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Affine surfaces containing cylinderlike open sets

By

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Introduction.

Let k be an algebraically closed field of characteristic zero, and let X be an affine surface defined over k such that $X\times A_k^1\cong A_k^3$, where A_k^n denotes the n-dimensional affine space. A cancellation problem asks whether or not X is isomorphic to A_k^2; we call this problem *Zariski's problem*. It is easily ascertained that X is a nonsingular rational surface and the affine coordinate ring of X is a unique factorization domain without non-constant units. In [5; Th. 2] one of the authors proved that X is isomorphic to A_k^2 provided we can derive the following condition from the given isomorphism:

X has a nonempty open set U such that U is isomorphic to $C\times A_k^1$, where C is an affine nonsingular curve; we then say that X has a nonempty open set with *a structure of trivial A^1-bundle* or that X contains *a cylinderlike open set.*

On the other hand, Iitaka and Fujita proved in [2] that, under the condition $X\times A_k^1\cong A_k^3$, X has the logarithmic Kodaira dimension $\bar{\kappa}(X)=-\infty$. This is equivalent to the following condition:

Embed X into a nonsingular projective surface V as an open set so that each irreducible component of the boundary set $V-X=\bigcup_{i=1}^{r}C_i$ is nonsingular and $\bigcup_{i=1}^{r}C_i$ has only normal crossings. Let $D=\sum_{i=1}^{r}C_i$, which is a reduced effective divisor. Then $|m(D+K_V)|=\phi$ for every integer $m>0$, where K_V is the canonical divisor of V.

Then we may ask the following problem:

*Let V be a nonsingular projective (not necessarily rational) surface and let D be a reduced effective divisor on V such that $V-$*Supp(D) *is affine. Assume that $|m(D+K_V)|=\phi$ for every integer $m>0$. Does there exist a nonempty cylinderlike open set in $V-$*Supp(D)?

If each irreducible component of D is nonsingular and Supp(D) has only normal crossings the converse is true. Namely, if $V-$Supp(D) contains a nonempty cylinderlike open set then $|m(D+K_V)|=\phi$ for every integer $m>0$ (cf. Lemma 1.3). If the irregularity $q:=\dim H^1(V, \mathcal{O}_V)$ is positive then the answer to the above problem is affirmative (cf. Theorems 2.1 and 2.2). When $q=0$ then V is a rational surface because $|2K|=\phi$ is a consequence of the assumption. A main part of this atircle is devoted to solving the above problem when V is rational. The answer we obtained so far is affirmative (cf. Corollaries 4.3 and 4.5, Theorems 5.1, 5.9, 5.10 and 5.11). The case in which we cannot solve the problem is the following:

There exists an exceptional curve E of the first kind on V such that $|E+D+K|\neq\phi$ and the $(E+D+K)$-dimension $\kappa(E+D+K)$ equals 2; moreover, we may assume that $((D+K)^2)\leqq -2$. (Cf. the assertion (A) in 6.1 and Remark 6.2.)

Finally we shall notice an analogy between our problem and a classical criterion of ruledness; the condition that $|mK_V|=\phi$ for every integer $m>0$ implies that V is a ruled surface. Indeed, some useful results, e. g. Lemma 3.2, are proved after the ideas effective in proving Castelnuovo's criterion of rationality (cf. [4; Th. 49]).

In this article the ground field k is assumed to be an algebraically closed field of characteristic zero. Let V be a nonsingular projective surface and let D be a divisor on V. Then $|D|$ denotes the complete linear system defined by D. If $P_1, \cdots, P_r$ are points (including infinitely near points) on V and if $m_1, \cdots, m_r$ are positive integers then $|D|-(m_1P_1+\cdots+m_rP_r)$ denotes the linear subsystem of $|D|$ consisting of members of $|D|$ which pass through P_i's with multiplicity$\geqq m_i$. Let $f\colon V\rightarrow W$ be a finite morphism of nonsingular projective surfaces. For a curve C on V, $f(C)$ denotes the set-theoretic image of C on W; for a curve C' on W, $f^{-1}(C')$ denotes the set-theoretic inverse image of C'; for a divisor D on V, $f_*(D)$ denotes the direct image (as a cycle) of D by f; for a divisor D' on W, $f^*(D')$ denotes the total transform of D' by f and $f'(D')$ denotes the proper transform of D' by f. The other notations are as follows:

p_g (or $p_g(V)$): the geometric genus of V,
q (or $q(V)$): the irregularity of V,
K (or K_V): the canonical divisor of V,
P_m (or $P_m(V)$): the m-th pluri-genus of V, i. e. $P_m=\dim|mK|+1$
$(D\cdot D')$, (D^2): the intersection multiplicity of divisors,
$D\sim D'$: a divisor D is linearly equivalent to a divisor D',
$p_a(D)$: the arithmetic genus of D, i.e., $p_a(D)=\frac{1}{2}(D\cdot D+K)+1$,
$h^i(V, D)$ (or $h^i(D)$): $=\dim H^i(V, \mathcal{O}_V(D))$.

If D is an effective divisor such that every irreducible component of D is nonsingular and Supp(D) has only normal crossings the dual graph of D is obtained by assigning a vertex to each irreducible component of D and by connecting two vertices by an edge if the corresponding components meet each other; we assign one edge for each intersection point. In this article, an exceptional curve of the first kind means always an irreducible curve E with $p_a(E)=0$ and $(E^2)=-1$.

§ 1. Preliminaries.

Let us begin with the following

2.1. Lemma. *Let V be a nonsingular projective surface and let $D=\sum_{i=1}^{r} C_i$ be a reduced effective divisor on V. Let m be the number of connected components of* Supp (D). *Assume that each irreducible component of D is nonsingular. Then we have*

$$e(D):=m-r+\sum_{i<j}(C_i\cdot C_j)\geqq 0 .$$

More precisely, the equality holds if and only if D has only normal crossings and the dual graph of D is a tree.

Proof. Let $D_1+\cdots+D_m$ be the decomposition of D into connected components. Then $e(D)=e(D_1)+\cdots+e(D_m)$. Hence we may assume that D is connected.

Let $\sigma: V'\rightarrow V$ be a quadric transformation with center P which lies on Supp (D). Set $E:=\sigma^{-1}(P)$, $C'_i:=\sigma'(C_i)$ for $1\leqq i\leqq r$ and $D':=E+\sigma'(D)$. Then D' is a connected reduced divisor. We shall show that $e(D)\geqq e(D')$ and the equality holds if and only if there are at most two irreducible components of D passing through P. Indeed, we assume after a change of indices that $C_1, \cdots, C_n$ are all irreducible components of D passing through P. If $n=1$ we have clearly $e(D')=e(D)$. Suppose that $n>1$. Noting that $(C'_i\cdot C'_j)=(C_i\cdot C_j)-1$ for $1\leqq i<j\leqq n$ and $(C'_i\cdot E)=1$ for $1\leqq i\leqq n$, we have

$$e(D')=e(D)-\binom{n}{2}+n-1 .$$

Thus $e(D)\geqq e(D')$, and $e(D)=e(D')$ if and only if $n\leqq 2$.

Let $\rho: \bar{V}\rightarrow V$ be the shortest composition of quadric transformations such that the set-theoretic inverse image $\bar{D}:=\rho^{-1}(D)$ has only normal crossings. Then $\bar{D}$ is a connected reduced divisor, and $e(D)\geqq e(\bar{D})$, where the equality holds if and only if D itself has only normal crossings. Thus we may assume that D has only normal crossings.

Let Γ be the dual graph of D. Then r is the number of vertices of Γ. Let M be the number of edges of Γ, and set $e(\Gamma):=1-r+M$. Then $e(D)=e(\Gamma)$. We shall prove by induction on $M+r$ that $e(\Gamma)\geqq 0$ and $e(\Gamma)=0$ if and only if Γ is a tree. If Γ contains a loop (=a cyclic chain) then take one edge off the loop. The obtained graph Γ' is connected and $e(\Gamma')=e(\Gamma)-1$, which is nonnegative by inductive assumption. Thus we may assume that Γ contains no loops. Then Γ is a tree. Let Γ'' be the graph obtained from Γ by deleting a terminal vertex and an edge connecting it to some other vertex. Then $e(\Gamma'')=e(\Gamma)$, which is zero by inductive assumption. Q.E.D.

1.2. Lemma. (cf. Kodaira [3; Th. 2.2]). *Let V and D be as in Lemma* 1.1. *Then we have:*

$$\dim|D+K_V|=p_a(D)+p_g-q+m+t-2$$
$$=\sum_{i=1}^{r}p_a(C_i)+p_g-q+e(D)+t-1\,,$$

where $q=\dim H^1(V, \mathcal{O}_V)$ *and* $t=\dim \mathrm{Ker}\,\varphi$, φ *being the canonical homomorphism* $H^1(V, \mathcal{O}_V)\to H^1(D, \mathcal{O}_D)$ *induced by*

$$0\longrightarrow \mathcal{O}_V(-D)\longrightarrow \mathcal{O}_V\longrightarrow \mathcal{O}_D\longrightarrow 0\,.$$

1.3. Lemma. *Let* V *be a nonsingular projective surface and let* D *be a reduced effective divisor such that* $V-\mathrm{Supp}\,(D)$ *is affine. Consider the following four conditions:*

(1) *There exists a nonempty open set* U *in* $V-\mathrm{Supp}\,(D)$ *such that* U *has a structure of trivial* $\boldsymbol{A}^1$*-bundle;*

(2) *there exists an irreducible curve* C *on* V *such that* $C\not\subset \mathrm{Supp}\,(D)$ *and* $(C\cdot D+K)<0$;

(3) *for any divisor* A *on* V, $|A+m(D+K)|=\phi$ *for any sufficiently large integer* m;

(4) $|m(D+K)|=\phi$ *for every positive integer* m.

Then we have the implications: (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *If* D *has at worst nodal double points as singularities then we have:* (1) $\Rightarrow$ (2).

Proof. (3) $\Rightarrow$ (4) is clear. (2) $\Rightarrow$ (3): Since $C\not\subset\mathrm{Supp}\,(D)$ and $V-\mathrm{Supp}\,(D)$ is affine we have $(C\cdot D)>0$, whence $(C\cdot K)\leqq -2$ and $(C^2)=2p_a(C)-2-(C\cdot K)\geqq 0$. Since $(A+m(D+K)\cdot C)<0$ if $m>-(A\cdot C)/(D+K\cdot C)$ we know that $|A+m(D+K)|=\phi$ if $m>-(A\cdot C)/(D+K\cdot C)$.

Assuming that D has at worst nodal double points as singularities, we shall show the implication: (1) $\Rightarrow$ (2). Since U has a structure of $\boldsymbol{A}^1$-bundle, let C_0 be one of its fibers; C_0 is then isomorphic to $\boldsymbol{A}^1$. Let C be the closure of C_0 in V and let $P:=C\cap D$. Let ν be the multiplicity of C at P and let μ be the multiplicity of the curve D at P. Then $\mu\leqq 2$ by assumption. Let $\sigma: V'\to V$ be a quadric transformation with center P, and let $E:=\sigma^{-1}(P)$. Then we have:

$$\sigma^*(C)=\sigma'(C)+\nu E\,,\qquad \sigma^*(D)=\sigma'(D)+\mu E$$
$$\text{and}\quad K_{V'}\sim\sigma^*(K_V)+E\,.$$

Let $C':=\sigma'(C)$ and $D':=\sigma'(D)+E$. Then D' is a reduced effective divisor such that $V'-\mathrm{Supp}\,(D')=V-\mathrm{Supp}\,(D)$, which is, therefore, affine, and D' has at worst nodal double points as singularities. Moreover we have:

$$(C'\cdot D'+K_{V'})=(C\cdot D+K_V)+\nu(2-\mu)\,.$$

Since $\mu\leqq 2$ we have: $(C\cdot D+K_V)\leqq(C'\cdot D'+K_{V'})$. Hence, by repeating this process if necessary we may assume that $(C\cdot D)=1$ and $(C\cdot C')=0$ if C' is the closure in V of another fiber $C_0'(\neq C_0)$ of the $\boldsymbol{A}^1$-bundle U. Then C moves in an algebraic pencil Λ on V, which has no base points. Thus C is a nonsingular rational curve, and we have:

$$(C\cdot D+K)=(C\cdot C+D+K)=(C\cdot D)+(C\cdot C+K)=-1\,.$$

Hence we get $(C\cdot D+K_V)<0$ in general. Q. E. D.

1.4. In the previous lemma, assume that V is relatively minimal and $D=0$. Then the four conditions are equivalent to each other, and they are, in fact, different characterizations of a ruled surface (cf. Mumford [7; pp. 326-330]). In the section below, we shall consider when the four conditions in Lemma 1.3 are equivalent to each other.

§ 2. Case of irrational ruled surfaces.

2.1. Theorem. *Let V be a nonsingular projective surface with irregularity $q>1$ and let D be a reduced effective divisor on V such that $|D+K|=\phi$ and $|mK|=\phi$ for every integer $m>0$. Then $V-\mathrm{Supp}\,(D)$ has a nonempty open set U which has a structure of trivial $\boldsymbol{A}^1$-bundle.*

Proof. Since $P_m(V)=0$ for every integer $m>0$, V is a ruled surface. Let $f\colon V\to B:=f(V)\hookrightarrow \mathrm{Alb}\,(V)$ be the Albanese mapping of V; then B is a nonsingular curve of genus q, and the ruling of V is given by f. Write $D=\sum_{i=1}^{r} C_i$. Suppose that $(D\cdot\Gamma)=0$ for a general fiber Γ of f. Then every irreducible component C_i of D is contained in a fiber of f. In this case, the existence of a nonempty open set U as stated as above is clear. Thus we assume that $(D\cdot\Gamma)>0$. Then there exists an irreducible component, say C_1, of D such that $(C_1\cdot\Gamma)>0$; we have then $q\leqq$geometric genus of $C_1\leqq p_a(C_1)$. By virtue of Lemma 1.2 we have:

$$q\leqq \sum_{i=1}^{r} p_a(C_i)=q-t-e(D)\leqq q-t\,,$$

whence we conclude the following:

(1) C_1 is a nonsingular curve of genus q, and other components $C_2, \cdots, C_r$ are contained in fibers of f;

(2) D has only normal crossings and the dual graph of D is a tree.

Now consider the morphism $\varphi:=f|_{C_1}\colon C_1\to B$, which is a surjective morphism. Applying Hurwitz's formula to φ we know that φ is an isomorphism. Hence C_1 is a cross-section of f, and the existence of a nonempty open set U as stated as above is clear. Q. E. D.

2.2. Thorem. *Let V be a nonsingular projective surface with irregularity 1 and let D be a reduced effective divisor on V such that $|m(D+K)|=\phi$ for every integer $m>0$. Then there exists a nonempty open set U in $V-\mathrm{Supp}\,(D)$ such that U has a structure of trivial $\boldsymbol{A}^1$-bundle.*

Proof. Since $P_m(V)=0$ for every integer $m>0$, V is a ruled surface, whose ruling is given by the Albanese mapping $f\colon V\to A:=\mathrm{Alb}\,(V)$, where A is a nonsingular elliptic curve. As in the proof of Theorem 2.1 there are two cases to be considered. Write $D=\sum_{i=1}^{r} C_i$. If every irreducible component C_i of D is contained in a fiber of f, then the existence of a nonempty open set U as stated as above is clear. Assume that not all irreducible components of D are contained in fibers of f. Then there exists a unique irreducible component, say C_1, such that $\varphi:=f|_{C_1}: C_1 \to A$ is a surjective morphism of nonsingular elliptic curves and that the other components $C_2, \cdots, C_r$ are contained in fibers of f. The morphism φ is, in fact, an unramified morphism of degree $n>0$. If $n=1$ then C_1 is a cross-section of f, and the existence of a nonempty open set U as stated as above is clear. Thus we shall assume below that $n>1$.

Set $A':=C_1$. Since A' is an abelian variety of dimension 1, we may assume that φ is a homomorphism of abelian varieties. Let $G:=\mathrm{Ker}\,\varphi$, and let $V':=V\times_A A'$. Then V' is a nonsingular projective surface; indeed, if ψ: $V'\to V$ and f': $V'\to A'$ are the first and second projections, respectively, we have the following cartesian diagram,

$$\begin{array}{ccc} V' & \xrightarrow{\ \psi\ } & V \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ A' & \xrightarrow{\ \varphi\ } & A \end{array},$$

where ψ is a finite unramified morphism. More precisely, G acts freely on V' via translations on A' and V is isomorphic to the quotient variety V'/G. Therefore we know that $K_{V'}=\psi^*K_V$ and $P_m(V')=0$ for every integer $m>0$. Hence V' is a ruled surface whose ruling is given by the Albanese mapping f': $V'\to A'$. Let C' be a cross-section of f' defined by

$$C':=\{(a', a');\ a'\in A'\}.$$

Then $\psi^*(C_1)=\sum_{\sigma\in G}\sigma(C')$, where

$$\sigma(C')=\{(a', a'+\sigma);\ a'\in A'\}$$

for $\sigma\in G\subset A'$. Hence $\sigma(C')\cap\tau(C')=\phi$ for distinct $\sigma, \tau\in G$. Since $|m(C_1+K_V)|=\phi$ for every integer $m>0$ we know that $|m(\psi^*(C_1)+K_{V'})|=\phi$ for every integer $m>0$. Thus we obtain

$$\begin{aligned} -1=\dim\Big|\sum_{\sigma\in G}\sigma(C')+K_{V'}\Big| &= p_a\Big(\sum_{\sigma\in G}\sigma(C')\Big)+p_g-q+n+t'-2 \\ &= n+t'-2, \end{aligned}$$

where $n=|G|$, $p_a(\sum_{\sigma\in G}\sigma(C'))=1$ and $t'=\dim \mathrm{Ker}(H^1(V', \mathcal{O}_{V'})\to H^1(\sum\sigma(C'), \mathcal{O}_{\sum\sigma(C')}))$ (cf. Lemma 1.2). Hence we have $n+t'=1$, which is a contradiction because $n>1$ and $t'\geqq 0$. Q.E.D.

2.3. By virtue of Theorems 2.1 and 2.2, we know that the four conditions of Lemma 1.3 are equivalent to each other.

§ 3. Case of rational surfaces - I.

In this section, V denotes a nonsingular projective rational surface and D denotes a reduced effective divisor on V.

3.1. Lemma. *Assume that* $|D+K|=\phi$. *Then every irreducible component of* D *is a nonsingular rational curve,* D *has only normal crossings, and the dual graph of* D *is a tree. Conversely, these three conditions imply* $|D+K|=\phi$.

Proof. Write $D=\sum_{i=1}^{r} C_i$. Then, with the notations of Lemma 1.2, we have:

$$\sum_{i=1}^{r} p_a(C_i) = -e(D),$$

where $t=0$ because $q=0$. Since $e(D)\geqq 0$ we have: $p_a(C_i)=0$ for $1\leqq i\leqq r$ and $e(D)=0$. Then our assertions follow from Lemma 1.1. The converse follows from Lemmas 1.1 and 1.2. Q. E. D.

3.2. Lemma. *Let* V *and* D *be as above. Assume that the following conditions are satisfied:*

(1) $V-\mathrm{Supp}(D)$ *is affine;*

(2) *there exists an irreducible curve* C *such that* $(C^2)\geqq 0$ *and* $|C+D+K|=\phi$.

Then $V-\mathrm{Supp}(D)$ *has a nonempty open set* U *which has a structure of trivial* $\boldsymbol{A}^1$*-bundle.*

Proof. (Cf. Kodaira [4; Th. 49]). Our proof consists of several subparagraphs.

3.2.1. Since $|C+K|=\phi$, C is a nonsingular rational curve (cf. the above lemma). Then the Riemann-Roch theorem implies

$$\dim |C| \geqq (C^2)+1 \geqq 1.$$

Replacing C by a general member of $|C|$ if necessary, we may assume that $C \not\subset \mathrm{Supp}(D)$. Since $V-\mathrm{Supp}(D)$ is affine, we have $(C\cdot D)>0$, whence $C+D$ is connected. By virtue of Lemma 3.1, we have, in fact, $(C\cdot D)=1$. Namely C meets only one irreducible component, say D_1, of D transversally at a single point.

3.2.2. Set

$$N := \{C\,;\ \text{irreducible curve with } (C^2)\geqq 0 \text{ and } |C+D+K|=\phi\}.$$

Then $N\neq\phi$ by assumption. Let $m_0=\min_{C\in N}(C^2)$, and set

$$M := \{C\in N\,;\ (C^2)=m_0\}.$$

Now fix a very ample divisor A and let $\delta := \min_{C\in M}(C\cdot A)$. Let C_0 be a member of M with $(C_0\cdot A)=\delta$. Then we claim that the following assertion holds:

Every member C' *of* $|C_0|$ *is either a nonsingular rational curve or a reducible curve* $C'=\sum n_i C_i$ *such that* $p_a(C_i)=0$ *and* $(C_i^2)<0$ *for every* i *and that at least one irreducible component of* C' *is an exceptional curve of the first kind.*

Proof. Write $C'=\sum_{i=1}^{r} n_i C_i$. Since

$$\dim |\sum_{i=1}^{r} C_i+K| \leqq \dim |C_0+K|=-1,$$

we know that $p_a(C_i)=0$ for every i (cf. Lemma 3.1). Assume that one irreducible component, say C_1, of C' has $(C_1^2)\geqq 0$. Then, since $\dim|C_1+D+K| \leqq \dim|C_0+D+K|=-1$ we have $C_1\in N$, whence $(C_1^2)\geqq m_0$. Moreover, we have

$$m_0=(C_0^2)=\sum_i n_i(C_i\cdot C_0)\geqq n_1(C_1\cdot C_0)=\sum_i n_1 n_i(C_1\cdot C_i)\geqq n_1^2(C_1^2)\geqq n_1^2 m_0 .$$

Hence we have $(C_1^2)=m_0$, which implies $C_1\in M$. Then we have

$$(A\cdot C_1)\geqq \delta=(A\cdot C_0)=n_1(A\cdot C_1)+\sum_{i\neq 1} n_i(A\cdot C_i).$$

Since $(A\cdot C_i)>0$ for every i, we know that $C'=\sum_{i=1}^{r} n_i C_i=C_1$. Therefore, if C' is irreducible then C' is a nonsingular rational curve; if C' is reducible then $p_a(C_i)=0$ and $(C_i^2)<0$ for every irreducible component C_i of C'. On the other hand, since

$$\sum_i n_i(K\cdot C_i)=(K\cdot C_0)=-2-(C_0^2)\leqq -2,$$

there exists an irreducible component, say C_1, of C' such that $(K\cdot C_1)<0$. Since $(C_1^2)<0$ if C' is reducible, C_1 is an exceptional curve of the first kind.

3.2.3. Note that $\dim|C_0|\geqq (C_0^2)+1\geqq 1$. We claim that $\dim|C_0|\leqq 2$.

Proof. We shall show that the union of the supports of reducible members of $|C_0|$ is a proper closed subset R of V. Let $C'=\sum_{i=1}^{r} n_i C_i$ be a reducible member of $|C_0|$. Then $p_a(C_i)=0$, $(C_i^2)<0$ and $0<(A\cdot C_i)<\delta=(A\cdot C_0)$ for $1\leqq i\leqq r$. For an irreducible component C_i of C', let Δ be an irreducible component of the Chow variety of nonsingular rational curves on V with degree $(C_i\cdot A)$ such that $C_i\in\Delta$. If $\dim\Delta>0$ then $(C_i^2)\geqq 0$, which contradicts $(C_i^2)<0$. Hence, $\dim\Delta=0$. Thus, there exist only finitely many reducible members in $|C_0|$, and R is a proper closed subset of V. Suppose that $\dim|C_0|\geqq 3$. Let P be a point not in R, and let C'' be a member of $|C_0|$ such that C'' passes through P with multiplicity$\geqq 2$. Then C'' is a singular irreducible curve, which contradicts the assertion in 3.2.2. Therefore, we have $\dim|C_0|\leqq 2$.

3.2.4. By virtue of 3.2.3, we have

$$0\leqq (C_0^2)\leqq \dim|C_0|-1\leqq 1.$$

Thus, replacing C by C_0 if necessary we may assume that $(C^2)=0$ or 1. We consider these two cases separately.

Case A. If $(C^2)=0$ then $|C|$ has no base points, and the morphism $\varphi:=\Phi_C: V\to \boldsymbol{P}^1$ has the following properties:

(1) General fibers of φ are nonsingular rational curves;

(2) D has an irreducible compenent D_1 such that $(D_1\cdot C)=1$ and $(D-D_1\cdot C)=0$; this implies that D_1 is a cross-section of φ and $D-D_1$ consists of components contained in fibers of φ. Then it is easy to see that there exists an open set $U(\neq\phi)$ in $V-\mathrm{Supp}(D)$ such that $\varphi|_U: U\to\varphi(U)$ is a trivial $\boldsymbol{A}^1$-bundle.

Case B. Suppose that $(C^2)=1$. Then $\dim|C|=2$. Since $(C\cdot D)=1$ there exists an irreducible component D_1 of D such that $(C\cdot D_1)=1$ and $(C\cdot D-D_1)=0$. Let P be a general point on D_1 and let $L:=|C|-P$. Then $\dim L=1$. Let $\sigma: V'\to V$ be a quadric transformation of V with center P, let $E=\sigma^{-1}(P)$ and let $L'=\sigma' L$. Then $\dim L'=1$ and $(C'^2)=0$ for a member C' of L'. Then the morphism $\varphi':=\Phi_{L'}: V'\to\boldsymbol{P}^1$ has the following properties:

(1) General fibers of φ' are nonsingular rational curves;

(2) E is a cross-section of φ';

(3) $V'-(E\cup\sigma'(D))\cong V-D$.

Hence there exists a nonempty open set U in $V-\mathrm{Supp}(D)$ such that $\varphi'|_U: U\to\varphi'(U)$ is a trivial $\boldsymbol{A}^1$-bundle. This completes a proof of Lemma 3.2.

3.3. Lemma. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor such that $|D+K|=\phi$ and $V-\mathrm{Supp}(D)$ is affine. Assume that there exists a terminal component C of D with $(C^2)\geqq 0$. Then $V-\mathrm{Supp}(D)$ has a non-empty open set U which has a structure of trivial $\boldsymbol{A}^1$-bundle.*

Proof. Note that $p_a(C)=0$ and $\dim|C|\geqq(C^2)+1$. Let $P_0:=C\cap(D-C)$ when $D\neq C$; let P_0 be any point on C when $D=C$. Let $n:=(C^2)$. If $n>0$, let $P_1,\cdots,P_{n-1}$ be points on C such that P_i is an infinitely near point of P_{i-1} of order one for $1\leqq i<n$. Let Λ be the linear subsystem of $|C|$ consisting of members which pass through $P_0, P_1,\cdots,P_{n-1}$, i.e.,

$$\Lambda:=|C|-(P_0+P_1+\cdots+P_{n-1}).$$

Then $\dim\Lambda\geqq 1$.

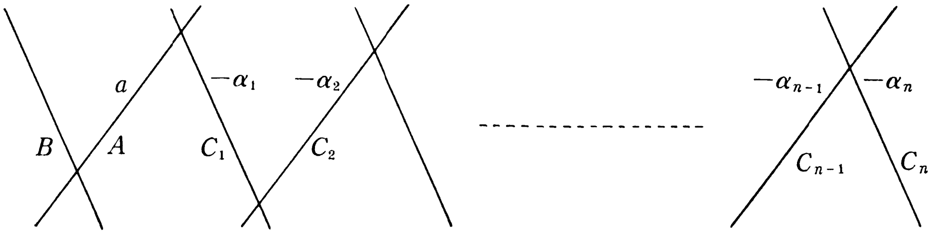
Let $V_0:=V$ and let $\sigma_i: V_i\to V_{i-1}$ be the quadric transformation with center P_{i-1} for $1\leqq i\leqq n$. Let $\sigma:=\sigma_1\cdots\sigma_n$, let $E_i:=(\sigma_{i+1}\cdots\sigma_n)'(\sigma_i^{-1}(P_{i-1}))$ for $1\leqq i\leqq n$, and let $C':=\sigma'(C)$. Then $(C'^2)=0$ and $C'\in\sigma'\Lambda$. Hence $\dim\Lambda=1$, i.e., Λ is a linear pencil. Let $D':=\sigma'(D)+E_1+\cdots+E_n$. Then D' is a reduced effective divisor on $V':=V_n$ such that $V'-\mathrm{Supp}(D')\cong V-\mathrm{Supp}(D)$. Note that $(C'\cdot D')=1$. This is clear if $n>0$; if $n=0$, D is reducible because $(D^2)>0$ otherwise and we get $(C\cdot D)=1$ by virtue of Lemma 3.1. Let $\varphi':=\Phi_{\sigma'\Lambda}: V'\to\boldsymbol{P}^1$. Then φ' has the following properties:

(1) General fibers of φ' are nonsingular rational curves;

(2) there exists an irreducible component D_1' of D' such that D_1' is a cross-section and $D'-D_1'$ consists of components contained in fibers of φ'.

It is now clear that $V-\mathrm{Supp}(D)$ has a nonempty open set U which has a structure of trivial $\boldsymbol{A}^1$-bundle. Q.E.D.

3.4. Lemma. *Let V be a nonsingular projective rational surface and let D be a* *reduced effective divisor such that* $|D+K|=\phi$ *and* $V-\mathrm{Supp}(D)$ *is affine. Assume that* $\mathrm{Supp}(D)$ *contains a chain of curves,*



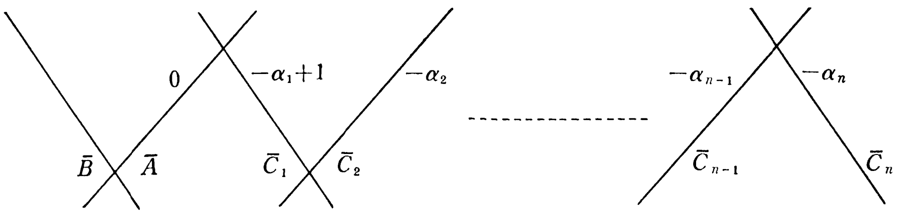
where:

(1) *There are no other curves in* $\mathrm{Supp}(D)$ *which intersect some of* $A, C_1, C_2, \cdots, C_n$;

(2) $a:=(A^2)\geqq 0$ *and* $\alpha_i:=-(C_i^2)>0$ *for* $1\leqq i\leqq n$.

Then $V-\mathrm{Supp}(D)$ *has a nonempty open set* U *which has a structure of trivial* $\boldsymbol{A}^1$-*bundle.*

Proof. By virtue of Lemma 3.1 we know that every irreducible component of D is a nonsingular rational curve, D has only normal crossings and the dual graph of D is a tree. Let $P_0:=A\cap B$, and let $P_1, \cdots, P_a$ be points on the curve A such that P_i is an infinitely near point of P_{i-1} of order one for $1\leqq i\leqq a$. Let $\sigma: V'\rightarrow V$ be the composition of quadric transformations with centers $P_0, P_1, \cdots, P_a$, and let E_i be the proper transform on V' of the exceptional curve which arises from the quadric transformation with center P_{i-1} for $1\leqq i\leqq a+1$. Then $(E_i^2)=-2$ for $1\leqq i\leqq a$ and $(E_{a+1}^2)=-1$. Let $A':=\sigma'(A)$. Then $(A'^2)=-1$. Let $\tau: V'\rightarrow \bar{V}$ be the contraction of A', and let $\bar{A}:=\tau(E_{a+1})$, $\bar{B}:=\tau(E_a)$ and $\bar{C}_i:=\tau(\sigma'(C_i))$ for $1\leqq i\leqq n$. Then we have the following chain of curves,



Let $\bar{D}:=(\tau\sigma^{-1})'(D)+\sum_{i=1}^{a+1}\tau(E_i)$. Then it is easy to show that:

(1) $\bar{V}-\mathrm{Supp}(\bar{D})\cong V-\mathrm{Supp}(D)$, whence $\bar{V}-\mathrm{Supp}(\bar{D})$ is affine;

(2) $|\bar{D}+K_{\bar{V}}|=\phi$ (cf. Lemma 3.1).

We repeat the above process α_1-times, after which the proper transform of C_1 has self-intersection multiplicity 0 and we obtain a similar situation as the one we started with; however, the number of curves $C_1, \cdots, C_n$ is one less than the original one. Hence, proceeding by induction on n we obtain a nonsingular projective rational surface $\tilde{V}$ and a reduced effective divisor $\tilde{D}$ such that:

(i) $\tilde{V}-\mathrm{Supp}(\tilde{D})\cong V-\mathrm{Supp}(D)$, which is affine;

(ii) $|\tilde{D}+K_{\tilde{V}}|=\phi$;

(iii) $\mathrm{Supp}(\tilde{D})$ has a terminal component with non-negative self-intersection multiplicity.

Then our assertion follows from Lemma 3.3. Q.E.D.

§4. Case of rational surfaces - II.

In this section, let V be a nonsingular projective rational surface and let D be a reduced effective divisor such that $|2(D+K_V)|=\phi$ and $V-\mathrm{Supp}(D)$ is affine.

4.1. Lemma. *Let V and D be as above. Then the following assertions hold true.*

(1) $\dim|-(D+K)|\geqq((D+K)^2)+1$. *Hence* $\dim|-(D+K)|\geqq 0$ *if* $((D+K)^2)\geqq -1$.

(2) *Suppose that* $|-(D+K)|\neq\phi$. *Let* $D'\in|-(D+K)|$. *Then* $D'>0$; *every irreducible component of D' is a nonsingular rational curve; D' has only normal crossings; the dual graph of any connected component of* $\mathrm{Supp}(D')$ *is a tree.*

Proof. Since $V-\mathrm{Supp}(D)$ is affine, D is connected. Then, $p_a(D)=0$ by Lemma 1.2. Hence $(D+K\cdot D)=-2$. Since $|2K+D|+D\subseteq|2(D+K)|=\phi$, we have $|D+2K|=\phi$. Then the Riemann-Roch theorem yields

$$\dim|-(D+K)|\geqq\frac{1}{2}(D+K\cdot D+2K)=((D+K)^2)+1\,.$$

Suppose that $|-(D+K)|\neq\phi$. Let $D'\in|-(D+K)|$. Then $D'>0$ because $|D+K|=\phi$. Furthermore, $|D'+K|=|-D|=\phi$. Then the remaining assertions follow from Lemma 3.1. Q.E.D.

4.2. Theorem. *Let V and D be as above. Assume that* $\dim|-(D+K)|>0$. *Then* $V-\mathrm{Supp}(D)$ *has a nonempty open set U which has a structure of trivial $\boldsymbol{A}^1$-bundle.*

Our proof consists of several subparagraphs below.

4.2.1. *Suppose that* $\dim|-(D+K)|>2$. *Then the assertion holds.*

Proof. Let $D'\in|-(D+K)|$. Then $D'>0$. Let F be the fixed part of the linear system $|D'|$, and write $|D'|=|D''|+F$. Suppose that $|D''|$ is composed of a pencil Λ. Then Λ is a linear pencil, and $D''\sim rL$ for $L\in\Lambda$ and an integer $r\geqq 2$. Since $(L^2)\geqq 0$ and $|L+D+K|=|-(r-1)L-F|=\phi$, the assertion follows from Lemma 3.2. Thus we may assume that general members of $|D''|$ are irreducible; by virtue of Lemma 4.1, irreducible members of $|D''|$ are nonsingular rational curves. Suppose that $\sum_i n_iC_i$ is a reducible member of $|D''|$. If $(C_i^2)\geqq 0$ for some i then we have

$$|C_i+D+K|=|-\sum_{j\neq i}n_jC_j-(n_i-1)C_i-F|=\phi\,,$$

and the assertion follows from Lemma 3.2. Thus, the remaining case is this:

If $\sum_i n_i C_i$ is a reducible member of $|D''|$ then $(C_i^2)<0$ for every i. This implies that $|D''|$ has finitely many reducible members (cf. the proof of 3.2.3). Since irreducible members of $|D''|$ are nonsingular, we have $\dim|D''|\leqq 2$, which is a contradiction. Thus the assertion of Theorem 4.2 holds if $\dim|-(D+K)|>2$.

Q. E. D.

In the following we may assume that $\dim|-(D+K)|\leqq 2$; thence $((D+K)^2)\leqq 1$. Let $d:=1-((D+K)^2)$; then $d\geqq 0$. Since $(D+K\cdot D)=-2$ (cf. Lemma 1.2) we obtain

$$(K^2)=(D^2)+5-d\,.$$

Since V is rational we have $(K^2)\leqq 9$; $(K^2)=9$ if and only if V is isomorphic to $\boldsymbol{P}^2$; $(K^2)=8$ if and only if V is a Hirzebruch surface $F_n:=\mathrm{Proj}\,(\mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(n))$ with $n\geqq 0$.

4.2.2. *If $(K^2)=9$ then V is isomorphic to $\boldsymbol{P}^2$ and D is a conic. In this case the assertion of Theorem 4.2 holds.*

Proof. V is isomorphic to $\boldsymbol{P}^2$ as remarked above, and $(D^2)=4+d$. Write $D\sim aH$ for an integer $a>0$ and a hyperplane H on $\boldsymbol{P}^2$. Then $a<3$ because $|D+K|=\phi$, whence we obtain $4\leqq 4+d=a^2<9$. Then $d=0$ and $a=2$. Thus D is a conic. If D is reducible, $D=l_1+l_2$ with distinct two lines l_1 and l_2. If D is irreducible then $p_a(D)=0$ and $(D^2)=4$. In both cases, Lemma 3.3 shows that $V-\mathrm{Supp}\,(D)$ has a nonempty open set with a structure of trivial $\boldsymbol{A}^1$-bundle.

Q. E. D.

4.2.3. We consider next the case where $(K^2)=8$. Then V is a Hirzebruch surface F_n with $n\geqq 0$. Let $\pi\colon V\to\boldsymbol{P}^1$ be the fibration by $\boldsymbol{P}^1$ which defines a ruling on V. Let M and l be the minimal section and a fiber of π, respectively; for $n=0$ we fix one fibration by $\boldsymbol{P}^1$ and a cross-section for π as M. Then $(M^2)=-n$, $(M\cdot l)=1$ and $(l^2)=0$; $K_V\sim -2M-(n+2)l$. Write an effective divisor D as $D\sim aM+bl$ with integers a, $b\geqq 0$. Since $(D^2)=3+d$ and $p_a(D)=0$, we obtain

$$(D^2)=a(2b-an)=3+d\,,$$

$$2p_a(D)=(a-1)\{2(b-1)-an\}=0\,.$$

Hence either $a=1$ or $2b-an=2$.

4.2.3.1. *If $a=1$ the assertion of Theorem 4.2 holds.*

Proof. If $a=1$ we have $b=(n+3+d)/2$. Then

$$D\sim M+\left(\frac{n+3+d}{2}\right)l \quad\text{and}\quad D+K\sim -M-\left(\frac{n+1-d}{2}\right)l\,.$$

Note that $(l^2)=0$ and $|l+D+K|=\left|-M-\left(\frac{n-1-d}{2}\right)l\right|=\phi$ because $(l\cdot l+D+K)=-1$. Hence the assertion of Theorem 4.2 follows from Lemma 3.2. Q. E. D.

4.2.3.2. *Assume that* $2b-an=2$. *Then either* $n=0$ *or* $d=1$, *and the assertion of Theorem* 4.2 *holds.*

Proof. We have $a=(3+d)/2$, $b=(4+3n+dn)/4$ and

$$D\sim\left(\frac{3+d}{2}\right)M+\left\{\left(\frac{3+d}{4}\right)n+1\right\}l.$$

Hence $d\equiv 1 \pmod 2$. Since $d\geqq 0$ we have $d\geqq 1$. Since $(D\cdot M)=1-\left(\frac{3+d}{4}\right)n$ we know that $(D\cdot M)<0$ if $n\geqq 1$ and $(D\cdot M)=1$ if $n=0$. If $n=0$ we may change the roles of l and M and assume that $D\sim M+\left(\frac{3+d}{2}\right)l$; then the assertion of Theorem 4.2 follows from Lemma 3.2 because $(l^2)=0$ and $|l+D+K|=\phi$. Thus we assume that $n\geqq 1$. Assume that $d>1$. Then D is of the form $D=M+M'$, where M' is a reduced effective divisor such that

$$M'\sim\left(\frac{1+d}{2}\right)M+\left\{\left(\frac{3+d}{4}\right)n+1\right\}l.$$

Since D is a reduced connected divisor we must have:

$$(M\cdot M')=\frac{4-(d-1)n}{4}>0,$$

which is impossible because $d>1$, $n>0$ and $(M\cdot M')$ is an integer. Therefore $d=1$ if $n\geqq 1$. Then $a=2$, $b=n+1$ and $D\sim 2M+(n+1)l$. Note that $(D\cdot M)=1-n<0$ if $n\geqq 2$ and $(D\cdot M)=0$ if $n=1$. Since $V-\mathrm{Supp}(D)$ is affine by assumption we have $D=M+D'$, where D' is a reduced effective divisor such that $D'\sim M+(n+1)l$. Let r be the number of irreducible components of D. Then $r\geqq 2$. If $r\geqq 3$ it is easy to see that $r=3$ and $D=M+l+M'$, where M' is an irreducible prime curve such that $M'\sim M+nl$; the dual graph of D is given by

$$\begin{array}{ccccc} -n & & 0 & & n \\ \circ & \text{——} & \circ & \text{——} & \circ \\ M & & l & & M' \end{array}$$

If $r=2$ then $D=M+M'$, where M' is an irreducible curve such that $M'\sim M+(n+1)l$; the dual graph of D is given by

$$\begin{array}{ccc} -n & & n+2 \\ \circ & \text{——} & \circ \\ M & & M' \end{array}.$$

In each of these cases the assertion of Theorem 4.2 follows from Lemma 3.3.

Q. E. D.

4.2.4. *If* $(K^2)<8$ *the assertion of Theorem* 4.2 *holds.*

Proof. Let $s:=8-(K^2)$. Then $s\geqq -1$. We shall prove the assertion by induction on $s+d$. The assertion holds in the following cases: $d<0$ (cf. 4.2.1), $s=-1$ (cf. 4.2.2) and $s=0$ (cf. 4.2.3). By assumption we have $s>0$. Then V is obtained from some Hirzebruch surface $F_n (n\geqq 0)$ by a composition of s quadric transformations. Hence there exists an exceptional curve E of the first kind. We shall show that $|E+D+K|=\phi$. In fact, if $|E+D+K|\neq\phi$ then $|E|\supseteq|E+D+K|+|-(D+K)|$, which is absurd because $\dim|-(D+K)|>0$. Since $p_a(D)=p_a(E)=0$ the Riemann-Roch theorem gives

$$-2\geqq(E+D+K\cdot E+D)=-4+2(D\cdot E),$$

whence $(D\cdot E)\leqq 1$. On the other hand, if $E\not\subset\mathrm{Supp}(D)$ then $(D\cdot E)>0$ because $V-\mathrm{Supp}(D)$ is affine; if $E\subset\mathrm{Supp}(D)$ and E is not a terminal component of D then $(D\cdot E)=(E^2)+(D-E\cdot E)>0$; if $E\subset\mathrm{Supp}(D)$ and E is a terminal component of D then $(D\cdot E)=0$. Therefore we have $(D\cdot E)=0$ or 1, where $(D\cdot E)=0$ if and only if E is a terminal component of D.

Let $\sigma: V\to\bar{V}$ be the contraction of E and let $\bar{D}=\sigma_*(D)$. Then $\bar{D}$ is a reduced effective divisor on $\bar{V}$ such that

$$\sigma^*(\bar{D})=\begin{cases} D & \text{if } (D\cdot E)=0 \\ D+E & \text{if } (D\cdot E)=1. \end{cases}$$

Furthermore, since $K_V\sim\sigma^*(K_{\bar{V}})+E$ we have

$$D+K_V\sim\begin{cases} \sigma^*(\bar{D}+K_{\bar{V}})+E & \text{if } (D\cdot E)=0 \\ \sigma^*(\bar{D}+K_{\bar{V}}) & \text{if } (D\cdot E)=1. \end{cases}$$

We shall consider the cases $(D\cdot E)=0$ and $(D\cdot E)=1$ separately.

4.2.4.1. If $(D\cdot E)=0$ we have the following:

(i) $|2(\bar{D}+K_{\bar{V}})|=\phi$ and $\bar{V}-\mathrm{Supp}(\bar{D})\cong V-\mathrm{Supp}(D)$, which is affine;

(ii) $\dim|-(\bar{D}+K_{\bar{V}})|>0$ since $-\sigma^*(\bar{D}+K_{\bar{V}})\sim-(D+K_V)+E$;

(iii) $((\bar{D}+K_{\bar{V}})^2)=((D+K_V)^2)+1=2-d$, $(K_{\bar{V}}^2)=(K_V^2)+1$ and $(\bar{D}^2)=(D^2)$, whence, if we put $\bar{s}:=8-(K_{\bar{V}}^2)$ and $\bar{d}:=1-((\bar{D}+K_{\bar{V}})^2)$, we have $\bar{s}+\bar{d}=s+d-2$.

By inductive assumption, $\bar{V}-\mathrm{Supp}(\bar{D})$ (hence $V-\mathrm{Supp}(D)$) has a nonempty open set which has a structure of trivial A^1-bundle. Thus the assertion of Theorem 4.2 holds in this case.

4.2.4.2. If $(D\cdot E)=1$ we have the following:

(i) $|2(\bar{D}+K_{\bar{V}})|=\phi$ and $\bar{V}-\mathrm{Supp}(\bar{D})\subseteq V-\mathrm{Supp}(D)$, where $\bar{V}-\mathrm{Supp}(\bar{D})$ is affine*;

(ii) $\dim|-(\bar{D}+K_{\bar{V}})|>0$;

(iii) $((\bar{D}+K_{\bar{V}})^2)=((D+K_V)^2)=1-d$, $(K_{\bar{V}}^2)=(K_V^2)+1$ and $(\bar{D}^2)=(D^2)+1$, whence $\bar{s}+\bar{d}=s+d-1$.

Then we know by inductive assumption that $\bar{V}-\mathrm{Supp}(\bar{D})$ (hence $V-\mathrm{Supp}(D)$) has a nonempty open set which has a structure of trivial A^1-bundle. Thus the assertion of Theorem 4.2 holds. This completes a proof of Theorem 4.2.

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* This is clear if $E\subset\mathrm{Supp}(D)$. If $E\not\subset\mathrm{Supp}(D)$, let A be an effective ample divisor on V such that $\mathrm{Supp}(A)=\mathrm{Supp}(D)$. Let $\bar{A}=\sigma_*(A)$. Then, by Nakai's criterion of ampleness, $\bar{A}$ is an effective ample divisor on $\bar{V}$ such that $\mathrm{Supp}(\bar{A})=\mathrm{Supp}(\bar{D})$.

4.3. Corollary. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor such that $V-\mathrm{Supp}(D)$ is affine, $|2(D+K)|=\phi$ and $((D+K)^2)\geqq 0$. Then $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle.*

Proof. Follows from Lemma 4.1 and Theorem 4.2.

4.4. Corollary. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V such that $|2(D+K)|=\phi$. Assume that D is ample. Then $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle.*

Proof. Since D is ample $V-\mathrm{Supp}(D)$ is affine. If $((D+K)^2)\geqq 0$ then we are done by virtue of Corollary 4.3. Thus we may assume that $((D+K)^2)<0$. We shall show that, for any effective ample divisor A on V, we have

$$|A+m_A(D+K)|\neq\phi \quad \text{and} \quad |A+(m_A+1)(D+K)|=\phi$$

for some integer $m_A\geqq 0$. Indeed, take an integer $m>0$ so that $m>-(A\cdot D+K)/((D+K)^2)$. Then $(A+m(D+K)\cdot D+K)<0$. Suppose that $|A+m(D+K)|\neq\phi$. Let $\sum_i n_iC_i$ be a member of $|A+m(D+K)|$. Since $\sum_i n_i(C_i\cdot D+K)<0$ there exists an irreducible component, say C_1, such that $(C_1\cdot D+K)<0$. Then we have

$$(A+m'(D+K)\cdot C_1)<0 \qquad \text{for} \quad m'>-(A\cdot C_1)/(D+K\cdot C_1)\,.$$

Besides, $(C_1\cdot D)>0$ because D is ample. Since $(C_1\cdot D+K)<0$ we obtain $(C_1\cdot K)\leqq -2$, whence $(C_1^2)\geqq 0$. This implies that $|A+m'(D+K)|=\phi$ for $m'>-(A\cdot C_1)/(D+K\cdot C_1)$. Thus we find an integer m_A as claimed above.

Suppose that $A+m_A(D+K)\sim 0$ for every effective ample divisor A. Then we can see easily that every divisor is a multiple of $D+K$ up to linear equivalence. Namely, $\mathrm{Pic}(V)\cong\boldsymbol{Z}[D+K]$. Since V is rational this implies that V is isomorphic to $\boldsymbol{P}^2$ and $D+K\sim -H$, where H is a hyperplane. Then $((D+K)^2)=1$ which contradicts the assumption $((D+K)^2)<0$. Thus we have an effective divisor D_A in $|A+m_A(D+K)|$ for some effective ample divisor A. Write $D_A=\sum_i n_iC_i$. If $(C_i^2)\geqq 0$ for some i then the stated assertion holds true because $|D_A+D+K|=\phi$ implies $|C_i+D+K|=\phi$ (cf. Lemma 3.2). Therefore assume that $(C_i^2)<0$ for every i.

By a simple computation we have

$$m_A^2((D+K)^2)=(2m_A+1)(D+K\cdot D_A)-(D_A^2)-(D+K\cdot D_A)+(A^2)\,.$$

Note that C_i is a nonsingular rational curve for every i. Since $(C_i^2)<0$ we have $(C_i\cdot K)=-2-(C_i^2)\geqq -1$. On the other hand, $(D\cdot C_i)>0$ because D is ample. Hence we have

$$(D+K\cdot C_i)\geqq 0 \qquad \text{for every } i, \text{ and} \quad (D+K\cdot D_A)\geqq 0\,.$$

Furthermore, the Riemann-Roch theorem gives

$$-1=\dim|D_A+D+K|\geqq\frac{1}{2}\{(D_A+D+K\cdot D_A)+(D_A\cdot D)\}-1,$$

whence we obtain

$$-(D_A+D+K\cdot D_A)\geqq(D_A\cdot D)=\sum_i n_i(C_i\cdot D)>0.$$

Since $(A^2)>0$ we obtain $m_A^2((D+K)^2)>0$. This contradicts the assumption $((D+K)^2)<0$. Q. E. D.

4.5. Corollary. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor. Assume that V is relatively minimal, $V-\mathrm{Supp}(D)$ is affine and $|2(D+K)|=\phi$. Then $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$- bundle.*

Proof. We may assume that $((D+K)^2)<0$ since our assertion follows from Corollary 4.3 in case $((D+K)^2)\geqq 0$. Since $p_a(D)=0$ we have $(K^2)<4+(D^2)$. Suppose that V is isomorphic to $\boldsymbol{P}^2$. Write $D\sim aH$ for a hyperplane H and an integer $a>0$. Then $a^2\geqq 6$, whence $a\geqq 3$; this is a contradiction because $D+K\sim(a-3)H\geqq 0$. Next suppose that V is a Hirzebruch surface F_n for $n\geqq 0$. With the same notations as in 4.2.3, write $D\sim aM+bl$ for integers $a, b\geqq 0$. Then, since $(K^2)=8$ we have

$$(D^2)=a(2b-an)\geqq 5$$

$$2p_a(D)=(a-1)\{2(b-1)-an\}=0.$$

If $a=1$ then $D\sim M+bl$ and our assertion holds. Suppose that $2b-an=2$. Then we have $2a\geqq 5$, whence $a\geqq 3$ and $b=(an+2)/2$. If $n=0$ then $D\sim l+aM$ and our assertion holds. Assume $n>0$. Then $(D\cdot M)=(2-an)/2<0$. Hence $D=M+D'$, where D' is a reduced effective divisor such that $D'\sim(a-1)M+bl$. Since D is reduced and connected we must have $(D'\cdot M)=(2+2n-an)/2>0$, which is impossible because $(D'\cdot M)$ is an integer. Q. E. D.

§5. Case of rational surfaces - III.

5.1. Thorem. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V such that $V-\mathrm{Supp}(D)$ is affine, $|2(D+K)|=\phi$ and $\dim|-(D+K)|=0$. Then $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle.*

The theorem will be proved in the following paragraphs 5.2∼5.6.

5.2. We shall show that we may assume the following additional condition:

$-(D+K)\sim E$, where E is an exceptional curve of the first kind.

Proof. By virtue of Corollary 4.5, we may assume that V is not relatively minimal. Hence V has an exceptional curve E of the first kind. Assume $|E+D+K|\neq\phi$. If $E+D+K\sim 0$ then $-(D+K)\sim E$; this is the desired condition. Suppose $E+D+K\not\sim 0$. Then since $|E|\supseteq|E+D+K|+|-(D+K)|$ and $D+K\not\sim 0$

we have a contradiction. Now assume $|E+D+K|=\phi$. As in 4.2.4 we have $(D\cdot E)=0$ or 1. Let $\sigma: V\to\bar{V}$ be the contraction of E and let $\bar{D}=\sigma_*(D)$. We shall consider the cases $(D\cdot E)=0$ and $(D\cdot E)=1$ separately.

Case $(D\cdot E)=0$. As in 4.2.4.1, we know the following:

(i) $|2(\bar{D}+K_{\bar{V}})|=\phi$ and $\bar{V}-\mathrm{Supp}(\bar{D})\cong V-\mathrm{Supp}(D)$, which is affine;

(ii) $\dim|-(\bar{D}+K_{\bar{V}})|\geqq 0$ since $-\sigma^*(\bar{D}+K_{\bar{V}})\sim-(D+K_V)+E$.

If $\dim|(\bar{D}+K_{\bar{V}})|>0$, $\bar{V}-\mathrm{Supp}(\bar{D})$ (hence $V-\mathrm{Supp}(D)$) contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle (cf. Theorem 4.2), and thus we are done. If $\dim|-(\bar{D}+K_{\bar{V}})|=0$ then $\bar{V}$ and $\bar{D}$ satisfy the same conditions as V and D do, while $(K_{\bar{V}}^2)=(K_V^2)+1$.

Case $(D\cdot E)=1$. As in 4.2.4.2, we have the following:

(i) $|2(\bar{D}+K_V)|=\phi$ and $\bar{V}-\mathrm{Supp}(\bar{D})\subseteq V-\mathrm{Supp}(D)$, where $\bar{V}-\mathrm{Supp}(\bar{D})$ is affine;

(ii) $\dim|-(\bar{D}+K_{\bar{V}})|=0$ since $-\sigma^*(\bar{D}+K_{\bar{V}})\sim-(D+K_V)$.

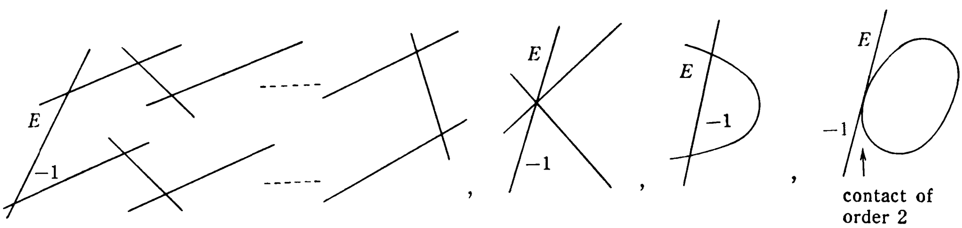
Thus $\bar{V}$ and $\bar{D}$ satisfy the same conditions as V and D do, while $(K_{\bar{V}}^2)=(K_V^2)+1$.

By repeating the above process finitely many times, either we know that $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle, or we are reduced to the case where the additional condition that $-(D+K)\sim E$ is satisfied for some exceptional curve E of the first kind.

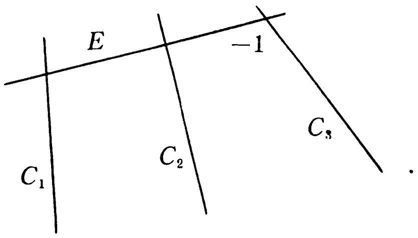
Q. E. D.

5.3. Lemma. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V such that $V-\mathrm{Supp}(D)$ is affine and $E+D+K\sim 0$ for some exceptional curve E of the first kind. Then we have the following:*

(1) *Assume that $E\not\subset\mathrm{Supp}(D)$; then $\mathrm{Supp}(D)$ is a linear chain of nonsingular rational curves, and $\mathrm{Supp}(D)\cup E$ has one of the following configurations;*



(2) *Assume that $E\subset\mathrm{Supp}(D)$; then $\mathrm{Supp}(D)$ consists of nonsingular rational curves E, C_1, C_2 and C_3, and the configuration of $\mathrm{Supp}(D)$ is given as follows,*



Proof. Since $D+K\sim -E$, we have $|2(D+K)|=\phi$ and $\dim|-(D+K)|=0$. Hence we know that $(D\cdot D+K)=-2$, every irreducible component of D is a nonsingular rational curve, D has only normal crossings, and the dual graph of D is a tree (cf. Lemma 3.1). Moreover, we have $(D\cdot E)=-(D\cdot D+K)=2$.

Assume that $E\not\subset \mathrm{Supp}(D)$. Let D_1 be an irreducible component of D. Then we have

$$0\geqq -(D_1\cdot E)=(D_1\cdot D+K)=(D_1\cdot D-D_1)+(D_1\cdot D_1+K)=(D_1\cdot D-D_1)-2,$$

where $(D_1\cdot D-D_1)>0$ because D is connected. This implies the following:

(i) If D_1 is not a terminal component of D then D_1 meets exactly two other components of D, and D_1 does not meet E;

(ii) If D_1 is a terminal component then D_1 meets E transversally at a single point if $D-D_1\neq\phi$.

Therefore $\mathrm{Supp}(D)$ is a linear chain and $\mathrm{Supp}(D)\cup E$ has one of the listed configurations.

Assume next that $E\subset \mathrm{Supp}(D)$. Since $(D\cdot E)=2$, $\mathrm{Supp}(D)$ has three irreducible components C_1, C_2, C_3 meeting E. We shall show that these curves are terminal components of D. Indeed, since

$$-1=-(C_1\cdot E)=(C_1\cdot D-C_1)-2=(C_1\cdot E)+(C_1\cdot D-C_1-E)-2$$
$$=(C_1\cdot D-C_1-E)-1,$$

we have $(C_1\cdot D-C_1-E)=0$. This implies that C_1 is a terminal component. The same argument applies to C_2 and C_3. Q. E. D.

5.4. *Let V, D and E be as in* 5.3. *Then either $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle, or we are reduced to the case where the following conditions are satisfied:*

(1) *There is no nonsingular rational curve F (other than E if $E\not\subset\mathrm{Supp}(D)$) on V such that $F\not\subset \mathrm{Supp}(D)$ and $(F^2)<0$;*

(2) *D does not contain any exceptional component (except E if $E\subset\mathrm{Supp}(D)$).*

Proof. Let F be a nonsingular rational curve (other than E if $E\not\subset\mathrm{Supp}(D)$) such that $F\not\subset\mathrm{Supp}(D)$ and $(F^2)<0$. Then we have

$$0\geqq -(F\cdot E)=(F\cdot D+K)\geqq (F\cdot D)-1,$$

where $(F\cdot D)>0$ because $V-\mathrm{Supp}(D)$ is affine, and where $(F\cdot K)=-2-(F^2)\geqq -1$. Hence $(F\cdot D)=1$, $(F\cdot K)=-1$ and $(F\cdot E)=0$. This implies that F is an exceptional curve of the first kind, F meets D transversally at a single point and F does not meet E. Let $\sigma\colon V\to\bar{V}$ be the contraction of F, let $\bar{D}=\sigma_*(D)$ and $\bar{E}=\sigma_*(E)$. Then we have the followihg:

(i) $\bar{V}-\mathrm{Supp}(\bar{D})\subset V-\mathrm{Supp}(D)$ and $\bar{V}-\mathrm{Supp}(\bar{D})$ is affine;

(ii) $\bar{E}$ is an exceptional curve of the first kind such that $\bar{D}+K_{\bar{V}}\sim -\bar{E}$ (cf. 4.2.4.2).

Thus we may contract F without loss of generality.

Suppose that C is an exceptional component of D (other than E if $E \subset \mathrm{Supp}(D)$). Suppose that C is not a terminal component of D. Then $(C \cdot D)=1$ and $(C \cdot E)=0$. Let $\sigma: V \to \bar{V}$ be the contraction of C, let $\bar{D}=\sigma_*(D)$ and let $\bar{E}=\sigma_*(E)$. Then the above two conditions (i) and (ii) are satisfied (cf. 4.2.4.2). Thus we may contract C without loss of generality. Suppose now that C is a terminal component of D. Then $(C \cdot E)=1$ by Lemma 5.3. Let $\sigma: V \to \bar{V}$ be the contraction of C, let $\bar{D}=\sigma_*(D)$ and let $\bar{E}=\sigma_*(E)$. Then we have the following:

(i)′ $\bar{V}-\mathrm{Supp}(\bar{D}) \cong V-\mathrm{Supp}(D)$, which is affine;

(ii)′ $\bar{E}$ is a nonsingular rational curve such that $(\bar{E}^2)=0$ and $\bar{D}+K_{\bar{V}} \sim -\bar{E}$, whence $|2(\bar{D}+K_{\bar{V}})|=\phi$ and $((\bar{D}+K_{\bar{V}})^2)=0$ (cf. 4.2.4.1).

By virtue of Corollary 4.3, $\bar{V}-\mathrm{Supp}(\bar{D})$ (hence $V-\mathrm{Supp}(D)$) contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle. Q.E.D.

5.5. *Proof of Theorem* 5.1. We shall assume that the additional conditions (1), (2) of 5.4 are satisfied by V, D and E. First, we consider the case where $E \not\subset \mathrm{Supp}(D)$. If some irreducible component of D has nonnegative self-intersection multiplicity then $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle (cf. Lemmas 3.3 and 3.4). Assume that each irreducible component of D has negative self-intersection multiplicity, which is then $\leqq -2$ by virtue of the condition (2) of 5.4. Then the intersection matrix of D is negative definite (cf. Mumford [6; p. 6]). This contradicts the assumption that $\mathrm{Supp}(D)$ is a support of an ample divisor.

Next we consider the case where $E \subset \mathrm{Supp}(D)$ (cf. the figure in Lemma 5.3). If $(C_i^2) \geqq 0$ for some i $(=1, 2, 3)$ then $V-\mathrm{Supp}(D)$ contains a nonempy open set which has a structure of trivial $\boldsymbol{A}^1$-bundle. Assume that $(C_i^2)<0$ for $i=1, 2, 3$. Then $(C_i^2) \leqq -2$ for $i=1, 2, 3$ by the condition (2) of 5.4. The condition (1) of 5.4 implies that E is the unique exceptional curve of the first kind on V. Let $\sigma: V \to \bar{V}$ be the contraction of E and let $\bar{C}_i=\sigma_*(C_i)$ for $i=1, 2, 3$. Then $(\bar{C}_i^2) \leqq -1$ for $i=1, 2, 3$. If $(\bar{C}_i^2) \leqq -2$ for $i=1, 2, 3$ then $\bar{V}$ is relatively minimal. This is a contradiction because there is at most one irreducible curve with negative self-intersection multiplicity on a relatively minimal rational surface (cf. Lemma 5.6 below). Hence one of $\bar{C}_i$'s, say $\bar{C}_1$, satisfies $(\bar{C}_1^2)=-1$. Let $\tau: \bar{V} \to \tilde{V}$ be the contraction of $\bar{C}_1$, and let $\tilde{C}_i=\tau_*(\bar{C}_i)$ for $i=2, 3$. Then $\tilde{C}_2$ and $\tilde{C}_3$ are nonsingular rational curves meeting each other at a single point with multiplicity 2, and $(\tilde{C}_i^2) \leqq 0$ for $i=2, 3$. Such a pair of curves does not exist on a relatively minimal rational surface (cf. Lemma 5.6 below). Then one of $\tilde{C}_2$ and $\tilde{C}_3$, say $\tilde{C}_2$, must be an exceptional curve of the first kind. Let $\rho: \tilde{V} \to W$ be the contraction of $\tilde{C}_2$, and let $C^*=\rho_*(\tilde{C}_3)$. Then C^* is an irreducible rational curve with only one ordinary cusp, and $(C^{*2}) \leqq 4$. Moreover, we know that W is a relatively minimal rational surface and that C^* is ample. This is impossible by virtue of Lemma 5.6 below. Q.E.D.

5.6. Lemma. *Let V be a relatively minimal rational surface. Then we have the following:*

(1) *There is at most one irreducible curve with negative self-intersection*

multiplicity;

(2) *there is no pair of distinct irreducible curves* C_1, C_2 *such that* $(C_i^2)\leqq 0$ $(i=1, 2)$ *and* $(C_1\cdot C_2)\geqq 2$;

(3) *there is no irreducible rational curve* C *such that* C *has only one ordinary cusp,* $(C^2)\leqq 4$ *and* $V-C$ *is affine.*

Proof. (1) Let C be an irreducible curve on V such that $(C^2)\leqq 0$. Then V is a Hirzebruch suface F_n with $n\geqq 0$ and $n\neq 1$. With the notations of 4.2.3, write $C\sim aM+bl$ with integers $a, b\geqq 0$. Then $(C^2)=a(2b-an)\leqq 0$. If $a=0$ then $C\sim l$ and $(C^2)=0$. If $a>0$ then $2b-an\leqq 0$. If $b>0$, i.e., $C\nsim M$ then $(C\cdot M)=b-an<0$, which is a contradiction. Hence $C\sim M$ if $a>0$. Thus $(C^2)<0$ if and only if $n\geqq 2$ and $C=M$; $(C^2)=0$ if and only if $C\sim l$. Therefore there exists at most one irreducible curve C with $(C^2)<0$.

(2) Let C_1 and C_2 be two distinct irreducible curves such that $(C_i^2)\leqq 0$ for $i=1, 2$. Then $(C_1\cdot C_2)=0$ or 1 by the above observation. Thus the second statement holds true.

(3) Let C be an irreducible rational curve C such that C has only one ordinary cusp, $(C^2)\leqq 4$ and C is ample. Then $p_a(C)=1$. If V is isomorphic to $\boldsymbol{P}^2$ then C is a conic and $p_a(C)=0$, which is absurd. Thus V is a Hirzebruch surface F_n with $n\geqq 0$ and $n\neq 1$. Write $C\sim aM+bl$ with integers $a, b\geqq 0$. Then we have

$$0<(C^2)=a(2b-an)\leqq 4\,,$$

$$2p_a(C)=(a-1)\{2(b-1)-an\}=2\,.$$

From the second equality, we have either $a=2$ and $2b-an=4$ or $a=3$ and $2b-an=3$. But neither case satisfies the first inequality. Hence follows the validity of the statement (3). Q.E.D.

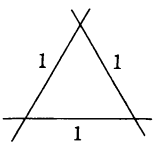
5.7. Let V, D and E be as in 5.3. Let $\sigma: V\to\bar{V}$ be the contraction of E and let $\bar{D}=\sigma_*(D)$. Then $\bar{D}$ is a reduced effective divisor such that every irreducible component of $\bar{D}$ is a (not necessarily nonsingular) rational curve, $\bar{D}+K_{\bar{V}}\sim 0$ and $\bar{V}-\mathrm{Supp}(\bar{D})$ is affine. If $\bar{D}$ has only normal crossings and there are no nowhere zero regular functions on $\bar{V}-\mathrm{Supp}(\bar{D})$, $\bar{V}-\mathrm{Supp}(\bar{D})$ is called *a logarithmic* $K3$-*surface* (cf. Iitaka [1]). Let F be an exceptional curve of the first kind on $\bar{V}$. Then $(F\cdot\bar{D})=1$. Hence either $F\not\subset\mathrm{Supp}(\bar{D})$ and F meets $\bar{D}$ transversally at a single point, or F is an irreducible component meeting one or two other components of $\bar{D}$. Let $\tau: \bar{V}\to\tilde{V}$ be the contraction of F and let $\tilde{D}=\tau_*(\bar{D})$. Then $\tilde{D}$ is a reduced effective divisor such that every irreducible component of $\tilde{D}$ is a rational curve, $\tilde{D}+K_{\tilde{V}}\sim 0$ and $\tilde{V}-\mathrm{Supp}(\tilde{D})$ is affine (cf. 4.2.2.2). If $F\not\subset\mathrm{Supp}(\bar{D})$ then τ is called *a half-point detachment;* if $F\subset\mathrm{Supp}(\bar{D})$ then τ is called *a canonical contraction.* The inverse transformation is called *a half-point attachment and a canonical blowing-up,* respectively (cf. [1]). Repeating these transformations (half-point detachments or canonical contractions) we obtain a relatively minimal rational surface W and a reduced effective divisor G such that every irreducible component of G is a (not necessarily nonsingular) rational curve, $G+K_W\sim 0$ and $W-\text{Supp}(G)$ is affine. Conversely, V and D are regained by the following process:

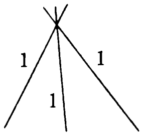
(1) Obtain $\bar{V}$ and $\bar{D}$ by repeating half-point attachments or canonical blowing-ups;

(2) let $\sigma: V\to\bar{V}$ be a quadric transformation with center P which is a double point or a triple point of $\bar{D}$, let $E=\sigma^{-1}(P)$ and let $D=\sigma^*(\bar{D})-2E$. Hence $E\not\subset\text{Supp}(D)$ if P is a double point of $\bar{D}$ and $E\subset\text{Supp}(D)$ if P is a triple point of $\bar{D}$.

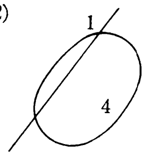
It is not hard to classify divisors G on relatively minimal rational surfaces W satisfying the above conditions. Those are given in the following list (cf. [1]):

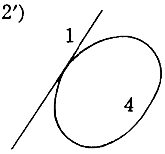
Case $W=\boldsymbol{P}^2$.

1)


1′)


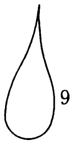
$G=l_1+l_2+l_3$
$l_i=a$ line

2)


2′)


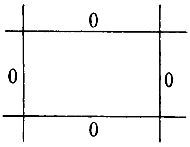
$G=l+Q$
$Q=a$ conic

3)

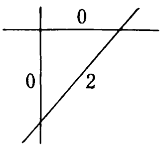

3′)


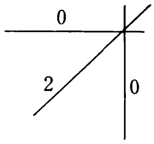
$G=a$ cubic

Case $W=\boldsymbol{P}^1\times\boldsymbol{P}^1$.

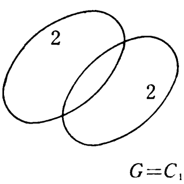
4)


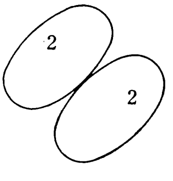
$G=M_1+M_2+l_1+l_2$
$M_1\sim M_2\sim M$
$l_1\sim l_2\sim l$

5)


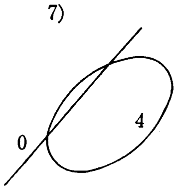
5′)


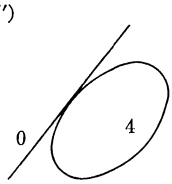
$G=M+l+C$
$C\sim M+l$

6)


6′)


$G=C_1+C_2$
$C_1\sim C_2\sim M+l$

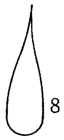
7)


7′)


$G=M+M'$
$M'\sim M+2l$

8)

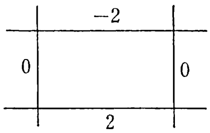
8′)



$G \sim 2M+2l$

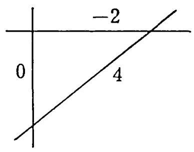
Case $W=F_2$.

9)

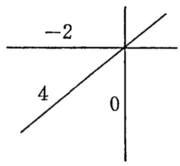


$G=M+M'+l_1+l_2$
$M' \sim M+2l,\ l_1 \sim l_2 \sim l$

10)

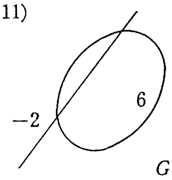


10′)

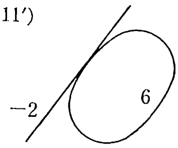


$G=M+l+M'$
$M' \sim M+3l$

11)

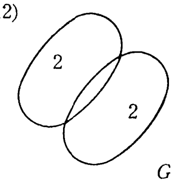


11′)

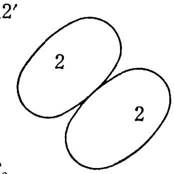


$G=M+M'$
$M' \sim M+4l$

12)

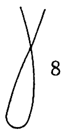


12′

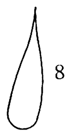


$G=C_1+C_2$
$C_1 \sim C_2 \sim M+2l$

13)



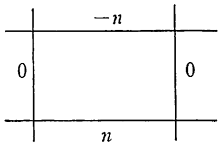
13′)



$G \sim 2M+4l$

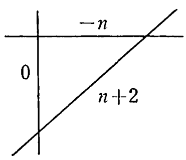
Case $W=F_n$ with $n \geqq 3$.

14)

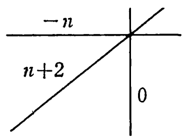


$G=M+M'+l_1+l_2$
$M' \sim M+nl$
$l_1 \sim l_2 \sim l$

15)

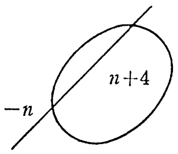


15′)

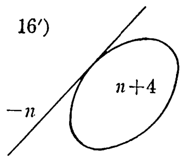


$G=M+M'+l$
$M' \sim M+(n+1)l$

16)



16′)



$G=M+M'$
$M' \sim M+(n+2)l$

5.8. By virtue of Theorems 4.2 and 5.1, we know that the four conditions in Lemma 1.3 are equivalent to each other if V is rational, $V-\mathrm{Supp}(D)$ is affine and $\dim|-(D+K)|\geqq 0$. Note that $\dim|-(D+K)|\geqq 0$ if $((D+K)^2)\geqq -1$ (cf. Lemma 4.1). In the following, we shall give a partial result in the case where $((D+K)^2)\leqq -2$. We shall first remark the following:

Lemma. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V such that $|D+K_V|=\phi$. Assume that $|E+D+K|\neq\phi$ for an exceptional curve E of the first kind. Let $\sigma: V\rightarrow\bar{V}$ be the contraction of E and let $\bar{D}=\sigma_*(D)$. Let m be a positive integer. Then*

$$\dim|m(E+D+K_V)|=0 \text{ if and only if } \dim|m(\bar{D}+K_{\bar{V}})|=0\,.$$

Proof. Assume that $\dim|m(E+D+K_V)|=0$. We claim that $(D\cdot E)=2$. Indeed, if $E+D+K_V\sim 0$ then $(D\cdot E)=2$. Assume that $E+D+K_V\sim\sum_i n_iC_i>0$. Since

$$0=\dim|E+D+K_V|\geqq\frac{1}{2}(E+D+K\cdot E+D)=(D\cdot E)-2\,,$$

we have $(D\cdot E)\leqq 2$. Assume that $(D\cdot E)\leqq 1$. Then E is an irreducible component of $\sum n_iC_i$ because $(E\cdot E+D+K)\leqq -1$, whence $|D+K_V|\neq\phi$, a contradiction. Therefore, $(D\cdot E)=2$. Then we have

$$\sigma^*(\bar{D})=D+2E \quad\text{and}\quad E+D+K_V\sim\sigma^*(\bar{D}+K_{\bar{V}})\,.$$

Hence we have: $\dim|m(\bar{D}+K_{\bar{V}})|=\dim|m(E+D+K_V)|=0$.

Next assume that $\dim|m(\bar{D}+K_{\bar{V}})|=0$. Let $n=(D\cdot E)$. Then $n\geqq 2$ as in the above argument. Then we have

$$\sigma^*(\bar{D})=D+nE \quad\text{and}\quad \sigma^*(\bar{D}+K_{\bar{V}})\sim(E+D+K_V)+(n-2)E\,.$$

Hence we have

$$0\leqq\dim|m(E+D+K_V)|\leqq\dim|m(\bar{D}+K_{\bar{V}})|=0\,. \qquad \text{Q.E.D.}$$

5.9. Theorem. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V such that $V-\mathrm{Supp}(D)$ is affine, $((D+K)^2)\leqq -2$ and $|2(D+K)|=\phi$. Assume that $\dim|2(E+D+K)|=\dim|E+D+K|=0$ for an exceptional curve E of the first kind. Then there exists a birational morphism ρ from V onto a nonsingular projective surface $\tilde{V}$ such that the following conditions are satisfied:*

(1) *E does not meet any exceptional curve of ρ, whence $\tilde{E}=\rho(E)$ is an exceptional curve of the first kind;*

(2) *let $\tilde{D}=\rho_*(D)$; then $\tilde{E}+\tilde{D}+K_{\tilde{V}}\sim 0$;*

(3) *$\tilde{V}-\mathrm{Supp}(\tilde{D})\subseteq V-\mathrm{Supp}(D)$, and $\tilde{V}-\mathrm{Supp}(\tilde{D})$ is affine.*

Therefore, $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle.

Proof. By assumption, $|E+D+K|\neq\phi$. If $E+D+K\sim 0$ then we have only to take $\tilde{V}=V$ and $\rho=$ the identity morphism. Thus we assume that $E+D+K$

$\sim\sum_i n_iC_i>0$. As in the proof of Lemma 5.8, we have $(D\cdot E)=2$. Hence $\sum_i n_i(C_i\cdot E)=0$. Note that E is not an irreducible component of $\sum_i n_iC_i$ because $|D+K|=\phi$. Hence E does not meet any component C_i.

We shall show that some component of $\sum_i n_iC_i$ is an exceptional curve of the first fiind. Since $(E\cdot E+D+2K)=-1$, E is a fixed component of $|E+D+2K|$ and $|D+2K|\neq\phi$ if $|E+D+2K|\neq\phi$. This is a contradiction because $D+|D+2K|\subseteq|2(D+K)|=\phi$. Hence we have $|E+D+2K|=\phi$. Therefore we have $|\sum_i C_i+K|=\phi$, which implies by virtue of Lemma 3.1 that every irreducible component C_i is a nonsingular rational curve. Then we have

$$\dim|C_i|\geqq\frac{1}{2}(C_i\cdot C_i-K)=(C_i^2)+1\,.$$

Since $\dim|E+D+K|=\dim|\sum_i n_iC_i|=0$ this implies that $(C_i^2)<0$ for every i. On the other hand, since $((D+K)^2)\leqq-2$ and $(D\cdot D+K)=-2$, we have $(D+K\cdot K)\leqq 0$. Hence we have

$$\sum_i n_i(C_i\cdot K)=(E+D+K\cdot K)=-1+(D+K\cdot K)\leqq-1\,.$$

Thus there exists an irreducible component of $\sum_i n_iC_i$, say C_1, such that $(C_1\cdot K)<0$. Since $(C_1^2)<0$, C_1 is an exceptional curve of the first kind.

We shall show that $|C_1+D+K|=\phi$. Assume the contrary. Then $C_1+D+K\sim F>0$ because $((D+K)^2)\leqq-2$, and $E+F\in|E+C_1+D+K|$. Note that

$$(n_1+1)C_1+\sum_{i\neq 1}n_iC_i\in|E+C_1+D+K|\,.$$

Since every component C_i is disjoint from E, this implies that $|E+C_1+D+K|$ contains two distinct members $E+F$ and $(n_1+1)C_1+\sum_{i\neq 1}n_iC_i$. This is a contradiction because we have

$$\dim|E+C_1+D+K|\leqq\dim|2(E+D+K)|=0\,.$$

Therefore, $|C_1+D+K|=\phi$, which implies $0\leqq(C_1\cdot D)\leqq 1$.

Let $\sigma_1\colon V\to V_1$ be the contraction of C_1, let $D_1=(\sigma_1)_*(D)$ and let $E_1=(\sigma_1)_*(E)$. Then we shall show that the following conditions are satisfied:

(i) $V_1-\mathrm{Supp}(D_1)\subseteq V-\mathrm{Supp}(D)$, and $V_1-\mathrm{Supp}(D_1)$ is affine;

(ii) $|2(D_1+K_{V_1})|=\phi$;

(iii) $((D_1+K_{V_1})^2)=((D+K_V)^2)$ if $(C_1\cdot D)=1$ and $((D_1+K_{V_1})^2)=((D+K_V)^2)+1$ if $(C_1\cdot D)=0$;

(iv) E_1 is an exceptional curve of the first kind on V_1 and $\dim|E_1+D_1+K_{V_1}|=\dim|2(E_1+D_1+K_{V_1})|=0$.

The first three assertions can be proved by the same fashion as in 4.2.4. We shall prove the assertion (iv). Since $(E\cdot C_1)=0$, E_1 is an exceptional curve of the first kind on V_1. Moreover, we have

$$E+D+K_V\sim\sigma_1^*(E_1+D_1+K_{V_1})+\delta C_1\,,$$

where $\delta=1$ if $(C_1\cdot D)=0$ and $\delta=0$ if $(C_1\cdot D)=1$. Then it is easy to see that $\dim|E_1+D_1+K_{V_1}|=\dim|2(E_1+D_1+K_{V_1})|=0$.

The above process can be repeated if $((D_1+K_{V_1})^2)\leqq-2$. Thus, repeating these processes finitely many times, we have a birational morphism $\rho: V\to\tilde{V}$ from V onto a nonsingular projective surface $\tilde{V}$ such that if we set $\tilde{D}=\rho_*(D)$ and $\tilde{E}=\rho_*(E)$ then the following conditions are satisfied:

1) $\tilde{V}-\mathrm{Supp}(\tilde{D})\subseteq V-\mathrm{Supp}(D)$, and $\tilde{V}-\mathrm{Supp}(\tilde{D})$ is affine;

2) $|2(\tilde{D}+K_{\tilde{V}})|=\phi$;

3) $((\tilde{D}+K_{\tilde{V}})^2)=-1$;

4) $\tilde{E}$ is an exceptional curve of the first kind on $\tilde{V}$, and $\dim|\tilde{E}+\tilde{D}+K_{\tilde{V}}|=\dim|2(\tilde{E}+\tilde{D}+K_{\tilde{V}})|=0$.

Then we shall show that $\tilde{E}+\tilde{D}+K_{\tilde{V}}\sim 0$. Indeed, assume that $\tilde{E}+\tilde{D}+K_{\tilde{V}}\sim F>0$. Then every irreducible component of F is distinct from $\tilde{E}$ because $|\tilde{D}+K_{\tilde{V}}|=\phi$. On the other hand, $\dim|-(\tilde{D}+K_{\tilde{V}})|\geqq 0$ by virtue of Lemma 4.1. Hence $-(\tilde{D}+K_{\tilde{V}})\sim G\geqq 0$. Then $\tilde{E}\sim F+G$, which is a contradiction. Therefore, $\tilde{E}+\tilde{D}+K_{\tilde{V}}\sim 0$. Then Theorem 5.1 implies that $\tilde{V}-\mathrm{Supp}(\tilde{D})$ (hence $V-\mathrm{Supp}(D)$) contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle. Q. E. D.

5.10. Theorem. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V such that $V-\mathrm{Supp}(D)$ is affine, $((D+K)^2)\leqq-2$ and $|m(D+K)|=\phi$ for every integer $m>0$. Assume that there exists an exceptional curve E of the first kind satisfying the following two conditions:*

(1) $\dim|E+D+K|=0$,

(2) *$(E+D+K)$-dimension $\kappa(E+D+K)=1$.**

Then $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle.

Proof. We shall obtain a contradiction by assuming that the assertion is false. Our argument consists of four steps.

(I) Let $\sum_i n_iC_i$ be an effective divisor in $|E+D+K|$. Then every component C_i is distinct from E, for, if otherwise, $|D+K|\neq\phi$ which contradicts the assumption. By the Riemann-Roch theorem we have $\dim|E+D+K|\geqq(E\cdot D)-2$, whence $(E\cdot D)\leqq 2$ because $\dim|E+D+K|=0$ by assumption. Suppose $(E\cdot D)\leqq 1$. Then the following sequence

$$0\longrightarrow H^0(V,\mathcal{O}_V(D+K))\longrightarrow H^0(V,\mathcal{O}_V(E+D+K))\longrightarrow H^0(E,\mathcal{O}_E((E\cdot D)-2))$$

implies that $|E+D+K|=\phi$. Since $\dim|E+D+K|=0$ we have $(E\cdot D)=2$.

Now consider $|E+D+2K|$. Since $(E\cdot E+D+2K)=-1$ and $|D+2K|=\phi$, we know that $|E+D+2K|=\phi$. Thence $|\sum_i C_i+K|=\phi$, which implies that every component C_i is a nonsingular rational curve. Since $\dim|C_i|\geqq(C_i^2)+1$ (cf. 3.2.1),

* For any integer $m>0$, let φ_m be the rational mapping from V to $\boldsymbol{P}^{N(m)}$ defined by $|m(E+D+K)|$, where $N(m)=\dim|m(E+D+K)|$. Then $(E+D+K)$-dimension $\kappa(E+D+K)$ is defined as $\sup_{m>0}\dim\varphi_m(V)$. If $|m(E+D+K)|=\phi$ for every integer $m>0$, we set $\kappa(E+D+K)=-\infty$.

we know that $(C_i^2)<0$ for every component C_i. On the other hand, the assumption $((D+K)^2)\leqq -2$ implies that $b:=-(D+K\cdot K)\geqq 0$. Hence we have

$$-1-b=(E+D+K\cdot K)=\sum_i n_i(C_i\cdot K)<0\,.$$

This implies that $(C_1\cdot K)<0$ for some component, say C_1. Then C_1 is an exceptional curve of the first kind. Furthermore, we know that $(C_i\cdot E)=0$ for every component C_i, because $(E\cdot D)=2$ and $C_i\neq E$. Hence $C_i\cap E=\phi$.

(II) By virtue of Theorem 5.9, we have $\dim|2(E+D+K)|>0$. Write $|2(E+D+K)|=|X|+F$, where F is the fixed part. Suppose $(X^2)>0$. Then Zariski's lemma (cf. Zariski [8]) implies that $|mX|$ has no base points for sufficiently large integer $m>0$; this implies that $|mX|$ is not composed of a pencil and $\dim \Phi_{|mX|}(V)=2$, which contradicts the assumption $\kappa(E+D+K)=1$. Therefore, we have $(X^2)=0$. We may write $X\sim rC$, where $|C|$ is an irreducible linear pencil. Since $2\sum_i n_iC_i\in|2(E+D+K)|$ we have $F\leqq 2\sum n_iC_i$. Since $(C_i\cdot E)=0$ for every i, we have $(F\cdot E)=0$. Hence we have

$$0=2(E\cdot E+D+K)=r(C\cdot E)+(F\cdot E)=r(C\cdot E)\,.$$

Thus $(C\cdot E)=0$. Namely, E is contained in a member of $|C|$; $C'+E\in|C|$ for some effective divisor C'. If $r\geqq 2$ then $2(D+K)\sim rC'+(r-2)E+F>0$, which contradicts the assumption $|2(D+K)|=\phi$. Hence we know that $r=1$. In a similar fashion, we can show that $E\not\subset \mathrm{Supp}\,(C')$. Thus we have

$$E+2(D+K)\sim C'+F \quad\text{and}\quad \dim|E+2(D+K)|=0\,.$$

We shall show that $\dim|E+m(D+K)|\leqq 0$ for every integer $m>0$. Since $\kappa(E+D+K)=1$, the moving part of $|m(E+D+K)|$ is composed of the pencil $|C|$, and we can write $|m(E+D+K)|=|t_mC|+F_m$, where F_m is the fixed part and $t_m>0$. Then $t_m\leqq m-1$, for, if otherwise, $m(D+K)\sim(t_m-m)C+mC'+F_m>0$, which contradicts the assumption $|m(D+K)|=\phi$. Then we have

$$(m-t_m)E+m(D+K)\sim t_mC'+F_m\,.$$

Since it is clear that $\dim|t_mC'+F_m|=0$, we have

$$\dim|E+m(D+K)|\leqq 0\,.$$

(III) In this step we shall show that, after replacing E if necessary by an exceptional curve of the first kind satisfying the same conditions as E does in the statement, there exists an exceptional curve Z of the first kind such that $(E\cdot Z)=0$ and $|Z+D+K|=\phi$. Suppose the contrary: If E' and Z are exceptional curves of the first kind such that E' satisfies the same conditions as E does in the statement and that $(E'\cdot Z)=0$, then $|Z+D+K|\neq\phi$. By making a slight change of notations in the step (I), we write

$$E+D+K\sim\sum_i n_i^{(1)}C_i^{(1)}>0\,,$$

where $(E\cdot C_i^{(1)})=0$ for every index i and $C_1^{(1)}$ is an exceptional curve of the first kind. By assumption, we have

$$|C_1^{(1)}+D+K|\neq\phi\,.$$

Since we have

$$E+2(D+K)\sim(n_1^{(1)}-1)C_1^{(1)}+\sum_{i\neq 1}n_i^{(1)}C_i^{(1)}+(C_1^{(1)}+D+K)\qquad\cdots(R_2)$$

and $\dim|E+2(D+K)|=0$, we have

$$\dim|C_1^{(1)}+D+K|=0\quad\text{and}\quad(C_1^{(1)}\cdot D)=2\,.$$

On the other hand, we have $\dim|2(C_1^{(1)}+D+K)|>0$, for, if otherwise, $V-$ Supp(D) contains a nonempty open set whicn has a structure of trivial $\boldsymbol{A}^1$-bundle (cf. Theorem 5.9). The relation (R_2) above implies $\kappa(C_1^{(1)}+D+K)=1$. The same argument as in the step (II) shows that

$$|2(C_1^{(1)}+D+K)=|C|+(\text{the fixed part})\quad\text{and}\quad(C_1^{(1)}\cdot C)=0\,,$$

where $|C|$ is the same pencil as constructed in the step (II).

Write

$$C_1^{(1)}+D+K\sim\sum_i n_i^{(2)}C_i^{(2)}>0\,,$$

where $(C_1^{(1)}\cdot C_i^{(2)})=0$ for every index i and some component, say $C_1^{(2)}$, is an exceptional curve of the first kind. Then we have,

$$\begin{aligned}E+3(D+K)\sim(n_1^{(1)}-1)C_1^{(1)}+\sum_{i\neq 1}n_i^{(1)}C_i^{(1)}+(n_1^{(2)}-1)C_1^{(2)}+\sum_{i\neq 1}n_i^{(2)}C_i^{(2)}\\ +(C_1^{(2)}+D+K)\qquad\qquad\cdots(R_3)\,.\end{aligned}$$

The relation (R_2) implies that $E\neq C_i^{(2)}$ for every index i because $|2(D+K)|=\phi$, and the relation (R_3) implies that

$$\dim|C_1^{(2)}+D+K|=0,\ (C_1^{(2)}\cdot D)=2\quad\text{and}\quad\kappa(C_1^{(2)}+D+K)=1\,,$$

because $|C_1^{(2)}+D+K|\neq\phi$ (since $C_1^{(1)}$ satisfies the same conditions as E does in the statement and $(C_1^{(1)}\cdot C_1^{(2)})=0$) and $\dim|E+3(D+K)|\leqq 0$. By the same argument as above, we have

$$|2(C_1^{(2)}+D+K)|=|C|+(\text{the fixed part})\quad\text{and}\quad(C_1^{(2)}\cdot C)=0\,.$$

Thus, proceeding inductively we obtain a sequence $\{C_1^{(1)}, C_1^{(2)}, \cdots\}$ of exceptional curves of the first kind satisfying the following conditions for $j=1, 2, \cdots$:

(1) $C_1^{(j-1)}+D+K\sim\sum_i n_i^{(j)}C_i^{(j)}$, where $C_1^{(0)}:=E$,

(2) $E+(j+1)(D+K)\sim\sum_{s=1}^{j}\{(n_1^{(s)}-1)C_1^{(s)}+\sum_{i\neq 1}n_iC_i^{(s)}\}+(C_1^{(j)}+D+K)\cdots(R_{j+1})$,

(3) $\dim|C_1^{(j)}+D+K|=0$, $(C_1^{(j)}\cdot D)=2$ and $\kappa(C_1^{(j)}+D+K)=1$,

(4) $|2(C_1^{(j)}+D+K)|=|C|+(\text{the fixed part})$, and $(C_1^{(j)}\cdot C)=0$,

(5) $C_1^{(t)}+(j+1-t)(D+K)\sim \sum_{s=t+1}^{j}\{(n_1^{(s)}-1)C_1^{(s)}+\sum_{i\neq 1} n_i^{(s)}C_i^{(s)}\}$

$+(C_1^{(j)}+D+K)$

and $C_1^{(t)}\neq C_1^{(j)}$ for $0\leqq t<j$.

Let $f: V\rightarrow \boldsymbol{P}^1$ be the morphism defined by the pencil $|C|$. Then, by virtue of the conditions (4) and (5), the curves in the sequence $\{C_1^{(1)}, C_1^{(2)}, \cdots\}$ are mutually distinct (hence infinitely many) exceptional curves of the first kind, each of which is contained in a fiber of f. This is a contradiction. Therefore, we know that there exists an exceptional curve Z of the first kind such that $(E\cdot Z)=0$ and $|Z+D+K|=\phi$, provided one replaces E (if necessary) by some $C_1^{(i)}$.

(IV) Let $\sigma: V\rightarrow \bar{V}$ be the contraction of Z, let $\bar{D}=\sigma_*(D)$ and let $\bar{E}=\sigma_*(E)$. Since $(E\cdot Z)=0$, $\bar{E}$ is an exceptional curve of the first kind on $\bar{V}$. As in the proof of 4.2.4, we have $(D\cdot Z)=0$ or 1; indeed, we have

$$\sigma^*(\bar{D})=\begin{cases} D & \text{if } (D\cdot Z)=0 \\ D+Z & \text{if } (D\cdot Z)=1 \end{cases}$$

and

$$E+D+K_V\sim\begin{cases} \sigma^*(\bar{E}+\bar{D}+K_{\bar{V}})+Z & \text{if } (D\cdot Z)=0 \\ \sigma^*(\bar{E}+\bar{D}+K_{\bar{V}}) & \text{if } (D\cdot Z)=1\,. \end{cases}$$

Then we have the following:

(i) $|m(\bar{D}+K_{\bar{V}})|=\phi$ for every integer $m>0$, and $\bar{V}-\mathrm{Supp}(\bar{D})$ is affine;

(ii) $\dim|\bar{E}+\bar{D}+K_{\bar{V}}|=0$;

(iii) $\kappa(\bar{E}+\bar{D}+K_{\bar{V}})=1$;

(iv) $(K_{\bar{V}}^2)=(K_V^2)+1$;

(v) $\bar{V}-\mathrm{Supp}(\bar{D})$ does not contain any nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle.

The assertion (i) is proved in the same fashion as in 4.2.4. We shall prove the assertions (ii) and (iii) in the case $(D\cdot Z)=0$. Note that Z is a fixed component of $|E+D+K_V|$ because $(Z\cdot E+D+K_V)=-1$. Hence $\dim|\bar{E}+\bar{D}+K_{\bar{V}}|=\dim|E+D+K_V-Z|=0$. On the other hand, we have clearly $0\leqq\kappa(\bar{E}+\bar{D}+K_{\bar{V}})\leqq\kappa(E+D+K_V)=1$. If $\kappa(\bar{E}+\bar{D}+K_{\bar{V}})=0$ then Theorem 5.9 implies that $\bar{V}-\mathrm{Supp}(\bar{D})$ (hence $V-\mathrm{Supp}(D)$) contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle, which is a contradiction. Hence $\kappa(\bar{E}+\bar{D}+K_{\bar{V}})=1$.

Thus we obtained the same situation on $\bar{V}$ as the one on V which we started with, except that $(K_{\bar{V}}^2)=(K_V^2)+1$. Therefore we can apply the same arguments as in the steps (I)$\sim$(III) infinitely many times. However, this is impossible because $(K^2)\leqq 8$ or 9. Thus we proved the assertion in the theorem.

5.11. Theorem. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V such that $V-\mathrm{Supp}(D)$ is affine and $|D+K|=\phi$. Assume that there exists an exceptional curve E of the first kind satisfying the following conditions:*

(1) $r := \dim|E+D+K| > 0$,

(2) $\kappa(E+D+K)=1$.

Then $V-\mathrm{Supp}(D)$ *contains a nonempty open set which has a structure of trivial* $\boldsymbol{A}^1$*-bundle.*

Proof. Our proof consists of three steps.

(I) Note that $h^0(D+K)=h^2(D+K)=0$. Hence we have

$$-h^1(D+K)=\frac{1}{2}(D+K\cdot D)+1=0.$$

Then, from an exact sequence

$$0 \longrightarrow \mathcal{O}_V(D+K) \longrightarrow \mathcal{O}_V(E+D+K) \longrightarrow \mathcal{O}_E((E\cdot D)-2) \longrightarrow 0,$$

we obtain an isomorphism

$$H^0(V, \mathcal{O}_V(E+D+K)) \longrightarrow H^0(E, \mathcal{O}_E((E\cdot D)-2)).$$

Hence $r=(E\cdot D)-2$.

Write $|E+D+K|=|A|+F$, where F is the fixed part. By virtue of the above isomorphism, we know that any irreducible component of F does not meet E, whence $(E\cdot F)=0$. Now suppose $(A^2)>0$. By Zariski's lemma (cf. Zariski [8; Th. 6.1]), $|mA|$ has no base points for sufficiently large integers $m>0$. Since $(A^2)>0$ then $|mA|$ is not composed of a pencil. This implies that $\kappa(E+D+K)=2$, which contradicts the condition (2) above. Therefore we have $(A^2)=0$. This implies that $|A|$ is written in the form $|A|=|rC|$, where $|C|$ is an irrecucible linear pencil on V.

(II) Since $E+D+K\sim rC+F$ and $(E\cdot F)=0$, we have

$$r=(E\cdot E+D+K)=(E\cdot rC+F)=r(E\cdot C),$$

whence $(E\cdot C)=1$. For every integer $m>0$, write

$$|m(E+D+K)|=|A_m|+F_m,$$

where F_m is the fixed part. If $(A_m^2)>0$ we reach to a contradiction as in the first step. Hence $(A_m^2)=0$ and $|A_m|$ is composed of a pencil. Since $|mrC|+mF \subseteq |m(E+D+K)|$, $|A_m|$ is, in fact, composed of the pencil $|C|$. Write $|A_m|=|t_mC|$ with $t_m \geqq 1$. Then we have

$$t_m+(F_m\cdot E)=mr,\ t_m \geqq mr, \quad \text{and} \quad F_m \leqq mF.$$

Hence we have: $t_m=mr$ and $F_m=mF$.

(III) We shall show that $|C+D+K|=\phi$; then, since $(C^2)=0$, Lemma 3.2 implies the validity of our theorem. Suppose $|C+D+K|\neq\phi$. By the second step, we have

$$|(r+1)(E+D+K)|=|r(r+1)C|+(r+1)F,$$

where $(r+1)F$ is the fixed part. Note that $|r(r+1)C-E|=\phi$, because, if otherwise, $(C\cdot r(r+1)C-E)\geqq 0$, which contradicts $(C\cdot r(r+1)C-E)=-(C\cdot E)=-1$.

This implies that any member of $|r(r+1)C|$ does not contain E. Since any irreducible component of F does not meet E, we know that any member of $|(r+1)(E+D+K)|$ does not contain E. However, if $|C+D+K|\neq\phi$, we have

$$(r+1)(E+D+K)\sim rB+rE+F,$$

where $B\in|C+D+K|$. This is a contradiction. Thus, $|C+D+K|=\phi$, and we are done.

§ 6. Further results and remarks.

6.1. We shall consider the following

Assertion (A). *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V. Assume that the following conditions are satisfied:*

(1) $V-\mathrm{Supp}(D)$ *is affine;*

(2) $|m(D+K)|=\phi$ *for every integer* $m>0$;

(3) $((D+K)^2)\leqq-2$;

(4) *there exists an exceptional curve E of the first kind such that* $|E+D+K|\neq\phi$ *and* $\kappa(E+D+K)=2$.

Then $|E+n(D+K)|=\phi$ *for some integer* $n>0$.

6.2. Remark. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V such that $V-\mathrm{Supp}(D)$ is affine and $|m(D+K)|=\phi$ for every integer $m>0$. If the assertion (A) is true then $V-\mathrm{Supp}(D)$ contains a nonempty open set which has a structure of trivial $\boldsymbol{A}^1$-bundle.*

Proof. We shall proceed by induction on $-(K^2)$, where $-(K^2)\geqq-8$ or -9. If V is relatively minimal, our assertion follows from Corollary 4.5. Thus we shall assume that V is not relatively minimal.

(I) Our assertion is true if $((D+K)^2)\geqq-1$ (cf. Corollary 4.3 and Theorem 5.1*). Thus we have only to consider the case where $((D+K)^2)\leqq-2$. Since V is not relatively minimal, there exists an exceptional curve E of the first kind on V. Consider a linear system $|E+D+K|$. If $|E+D+K|=\phi$ then $(D\cdot E)=0$ or 1. Let $\sigma: V\to\bar{V}$ be the contraction of E and let $\bar{D}=\sigma_*(D)$. Then $\bar{V}-\mathrm{Supp}(\bar{D})$ is affine, and $|m(\bar{D}+K_{\bar{V}})|=\phi$ for every integer $m>0$ (cf. 4.2.4). Since $-(K_{\bar{V}}^2)=-(K_{\bar{V}}^2)-1$ we are done by inductive assumption. Suppose $|E+D+K|\neq\phi$. If $\kappa(E+D+K)\leqq1$ our assertion holds (cf. Theorems 5.9, 5.10 and 5.11). Thus we are reduced to the situation as in the assertion (A), which we shall consider in the next step.

(II) Suppose that the assertiohn (A) is true. Since $|E+D+K|\neq\phi$, we may assume that $|E+(n-1)(D+K)|\neq\phi$. Let $\sum_i n_iC_i\in|E+(n-1)(D+K)|$; since $((D+K)^2)\leqq-2$ implies $(D+K\cdot K)\leqq0$, we have $(E+(n-1)(D+K)\cdot K)\leqq-1$, whence

* As remarked in 5.8, $\dim|-(D+K)|\geqq0$ if $((D+K)^2)\geqq-1$.

$\sum_i n_i C_i > 0$. Then C_i is a nonsingular rational curve such that $|C_i+D+K|=\phi$ for every index i. If $(C_i^2)\geqq 0$ then our assertion holds by virtue of Lemma 3.2. Therefore we may assume that $(C_i^2)<0$ for every index i. On the other hand, we have

$$(K\cdot E+(n-1)(D+K))=\sum_i n_i(C_i\cdot K)<0\,,$$

whence $(C_1\cdot K)<0$ for some component, say C_1. Then C_1 is an exceptional curve of the first kind such that $|C_1+D+K|=\phi$. Let $\sigma\colon V\to \bar{V}$ be the contraction of C_1 and let $\bar{D}=\sigma_*(D)$. Then $\bar{V}-\mathrm{Supp}\,(\bar{D})$ is affine, and $|m(\bar{D}+K_{\bar{V}})|=\phi$ for every integer $m>0$ (cf. 4.2.4). Since $-(K_{\bar{V}}^2)=-(K_V^2)-1$ we are done by inductive assumption. Q. E. D.

6.3. Theorem. *Let V be a nonsingular projective rational surface and let D be a reduced effective divisor on V such that $V-\mathrm{Supp}\,(D)$ is affine and that D has at worst nodal double points as singularities. Then the conditions* (1), (2) *and* (3) *in the statement of Lemma* 1.3 *are equivalent to each other.*

Proof. The implication (1) ⇒ (2) ⇒ (3) is shown in Lemma 1.3. We shall prove the implication (3) ⇒ (1). We shall proceed by induction on $-(K^2)$. It is easily seen from the proof of Remark 6.2 that we have only to prove the following

Assertion (B). *Let V and D be as above. Assume that there exists an exceptional curve E of the first kind such that $|E+D+K|=\phi$. Let $\sigma\colon V\to\bar{V}$ be the contraction of E and let $\bar{D}=\sigma_*(D)$. If the condition* (3) *is satisfied on V then the condition* (3) *is satisfied on $\bar{V}$. Namely, for any divisor $\bar{A}$ on $\bar{V}$, we have $|\bar{A}+m(\bar{D}+K_{\bar{V}})|=\phi$ for any sufficiently large integer $m>0$.*

Proof. Indeed, we have $|\sigma^*(\bar{A})+m(D+K)|=\phi$ for any sufficiently large integer $m>0$. Suppose $(D\cdot E)=0$. Then we have

$$D+K\sim\sigma^*(\bar{D}+K_{\bar{V}})+E \quad\text{and}\quad |\sigma^*(\bar{A}+m(\bar{D}+K_{\bar{V}}))|+mE\subseteq|\sigma^*(\bar{A})+m(D+K)|.$$

Suppose $(D\cdot E)=1$. Then we have

$$D+K\sim\sigma^*(\bar{D}+K_{\bar{V}}) \quad\text{and}\quad |\sigma^*(\bar{A}+m(\bar{D}+K_{\bar{V}}))|=|\sigma^*(\bar{A})+m(D+K_V)|.$$

Hence $|\bar{A}+m(\bar{D}+K_{\bar{V}})|=\phi$ for any sufficiently large integer $m>0$. Thus the assertion (B) is verified. Needless to say, the assertion (A) is true if the condition (3) of Lemma 1.3 is satisfied. Q. E. D.

6.4. We shall give an example which satisfies the conditions of the assertion (A):

Example. Let $V_0:=\boldsymbol{P}^1\times\boldsymbol{P}^1$, and let C_1 and C_2 be irreducible nonsingular curves such that:

(1) $C_1\sim M+2l$ and $C_2\sim 2M+l$ (cf. 4.2.3 for the notations);

(2) C_1 and C_2 intersect each other in distinct five points $P_1, \cdots, P_5$;

(3) if $M_i \sim M$, $l_i \sim l$, and $P_i = M_i \cap l_i$ then $i(M_i, C_1; P_i) = i(l_i, C_2; P_i) = 1$* for $1 \leqq i \leqq 5$.

Let $\sigma: V \to V_0$ be the composite of quadric transformations with centers $P_1, \cdots, P_5$, let $C_j' := \sigma'(C_j)$ $(j=1, 2)$ and let $D_i := \sigma^{-1}(P_i)$ $(i=1, \cdots, 5)$. Let $D = C_1' + \sum_{i=1}^{5} D_i$ and let $E = C_2'$. Then $V - \mathrm{Supp}(D) \cong V_0 - C_1$, which is affine and contains a cylinderlike open set. Hence $|m(D+K)| = \phi$ for every integer $m > 0$. Moreover, it is easy to show that $((D+K)^2) = -5$ and $E + D + K \sim \sigma^*(M+l)$. Hence V and D satisfy the conditions of the assertion (A). In this example, the assertion (A) is valid.

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Added in proof. In January, 1979, T. Fujita proved that the implication (4) $\Rightarrow$ (3) in Lemma 1.3 holds true under the assumption that D has only normal crossings as singularities. Thus, the four conditions of Lemma 1.3 are equivalent to each other under the same assumption. Therefore, Zariski's Problem is now answered in the affirmative; it is true even over an arbitrary field k of characteristic zero by virtue of T. Kambayashi [On the absence of nontrivial forms of the affine plane, J. Alg. **35** (1975), 449-456].

* $i(M_i, C_1; P_i)$ = the local intersection multiplicity of M_i, C_1 at P_i, etc.