J. Math. Kyoto Univ. (JMKYAZ)
20-1 (1980) 179-190

# On the $\mathcal{E}$-well posedness for the Goursat problem with constant coefficients

By

Tatsuo NISHITANI

(Received Nov. 29, 1978)

## 1. Introduction.

Recently, Y. Hasegawa [1] investigated the Goursat problem for the general equations in the case when the initial hyperplane is simple characteristic.

In this paper, we consider the case when the initial hyperplane is simple characteristic or the case that the differential operator is hyperbolic with respect to some direction. And, from somewhat different point of view, we shall give a necessary and sufficient condition for the $\mathcal{E}$-well posedness of the Goursat problem with constant coefficients.

We state problems, assumptions, and results. We consider the following problem

$$(P)\qquad \begin{cases} A(D_t, D_x, D_y)u=0 & t>0,\ x\in R^1,\ y\in R^n \\ D_x^i u=g_i(t, y) & t>0,\ y\in R^n,\ x=0,\ i=0, 1, \cdots l-1 \\ D_t^j u=h_j(x, y) & t=0,\ x\in R^1,\ y\in R^n,\ j=0, 1, \cdots, m-l-1 \\ \left(D_t=\dfrac{1}{i}\dfrac{\partial}{\partial t},\ D_x=\dfrac{1}{i}\dfrac{\partial}{\partial x},\ D_y=\left(\dfrac{1}{i}\dfrac{\partial}{\partial y_1}, \cdots, \dfrac{1}{i}\dfrac{\partial}{\partial y_n}\right)\right) & \end{cases}$$

where we impose on the data $(g_i, h_j)$ the following compatibility condition.

$$(C)\qquad D_x^i h_j(0, y)=D_t^j g_i(0, y)$$
$$\text{for}\quad i=0, 1, \cdots, l-1,\ j=0, 1, \cdots, m-l-1,\ y\in R^n.$$

We assume that $A$ is a differential operator of order $m$ with constant coefficients, and written as follows

$$(1.1)\qquad A(\tau, \zeta, \eta)=\sum_{j=l}^{m} C_j(\zeta, \eta)\tau^{m-j}=\sum_{j=0}^{k} D_{m-j}(\tau, \eta)\zeta^j$$

where $C_j(\zeta, \eta)$ is a polynomial of order$\leqq j$, and $C_l^0(1, 0)=1$ ($C_l^0$ means the homogeneous part of degree $l$ of $C_l$).

**Remark.** If the hyperplane $t=0$ is simple characteristic for $A$, then $A$ is written in the form (1.1) always. In other case, if we assume that $A$ is hyperbolic with respect to some direction, and $t=0$ is a characteristic hyperplane, then $A$ has the form (1.1).

We shall say that the problem $(P)$ is $\mathcal{E}$-well posed, if for every $g_i \in C^\infty(\overline{R^1_+}\times R^n)$, $h_j \in C^\infty(R^1\times R^n)$ with compatibility condition $(C)$, there exists a unique solution $u\in C^\infty(\overline{R^1_+}\times R^1\times R^n)$. Then we have

**Theorem 1.1.** *In order that $(P)$ is $\mathcal{E}$-well posed, it is necessary and sufficient that the following condition $(G)$ is fulfilled.*

$$(G)\qquad \begin{cases} \text{There exists a positive constant } \varepsilon>0, \text{ such that for every } \delta \text{ with} \\ 0<|\delta|\leqq\varepsilon,\ A(\tau, \zeta, \eta) \text{ is hyperbolic with respect to } (1, \delta, 0). \end{cases}$$

From the condition $(G)$, we have more concrete necessary conditions. When $l=1$, the next theorem is obtained in [1] by constructing the exact analytic solutions.

**Theorem 1.2.** *If $(P)$ is $\mathcal{E}$-well posed, then the principal part $A_m$ of $A$ is decomposed as follows:*

$$A_m(\tau, \zeta, \eta)=C_l^0(\zeta, \eta)Q_{m-l}(\tau, \zeta, \eta)$$

*where $Q_{m-l}$ is a homogeneous polynomial of degree $m-l$ which is hyperbolic with respect to $(1, 0, 0)$.*

From the hyperbolicity of $A$ with respect to $(1, \delta, 0)$ and the theorem 1.2, we have

**Corollary 1.1.** *If $(P)$ is $\mathcal{E}$-well posed, then $A$ is written as follows:*

$$A(\tau, \zeta, \eta)=\tilde{C}_l(\zeta, \eta)\tilde{Q}_{m-l}(\tau, \zeta, \eta)+R_{m-2}(\tau, \zeta, \eta) \tag{1.2}$$

*where $\tilde{C}_l$, $\tilde{Q}_{m-1}$ is the polynomial with principal part $C_l^0$, $Q_{m-l}$ respectively, and $R_{m-2}$ is a polynomial of degree at most $m-2$.*

**Theorem 1.3.** (Hasegawa [1]) *Let $l=1$, and assume that $Q_{m-1}$ is strictly hyperbolic with respect to $(1, 0, 0)$. Then for the Goursat problem $(P)$ is $\mathcal{E}$-well posed, it is necessary and sufficient that the decomposition condition (1.2) holds.*

*Proof.* It will suffice to prove the sufficiency. Since $Q_{m-1}$ is strictly hyperbolic with respect to $(1, 0, 0)$, we have

$$R_{m-2}(\lambda+i\tau, \xi+i\delta\tau, \eta)(Q_{m-1}(\lambda+i\tau, \xi+i\delta\tau, \eta))^{-1}=O(|\tau|^{-1})$$
$$(\tau\in R,\ |\tau|\longrightarrow\infty)$$

for sufficiently small $\delta$. If we note (1.2), this shows that $A$ satisfies the condition $(G)$.

## 2. Necessity of the condition $(G)$.

If $(P)$ is well posed, then by the closed graph theorem, there are positive constants $T$, $X$, $Y$, $C$ and integer $N$, such that

$$\text{(2.1)} \qquad |u(1, \mp 1, 0)| \leqq C \sum_{j=0}^{m-l-1} \sup_{\substack{|x| \leqq X, |y| \leqq Y \\ k+|\alpha| \leqq N}} |D_x^k D_y^\alpha h_j(x, y)| + C \sum_{i=0}^{l-1} \sup_{\substack{0 \leqq t \leqq T, |y| \leqq Y \\ h+|\alpha| \leqq N}} |D_t^k D_y^\alpha g_i(t, y)|$$

holds for every solutions of $(P)$.

**Lemma 2.1.** *If* $(P)$ *is well posed, then there exists* $\varepsilon > 0$ *such that for every* $\delta$ $(0<\delta<\varepsilon)$, *there is a* $P_\delta > 0$ *with*

$$\text{(2.2)} \qquad A(\tau, \zeta, \eta) \neq 0 \quad \textit{for} \quad \operatorname{Im} \tau < -P_\delta(\log(1+|\tau|+|\zeta|+|\eta|)+1)$$
$$|\operatorname{Im} \zeta| = \delta |\operatorname{Im} \tau|, \ \eta \in R^n.$$

*Proof.* We take $\varepsilon$ such that $0<\varepsilon X<1$, and show that this lemma holds with this $\varepsilon$. We assume that there is a $\delta>0$ $(0<\delta<\varepsilon)$ such that for every $p$ there exists $\tau_p, \zeta_p, \eta_p$ with $A(\tau_p, \zeta_p, \eta_p)=0$ and

$$\operatorname{Im} \tau_p < -p(\log(1+|\tau_p|+|\zeta_p|+|\eta_p|)+1),$$
$$|\operatorname{Im} \zeta_p| = \delta |\operatorname{Im} \tau_p|, \ \eta_p \in R^n.$$

For fixed $\eta_p$, the equation $A(\tau, \zeta, \eta_p)=0$ has $l$ roots which are bounded and separated from other roots when $\tau \to \infty$. Therefore we construct the solution of the following problem by Fourier-Laplace transformation.

$$\begin{cases} A(D_t, D_x, \eta_p) v_p(t, x) = 0 \\ D_x^i v_p(t, 0) = a(t) \zeta_p^i e^{i\tau_p t} \qquad i=0, 1, \cdots l-1 \\ v_p(t, x) \equiv 0 \qquad \text{if} \quad t \leqq 1 \end{cases}$$

where $a(t)$ is a $C^\infty$-function on $R$ with $a(t) \equiv 0$ if $t \leqq 1$ and $a(t) \equiv 1$ if $1+2^{-1}\delta \leqq t$.

Now set

$$u_p(t, x, y) = (e^{i\tau_p t} e^{i\zeta_p x} - v_p(t, x)) e^{i\eta_p y}$$

then it follows that

$$\begin{cases} A(D_t, D_x, D_y) u_p(t, x, y) = 0 \\ D_x^i u_p(t, 0, y) = (1-a(t)) \zeta_p^i e^{i\tau_p t} e^{i\eta_p y} \qquad i=0, 1, \cdots l-1 \\ D_t^j u_p(0, x, y) = \tau_p^j e^{i\zeta_p x} e^{i\eta_p y} \qquad j=0, 1, \cdots, m-l-1 \\ u_p(1, \pm 1, 0) = e^{i\tau_p} e^{\pm i\zeta_p} \end{cases}$$

Since the support of $(1-a(t))$ is contained in $t \leqq 1+2^{-1}\delta$, and $-\operatorname{Im} \tau_p > p(\log(1+|\tau_p|+|\zeta_p|+|\eta_p|)+1)$, $|\operatorname{Im} \zeta_p| = \delta |\operatorname{Im} \tau_p|$, the solutions $u_p$ violate the inequality (2.1) when $p \to \infty$.

The next lemma follows from Seidenberg-Tarski's lemma ([5] c. f. Appendix in [3]).

**Lemma 2.2.** *Assume that there are constants $\delta>0$, $P>0$ such that*

$$A(\tau,\zeta,\eta)\neq 0 \quad \textit{when} \quad \operatorname{Im}\tau<-P(\log(1+|\tau|+|\zeta|+|\eta|)+1)$$
$$|\operatorname{Im}\zeta|=\delta|\operatorname{Im}\tau|,\ \eta\in R^n$$

*then there exists $M>0$ with*

$$A(\tau,\zeta,\eta)\neq 0 \quad \textit{for} \quad \operatorname{Im}\tau<-M,\ |\operatorname{Im}\zeta|=\delta|\operatorname{Im}\tau|. \tag{2.3}$$

This lemma shows the necessity of the condition $(G)$.

## 3. Investigations of the condition $(G)$.

First we improve the condition $(G)$ more useful form.

**Lemma 3.1.** *Assume the condition $(G)$, then there exists $\varepsilon_1>0$ such that for every $\delta$ $(0<\delta\leqq\varepsilon_1)$, there is a costant $P_\delta>0$ with*

$$A(\tau,\zeta,\eta)\neq 0 \quad \textit{for} \quad \operatorname{Im}\tau<-P_\delta(|\operatorname{Im}\eta|+1),\ \delta\leqq|\operatorname{Im}\zeta|(|\operatorname{Im}\tau|)^{-1}\leqq\varepsilon_1. \tag{3.1}$$

*Proof.* First we take $r>0$, so that $A_m(1,\xi,\eta)\neq 0$ for $\delta\leqq|\xi|\leqq\varepsilon$, $|\eta|\leqq r$, $\xi\in R$, $\eta\in R^n$. From the assumption, for every $s$ $(0<s\leqq\varepsilon)$, there exists $\tau_s>0$ such that

$$A(\lambda+i\tau+i\sigma,\ \xi\pm is(\tau+\sigma),\ \eta+i\theta\sigma)\neq 0$$

when $\tau<-\tau_s$, $\sigma\leqq 0$, $|\theta|\leqq r$, $\lambda$, $\xi\in R$, $\theta$, $\eta\in R^n$ (c.f. Theorem 5.5.4 in [3]).

For any fixed $\eta\in C^n$, we choose $\sigma\leqq 0$ so that $\operatorname{Im}\eta=\theta\sigma$ with $|\theta|\leqq r$. Then it follows that

$$A(\tau,\zeta,\eta)\neq 0 \quad \text{for} \quad \operatorname{Im}\tau<-\tau_s-r^{-1}|\operatorname{Im}\eta|,\ s|\operatorname{Im}\tau|=|\operatorname{Im}\zeta|. \tag{3.2}$$

Thus it will suffice to show that $\tau_s$ are bounded from above in $\delta\leqq s\leqq\varepsilon_1$ with some fixed constant $\varepsilon_1>0$.

We set

$$T_\rho=\{(\tau,\zeta,\eta,\nu)\in C^{n+2}\times R\,;\,A(\tau,\zeta,\eta)=0,\ \rho^2|\operatorname{Im}\zeta|^2=|\operatorname{Im}\tau|^2,\ |\operatorname{Im}\eta|^2=\nu^2,\ \nu\geqq 0\}$$

By Seidenberg-Tarski's lemma, it follows that

$$\sup_{T_\rho}(-\operatorname{Im}\tau-r^{-1}\nu)\equiv+\infty,\ \text{ or }\ =C\rho^a(1+o(1)) \quad \text{when} \quad \rho\geqq\rho_1$$

But (3.2) shows the impossibility of the former case, thus we obtain (3.1).

From now on, we assume (3.1). The following lemma is easily verified.

**Lemma 3.2.** *With the same constants in lemma 3.1, it follows that*

$$\begin{aligned} &C_l(\zeta,\eta)\neq 0 && \textit{if} \quad |\operatorname{Im}\zeta|\geqq\delta P_\delta(|\operatorname{Im}\eta|+1)\\ &D_{m-k}(\tau,\eta)\neq 0 && \textit{if} \quad \operatorname{Im}\tau<-P_\delta(|\operatorname{Im}\eta|+1)\end{aligned} \tag{3.3}$$

*Especially, $C_l(\zeta,\eta)$ is hyperbolic with respect to $(1, 0)$.*

**Lemma 3.3.** *The equation $A_m(\tau, \zeta, \eta)=0$ has only real roots $\tau$ for any real $(\zeta, \eta)\in R^{n+1}$ with $C_l^0(\zeta, \eta)\neq 0$.*

*Proof.* We assume that there exist $\hat{\tau}, \hat{\zeta}, \hat{\eta}$ such that

$$C_l^0(\hat{\zeta}, \hat{\eta})\neq 0, \quad A_m(\hat{\tau}, \hat{\zeta}, \hat{\eta})=0, \quad \operatorname{Im}\hat{\tau}\neq 0, \quad (\hat{\zeta}, \hat{\eta})\in R^{n+1}.$$

Choose $\hat{\varepsilon}>0$ sufficiently small so that the equation $A_m(\tau, \hat{\zeta}+i\hat{\varepsilon}, \hat{\eta})=0$ hsa a solution $\tau$ with $|\tau-\hat{\tau}|<2^{-1}|\operatorname{Im}\hat{\tau}|$ and $C_l^0(\hat{\zeta}+i\hat{\varepsilon}, \hat{\eta})\neq 0$, $\hat{\varepsilon}/|\operatorname{Im}\hat{\tau}|<2^{-1}\varepsilon$. Here we consider the solution of the following equation in $\tau$.

$$A(\tau, \lambda\hat{\zeta}+i\lambda\hat{\varepsilon}, \lambda\hat{\eta})=\lambda^m\{A_m(\sigma, \hat{\zeta}+i\hat{\varepsilon}, \hat{\eta})+\lambda^{-1}A_{m-1}(\sigma, \hat{\zeta}+i\hat{\varepsilon}, \hat{\eta})+\cdots+\lambda^{-m}A_0\}=0$$

(where we used the notation $\tau=\lambda\sigma$). For sufficiently large $\lambda$, this equation has a solution $\tau(\lambda)$ such that $\hat{\delta}\leqq|\lambda\hat{\varepsilon}|(|\operatorname{Im}\tau(\lambda)|)^{-1}\leqq\varepsilon_1$ with some constant $\hat{\delta}>0$. When $\lambda\to+\infty$ or $\lambda\to-\infty$ these solutions violate the condition (3.1).

**Lemma 3.4.** *There are constants $\delta>0$, $P>0$ such that*

$$A(\lambda+i\tau+i\sigma, \xi\pm i\delta\tau, \eta)\neq 0 \tag{3.4}$$
$$\textit{for}\quad \operatorname{Re}\tau<-P(|\operatorname{Im}\eta|+1), \ \operatorname{Re}\sigma\leqq 0, \ \lambda, \xi\in R, \ \eta\in C.$$

*Proof.* From lemma 3.3, we can take $\delta>0$ $(0<\delta\leqq\varepsilon_1)$ so that the equation $A_m(1+u, \pm\delta, 0)=0$ has only negative zeros $u$. And, in virtue of lemma 3.2 and (3.1), there is a $P>0$ such that

$$C_l(\xi\pm i\delta\tau, \eta)\neq 0, \quad A(\lambda+i\tau, \xi\pm i\delta\tau, \eta)\neq 0 \tag{3.5}$$
$$\text{for}\quad \operatorname{Re}\tau<-P(|\operatorname{Im}\eta|+1), \ \lambda, \xi\in R, \ \eta\in C^n$$

We shall prove this lemma with above constants $\delta$, $P$. The inequality (3.4) follows from (3.5) when $\operatorname{Re}\sigma=0$. We shall study the zeros of $A(\lambda+i\tau+i\sigma, \xi\pm i\delta\tau, \eta)=0$ considered as a polynomial in $\sigma$, when $\tau$ varies in the half plane $\operatorname{Re}\tau<-P(|\operatorname{Im}\eta|+1)$. If we note that the coefficient of $\sigma^{m-l}$ never vanish when $\operatorname{Re}\tau<-P(|\operatorname{Im}\eta|+1)$, it will suffice to prove that there are no zeros with $\operatorname{Re}\sigma<0$ when $\tau$ is a sufficient large negative number. With the notation $\sigma=u\tau$, the equation $A(\lambda+i\tau+i\sigma, \xi\pm i\delta\tau, \eta)=0$ converges to the equation $A_m(1+u, \pm\delta, 0)=0$ when $\tau\to-\infty$. This equation has only negative zeros from the assumption, and which completes the proof.

**Lemma 3.5.** *There are constants $\delta_1$, $P_1>0$ such that if* $\operatorname{Im}\tau<-M(|\operatorname{Im}\eta|+1)$, *$M>P_1$, the equation $A(\tau, \zeta, \eta)=0$ in $\zeta$ has exactly $l$ roots with $|\operatorname{Im}\zeta|<\delta_1 M(|\operatorname{Im}\eta|+1)$.*

*Proof.* We set $\delta_1=\delta$ ($\delta$ is the constant in lemma 3.4), and take $P_1$ so that the equation $C_l(\zeta, \eta)=0$ has $l$ roots with $|\operatorname{Im}\zeta|<\delta_1 P_1(|\operatorname{Im}\eta|+1)$, $D_{m-k}(\tau, \eta)$ never vanish when $\operatorname{Im}\tau<-P_1(|\operatorname{Im}\eta|+1)$ and $P_1>P$ ($P$ is the same constant in lemma 3.4).

We consider the zeros of $A(\tau, \zeta, \eta)$ regarded as a polynomial in $\zeta$, when $\operatorname{Im}\tau<-M(|\operatorname{Im}\eta|+1)$. In virtue of lemma 3.4, it follows that

(3.6) $\qquad A(\tau, \zeta, \eta)\neq 0 \quad \text{if} \quad |\mathrm{Im}\,\zeta|=\delta M(|\mathrm{Im}\,\eta|+1),\ \mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1).$

For a fixed $\eta\in C^n$, the zeros of $A(\tau, \zeta, \eta)=0$ are expressed in Puiseux series in the form

$$\zeta(\tau)=\sum_{j=N}^{-\infty} A_j(\tau^{1/p})^j$$

when $|\tau|>K(\eta)$ where $K(\eta)$ is a some constant which depends on $\eta$. We assume that $|\mathrm{Im}\,\zeta(\tau)|<\delta M(|\mathrm{Im}\,\eta|+1)$ holds at some $\tau$ with $\mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1)-K(\eta)$, then in virtue of the continuity of $\zeta(\tau)$ and (3.6), it follows that $|\mathrm{Im}\,\zeta(\tau)|<\delta M(|\mathrm{Im}\,\eta|+1)$ is also valid in the half plane $\mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1)-K(\eta)$ In this case the coefficients $A_j$ must vanish when $1\leqq j\leqq N$. In fact, if $A_j$ is the first non-vanishing coefficient and $1\leqq j\leqq N$, $\zeta(\tau)$ will behave asymptotically as $A_j(\tau^{1/p})^j$ when $\tau\to\infty$. But, $\mathrm{Im}\,A_j(\tau^{1/p})^j$ is bounded only on several lines, which contradicts to the boundedness of $\mathrm{Im}\,\zeta(\tau)$ in $\mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1)-K(\eta)$. Hence $\zeta(\tau)=O(1)$ which means that, if $\mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1)-K(\eta)$, the zeros of $A(\tau, \zeta, \eta)$ with $|\mathrm{Im}\,\zeta|<\delta M(|\mathrm{Im}\,\eta|+1)$ must be bounded by some constant $B(\eta)$.

Now we shall give the proof of this lemma. Note that the coefficient of $\zeta^k$ is $D_{m-k}(\tau, \eta)$ which shows that the zeros vary continuously with $\tau$ when $\mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1)$. Thus it is sufficient to show that the equation $A(\tau, \zeta, \eta)=0$ has exactly $l$ roots with $|\mathrm{Im}\,\zeta|<\delta M(|\mathrm{Im}\,\eta|+1)$ when $\mathrm{Im}\,\tau$ is a sufficiently large negative number.

From (1.1) the equation $A(\tau, \zeta, \eta)=0$ has $l$ roots which converge to those of $C_l(\zeta, \eta)=0$ when $\mathrm{Im}\,\tau\to-\infty$. On the other hand, the zeros of $A(\tau, \zeta, \eta)$ with $|\mathrm{Im}\,\zeta|<\delta M(|\mathrm{Im}\,\eta|+1)$ are bounded when $\mathrm{Im}\,\tau\to-\infty$ which shows that there are no other roots with $|\mathrm{Im}\,\zeta|<M(|\mathrm{Im}\,\eta|+1)$. This completes the proof.

**Corollary 3.1.** *There are constants $\delta$, $P$ and integers $\mu$, $\nu$ such that for every $M>P$, if $\mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1)$, the equation $A(\tau, \zeta, \eta)=0$ has $\mu$ roots $\zeta_j^+$ with $\mathrm{Im}\,\zeta_j^+>\delta M(|\mathrm{Im}\,\eta|+1)$, $l$ roots $\zeta_j^0$ with $|\mathrm{Im}\,\zeta_j^0|<\delta P(|\mathrm{Im}\,\eta|+1)$ and $\nu$ roots $\zeta_j^-$ with $\mathrm{Im}\,\zeta_j^-<-\delta M(|\mathrm{Im}\,\eta|+1)$.*

Finally we shall give the proof of theorem 1.2. To prove this theorem, we use the following lemma.

**Lemma 3.6.** *Let*

$$P(\sigma, \zeta)=\sum_{j=l}^{m} a_j(\zeta)\sigma^{j-l}$$

*be a polynomial in $(\sigma, \zeta)$. We assume that the degree of $a_j(\zeta)$ is at most $j$, and $a_l(\zeta)$ is a monic polynomial of degree $l$. Then, in order that there is no root $\zeta(\sigma)$ of $P(\sigma, \zeta)=0$ which is bounded when $|\sigma|\to 0$, and is not constant, it is necessary and sufficient that every $a_j(\zeta)$ is divisible by $a_l(\zeta)$.*

*Proof.* If every $a_j(\zeta)$ is divisible by $a_l(\zeta)$, then $P(\sigma, \zeta)$ is written as follows

$$P(\sigma, \zeta)=a_l(\zeta)\Big\{1+\sum_{j=l+1}^{m}\tilde{a}_j(\zeta)\sigma^{j-l}\Big\}.$$

Therefore, sufficiency is trivial. To prove the necessity we note that $P(\sigma, \zeta)=0$ has at least one root $\zeta(\sigma)$ which is expressed in the form

$$\zeta(\sigma)=\sum_{j=0}^{\infty} A_j(\sigma^{1/p})^j$$

in $0<|\sigma|<c$ with some constant $c$. By the assumption, it follows that $\zeta(\sigma)=A_0$, then we get $a_j(A_0)=0$, $j=l+1, \cdots, m$. Hence $P(\sigma, \zeta)$ is written in the form

$$P(\sigma, \zeta)=(\zeta-A_0)\sum_{j=l}^{m} a_j^1(\zeta)\sigma^{j-l}$$

Hence, if we note that $a_l^1(\zeta)$ is a monic polynomial of order $l-1$, the proof carried out by the iterations of this argument.

*Proof of theorem* 1.2. Since the coefficient of $\zeta^l$ in $C_l^0(\zeta, \eta)$ is equal to 1, it is sufficient to show that every $C_j^0(\zeta, \eta)$ is divisible by $C_l^0(\zeta, \eta)$ as a polynomial in $\zeta$ for any fixed $\eta\in R^n$.

Assume that there exist $j$ $(l+1\leqq j\leqq m)$ and $\hat{\eta}\in R^n$ such that $C_j^0(\zeta, \hat{\eta})$ is not divisible by $C_l^0(\zeta, \hat{\eta})$, and study the solutions of the equation

$$\tau^{l-m}A_m(\tau, \zeta, \hat{\eta})=\sum_{j=l}^{m} C_j^0(\zeta, \hat{\eta})\tau^{l-j}=0. \tag{3.7}$$

Hence it follows from lemma 3.7 that the equation (3.7) has a solution $\zeta(\tau)$ which is holomorphic, bounded and not constant for $\operatorname{Im}\tau<-R$, where $R$ is a some constant. Thus we can take $\hat{\tau}$ such that $\operatorname{Im}\hat{\tau}<\delta^{-1}B$ $(<-R)$, $\operatorname{Im}\zeta(\hat{\tau})=c$ $(\neq 0)$, where $B$ is a bound of $|\zeta(\tau)|$ and $\delta$ is a constant with $0<\delta<\varepsilon_1$ ($\varepsilon_1$ is the constant in lemma 3.1).

Now we consider the zeros of the equation

$$A(\lambda\hat{\tau}, \zeta, \lambda\hat{\eta})=\lambda^m\{A_m(\hat{\tau}, s, \hat{\eta})+\lambda^{-1}A_{m-1}(\hat{\tau}, s, \hat{\eta})+\cdots+\lambda^{-m}A_0\}=0$$

where we have used the notation $\tau=\lambda s$. For sufficiently large $\lambda$, this equation has a solution $\zeta(\lambda)$ which satisfies $\hat{\delta}\leqq|\operatorname{Im}\zeta(\lambda)|(|\operatorname{Im}\lambda\hat{\tau}|)^{-1}\leqq\delta$ with some constant $\hat{\delta}>0$. When $\lambda\to+\infty$, this solutions violate the condition (3.1).

## 4. Existence of the solution of $(P)$.

From lemma 3.4, it follows immediately

**Proposition 4.1.** *Let $\delta$, $P$ be the same constant in lemma* 3.4. *If $A(\tau, \zeta, \eta)=0$ and $|\operatorname{Im}\zeta|>\delta P(|\operatorname{Im}\eta|+1)$, then it follows that $-\operatorname{Im}\tau<\delta^{-1}|\operatorname{Im}\zeta|$.*

First we solve the following ordinary differential equation.

$$\begin{cases} \left\{D_t^{m-l}+\sum_{j=l+1}^{m}(C_l(\zeta, \eta))^{-1}C_j(\zeta, \eta)D_t^{m-j}\right\}w(t, \zeta, \eta)=0 \\ D_t^j w(0, \zeta, \eta)=\hat{h}_j(\zeta, \eta) \qquad j=0, 1, \cdots, m-l-1 \end{cases}$$

where $h_j(\zeta, \eta)$ is the Fourier-Laplace transform of $h_j(x, y)\in C_0^\infty(R^1\times R^n)$. We next define $W^\pm(t, x, y)$ by

$$W^{\pm}(t, x, y)=\int_{\operatorname{Im}\eta_j=\gamma_j} e^{iy\eta}d\eta\int_{\operatorname{Im}\zeta=\pm\Gamma} e^{i\zeta x}w(t, \zeta, \eta)d\zeta$$

where $\gamma=(\gamma_1, \cdots, \gamma_n)$ with $\Gamma>2\delta P(|\gamma|+1)$.

Taking into account of the support of $W^{\pm}$, by use of the suitable partition of unity, we can solve the following problem.

$$\begin{cases} A(D_t, D_x, D_y)W(t, x, y)=0 & t>0\,,\ x\in R\,,\ y\in R^n \\ D_t^j W(0, x, y)=h_j(x, y) & x\in R\,,\ y\in R^n\,,\ j=0, 1, \cdots, m-l-1\,. \end{cases}$$

**Lemma 4.1.** *There exists constants $\delta>0$, $P_2>0$ such that for every $M>P_2$, there are constants $C>0$, $\rho_1>0$ and rational number $a$ with*

$$A(\tau, \zeta, \eta)\neq 0 \qquad \textit{for}\quad \operatorname{Im}\tau<-M(|\operatorname{Im}\eta|+1)\,,\ |\operatorname{Im}\zeta|<\delta M(\operatorname{Im}\eta|+1)\,,$$
$$|\operatorname{Re}\zeta|>C|\eta|^a\,,\ |\eta|\geqq\rho_1\,.$$

*Proof.* From lemma 3.2 and lemma 3.5, we can take $P_2>0, \delta>0$ so that the both equations $C_l(\zeta, \eta)=0$ and $A(\tau, \zeta, \eta)=0$ have exactly $l$ roots with $|\operatorname{Im}\zeta|<\delta P_2(|\operatorname{Im}\eta|+1)$, and $|D_{m-k}(\tau, \eta)|\geqq c$ with some constant $c>0$ for $\operatorname{Im}\tau<-P_2(|\operatorname{Im}\eta|+1)$. We shall prove this lemma with these constants $P_2$, $\delta$.

We set

$$T_\rho=\{(\tau, \zeta, \eta, \nu)\in C^{n+2}\times R\,;\, A(\tau, \zeta, \eta)=0,\ |\eta|^2\leqq\rho^2,\ |\operatorname{Im}\zeta|<\delta M(\nu+1),$$
$$\operatorname{Im}\tau<-M(\nu+1),\ |\operatorname{Im}\eta|^2=\nu^2,\ \nu\geqq 0\}$$

Since lemma 3.5 assures that $T_\rho\neq\phi$, from Seidenberg-Tarski's lemma, it follows that

$$M(\rho)=\sup_{T_\rho}(\operatorname{Re}\zeta)\equiv+\infty \quad\text{or}\quad M(\rho)=c\rho^a(1+o(1)) \qquad\text{for}\quad \rho\geqq\rho_1$$

where $\rho_1$, $c$ are some positive contants and $a$ is a some rational number. Therefore it will suffice to show the impossibility of the former case.

We assume that $M(\rho)\equiv+\infty$ for $\rho\geqq\rho_1$, then there exist $\tau_p$, $\eta_p$, $\zeta_p$ such that $A(\tau_p, \zeta_p, \eta_p)=0$, $|\eta_p|^2\leqq\rho_1^2$, $|\operatorname{Im}\zeta_p|<\delta M(|\operatorname{Im}\eta_p|+1)$, $\operatorname{Im}\tau_p<-M(|\operatorname{Im}\eta_p|+1)$, and $\operatorname{Re}\zeta_p\to+\infty$ when $p\to\infty$. Here, choosing a subsequence if necessary, we may assume that $\eta_p\to\hat\eta$ when $p\to+\infty$.

In virtue of $|D_{m-k}(\tau_p, \eta_p)|\geqq c$ and the boundedness of $\eta_p$, it follows that $|\zeta_p|\leqq C(1+|\tau_p|)^N$ with some constant $C$ and integer $N$. This shows that $|\tau_p|\to\infty$ when $p\to\infty$. If $\operatorname{Im}\tau_p<-M(|\operatorname{Im}\eta_p|+1)$ the equation $A(\tau_p, \zeta, \eta_p)=0$ has exactly $l$ roots with $|\operatorname{Im}\zeta|<\delta M(|\operatorname{Im}\eta_p|+1)$. On the other hand, if $|\tau_p|$ is sufficiently large and $\eta_p$ are in a some neighborhood of $\hat\eta$, the equation $A(\tau_p, \zeta, \eta_p)=0$ has $l$ bounded roots with $|\operatorname{Im}\zeta|<\delta M(|\operatorname{Im}\eta_p|+1)$. This shows that the zeros of $A(\tau_p, \zeta, \eta_p)$ with $|\operatorname{Im}\zeta|<\delta M(|\operatorname{Im}\eta_p|+1)$ are bounded when $\operatorname{Im}\tau_p<-M(|\operatorname{Im}\eta_p|+1)$ and $p\to+\infty$. But this is a obvious contradiction.

From this lemma, we have

**Proposition 4.2.** *For every $\eta\in C^n$, there is a simple closed curve $C_\eta$ in $\zeta$ plane which possesses the following properties:*

(1) $|\mathrm{Im}\,\zeta| \leqq \delta M(|\mathrm{Im}\,\eta|+1)$ *if* $\zeta \in C_\eta$

(2) $C_\eta$ *encloses* $\{\zeta_j^0\}_{1\le j\le l}$ *if* $\mathrm{Im}\,\tau < -M(|\mathrm{Im}\,\eta|+1)$

(3) $A(\tau, \zeta, \eta) \neq 0$ *if* $\zeta \in C_\eta$ *and* $\mathrm{Im}\,\tau < -M(|\mathrm{Im}\,\eta|+1)$

(4) *The length of* $C_\eta$ *is bounded by* $C(1+|\eta|)^N$.

Let us denote

$$\mathcal{P}_k(\tau, \eta; \zeta) = \sum_{j=1}^{l} R_{jk}(\tau, \eta)\zeta^{j-1}\{A_0(\tau, \eta; \zeta)\}^{-1}$$

where $R_{jk}(\tau, \eta)$ is $(k, j)$-cofactor of $\left((2\pi i)^{-1}\oint_{C_\eta}\zeta^{i+j-2}(A_0(\tau, \zeta; \eta))^{-1}d\zeta\right)_{1\le i,j\le l}$ and $A_0(\tau, \eta; \zeta) = \prod_{j=1}^{l}(\zeta - \zeta_j^0(\tau, \eta))$. Then the solution of the problem

$$\begin{cases} A(\tau, D_x, \eta)v(x, \tau, \eta) = 0 \\ D_x^i v(0, \tau, \eta) = \hat{g}_i(\tau, \eta) \qquad i=0, 1, \cdots, l-1 \end{cases}$$

is given by

$$v(x, \tau, \eta) = \sum_{i=1}^{l}(2\pi i)^{-1}\oint_{C_\eta} e^{i\zeta x}\mathcal{P}_k(\tau, \eta; \zeta)d\zeta\,\hat{g}_i(\tau, \eta).$$

For $\theta_i(t, y) \in C_0^\infty(\overline{R_+^1} \times R^n)$ with $D_t^j\theta_i(0, y) = 0$ $(j=0, 1, \cdots, m-l-1)$, we define $\hat{\theta}_i(\tau, \eta)$ and $V(t, x, y)$ by

$$\hat{\theta}_i(\tau, \eta) = \int_0^\infty e^{i\tau t}dt\int e^{iy\eta}\theta_i(t, y)dy$$

$$V(t, x, y) = \int_{\mathrm{Im}\,\eta_j=\gamma_j} e^{iy\eta}d\eta\int_{\mathrm{Im}\,\tau=-\Gamma} e^{i\tau t}v(x, \tau, \eta)d\tau$$

where $\gamma=(\gamma_1, \cdots, \gamma_n)$ and $\Gamma > M(|\gamma|+1)$. Then, using the partition of unity we can solve the following problem.

$$\begin{cases} A(D_t, D_x, D_y)V(t, x, y) = 0 & t>0,\ x\in R,\ y\in R^n \\ D_x^i V(t, 0, y) = \theta_i(t, y) & t>0,\ y\in R^n,\ i=0, 1, \cdots, l-1 \\ D_t^j V(0, x, y) = 0 & x\in R,\ y\in R^n,\ j=0, 1, \cdots, m-l-1. \end{cases}$$

For the brevity, we denote by $C_{x,y}^\infty C_t^{m-l}(C_l)$ the set of all $f$ such that $D_t^j D_x^i D_y^\alpha f(t, x, y)$ exists and continuous for any $i$, $\alpha$ and for $j=0, 1, \cdots, m-l-1$ and that $D_t^j D_x^i D_y^\alpha(C_l(D_x, D_y)f(t, x, y))$ exists and continuous for all $i$, $\alpha$ and for $j=0, 1, \cdots, m-l$. Then it follows that $V(t, x, y)$ belongs to $C_{x,y}^\infty C_t^{m-l}(C_l)$.

If we set $\theta_i = g_i - D_x^i W(t, 0, y)$, the compatibility condition $(C)$ shows that $D_t^j\theta_i(0, y) = 0$ $(j=0, 1, \cdots, m-l-1)$, and thus the solution $u$ which belongs to $C_{x,y}^\infty C_t^{m-l}(C_l)$ of $(P)$ is given by $u = W + V$.

**Proposition 4.3.** *For any given* $g_i \in C^\infty(\overline{R_+^1} \times R^n)$, $h_j \in C^\infty(R^1 \times R^n)$ *with compatibility condition* $(C)$, *there is a solution of* $(P)$ *which belongs to* $C_{x,y}^\infty C_t^{m-l}(C_l)$.

## 5. Uniqueness of the solution of $(P)$.

Let us denote

$$A_+(\tau, \zeta, \eta)=\prod_{j=1}^{\mu}(\zeta-\zeta_j^+), \qquad A_-(\tau, \zeta, \eta)=\prod_{j=1}^{\nu}(\zeta-\zeta_j^-)$$

$$A_+(\tau, \zeta, \eta)=\zeta^{\mu}+a_1^+(\tau, \eta)\zeta^{\mu-1}+\cdots+a_{\mu}^+(\tau, \eta)$$

$$A_-(\tau, \zeta, \eta)=\zeta^{\nu}+a_1^-(\tau, \eta)\zeta^{\nu-1}+\cdots+a_{\nu}^-(\tau, \eta)$$

and consider the following problems

$$(P)_+ \qquad \begin{cases} A_+(\tau, D_x, \eta)u_j(x, \tau, \eta)=0 \\ D_x^{i-1}u_j(0, \tau, \eta)=\delta_{ij} \qquad i, j=1, 2, \cdots, \mu \end{cases}$$

$$(P)_- \qquad \begin{cases} A_-(\tau, D_x, \eta)u_{\mu+j}(x, \tau, \eta)=0 \\ D_x^{\mu+i-1}u_{\mu+j}(0, \tau, \eta)=\delta_{ij} \qquad i, j=1, 2, \cdots, \nu \end{cases}$$

where $\delta_{ij}$ is the Kronecker's delta.

We set

$$W(\tau, \eta)=\det(D_x^{i-1}u_j(0, \tau, \eta))_{1\leq i, j\leq \mu+\nu}$$

then we have

**Lemma 5.1.** *There exists positive constants $C$, $a$, $M$ such that*

$$(5.1) \qquad |W(\tau, \eta)|\geqq C(1+|\tau|+|\eta|)^{-a} \qquad \textit{for} \quad \operatorname{Im}\tau<-M(|\operatorname{Im}\eta|+1).$$

*Proof.* First we note that $W(\tau, \eta)\neq 0$, and there exists a polynomial $F$ of $\mu+\nu$ variables such that

$$(5.2) \qquad (a_{\nu}^-(\tau, \eta))^{\mu}W(\tau, \eta)=F(a_1^+, \cdots, a_{\mu}^+, a_1^-, \cdots, a_{\nu}^-).$$

Denote

$$T_{\rho}=\Big\{(\tau, \eta, \xi_1, \cdots, \xi_k, \nu)\in C^{k+n+1}\times R\,;\, D_{m-k}(\tau, \eta)\sum_{i=1}^{k}\xi_i$$

$$=-D_{m-k-1}(\tau, \eta), \cdots, D_{m-k}(\tau, \eta)\prod_{i=1}^{k}\xi_i$$

$$=(-1)^kD_m(\tau, \eta),\ \operatorname{Im}\tau<-M(\nu+1),\ |\eta|^2\leqq\rho^2,\ |\operatorname{Im}\eta|^2$$

$$=\nu^2,\ \nu\geqq 0,\ \operatorname{Im}\xi_1\geqq\cdots\geqq\operatorname{Im}\xi_{\mu}\geqq\cdots\geqq\operatorname{Im}\xi_{\mu+l+1}\geqq\cdots\geqq\operatorname{Im}\xi_k\Big\}$$

then Seidenberg-Tarski's lemma gives that

$$\operatorname*{Sup}_{T_\rho}\Big\{-\Big|F\Big(-\sum_{i=1}^{\mu}\xi_i, \cdots, (-1)^{\mu}\prod_{i=1}^{\mu}\xi_i, -\sum_{i=1}^{\nu}\xi_{\mu+l+1}, \cdots, (-1)^{\nu}\prod_{i=1}^{\nu}\xi_{\mu+l+i}\Big)\Big|^2\Big\}$$

$$=C\rho^{a}(1+o(1)) \qquad \text{when} \quad \rho\geqq\rho_1.$$

Hence, from (5.2) and the definition of $T_{\rho}$, it follows, with positive constants $C$, that

(5.3) $$\left|\left(\prod_{i=1}^{\nu}\xi_{\mu+l+i}\right)^{\mu}W(\tau,\eta)\right|\geqq C(1+|\eta|)^{-\alpha}\quad \text{if}\quad \mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1).$$

When $\mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1)$, we may assume that $|D_{m-k}(\tau,\eta)|\geqq c$ $(c>0)$ which shows that

$$|\xi_i|\leqq B(1+|\tau|+|\eta|)^K \qquad \text{If}\quad \mathrm{Im}\,\tau<-M(|\mathrm{Im}\,\eta|+1)$$

with some constants $B$ and $K$. Combining this estimate with (5.3), we get (5.1).

Here, we consider the following adjoint problem.

$$(P^*)\qquad \begin{cases} A^*(D_t, D_x, D_y)u^{\pm}=0 & x\gtrless 0,\ t\in R,\ y\in R^n \\ D_t^j u^{\pm}(T, x, y)=0 & x\gtrless 0,\ y\in R^n,\ j=0, 1, \cdots, m-l-1 \\ D_x^{i-1}u^+(t, 0, y)+D_x^{i-1}u^-(t, 0, y)=g_i(t, y) & i=1, 2, \cdots, \mu+\nu \end{cases}$$

where $A^*(\tau, \zeta, \eta)=\overline{A(\bar\tau, \bar\zeta, \bar\eta)}$ ($\bar a$ denotes the complex conjugate of $a$) and $g_j(t, y)$ $\in C_0^\infty(R^1\times R^n)$ with $g_j(t, y)\equiv 0$ for $t\geqq T$.

If we note

$$A^*(-\tau, -\zeta, -\eta)$$
$$=\overline{D_{m-k}(-\bar\tau, -\bar\zeta)}\,\overline{A_0(-\bar\tau, -\bar\zeta, -\bar\eta)}\,\overline{A_+(-\bar\tau, -\bar\zeta, \bar\eta)}\,\overline{A_-(-\bar\tau, -\bar\zeta, -\bar\eta)}$$

and

$$\left|\det\left((2\pi i)^{-1}\oint\bar\zeta^{\mu+i+j-2}\{A_-(-\bar\tau, -\bar\zeta, -\bar\eta)\}^{-1}d\bar\zeta\right)_{1\leq i, j\leq\nu}\right|=\left|\prod_{j=1}^{\nu}\zeta_j^-(-\bar\tau, -\bar\eta)\right|^{\mu}$$

then, from lemma 5.1 and corollary 3.1, we can construct the solutions $u^{\pm}\in C^\infty(R^1\times\overline{R^1_{\pm}}\times R^n)$ of $(P^*)$ such that $u^{\pm}=0$ for $t\geqq T$, $x\gtrless 0$ and for any fixed $T_1$, $u^{\pm}$ is identically zero for large $x\gtrless 0$ when $t\geqq T_1$ (c. f. [4]).

Now we shall prove the uniqueness of the solution $u\in C^\infty_{x,y}C_t^{m-l}(C_l)$ of $(P)$. Let $u$ be a solution of $(P)$ with zero data and $u^{\pm}$ be solutions of $(P^*)$ with support conditions metioned above, and rewite $A(D_t, D_x, D_y)$ in the following from

$$A(D_t, D_x, D_y)=D_t^{m-l}C_l(D_x, D_y)+\sum_{j=0}^{k}E_{m-j}(D_t, D_y)D_x^j$$

where the order of $E_{m-j}$ in $D_t$ is at most $m-l-1$.

Then, the integrations by parts gives

$$0=\sum_{j=1}^{\mu+\nu}\int dy\int_0^T B_j(D_t, D_x, D_y)u(t, 0, y)\overline{g_j(t, y)}dt$$

where $B_j(D_t, D_x, D_y)$ is a some differential polynomial whose order in $D_t$ is at most $m-l-1$. Hence, from the arbitrariness of $g_j$, it follows that $B_j(D_t, D_x, D_y)$ $u(t, 0, y)=0$ for $j=1, 2, \cdots, \mu+\nu$.

Next we make the analogous considerations using the solutions of the following problem

$$\begin{cases} A^*(D_t, D_x, D_y)v^{\pm}=0 & t<0,\ x\in R,\ y\in R^n \\ D_t^j v^{\pm}(0, x, y)=h_j^{\pm}(x, y) & x\in R,\ y\in R^n,\ j=0, 1, \cdots, m-l-1 \end{cases}$$

where $h_j^{\pm} \in C_0^\infty(R^1 \times R^n)$ and $v^{\pm}$ vanishes for large positive (resp. negative) $x$ when $t$ varies in every compact set. Then, we can conclude that $C_l(D_x, D_y)u(T, x, y) = 0$ for any fixed $T>0$, Since $D_x^i u(T, 0, y)=0$ for $i=0, 1, \cdots, l-1$ from the assumption, the hyperbolicity of $C_l$ shows that $u(T, x, y)\equiv 0$.

**Proprsition 5.1.** *The solution of* $(P)$ *which belongs to* $C_{x,y}^\infty C_t^{m-l}(C_l)$ *is unique.*

## 6. Smoothness of the solution of $(P)$.

In this section we remark that the solution of $(P)$ which belongs to $C_{x,y}^\infty C_t^{m-l}(C_l)$ becomes the infinitely differentiable solution in virtue of the uniqueness of the solution.

First we solve the following initial value problem.

$$\begin{cases} C_l(D_x, D_y)\mu(x, y) + \sum_{j=l+1}^{m} C_j(D_x, D_y)h_{m-j}(x, y) = 0 \\ D_x^i \mu(0, y) = D_t^{m-l} g_i(0, y) \qquad i=0, 1, \cdots, l-1. \end{cases}$$

By the hyperbolicity of $C_l$, this problem has a unique solution $\mu \in C^\infty(R^1 \times R^n)$.

Here, setting $\mu = h_{m-l}$, we consider the following problem.

$$(\tilde{P}) \qquad \begin{cases} D_t A(D_t, D_x, D_y)\tilde{u} = 0 \\ D_x^i \tilde{u}(t, 0, y) = g_i(t, y) \qquad i=0, 1, \cdots l-1 \\ D_t^j \tilde{u}(0, x, y) = h_j(x, y) \qquad j=0, 1, \cdots m-l \end{cases}$$

Note that $\tau A(\tau, \zeta, \eta)$ has the same form as (1.1) and satisfies the condition $(G)$. Moreover the choice of the initial data of $\mu$ shows that the compatibility condition $(C)$ which corresponds to $\tau A(\tau, \zeta, \eta)$ is satisfied. Hence, by proposition 4.3, there is a solution $\tilde{u} \in C_{x,y}^\infty C_t^{m+1-l}(C_l)$ of $(\tilde{P})$.

The equation defining $\mu$ means that $A(D_t, D_x, D_y)\tilde{u}|_{t=0} \equiv 0$ and then we get $A(D_t, D_x, D_y)\tilde{u} \equiv 0$. Thus it follows from the uniquness of the solution that $u = \tilde{u}$ and then $u$ belongs to $C_{x,y}^\infty C_t^{m+1-l}(C_l)$. The iterations of this arguments show that $u \in C^\infty(\overline{R_+^1} \times R^1 \times R^n)$.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

## References

[1] Y. Hasegawa: On the $C^\infty$-Goursat problem for equation with constant coefficients; to appear.
[2] Y. Hasegawa: On the $C^\infty$-Goursat problem for 2nd order equations with constant coefficients. Proc. Japan. Acad., **51** (1975) 516-519.
[3] L. Hörmander: Linear partial differential operators. Springer-Verlag (1964).
[4] R. Sakamoto: $\mathcal{E}$-well posedness for hyperbolic mixed problems with constant cofficients. J. Math. Kyoto Univ. **4** (1974) 93-118.
[5] A. Seidenberg: A new decision method for elementary algebra. Ann. Math. **60** (1954) 365-374.
[6] L. Svensson: Necessary and sufficient conditions for the hyperbolicity of polynomials with hyperbolic principal part. Ark. Mat. **8** (1969) 145-162.