta-U)=2p_c.$$

Among these four equations (4.5), (4.7), (4.8) and (4.9), the second is solely a consequence of the last two, as it is easily verified. (4.5) and (4.9) are equivalent to the two equations

$$(4.10)\quad (n+1-2p)\beta=2\{(nq+1)\alpha-(n-2)p_c\},$$

$$(4.11)\quad (n-1)(n+1-2p)pU=2[(n-1)(nq+1)p\alpha-\{n+1+n(n-3)p\}p_c].$$

Accordingly we have three independent equations (4.8), (4.10) and (4.11), which give V, β and U respectively in terms of α, p and p_c, provided $n+1-2p\neq 0$.

On account of the above circumstances we are naturally led to the classification of semi-C-reducible spaces as follows:

Definition. A semi-C-reducible Finsler space is called *of the first kind* or *of the second kind*, according as the characteristic scalar $p\neq(n+1)/2$ or $p=(n+1)/2$.

We continue the discussion of a space of the first kind. The equation (4.6) is written in the form

$$(4.6_1)\qquad C_i|_j = C^2Wh_{ij} + (C_ip_j + C_jp_i)/(n-1)p - (C_il_j + C_jl_i)/L + VC_iC_j,$$

where the coefficient W is given by

$$(4.12)\qquad (n+1-2p)W = p\alpha + p_c.$$

Substitution from (4.4) and (4.6_1) into (4.2) yields

$$(4.2_1)\qquad \begin{aligned} C_{ijk}|_h = \mathfrak{S}_{(ijk)}\{&(pWC^2h_{ij}h_{kh} + h_{ij}P^{(1)}_{kh} + h_{hi}P^{(2)}_{jk})/(n+1) \\ &+ [q/(n-1)pC^2]C_iC_jp_kC_h\} - [q/LC^2](C_iC_jC_kl_h \\ &+ C_jC_kC_hl_i + C_kC_hC_il_j + C_hC_iC_jl_k) \\ &+ [(1-np)/(n-1)pC^2]C_iC_jC_kp_h + q(3V-U)C_iC_jC_kC_h/C^2, \end{aligned}$$

where we put

$$(4.13)\qquad \begin{aligned} P^{(1)}_{ij} &= (C_ip_j + C_jp_i)/(n-1) - [p/L](C_il_j + C_jl_i) \\ &\quad + pVC_iC_j + C_ip_j, \\ P^{(2)}_{ij} &= (n+1)qWC_iC_j - [p/L](C_il_j + C_jl_i). \end{aligned}$$

Now the symmetry property $C_{ijk}|_h - C_{ijh}|_k = 0$ is written as

$$(4.14_1)\quad \mathfrak{A}_{(kh)}\{h_{ij}C_kp_h + h_{jk}(P^{(1)}_{ih} - P^{(2)}_{ih}) + h_{ik}(P^{(1)}_{jh} - P^{(2)}_{jh}) - (n+1)C_iC_jC_kp_h/C^2\} = 0.$$

Contraction of (4.14_1) by C^iC^j yields $n(n-3)(C_kp_h - C_hp_k) = 0$, so that the equation

$$(4.15)\qquad p_k = p_cC_k$$

must be satisfied, provided $n \geqq 4$.

It is seen from (4.8) and (4.12) that (4.15) implies $P^{(1)}_{ij} = P^{(2)}_{ij}$, and (4.2_1) is rewritten in the symmetric form

$$(4.16_1)\qquad \begin{aligned} C_{ijk}|_h = \mathfrak{S}_{(ijk)}\{&[(p\alpha + p_c)pC^2/(n+1)(n+1-2p)]h_{ij}h_{kh} \\ &+ h_{ij}P_{kh} + h_{hi}P_{jk}\} - [q/LC^2](C_iC_jC_kl_h \\ &+ C_jC_kC_hl_i + C_kC_hC_il_j + C_hC_iC_jl_k) + \gamma C_iC_jC_kC_h/C^2, \end{aligned}$$

where P_{ij} and γ are given by

$$(4.17)\qquad \begin{aligned} &P_{ij} = qWC_iC_j - [p/(n+1)L](C_il_j + C_jl_i), \\ &(n+1-2p)\gamma = \{n+1-(n+3)p\}q\alpha + \{(n+3)p - 2(n+1)\}p_c. \end{aligned}$$

We turn our discussion to a semi-C-reducible F^n $(n \geqq 4)$ of the second kind. In this case (4.5), (4.7), (4.8) and (4.9) are reduced to $\alpha = 0$ and $\beta = U$ only. (4.4) is of the form

$$(4.4_2)\qquad C^2|_j = C^2(\beta C_j - 2l_j/L).$$

Then (4.3) is reduced to a trivial equation. Substitution from (4.4_2) into (4.2) gives

$$(4.2_2)\qquad \begin{aligned} C_{ijk}|_h = &-(1/2)\mathfrak{S}_{(ijk)}\{h_{hi}(C_j l_k + C_k l_j)/L - h_{ij}C_k|_h \\ &+(n-1)C_iC_jC_k|_h/C^2\} + (n-1)(\beta C_h/2 - l_h/L)C_iC_jC_k/C^2. \end{aligned}$$

Therefore $C_{ijk}|_h - C_{ijh}|_k = 0$ is given by

$$(4.14_2)\qquad \begin{aligned} \mathfrak{A}_{(kh)}\{&h_{ik}Q^{(1)}_{jh} + h_{jk}Q^{(1)}_{ih} - [2(n-1)/LC^2]C_iC_jC_kl_h \\ &-(n-1)(C_jC_kC_i|_h + C_iC_kC_j|_h)/C^2\} = 0, \end{aligned}$$

where we put

$$(4.18)\qquad Q^{(1)}_{ij} = C_i|_j + (C_il_j + C_jl_i)/L.$$

We contract (4.14_2) by C^iC^h. Then it follows from (4.4_2) that

$$(4.19)\qquad \begin{aligned} C_j|_k = &-[\beta C^2/2(n-2)]h_{jk} - (C_jl_k + C_kl_j)/L \\ &+[(n-1)\beta/2(n-2)]C_jC_k. \end{aligned}$$

Finally, substitution from (4.19) into (4.2_2) yields the symmetric form of $C_{ijk}|_h$:

$$(4.16_2)\qquad \begin{aligned} C_{ijk}|_h = &\mathfrak{S}_{(ijk)}\{-[\beta C^2/4(n-2)]h_{ij}h_{kh} + h_{ij}Q_{kh} + h_{hi}Q_{jk}\} \\ &+[(n-1)/2LC^2](C_iC_jC_kl_h + C_jC_kC_hl_i + C_kC_hC_il_j \\ &+C_hC_iC_jl_k) - [(n^2-1)\beta/4(n-2)C^2]C_iC_jC_kC_h, \end{aligned}$$

where we put

$$(4.20)\qquad Q_{ij} = -(C_il_j + C_jl_i)/2L + [(n-1)\beta/4(n-2)]C_iC_j.$$

It is observed that (4.16_1) and (4.16_2) are of somewhat complicated form, but these lead to the T-tensor of rather simple form. In fact, if we put

$$\begin{aligned} &H_{hijk} = h_{hi}h_{jk} + h_{hj}h_{ki} + h_{hk}h_{ij}, \\ &H^{(c)}_{hijk} = \mathfrak{S}_{(ijk)}\{h_{hi}C_jC_k + h_{ij}C_kC_h\}, \qquad C^{(4)}_{hijk} = C_hC_iC_jC_k, \end{aligned}$$

it then follows from these equations and (2.1) that the T-tensor T_{hijk} is written in the form

$$(4.21)\qquad T_{hijk}/L = T^{(\tau)}_1 H_{hijk} + T^{(\tau)}_2 H^{(c)}_{hijk} + T^{(\tau)}_3 C^{(4)}_{hijk}, \qquad \tau = 1, 2,$$

where the coefficients $T^{(\tau)}_1$, $T^{(\tau)}_2$ and $T^{(\tau)}_3$ are given, according as the ordinal number $\tau =$ 1 or 2 of the kind, as follows:

$$(4.22_1)\qquad T^{(1)}_1 = (p\alpha + p_c)pC^2/(n+1)(n+1-2p), \quad T^{(1)}_2 = qW, \quad T^{(1)}_3 = \gamma/C^2,$$

$$(4.22_2)\qquad \begin{aligned} &T^{(2)}_1 = -\beta C^2/4(n-2), \qquad T^{(2)}_2 = \beta(n-1)/4(n-2), \\ &T^{(2)}_3 = -(n^2-1)\beta/4(n-2)C^2. \end{aligned}$$

Summarizing up all the above, we have

Proposition 1. (1) *The characteristic scalar p of a semi-C-reducible Finsler space* F^n $(n \geqq 4)$ *of the first kind must be such that* $\dot{\partial}_i p$ *is proportional to* C_i, *and the T-tensor of the* F^n *is written in the form* (4.21) $(\tau=1)$. (2) *As to a semi-C-reducible* F^n $(n \geqq 4)$ *of the second kind, the tensor* $C^i|_i$ *vanishes and the T-tensor is written in the form* (4.21) $(\tau=2)$.

We are concerned with the exceptional case $n=3$. Comparing (1.7) with (1.2) and paying attention to $h_{ij}=m_i m_j+n_i n_j$ and $C_i=Cm_i$, we get $J=0$ as the condition of semi-C-reducibility. Further we get $H=(1-p/4)LC$ and $I=pLC/4$. The space is of the second kind iff $p=2$, i.e., $H=I$. It is, however, easily shown ([12], § 29 of [17]) that $H=I$ and $J=0$ cause $S_{hijk}=0$ immediately. Thus we have

Proposition 2. *A three-dimensional Finsler space is semi-C-reducible, iff one of the main scalars J vanishes identically. The characteristic scalar p is equal to* $p=4I/(LC)$. *The space is of the second kind, iff* $H=I$, *and the v-curvature tensor* S_{hijk} *vanishes in this case.*

The T-tensor of F^3 is expressed by the scalar components $T_{\alpha\beta\gamma\delta}(\alpha, \beta, \gamma, \delta=1, 2, 3)$ with reference to the Moór frame. In the case of $J=0$ we obtain ((29.19′) and (29.22′) of [17]) $T_{1\beta\gamma\delta}=0$ and

$$T_{222\delta}=H_{;\delta}, \qquad T_{223\delta}=(H-2I)v_\delta,$$
$$T_{233\delta}=I_{;\delta}, \qquad T_{333\delta}=3Iv_\delta, \qquad (\delta=2, 3).$$

The symmetry property of $C_{ijk}|_h$ is written in the form (29.20′) of [17]; in the case of $J=0$ it is written as

$$(H-2I)v_2=H_{;3}, \quad (H-2I)v_3=I_{;2}, \quad 3Iv_2=I_{;3}.$$

We consider the condition for a semi-C-reducible F^n $(n \geqq 4)$ to be "T-tensor$=0$". In § 2 we already have (2.4) $(\alpha=0)$ and (2.6) $(\beta=0)$. It is obvious from (4.21) and (4.22_2) that $\beta=0$ is sufficient for the space of the second kind to be "T-tensor$=0$". In the case of the first kind, (4.10) yields $p_c=0$ and (4.15) does $p_k=0$. Conversely, $\alpha=p_k=0$ lead us to $W=0$ from (4.12) and $\gamma=0$ from (4.17), so that (4.21) and (4.22_1) yield $T_{hijk}=0$. Consequently we have

Theorem 5. *A necessary and sufficient condition for a semi-C-reducible Finsler space* F^n $(n \geqq 4)$ *to have the vanishing T-tensor is as follows*:

(1) *For the* F^n *of the first kind*: $C^i|_i=0$ *and the characteristic scalar p is a function of position alone.*

(2) *For the* F^n *of the second kind*: $C^2|_i C^i=0$.

The meaning of Theorem 5 is that the system of differential equations (2.2) is reduced to a single equation $C^i|_i=0$ or $C^2|_i C^i=0$ on the assumption of semi-C-reducibility, similarly to the case of C-reducibility (cf. (30.28) of [17]).

§ 5. Semi-C-reducible Landsberg spaces

As to a C-reducible space, it is known ([11], § 30 of [17]) that a C-reducible Landsberg space is a Berwald space. We shall try to generalize this theorem to the case of semi-C-reducible space. From (1.7) the h-covariant derivative $C_{jki|h}$ of C_{jki} is written in the form

$$C_{jki|h} = \mathfrak{S}_{(ijk)}\{h_{ij}(p_{|h}C_k + pC_{k|h})/(n+1) + qC_{i|h}C_jC_k/C^2\} - (p_{|h}/C^2 + qC^2{}_{|h}/C^4)C_iC_jC_k. \tag{5.1}$$

A Berwald space is characterized by the equation $C_{jki|h}=0$. As $C_{i|h}=0$ and $C^2{}_{|h}=0$ are derived from it, (5.1) leads immediately to $p_{|h}=0$; the characteristic scalar p being h-covariant constant. Conversely $p_{|h}=0$ and $C_{i|h}=0$ imply $C_{jki|h}=0$ by (5.1). Therefore we have

Proposition 3. *A semi-C-reducible Finsler space is a Berwald space iff the characteristic scalar p and the torsion vector C_i are h-covariant constant.*

Next we treat a semi-C-reducible F^n ($n \geqq 4$) which is a Landsberg space, i.e., the hv-curvature tensor

$$P_{ijkl} = C_{jkl|i} - C_{ikl|j} + C_{ikr}P_j{}^r{}_l - C_{jkr}P_i{}^r{}_l \tag{5.2}$$

vanishes. In this section we shall use the notations

$$p'_k = p_{|k}, \qquad p'_c = p_kC^k/C, \qquad \alpha' = C^i_{|i}/C, \qquad \beta' = C^2_{|i}C^i/C^2. \tag{5.3}$$

Contracting (5.1) by g^{hk} and paying attention to $C_{jkl|i} = C_{ikl|j}$ from $P_{ijkl}=0$, we obtain

$$\begin{aligned}(n+1-2p)C_{i|j} = {} & (p\alpha' + p'_c)Ch_{ij} + C_ip'_j + C_jp'_i \\ & - p_0(C_il_j + C_jl_i) + (n+1)q(C_iC^2_{|j} + C_jC^2_{|i})/2C^2 \\ & + (n+1)\{q(\alpha'-\beta') - p'_c\}C_iC_j/C.\end{aligned} \tag{5.4}$$

From one of the Bianchi identities ((17.17) of [17]) we have $S_{hijk|l}=0$ for a Landsberg space. Thus (1.8) implies

$$h_{hj}M_{ik|l} + h_{ik}M_{hj|l} - h_{hk}M_{ij|l} - h_{ij}M_{hk|l} = 0. \tag{5.5}$$

Contracting (5.5) by g^{hj} and putting $M = g^{ij}M_{ij}$, we have $(n-3)M_{ik|l} + h_{ik}M_{|l} = 0$. Further, contracting by g^{ik}, we have $M_{|l}=0$, so that $M_{ij|k}=0$ because of $n \geqq 4$. It follows from (1.9) that the last equation is written in the form

$$\frac{p}{2(n+1)}(2C^2p_{|k} + pC^2_{|k})h_{ij} + \left(\frac{2p}{n+1} + 1 - 2p\right)C_iC_jp_{|k} + p\left(1 - \frac{np}{n+1}\right)(C_{i|k}C_j + C_{j|k}C_i) = 0. \tag{5.6}$$

Contraction of (5.6) by C^k yields

$$\frac{pC^3}{2(n+1)}(2p'_c+p\beta')h_{ij}+C_iD_j+C_jD_i=0, \tag{5.7}$$

where we put

$$D_j=(Cp'_c/2)\left(\frac{2p}{n+1}+1-2p\right)C_j+(p/2)\left(1-\frac{np}{n+1}\right)C^2_{|j}. \tag{5.8}$$

It is observed from (5.7) that the rank of the matrix (h_{ij}) becomes less than three if $2p'_c+p\beta'\neq 0$, contradicting to $n\geqq 4$. Thus

$$2p'_c+p\beta'=0 \tag{5.9}$$

and $C_iD_j+C_jD_i=0$. The latter leads immediately to $D_j=0$, i.e.,

$$(Cp'_c/2)\left(\frac{2p}{n+1}+1-2p\right)C_j+(p/2)\left(1-\frac{np}{n+1}\right)C^2_{|j}=0. \tag{5.10}$$

Contraction of (5.10) by C^j and (5.9) yield $p'_c=0$ at once, so that $\beta'=0$ from (5.9), and (5.10) yields (I) $np\neq n+1$ and $C^2_{|j}=0$ or (II) $np=n+1$.

On the other hand, contraction of (5.6) by g^{ij} gives

$$C^2qp_{|k}+p(1-p/2)C^2_{|k}=0. \tag{5.11}$$

In the case (I) (5.11) is reduced to $p_{|k}=0$. Then (5.6) is reduced to $C_{i|k}C_j+C_{j|k}C_i=0$, so that $C_{i|k}=0$ by contraction by C^j.

In the case (II), (5.6) is immediately reduced to $C^2_{|k}=0$. Then we see from (1.9) that M_{ij} is proportional to h_{ij} and the space is $S3$-like, as it is shown from (1.8). Consequently we have

Proposition 4. *All the semi-C-reducible Landsberg spaces of dimension $n\geqq 4$ are divided into the following two classes:*

(I) $np\neq n+1, p_{|i}=0$ *and* $C_{i|j}=0$.

(II) $np=n+1$, $C^2_{|i}=0$ *and the space is $S3$-like.*

In the case (II), from (5.4) we have

$$(n-2)C_{i|h}=-\frac{\alpha'}{C}(C^2h_{hi}-C_iC_h). \tag{5.12}$$

Therefore $C_{i|h}=0$ is equivalent to $\alpha'=0$. From this fact and the last two propositions the following conclusion is obvious.

Theorem 6. (1) *A semi-C-reducible Landsberg space belonging to the class* (I) *of Proposition* 4 *is a Berwald space.*

(2) *A semi-C-reducible Landsberg space belonging to the class* (II) *of Proposition* 4 *is $S3$-like. It is a Berwald space, iff $C^i_{|i}$ vanishes identically.*

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