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Cauchy problem for weakly hyperbolic equations of second order

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§ 0. Introduction.

We are concerned with the Cauchy problem for weakly hyperbolic equations with characteristics of variable multiplicity. It is an interesting but difficult problem to obtain a condition for this Cauchy problem to be well-posed. Many papers have been devoted to this problem.

Comparing with the case of constant multiplicity, it is remarkable that in this case the solution may considerably lose its differentiability. The multiplicity of characteristics is not the only one which determines the extent of loss of differentiability, e. g. [1]. This suggests the complicated situation.

In this paper we consider the Cauchy problem:

$$\text{(C. P.)} \qquad \begin{aligned} &\{\partial_t^2+2a_1\partial_t\partial_x+a_2\partial_x^2-b_0\partial_t-b_1\partial_x-c\}\,u=f\,, \\ &u|_{t=t_0}=u_0(x),\ \partial_t u|_{t=t_0}=u_1(x)\,. \end{aligned}$$

We assume the equation to be weakly hyperbolic. It means in this paper that the characteristic roots, namely the roots of

$$\lambda^2+2a_1\lambda+a_2=0$$

are real-valued and smooth, which we denote by $\lambda_i(x, t)$, $i=1, 2$.

Our approach to this problem is based on the method of successive approximation. The obtained results are stated in § 1. The commutator $[\partial_t-\lambda_1\partial_x, \partial_t-\lambda_2\partial_x]$ plays an important role there.

Our idea is very simple. Roughly to say, it is as follows. We use two ways of inductive estimate, saying one be analytic type and another be hyperbolic type. We repeat first the analytic type estimate certain times, say N times, and after that time repeat the hyperbolic type estimate. It is important to choose N appropriately. This number N corresponds to the decrease of differentiability of the solution.

We note that the analoguous idea is found in [2].

§ 1. Statement of results.

1.1. The differential operators considered at first are of the form

$$L=\partial_t^2-\lambda^2\partial_x^2-a\partial_t-b\partial_x-c \tag{1.0}$$

$(x, t)\in\Omega=\{(x, t);\ -\infty<x<+\infty,\ 0\leqq t\leqq T\}$, $0<T<+\infty$, $\partial_t=\frac{\partial}{\partial t}$, $\partial_x=\frac{\partial}{\partial x}$. We suppose that λ, a, b and c belong to $\mathscr{B}(\Omega)$ and also suppose that λ is real-valued, namely the operator L is weakly hyperbolic. The Cauchy problem to be solved is

$$(1.1)\qquad \begin{aligned} &L[u]=f,\ t\geqq t_0\\ &u|_{t=t_0}=u_0(x),\ \partial_t u|_{t=t_0}=u_1(x)\end{aligned}$$

where $0\leqq t_0<T$.

As aforesaid our approach to this problem is based on the method of successive approximation. We may assume with no loss of generality that $u_0=u_1=0$. Let L_p and L' be as follows:

$$\begin{aligned} L_p[u]&=(\partial_t-\lambda\partial_x)(\partial_t+\lambda\partial_x)u\,,\\ L'[u]&=(\partial_t-\lambda\partial_x)(au)+(b+a\lambda+\lambda_t-\lambda\lambda_x)\partial_x u\\ &\quad+(c-a_t+\lambda a_x)u\,.\end{aligned}$$

We define u_i, $i=1, 2, \cdots$, by

$$L_p[u_i]=f_{i-1}\,,\qquad u_i|_{t=t_0}=\partial_t u_i|_{t=t_0}=0\,,$$

where $f_0=f$, $f_i=L'[u_i]$, $i=1, 2, \cdots$. It then comes into question whether the formal solution $\sum_{i=1}^{\infty} u_i$ converges or not.

Let π be a rectangular domain as follows:

$$\pi=\{(x, t);\ \alpha<x<\beta,\ 0\leqq t\leqq T\}\,,\ -\infty\leqq\alpha<\beta\leqq+\infty\,.$$

We assume the following conditions.

Condition (H_1): *There exist two functions k and $h\in\mathscr{B}(\pi)$ such that*

$$b=k\lambda_t+h\lambda\,,\qquad (x, t)\in\pi\,.$$

Condition (H_2^0):

(H_2^0-1) $\quad\lambda(x, t)\geqq 0\,,\qquad (x, t)\in\pi\,,$

(H_2^0-2) $\quad$ *there exists a constant C such that*

$$\lambda_t+C\lambda\geqq 0\,,\qquad (x, t)\in\pi\,.$$

We denote by $\varphi^{\pm}(x, t, s)$ respectively the solution of

$$\varphi_t\pm\lambda\varphi_x=0\,,\qquad \varphi|_{t=s}=x\,,$$

and define the domain D by

$$D=\{(x, t)\in\pi\,;\ \alpha<\varphi^{\pm}(x, t, 0)<\beta\}\,.$$

The following proposition holds.

Proposition 1. *Assume that the weakly hyperbolic operator L satisfies the condition H_1 and H_2^0. Then for any integer $m \geqq 0$, there exists a constant $\delta(m)$ such that for any t_0; $0 \leqq t_0 < T$, and any $f \in \mathscr{B}[D \cap \{t \geqq t_0\}]$, $\sum_{i=0}^{\infty} \{\sum_{p+q \leqq m, q \leqq 1} |\partial_x^p \partial_t^q u_i|\}$ converges uniformly in $D \cap \{t_0 \leqq t \leqq t_0 + \delta(m)\}$.*

We remark that even if $-\lambda$, instead of λ, satisfies the condition (H_2^0), the consequence of the above proposition remains true. Moreover, reading the proof of Proposition 1 carefully, one can see that it is able to replace the condition (H_2^0) with the following weaker one.

Condition (H_2^1): There exist at most countable numbers of open intervals $I_i = (\alpha_i, \beta_i)$, $\alpha \leqq \alpha_i < \beta_i \leqq \beta$, such that

$$(\alpha, \beta) \subset \overline{\bigcup_i I_i}, \qquad I_i \cap I_j = \phi \quad \text{if } i \neq j,$$

and that in each $\pi_i = I_i \times [0, T]$, either λ or $-\lambda$ satisfies the condition (H_2^0), provided that the constant C in (H_2^0-2) is independent of i.

It is useful to remark the followings: 1) If $\alpha_i = \beta_j$ for some i and j, and $\lambda \geqq 0$ in one of π_i and π_j, then we may suppose that $\lambda \leqq 0$ in the other, and in particular that $\lambda(\alpha_i, t) \equiv 0$, $0 \leqq t \leqq T$. 2) If $(x, t) \in D$, then $(\varphi^{\pm}(x, t, s), s) \in D$ for any s; $0 \leqq s \leqq t$. 3) If $\alpha < \alpha_i < \beta_i < \beta$, then $\pi_i \subset D$. Besides, if $(x, t) \in \pi_i$, then $(\varphi^{\pm}(x, t, s), s) \in \pi_i$ for any s; $0 \leqq s \leqq t$.

Well, the condition (H_2^1) is equivalent to

Condition (H_2): There exists a constant C such that

$$(\lambda^2)_t + C\lambda^2 \geqq 0, \qquad (x, t) \in \pi.$$

Proof. Evidently (H_2) follows from (H_2^1). Hence we show its inverse. We assume (H_2). At first we remark that if $\lambda(x_0, t_0) = 0$ at some point $(x_0, t_0) \in \pi$, then $\lambda(x_0, t) = 0$ for any $t \leqq t_0$. Suppose that $\lambda(x_0, T) \neq 0$ for some $x_0 \in (\alpha, \beta)$, then we can find the maximal open interval $I \subset (\alpha, \beta)$, containing x_0, such that $\lambda(x, T) \neq 0$ for any $x \in I$. The number of such intervals is at most countable. We denote them by $I_i = (\alpha_i, \beta_i)$. $(\alpha, \beta) - \overline{\bigcup I_i}$ is represented as a union of at most countable numbers of disjoint open intervals J_j. By the above remark, we see that $\lambda(x, t) \equiv 0$ in $J_j \times [0, T]$ for all j. Also we see that in each $I_i \times [0, T]$ either λ or $-\lambda$ satisfies the condition (H_2^0), because, if $\lambda(x, T) > 0$ (or < 0) for $x \in I_i$, then $\lambda(x, t) \geqq 0$ (or $\leqq 0$ respectively) in $I_i \times [0, T]$. (*q. e. d.*)

We see after all that if the weakly hyperbolic operator L satisfies the condition (H_1) and (H_2), then the same consequence as in Proposition 1 holds. Since the differentiability with respect to t of the obtained solution follows from the equation, we have

Theorem 1. *Assume that the weakly hyperbolic operator L satisfies the conditions (H_1) and (H_2). Then for any integer $m \geqq 2$, there exists a constant $\delta(m)$ such that for any t_0; $0 \leqq t_0 < T$, any $f \in \mathscr{B}[D \cap \{t \geqq t_0\}]$ and any $u_0, u_1 \in \mathscr{B}[D \cap \{t = t_0\}]$, there exists a solution $u \in \mathscr{B}^m[D \cap \{t_0 \leqq t \leqq t_0 + \delta(m)\}]$ of the Cauchy problem* (1.1).

Moreover there exists an integer $m_0 \geqq 2$ *such that for any* t_0; $0 \leqq t_0 < T$, *the solution* $u \in \mathscr{B}^{m_0}[D \cap \{t \geqq t_0\}]$ *of* (1.1) *is unique.*

1.2. Some remarks

(A). If we assume the following condition besides (H_1) and (H_2), then we may take $\delta(m) = +\infty$ in Theorem 1.

Condition (H_3): *There exists a function* $\sigma(t)$ *such that*

$(H_3\text{-}1)$ $\quad |\partial_x^p \lambda_t(x, t)| \leqq C_p \sigma(t)$, $(x, t) \in D$, *for any* p,

where C_p *is a constant which may depend on* p,

$$(H_3\text{-}2) \qquad \frac{\Sigma(t)}{\sigma(t)} |\lambda_t(x, t)| \leqq M |\lambda(x, t)|, \qquad (x, t) \in D,$$

where $\Sigma(t) = \int_0^t \sigma(s)\, ds$ *and* M *is a constant.*

Consulting the proof of Proposition 1, one can prove this easily.

(B). (H_2) and (H_3) are conditions concerning $\lambda(x, t)$. We shall give some examples.

(E.1): $\lambda(x, t) = p(t) q(x, t)$; $q(x, t) \geqq c_0 > 0$, $p(t) \geqq 0$, $p'(t) + Cp(t) \geqq 0$, c_0 *and* C *are constants.*

Putting $\sigma(t) = |p'(t)| + p(t)$, one can verify that this example satisfies the conditions (H_2) and (H_3).

(E.2): $\lambda(x, t) = p(t) q(x)$; $p(t) \geqq 0$, $p'(t) + Cp(t) \geqq 0$, $C = const.$.

This example also satisfies (H_2) and (H_3). In this case we put $\sigma(t) = |p'(t)|$.

(E.3): $\lambda(x, t) = p_1(t) q_1(x) + p_2(t) q_2(x)$; $p_i(t) \geqq 0$, $p_i'(t) \geqq 0$, $i = 1, 2$, $q_1(x) q_2(x) \geqq 0$.

This example satisfies (H_2). If we assume that

$$p_2'(t) \leqq \text{const.}\, p_1'(t), \quad \frac{p_1(t) p_2'(t)}{p_1'(t) p_2(t)} \leqq \text{const.},$$

then the condition (H_3) is also satisfied. For example, $\lambda(x, t) = t x^4 + \exp(-t^{-1}) x^2$ satisfies (H_2) but does not satisfy (H_3).

(C). In case $\lambda = t$, $a = c = 0$ and $b = constant$, the explicit solution is given in [1]. The case of $\lambda = t^m$ is considered in [2]. A sufficient condition of well-posedness is given in [3], which contains the example (E.1). O. A. Oleinik considered the weakly hyperbolic equations in many independent variables and not assuming the smoothness of characteristic roots, gave a sufficient condition of well-posedness, [4]. In case of two independent variables, it is as follows:

$$\alpha t b^2 \leqq A \lambda^2 + (\lambda^2)_t, \qquad \alpha \text{ and } A \text{ are constants } (>0).$$

These results are partly extended to higher order equations, [5], [6], [7]. Strongly hyperbolic equations are characterized in [8], [9].

1.3. Now we shall consider the Cauchy problem (1.1) for differential operators of more general form, namely for

$$\mathcal{L}=\partial_t^2+2a_1\partial_t\partial_x+a_2\partial_x^2-b_0\partial_t-b_1\partial_x-c\,. \tag{1.2}$$

We assume that the coefficients belong to $\mathcal{B}$. We denote by λ_1, λ_2 the roots of the characteristic equation

$$\tau^2+2a_1\tau+a_2=0\,.$$

Let us say the operator $\mathcal{L}$ be weakly hyperbolic if λ_i are real-valued and belong to $\mathcal{B}$, $i=1, 2$. Let $\chi(x, t, s)$ and $\varphi_i(x, t, s)$ stand respectively for the solution of

$$\begin{aligned}&\chi_t+a_1\chi_x=0\,, \qquad \chi|_{t=s}=x\,,\\ &\varphi_t-\lambda_i\varphi_x=0\,, \qquad \varphi|_{t=s}=x\,, \qquad i=1, 2\,.\end{aligned} \tag{1.3}$$

We introduce the so-called subprincipal symbol P_1^s, which is defined by

$$P_1^s=\frac{1}{2}\partial_t\partial_\tau P_2+\frac{1}{2}\partial_x\partial_\xi P_2-P_1\,,$$

where $P_2=\tau^2+2a_1\tau\xi+a_2\xi^2$, $P_1=-b_0\tau-b_1\xi$. Let π be the domain defined by

$$\pi=\{(x, t)\,|\,\alpha<\chi(x, t, 0)<\beta,\ 0\leqq t\leqq T\}\,, \qquad -\infty\leqq\alpha<\beta\leqq+\infty,\ T>0\,.$$

We assume the following two conditions.

Condition $(\mathcal{H}_1)$: *There exist two functions k and $h\in\mathcal{B}(\pi)$ such that*

$$P_1^s|_{\tau=-a_1\xi}=k\{\tau-\lambda_1\xi,\ \tau-\lambda_2\xi\}+h(\lambda_1-\lambda_2)\xi\,,$$

$(x, t)\in\pi$, $\xi\in\boldsymbol{R}$, *where* $\{\ ,\ \}$ *denotes the Poisson's bracket.*

Condition $(\mathcal{H}_2)$: *There exists a constant C such that*

$$\{\tau+a_1\xi,\ d\xi^2\}+Cd\xi^2\geqq 0\,, \qquad (x, t)\in\pi,\ \xi\in\boldsymbol{R}\,,$$

where $d=a_1^2-a_2$.

Let $\mathcal{D}$ be the domain defined by

$$\mathcal{D}=\{(x, t)\in\pi\,|\,\alpha<\varphi_i(x, t, 0)<\beta,\ i=1, 2\}\,.$$

Then we have the following theorem.

Theorem 2. *Assume that the weakly hyperbolic operator $\mathcal{L}$ satisfies the conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. Then for any integer $m\geqq 2$, there exists a constant $\delta(m)$ such that for any t_0; $0\leqq t_0<T$, any $f(x, t)\in\mathcal{B}[\mathcal{D}\cap\{t\geqq t_0\}]$ and any $u_0(x)$, $u_1(x)\in\mathcal{B}[\mathcal{D}\cap\{t=t_0\}]$, there exists a solution $u(x, t)\in\mathcal{B}^m[\mathcal{D}\cap\{t_0\leqq t\leqq t_0+\delta(m)\}]$ of the Cauchy problem (1.1) for $\mathcal{L}$. Moreover there exists an integer $m_0\geqq 2$ such that for any t_0; $0\leqq t_0<T$, the solution $u\in\mathcal{B}^{m_0}[\mathcal{D}\cap\{t\geqq t_0\}]$ of (1.1) for $\mathcal{L}$ is unique.*

Proof. By the transform of independent variables defined by

$$y=\chi(x, t, 0),\ s=t\,,$$

the operator $\mathcal{L}$ is transformed into

$$\tilde{\mathcal{L}}=\partial_s^2-(\lambda\chi_x)^2\partial_y^2-b_0\partial_s-\tilde{b}_1\partial_y-c\ ,$$

where $2\lambda=\lambda_2-\lambda_1$, $\tilde{b}_1=\{b_1+(a_1)_t+a_1(a_1)_x-b_0a_1+\lambda^2\chi_{xx}\chi_x^{-1}\}\chi_x$. This is the operator of the same form as L in (1.0), and satisfies the conditions (H_1) and (H_2). Here we remark that

$$\chi_x(x,\ t,\ 0)=\exp\{-\int_0^t(a_1)_x(\chi(x,\ t,\ s),\ s)ds\}>0\ .$$

(*q. e. d.*)

§ 2. Preparatory considerations.

2.1. To simplify the descriptions, we use some notations as follows.

$$A(f)=\int_0^t ds\int_0^s f\circ\varphi du\ ,\qquad A^{\pm}(f)=\int_0^t f\circ\varphi^{\pm}ds\ ,$$

where $f\circ\varphi=f(\varphi^-(\varphi^+(x,\ t,\ s),\ s,\ u),\ u)$, $f\circ\varphi^{\pm}=f(\varphi^{\pm}(x,\ t,\ s),\ s)$. By the formula of differential of a composite function,

$$\partial_x^p(f\circ\varphi)=\sum_{r=1}^p\sum_{\substack{r_1+\cdots+r_p=r\\ r_1+\cdots+pr_p=p}}C_{r_1\cdots r_p}\varphi_1^{r_1}\cdots\varphi_p^{r_p}(\partial_x^r f)\circ\varphi\ ,$$

where $\varphi_j=\partial_x^j\varphi$, $\varphi=\varphi^-(\varphi^+(x,\ t,\ s),\ s,\ u)$. We put

$$\Phi_{p,r}=\sum_{\substack{r_1+\cdots+r_p=r\\ r_1+\cdots+pr_p=p}}C_{r_1\cdots r_p}\varphi_1^{r_1}\cdots\varphi_p^{r_p}\ .$$

It holds that

$$\Phi_{p,r-1}\varphi_1+\partial_x\Phi_{p,r}=\Phi_{p+1,r}\ ,\qquad r=1,\ 2,\ \cdots,\ p+1\ , \tag{2.1}$$

where $\Phi_{p,0}=\Phi_{p,p+1}=0$. In particular, $\Phi_{p,p}=\varphi_1^p$.

We put

$$\partial_x^p A(f)=\sum_{r=1}^p\int_0^t ds\int_0^s\Phi_{p,r}(\partial_x^r f)\circ\varphi\ du=\sum_{r=1}^p A_{p,r}(\partial_x^r f)\ ,$$

$$\begin{aligned}\partial_x A_{p,r}(f)&=\int_0^t ds\int_0^s(\partial_x\Phi_{p,r})f\circ\varphi\ du+\int_0^t ds\int_0^s\Phi_{p,r}\varphi_1(\partial_x f)\circ\varphi\ du\\ &=A_{p,r,0}(f)+A_{p,r,1}(\partial_x f)\ .\end{aligned}$$

From (2.1) we have

$$A_{p,r,0}+A_{p,r-1,1}=A_{p+1,r}\ ,\qquad r=1,\ 2,\ \cdots,\ p+1\ . \tag{2.2}$$

$\Phi^{\pm}_{p,r}$, $A^{\pm}_{p,r}$, $A^{\pm}_{p,r,0}$ and $A^{\pm}_{p,r,1}$ are defined in parallel, and they satisfy the same relations as (2.1) and (2.2).

By the way, under the condition (H_1) the operator L' is expressed in the form

$$L'[u_{i-1}]=(\partial_t-\lambda\partial_x)(au_{i-1})+(k\lambda_t+h\lambda)\partial_x u_{i-1}+cu_{i-1}$$

where k and h are different from those in (H_1) and c from that of (1.0). Hence we have

$$(2.3)\quad \begin{aligned} u_i &\equiv B[u_{i-1}] = A^+[a u_{i-1}] + A[(k\lambda_t + h\lambda)\partial_x u_{i-1} + c u_{i-1}]\,, \\ \partial_x^p u_i &= \sum_{r,r_1} A_{p,r}\Big[\binom{r}{r_1}(k\lambda_t + h\lambda)^{(r_1)}\partial_x^{r_2+1} u_{i-1}\Big] \\ &\quad + \sum_{r,r_1} A^+_{p,r}\Big[\binom{r}{r_1} a^{(r_1)}\partial_x^{r_2} u_{i-1}\Big] + \sum_{r,r_1} A_{p,r}\Big[\binom{r}{r_1} c^{(r_1)}\partial_x^{r_2} u_{i-1}\Big] \end{aligned}$$

where $\sum_{r,r_1} = \sum_{r=1}^{p} \sum_{r_1+r_2=r}$, $a^{(r)} = \partial_x^r a$ and so on.

2.2. We put $w_{ij} = \dfrac{\lambda^i t^j}{i!\, j!}$.

$$\int_0^t w_{i-j+1,i}\circ\varphi^\pm ds = [w_{i-j+1,i+1}\circ\varphi^\pm]_0^t - \int_0^t \partial_s(\lambda\circ\varphi^\pm) w_{i-j,i+1}\circ\varphi^\pm ds\,.$$

Since that $\partial_s(\lambda\circ\varphi^\pm) = (\lambda_t + C\lambda - C\lambda \pm \lambda\lambda_x)\circ\varphi^\pm$ and that $\lambda_t + C\lambda \geqq 0$ from the condition (H_2^0),

$$\int_0^t w_{i-j+1,i}\circ\varphi^\pm ds \leqq w_{i-j+1,i+1} + \{C + \sup_D |\lambda_x|\}\,\frac{i-j+1}{i+1}\, t\int_0^t w_{i-j+1,i}\circ\varphi^\pm ds\,.$$

We take δ so small as $\{C+\sup|\lambda_x|\}\delta < 1$. Then we have

Lemma 1. *For any* $i=1, 2, \cdots$, *and any* $j=1, 2, \cdots, i+1$, *it holds that*

$$\int_0^t w_{i-j+1,i}\circ\varphi^\pm ds \leqq M_1 w_{i-j+1,i+1}\,, \qquad (x, t)\in D_\delta$$

where $D_\delta = D\cap\{0\leqq t\leqq\delta\}$, $M_1 = \{1-(C+\sup_D|\lambda_x|)\delta\}^{-1}$.

Since $|\lambda_t| \leqq \lambda_t + 2C\lambda$ from the condition (H_2^0),

$$\begin{aligned} \int_0^t \{|\lambda_t|\, w_{i-j,i+1}\}\circ\varphi^\pm ds &\leqq \int_0^t \partial_s(\lambda\circ\varphi^\pm) w_{i-j,i+1}\circ\varphi^\pm ds \\ &\quad + (2C + \sup_D|\lambda_x|)(i-j+1)\int_0^t w_{i-j+1,i+1}\circ\varphi^\pm ds \\ &\leqq \Big\{1 + (2C+\sup_D|\lambda_x|)\frac{i-j+1}{i+2} M_1\delta\Big\} w_{i-j+1,i+1}\,. \end{aligned}$$

Therefore we have

Lemma 2. *For any* $i=1, 2, \cdots$, *and any* $j=1, 2, \cdots, i$, *it holds that*

$$\int_0^t \{|\lambda_t|\, w_{i-j,i+1}\}\circ\varphi^\pm ds \leqq M_2 w_{i-j+1,i+1}$$

$(x, t)\in D_\delta$, *where* M_2 *is a constant independent of* i *and* j.

§ 3. Proof of Proposition 1.

3.1. We shall say that u_i has the estimate scheme S_n if 1) for any p, $\partial_x^p u_i$ is decomposable as a sum of n components $u_{i,j}^p$, $j=1, 2, \cdots, n$:

$$\partial_x^p u_i = \sum_{j=1}^{n} u_{i,j}^p\,,$$

each of which has the estimate as follows:

$$|u^p_{i,j}|\leqq const.\ w_{n-j,n+1}|f|_{p+n-j}, \qquad j=1, 2, \cdots, n, ^{*)}$$

and 2) for each j, $\partial_x u^p_{i,j}$ is decomposable as a sum of two components $u^{p+1}_{i,j,0}$ and $u^{p+1}_{i,j,1}$:

$$\partial_x u^p_{i,j}=u^{p+1}_{i,j,0}+u^{p+1}_{i,j,1},\ provided\ that\ u^{p+1}_{i,n,1}=0,$$

each of which has the estimate as follows:

$$|u^p_{i,j,k}|\leqq const.\ w_{n-j-k,n+1}|f|_{p+n-j-k}, \qquad k=0, 1,$$

and it holds the following relations:

$$u^{p+1}_{i,j,0}+u^{p+1}_{i,j-1,1}=u^{p+1}_{i,j}.$$

We can easily see that u_1 has the estimate scheme S_1. In fact, if we put $\partial^p_x u_1=u^p_{1,1}$ and $\partial_x u^p_{1,1}=u^{p+1}_{1,1,0}(=u^{p+1}_{1,1})$, then

$$|u^p_{1,1}|=|u^p_{1,1,0}|\leqq const.\ w_{0,2}|f|_p.$$

3.2. Now assuming that u_i has the estimate scheme S_n and taking (2.3) into account, we define the operators $B^p_j[u_i]$, $B^{p+1}_{j,0}[u_i]$ and $B^{p+1}_{j,1}[u_i]$ as follows:

$$\begin{aligned}B^p_j[u_i]=&A_{p,p}[(k\lambda_t+h\lambda)u^{p+1}_{i,j}]+\sum_{r,r_1}A^+_{p,r}\left[\binom{r}{r_1}a^{(r_1)}u^{r_2}_{i,j-1}\right]\\&+\sum_{r,r_1}{}'A_{p,r}\left[\binom{r}{r_1}(k\lambda_t+h\lambda)^{(r_1)}u^{r_2+1}_{i,j-1}\right]\\&+\sum_{r,r_1}A_{p,r}\left[\binom{r}{r_1}c^{(r_1)}u^{r_2}_{i,j-1}\right],\end{aligned}$$

where $\sum_{r,r_1}=\sum_{r=1}^{p}\sum_{r_1+r_2=r}$, $\sum'_{r,r_1}$ denotes the summation excepting the term of $(r, r_1)=(p, 0)$.

$$\begin{aligned}B^{p+1}_{j,0}[u_i]=&A_{p,p,1}[(k\lambda_t+h\lambda)u^{p+2}_{i,j,0}]+\sum_{r,r_1}A^+_{p,r,1}\left[\binom{r}{r_1}a^{(r_1)}u^{r_2+1}_{i,j-1,0}\right]\\&+\sum A^+_{p,r,1}\left[\binom{r}{r_1}a^{(r_1+1)}u^{r_2}_{i,j-1}\right]+\sum A^+_{p,r,0}\left[\binom{r}{r_1}a^{(r_1)}u^{r_2}_{i,j-1}\right]\\&+\sum{}'A_{p,r,1}\left[\binom{r}{r_1}(k\lambda_t+h\lambda)^{(r_1)}u^{r_2+2}_{i,j-1,0}+\binom{r}{r_1}(k\lambda_t+h\lambda)^{(r_1+1)}u^{r_2+1}_{i,j-1}\right]\\&+\sum{}'A_{p,r,0}\left[\binom{r}{r_1}(k\lambda_t+h\lambda)^{(r_1)}u^{r_2+1}_{i,j-1}\right]+\sum A_{p,r,0}\left[\binom{r}{r_1}c^{(r_1)}u^{r_2}_{i,j-1}\right]\\&+\sum A_{p,r,1}\left[\binom{r}{r_1}c^{(r_1)}u^{r_2+1}_{i,j-1,0}+\binom{r}{r_1}c^{(r_1+1)}u^{r_2}_{i,j-1}\right].\end{aligned}$$

$$\begin{aligned}B^{p+1}_{j,1}[u_i]=&A_{p,p,1}[(k\lambda_t+h\lambda)u^{p+2}_{i,j,1}+(k\lambda_t+h\lambda)^{(1)}u^{p+1}_{i,j}]\\&+A_{p,p,0}[(k\lambda_t+h\lambda)u^{p+1}_{i,j}]+\sum A^+_{p,r,1}\left[\binom{r}{r_1}a^{(r_1)}u^{r_2+1}_{i,j-1,1}\right]\end{aligned}$$

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*) $|f|_p=\sum_{q\leq p}\sup_{D_t}|\partial^q_x f(x,s)|$, $D_t=D\cap\{0\leqq s\leqq t\}$, $t\leqq\delta$.

$$+\Sigma' A_{p,r,1}\left[\binom{r}{r_1}(k\lambda_t+h\lambda)^{(r_1)}u_{i,j-1,1}^{r_2+2}\right]$$

$$+\Sigma A_{p,r,1}\left[\binom{r}{r_1}c^{(r_1)}u_{i,j-1,1}^{r_2+1}\right].$$

Then it is clear that

$$\partial_x^p B[u_i]=\sum_{j=1}^{n+1} B_j^p[u_i],$$

(3.1)

$$\partial_x B_j^p[u_i]=B_{j,0}^{p+1}[u_i]+B_{j,1}^{p+1}[u_i], \qquad B_{n+1,1}^{p+1}[u_i]=0.$$

Moreover, taking (2.2) into account, one can verify that

(3.2) $$B_{j,0}^{p+1}[u_i]+B_{j-1,1}^{p+1}[u_i]=B_j^{p+1}[u_i], \qquad j=1,2,\cdots,n+1.$$

3.3. Suppose that u_i has the estimate scheme S_n. If we define $u_{i+1,j}^p$, $u_{i+1,j,0}^{p+1}$ and $u_{i+1,j,1}^{p+1}$ as follows:

$$u_{i+1,j}^p=B_j^p[u_i], \qquad u_{i+1,j,k}^{p+1}=B_{j,k}^{p+1}[u_i]$$

$j=1, 2, \cdots n+1$, $k=0, 1$, then on account of (3.1) and (3.2),

$$\partial_x^p u_{i+1}=\sum_{j=1}^{n+1} u_{i+1,j}^p, \qquad \partial_x u_{i+1,j}^p=u_{i+1,j,0}^{p+1}+u_{i+1,j,1}^{p+1},$$

$$u_{i+1,n+1,1}^{p+1}=0, \qquad u_{i+1,j,0}^{p+1}+u_{i+1,j-1,1}^{p+1}=u_{i+1,j}^{p+1}.$$

Moreover each $u_{i+1,j}^p$, $u_{i+1,j,0}^{p+1}$, $u_{i+1,j,1}^{p+1}$ has the estimate as follows:

$$|u_{i+1,j}^p|\leqq\text{const. } w_{n+1-j,n+2}|f|_{p+n+1-j},$$

$$|u_{i+1,j,k}^{p+1}|\leqq\text{const. } w_{n+1-j-k,n+2}|f|_{p+1+n+1-j-k}, \quad k=0,1.$$

By means of Lemma 1 and Lemma 2, we can prove these inequalities. In fact, for example,

$$|A_{p,p}[k\lambda_t u_{i,j}^{p+1}]|\leqq\text{const. } |f|_{p+1+n-j}A[|\lambda_t|w_{n-j,n+1}]$$

$$\leqq\text{const. } |f|_{p+n+1-j}A^+[w_{n+1-j,n+1}]$$

$$\leqq\text{const. } |f|_{p+n+1-j}w_{n+1-j,n+2}.$$

Thus the following proposition has been obtained.

Proposition 3.1. *If u_i has the estimate scheme S_n, then u_{i+1} has the estimate scheme S_{n+1}.*

In consequnce of this proposition, we see that u_i has the estimate scheme S_i, because u_1 has the estimate scheme S_1.

3.4. We suppose again that u_i has the estimate scheme S_n. More precisely, let $K_{m,i}$ be such a constant as

$$|u_{i,j}^p|\leqq K_{m,i}w_{n-j,n+1}|f|_{p+n-j},$$

$$|u_{i,j,k}^{p+1}|\leqq K_{m,i}w_{n-j-k,n+1}|f|_{p+1+n-j-k},$$

$j=1, 2, \cdots, n$, $k=0, 1$, for any p; $p \leqq m-1+j$, and for any p; $p+1 \leqq m-1+j+k$, respectively, where m is an integer given arbitrarily ($\geqq 0$). This time let us define $u^{p}_{i+1,j}$, $u^{p+1}_{i+1,j,0}$ and $u^{p+1}_{i+1,j,1}$ as follows:

$$u^{p}_{i+1,1}=B^{p}_{1}[u_i]+B^{p}_{2}[u_i], \qquad u^{p}_{i+1,j}=B^{p}_{j+1}[u_i], \qquad j=2, \cdots, n,$$

$$u^{p+1}_{i+1,1,0}=B^{p+1}_{1,0}[u_i]+B^{p+1}_{1,1}[u_i]+B^{p+1}_{2,0}[u_i],$$

$$u^{p+1}_{i+1,j,0}=B^{p+1}_{j+1,0}[u_i], \qquad j=2, \cdots, n,$$

$$u^{p+1}_{i+1,j,1}=B^{p+1}_{j+1,1}[u_i], \qquad j=1, \cdots, n.$$

Then on account of (3.1) and (3.2),

$$\partial^{p}_{x}u_{i+1}=\sum_{j=1}^{n} u^{p}_{i+1,j}, \qquad \partial_x u^{p}_{i+1,j}=u^{p+1}_{i+1,j,0}+u^{p+1}_{i+1,j,1},$$

$$u^{p+1}_{i+1,n,1}=0, \qquad u^{p+1}_{i+1,j,0}+u^{p+1}_{i+1,j-1,1}=n^{p+1}_{i+1,j}.$$

We shall prove later that if $n \geqq 2$ there exists a constant $\rho=\rho(m, n, \delta)$, which depends on m, n and δ but not on i and $K_{m,i}$, such that

$$\text{(3.3)} \qquad \begin{aligned} &|u^{p}_{i+1,j}| \leqq \rho(m, n, \delta)K_{m,i}w_{n-j,n+1}|f|_{p+n-j}, \\ &|u^{p+1}_{i+1,j,k}| \leqq \rho(m, n, \delta)K_{m,i}w_{n-j-k,n+1}|f|_{p+1+n-j-k}, \end{aligned}$$

$j=1, 2, \cdots, n$, $k=0, 1$, for any p; $p \leqq m-1+j$, and for any p; $p+1 \leqq m-1+j+k$, respectively, and what is more, that if we take n large according to m and δ small according to m and n, then

$$\text{(3.4)} \qquad \rho(m, n, \delta)<1.$$

3.5. The following proposition is the direct consequence of the results obtained in 3.3 and 3.4.

Propositon 3.2. *For any integer $m \geqq 0$, if we take $n(\geqq 2)$ large according to m, and δ small according to m and n, then there exists a constant $\rho<1$ such that for any $p \leqq m$*

$$|\partial^{p}_{x}u_i| \leqq \rho^{i-n}K \sum_{q \leq m+n-1} \sup_{Dt}|\partial^{q}_{x}f|, \qquad (x, t) \in D_\delta,$$

$i=n, n+1, \cdots$, where K is a constant independent of i.

Now we shall consider the estimate of $\partial_t\partial^{p}_{x}u_i$. Because

$$\partial_t u_i=-\lambda\partial_x u_i+a u_{i-1}+A^{-}[(k\lambda_t+h\lambda)\partial_x u_{i-1}]+A^{-}[c u_{i-1}],$$

it is easily seen that for any $p \leqq m-1$

$$|\partial_t\partial^{p}_{x}u_i| \leqq \rho^{i-n}K' \sum_{q \leq m+n-1} \sup_{Dt}|\partial^{q}_{x}f|, \qquad (x, t) \in D_\delta,$$

$i=n, n+1, \cdots$, where K' is a constant independent of i.

Thus the proposition 1 has been shown.

3.6. Proof of (3.3) and (3.4).

(A). At first we consider the estimate of $B_1^p[u_i]=A_{p,p}[(k\lambda_t+h\lambda)u_{i,1}^{p+1}]$. Taking account of the relation: $u_{i,1}^{p+1}=u_{i,1,0}^{p-1}=\partial_x u_{i,1}^p-u_{i,1,1}^{p+1}$, we put

$$I=A_{p,p}[(k\lambda_t+h\lambda)\partial_x u_{i,1}^p]\,.$$

Because $\{(k\lambda_t+h\lambda)\partial_x u_{i,1}^p\}\circ\varphi=\varphi_s^{-1}\partial_s\{(k\lambda_t+h\lambda)u_{i,1}^p\circ\varphi\}-\{(k\lambda_t+h\lambda)_x u_{i,1}^p\}\circ\varphi$ and $\varphi_s=2\lambda(\varphi^+(x,t,s),s)\varphi_x^-(\varphi^+(x,t,s),s,u)$, if we put $\Phi_{p,p}\{2\varphi_x^-(\varphi^+(x,t,s),s,u)\}^{-1}=\Phi_p$, then we have

$$\begin{aligned}I=&[(\lambda\circ\varphi^+)^{-1}\int_0^s\Phi_p\{(k\lambda_t+h\lambda)u_{i,1}^p\}\circ\varphi\,du]_{s=0}^{s=t}\\&-\int_0^t(\lambda\circ\varphi^+)^{-1}[\Phi_p\{(k\lambda_t+h\lambda)u_{i,1}^p\}\circ\varphi]_{u=s}ds\\&-\int_0^t ds\int_0^s\partial_s\{\Phi_p(\lambda\circ\varphi^+)^{-1}\}\,\{(k\lambda_t+h\lambda)u_{i,1}^p\}\circ\varphi\,du\\&-\int_0^t ds\int_0^s\Phi_{p,p}\{(k\lambda_t+h\lambda)_x u_{i,1}^p\}\circ\varphi\,du\\=&I_1+I_2+I_3+I_4\,.\end{aligned}$$

By the lemmas 1 and 2,

$$\begin{aligned}&\int_0^s\Phi_p\{(k\lambda_t+h\lambda)u_{i,1}^p\}\circ\varphi\,du\\&\quad\leqq C_pK_{m,i}|f|_{p+n-1}A^-[(|k|_0|\lambda_t|+|h|_0\lambda)w_{n-1,n+1}]\circ\varphi^+\\&\quad\leqq C_p\{|k|_0M_2+|h|_0M_1\delta\}K_{m,i}|f|_{p+n-1}w_{n,n+1}\circ\varphi^+\,.\end{aligned}$$

Therefore

$$|I_1|\leqq C_pn^{-1}\{|k|_0M_2+|h|_0M_1\delta\}K_{m,i}w_{n-1,n+1}|f|_{p+n-1}\,.$$

In the same way,

$$|I_2|\leqq C_p\Big\{\frac{|k|_0M_2}{n-1}+\frac{|h|_0M_1\delta}{n+2}\Big\}K_{m,i}w_{n-1,n+1}|f|_{p+n-1}\,,$$

$$\begin{aligned}|I_3|\leqq&\Big\{\frac{C_p'|k|_0M_2M_1\delta}{n(n+2)}+\frac{C_p'|h|_0M_1^2\delta^2}{(n+3)(n+2)}+\frac{C_p|k|_0M_2^2}{n(n-1)}+\frac{C_p|h|_0M_1M_2\delta}{(n-1)(n+2)}\Big\}\\&\times K_{m,i}w_{n-1,n+1}|f|_{p+n-1}\,,\end{aligned}$$

$$|I_4|\leqq C_p''\frac{M_1^2\delta^2}{(n+3)(n+2)}K_{m,i}w_{n-1,n-1}|f|_{p+n-1}\,.$$

On the other hand,

$$\begin{aligned}&|A_{p,p}[(k\lambda_t+h\lambda)u_{i,1,1}^{p+1}]|\\&\quad\leqq C_p''\Big\{\frac{|k|_0M_2M_1\delta}{n+2}+\frac{|h|_0M_1^2\delta^2}{n+3}\Big\}K_{m,i}w_{n-1,n+1}|f|_{p+n-1}\,.\end{aligned}$$

Thus it has been shown that there exists a constant $\rho_1(m, n, \delta)$ such that for any $p \leqq m$

$$|B_1^p[u_i]| \leqq \rho_1(m, n, \delta) K_{m,i} w_{n-1,n+1} |f|_{p+n-1}, \qquad (x, t) \in D_\delta,$$

where $\rho_1(m, n, \delta)$ does not depend on i and $K_{m,i}$. Moreover we have seen that for any integer $m \geqq 0$, if we take n large according to m, then $\rho_1(m, n, \delta)$ becomes small as we wish.

(**B**) Next we consider the estimates of $B_{j+1}^p[u_i]$, $j=1, 2, \cdots, n$. By the lemmas 1 and 2, for any $p \leqq m-1+j$

$$|A_{p,p}[(k\lambda_t + h\lambda) u_{i,j+1}^{p+1}]|$$
$$\leqq \left\{ \frac{C_p |k|_0 M_2 M_1 \delta}{n+2} + \frac{C_p |h|_0 M_1^2 \delta^2}{n+3} \right\} K_{m,i} w_{n-j,n+1} |f|_{p+n-j},$$

$$\left| \sum_{r,r_1} A_{p,r}^+ \left[\binom{r}{r_1} a^{(r_1)} u_{i,j}^{r_2} \right] \right| \leqq \frac{C_p' M_1 \delta}{n+2} K_{m,i} w_{n-j,n+1} |f|_{p+n-j}.$$

The remaining terms are majorized by

$$\frac{C_p'' M_1^2 \delta^2}{(n+2)(n+3)} K_{m,i} w_{n-j,n+1} |f|_{p+n-j}.$$

Therefore there exist constants $\rho_{j+1}(m, n, \delta)$ such that

$$|B_{j+1}^p[u_i]| \leqq \rho_{j+1}(m, n, \delta) K_{m,i} w_{n-j,n+1} |f|_{p+n-j},$$

$(x, t) \in D_\delta$, for any $p \leqq m-1+j$, $j=1, 2, \cdots, n$. In addition, taking δ small if necessary, it is satisfied that

$$\rho_2(m, n, \delta) < \frac{1}{2}, \qquad \rho_{j+1}(m, n, \delta) < 1, \qquad j=2, \cdots, n.$$

Concerning $B_{j+1,0}^p[u_i]$ and $B_{j+1,1}^p[u_i]$, the situations are same. Thus (3.3) and (3.4) have been proved.

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References

[1] Chi Min-You, On the Cauchy problem for a class of hyperbolic equations with data given on the degenerate parabolic line, *Acta Math. Sinica* 8 (1958), 521-529.

[2] ———, On the Cauchy problem for second order hyperbolic equations in two variables with initial data on the parabolic degenerating line, *Acta Math. Sinica* **12** (1962), 68-76.

[3] A. B. Nersesjan, The Cauchy problem for degenerating hyperbolic equations of second order, *Dokl. Acad. Nauk SSSR* **166** (1966), 1288-1291.

[4] O. A. Oleinik, On the Cauchy problem for weakly hyperbolic equations, *Comm. P. A. M.* **23** (1970), 569-586.

[5] A. Menikoff, The Cauchy problem for weakly hyperbolic equations, *Amer. J. M.* **97** (1975), 548-558.

[6] K. Kitagawa et T. Sadamatsu, Sur une condition suffisante pour que le problème de Cauchy faiblement hyperbolique soit bien posé, *J. Math. Kyoto Uuiv.* **17** (1977), 465-499.

[7] T. Nishitani, On the Cauchy problem for weakly hyperbolic equations, *Comm. Part. Diff. Eq.* **3** (1978), 319-333.

[8] V. Ja. Ivrii, The Cauchy problem for non-strictly hyperbolic equations of second order, *Soviet Math. Dokl.* **12** (1971), 483-486.

[9] ———, Sufficient conditions of regularly and completely regularly hyperbolicity, *Trudy Moscow Math. Soc.* **33** (1975), 3-65.