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A remark on Garsia's integral test about sample continuity of L_p-processes

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§ 1. Introduction

First of all, we are concerned with a simple sufficient condition for essential continuity of a real function f defined on $D_N=\{(i_1 2^{-n}, \cdots, i_N 2^{-n}); n=0, 1, 2, \cdots, i_k=0, 1, \cdots, 2^n, 1 \leqq k \leqq N\}$. Set

$$\Delta_n(f)=\max_{|i-j|=1} |f(\boldsymbol{i}2^{-n})-f(\boldsymbol{j}2^{-n})|,$$

where $\boldsymbol{i}=(i_1, \cdots i_N)$, $\boldsymbol{j}=(j_1, \cdots, j_N)$, $1 \leqq i_k$, $j_k \leqq 2^n$, and $|\boldsymbol{x}|=\max_{1\leq k\leq N} |x_k|$ for $\boldsymbol{x}=(x_1, \cdots, x_N)$.

Then we have

Lemma 1. *If* $\sum_n^\infty \Delta_n(f)<+\infty$, *then there exists a continuous function* $\bar{f}$ *defined on* $I_N=[0, 1]^N$ *such that* $\bar{f}(\boldsymbol{x})=f(\boldsymbol{x})$ *for all* $\boldsymbol{x}\in D_N$.

From this lemma, we get very easily an integral test for sample continuity of stochastic processes with the help of Fubini's theorem and Hölder's inequality. (c. f. [4])

We shall say that a separable and measurable stochastic process $\{X(\boldsymbol{t}, \omega); \boldsymbol{t}\in I_N, \omega\in\Omega\}$ is an L_p-process if the sample path belongs to $L_p(I_N, d\boldsymbol{t})$ with probability 1. Then a stochastic version of Lemma 1 is the following:

Corollary 1. *If an* L_p*-process* $\{X(\boldsymbol{t}, \omega); \boldsymbol{t}\in I_N, \omega\in\Omega\}$ *with* $p\geqq 1$ *has a non-decreasing continuous function* $\sigma(h)$ *such that*

$$(E[|X(\boldsymbol{t}+\boldsymbol{h})-X(\boldsymbol{t})|^p])^{1/p}\leqq\sigma(|\boldsymbol{h}|),$$

and

$$\int_{+0}\sigma(\delta)\delta^{-(1+N/p)}d\delta<+\infty,$$

then the sample path is continuous with probability 1.

These arguments are due to Delporte [2] and this integral test is best possible in a sense at least when $N=1$ and $p\geqq 2$, ([5], [8]).

A sharper form than Corollary 1 is obtained by Garsia and others ([3], [9]) by virtue of the following real variable lemma: Set

$$Q_p(\delta, f)=\left(\int_{|s-t|<\delta}|f(s)-f(t)|^p ds dt\right)^{1/p}$$

for $f\in L_p(I_N, dt)$, $p\geqq 1$.

Lemma 2. ([3], [9]). *If*

$$\int_{+0} Q_p(\delta, f)\delta^{-(1+2N/p)}d\delta<+\infty,$$

then $f(t)$ is essentially continuous.

From this lemma, a sharper form than Corollay 1 is obtained again by Fubini's theorem and Hölder's inequality.

Corollary 2. *If an L_p-process $\{X(t, \omega); t\in I_N, \omega\in\Omega\}$ with $p\geqq 1$ satisfies*

$$\int_{+0}\left(\int_{|s-t|\leq\delta}E[|X(s)-X(t)|^p]ds dt\right)^{1/p}\delta^{-(1+2N/p)}d\delta<+\infty,$$

then the sample path is continuous with probability 1.

In this paper, we shall give a real analytical proof for Lemma 2, which is elementary and simpler than that of their combinatorial or Fourier analytical methods. In § 3, applying our real analytical method we shall obtain an integral test for differentiabililty of $f\in L_p([0, 1], dt)$ and of sample paths of L_p-processes, which is sharper than that of [7]. In § 4, we shall give some remarks

§ 2. Real analytical proof of Lemma 2.

Set

$$I_{n,i}=((i-1)2^{-n}, i2^{-n}], \quad \text{if} \quad i=2, \cdots, 2^n,$$
$$=[0, 2^{-n}], \quad \text{if} \quad i=1,$$
$$D_{n,i}=\prod_{k=1}^{N} I_{n,i_k} \quad \text{for} \quad i=(i_1, \cdots, i_N),$$
$$f_n(t)=2^{nN}\int_{D_{n,i}} f(u)du \quad \text{for} \quad t\in D_{n,i},$$

and

$$A_n(f)=\max_{|i-j|=1}|f_n(i2^{-n})-f_n(j2^{-n})|$$

Since $|i-j|=1$ and $(u, v)\in D_{n,i}\times D_{n,j}$ imply $|u-v|\leqq 2^{-n+1}$, we have

$$A_n(f)\leqq \max_{|i-j|=1} 2^{2nN}\int_{D_{n,i}\times D_{n,j}}|f(u)-f(v)|^p du dv)^{1/p}$$
$$\leqq 2^{2nN/p}Q_p(2^{-n+1}, f).$$

First we shall show that $f_n(t)$ converges uniformly to a continuous function $f_\infty(t)$. In fact $t\in D_{n,i}\cap D_{n+1,j}$ and $(u, v)\in D_{n,i}\times D_{n+1,j}$ imply $|u-v|\leqq 2^{-n}$, which yields

$$\sum_n^{\infty} |f_{n+1}(t)-f_n(t)|$$

$$\leqq \sum_n^{\infty} 2^{(2n+1)N/p}\Big(\int_{D_{n,i}\times D_{n,j}} |f(u)-f(v)|^p du dv\Big)^{1/p}$$

$$\leqq \sum_n^{\infty} 2^{(2n+1)N/p} Q_p(2^{-n}, f)$$

$$\leqq 2^{1+3N/p}\int_{+0}^{1} Q_p(\delta, f)\delta^{-(1+2N/p)} d\delta < +\infty\,.$$

Therefore there exists a limit function $f_\infty(t)$ of $\{f_n(t)\}$.

Next we shall show that $f_\infty(t)$ is continuous. Since $t\in D_{q,i}$, $s\in D_{q,j}$ and $2^{-q-1}\leqq |s-t|<2^{-q}$ imply $|i-j|\leqq 1$, it follows that

$$|f_q(s)-f_q(t)|\leqq A_q(f)\leqq 2^{2qN/p}Q_p(2^{-q+1}, f)\,.$$

Therefore we have

$$|f_\infty(s)-f_\infty(t)|\leqq \sum_{n=q}^{\infty} |f_{n+1}(s)-f_n(s)|$$

$$+\sum_{n=q}^{\infty} |f_{n+1}(t)-f_n(t)|+|f_q(s)-f_q(t)|$$

$$\leqq 2^{1+N/p}\sum_{n=q}^{\infty} 2^{2nN/p}Q_p(2^{-n}, f)+2^{2qN/p}Q_p(2^{-q+1}, f)$$

$$\leqq 4^{1+2N/p}\int_{+0}^{2^{-q+2}} Q_p(\delta, f)\delta^{-(1+2N/p)} d\delta$$

$$\leqq 4^{1+2N/p}\int_{+0}^{8|s-t|} Q_p(\delta, f)\delta^{-(1+2N/p)} d\delta\,.$$

Remark. The above modulus of continuity is slightly different from that of Garsia.

Finally we shall show that $f(t)=f_\infty(t)$ almost everywhere. It is sufficient to check that $f_n(t)$ converges to $f(t)$ in $L_p(I_N, dt)$-norm. In fact

$$\int_{I_N} |f(t)-f_n(t)|^p dt$$

$$=\sum_i \int_{D_{n,i}} |f(t)-2^{nN}\int_{D_{n,i}} f(u)du|^p dt$$

$$\leqq 2^{pnN}\sum_i \int_{D_{n,i}\times D_{n,i}} |f(t)-f(u)|^p du dt$$

$$\leqq 2^{pnN}Q_p(2^{-n}, f)^p$$

$$\leqq 2^{p+N}\Big(\int_{2^{-n}}^{2^{-n+1}} Q_p(\delta, f)\delta^{-(1+N/p)} d\delta\Big)^p \longrightarrow 0,$$

as $n\longrightarrow +\infty$. Q. E. D.

§ 3. An integral test for differentiability

Now we shall extend the idea of § 2 to obtain a sufficient condition for differentiability of $f\in L_p([0, 1], dt)$. Set

$$\theta_h f(t)=f(t+h),$$
$$\Delta_h^{(r)}f(t)=(\theta_h-\theta_0)^r f(t)$$
$$=\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f(t+kh),$$

and

$$Q_p^{(r)}(\delta, f)=\left(\int_0^\delta\int_0^{1-rh}|\Delta_h^{(r)}f(t)|^p dt\,dh\right)^{1/p}, \qquad (p\geqq 1).$$

Then we have

Lemma 3. *If*

$$\int_{+0} Q_p^{(r+1)}(\delta, f)\delta^{-(1+r+2/p)}d\delta<+\infty,$$

then there exists $\bar{f}$ having the r-th continuous derivative which coincides with f almost everywhere.

From this lemma, a sharper form than that of [7] is obtained by Fubini's theorem and Hölder's inequality.

Corollary 3. *If an L_p-process $\{X(t, \omega); 0\leqq t\leqq 1, \omega\in\Omega\}$ with $p\geqq 1$ satisfies*

$$\int_{+0}\left(\int_0^\delta\int_0^{1-(r+1)h}E[|\Delta_h^{(r+1)}X(t)|^p]dt\,dh\right)^{1/p}\delta^{-(1+r+2/p)}d\delta<\infty,$$

then the sample path $X(t, \omega)$ has the r-th continuous derivative with probability 1.

Proof of Lemma 3. Set

$$f_n^{(r)}(t)=f_n^{(r)}(i2^{-n})$$
$$=(r+2)2^{(r+2)n}\int_0^{2^{-n}}\int_{i2^{-n}}^{i2^{-n}+h}\Delta_h^{(r)}f(s)ds\,dh,$$

for $i2^{-n}\leqq t<(i+1)2^{-n}$ and $0\leqq i\leqq 2^n-(r+1)$,

$$f_n^{(r)}(t)=f_n^{(r)}(1-(r+1)2^{-n}), \qquad \text{for } 1-r2^{-n}\leqq t\leqq 1,$$

and

$$A_n^{(r)}(f)=\max_{1\leqq i\leqq 2^n-(r+1)}|f_n^{(r)}(i2^{-n})-f_n^{(r)}((i-1)2^{-n})|.$$

We remark that if $f(t)$ has the r-th continuous derivative $f^{(r)}(t)$, then $f_n^{(r)}(t)$ tends to $f^{(r)}(t)$ as $n\to+\infty$. Since we have

$$f_n^{(r)}(i2^{-n})=(r+2)2^{(r+2)n}\left\{\int_0^{2^{-n}}\int_0^{i2^{-n}}\Delta_h^{(r+1)}f(s)ds\,dh+\int_0^{2^{-n}}\int_0^h\Delta_h^{(r)}f(s)ds\,dh\right\},$$

it follows by Hölder's inequality that

$$A_n^{(r)}(f)=\left\{\max_{1\leq i\leq 2^n-(r+1)}(r+2)^p 2^{p(r+2)n}\left(\int_0^{2^{-n}}\int_{(i-1)2^{-n}}^{i2^{-n}}|\Delta_h^{(r+1)}f(s)|\,ds\,dh\right)^p\right\}^{1/p}$$
$$\leqq(r+2)2^{(r+2/p)n}Q_p^{(r+1)}(2^{-n}, f).$$

By an obvious formula

$$\Delta_h^{(r)}+\Delta_h^{(r)}\theta_{h/2}-2^{r+1}\Delta_{h/2}^{(r)}=\sum_{j=1}^{r}2^j(\theta_{h/2}+\theta_0)^{r-j}\Delta_{h/2}^{(r+1)},$$

it follows that for $i2^{-n}\leqq t<(2i+1)2^{-n-1}$,

$$|f_n^{(r)}(t)-f_{n+1}^{(r)}(t)|$$
$$=(r+2)2^{(r+2)n}\left|\int_0^{2^{-n}}\left\{\int_{i2^{-n}}^{i2^{-n}+h}\Delta_h^{(r)}f(s)ds-2^{r+1}\int_{i2^{-n}}^{i2^{-n}+h/2}\Delta_{h/2}^{(r)}f(s)ds\right\}dh\right|$$
$$=(r+2)2^{(r+2)n}\left|\int_0^{2^{-n}}\int_{i2^{-n}}^{i2^{-n}+h/2}\left(\sum_{j=0}^{r}2^j(\theta_{h/2}+\theta_0)^{r-j}\Delta_{h/2}^{(r+1)}\right)f(s)ds\,dh\right|$$
$$\leqq(r+2)^2 2^{r-3(1-1/p)+(r+2/p)n}Q_p^{(r+1)}(2^{-n-1}, f),$$

and for $(2i+1)2^{-n-1}\leqq t<(i+1)2^{-n}$,

$$|f_n^{(r)}(t)-f_{n+1}^{(r)}(t)|\leqq|f_n^{(r)}(i2^{-n})-f_{n+1}^{(r)}(i2^{-n})|+|f_{n+1}^{(r)}(2i2^{-n-1})-f_{n+1}^{(r)}((2i+1)2^{-n-1})|$$
$$\leqq(r+2)^2 2^{r-3(1-1/p)+(r+2/p)n}Q_p^{(r+1)}(2^{-n-1}, f)+A_{n+1}^{(r)}(f)$$
$$\leqq(r+2)^2 2^{r+3/p}2^{(r+2/p)n}Q_p^{(r+1)}(2^{-n-1}, f).$$

Therefore we have

$$\sum_{n=q}^{\infty}|f_n^{(r)}(t)-f_{n+1}^{(r)}(t)|$$
$$\leqq(r+2)^2 2^{r+3/p}\sum_{n=q}^{\infty}2^{(r+2/p)n}Q_p^{(r+1)}(2^{-n-1}, f)$$
$$\leqq 2(r+2)^2 2^{r+3/p}\int_{+0}^{2^{-q}}Q_p^{(r+1)}(\delta, f)\delta^{-(1+r+2/p)}d\delta<+\infty.$$

This implies that $f_n^{(r)}(t)$ converges uniformly on any compact subset of $[0, 1]$ to a limit function $f_\infty^{(r)}(t)$, $0\leqq t<1$.

Next we shall show that $f_\infty^{(r)}(t)$ is uniformly continuous, so it is extendable continuously till $t=1$. In fact, for $2^{-q-1}\leqq s-t<2^{-q}$ we have

$$|f_\infty^{(r)}(s)-f_\infty^{(r)}(t)|$$
$$\leqq\sum_{n=q}^{\infty}|f_{n+1}^{(r)}(s)-f_n^{(r)}(s)|+|f_q^{(r)}(s)-f_q^{(r)}(t)|+\sum_{n=q}^{\infty}|f_{n+1}^{(r)}(t)-f_n^{(r)}(t)|$$
$$\leqq 4(r+2)^2 2^{r+3/p}\int_{+0}^{2^{-q}}Q_p^{(r+1)}(\delta, f)\delta^{-(1+r+2/p)}d\delta+A_q^{(r)}(f)$$
$$\leqq 4(r+2)^2 2^{r+3/p}\int_{+0}^{2^{-q+1}}Q_p^{(r+1)}(\delta, f)\delta^{-(1+r+2/p)}d\delta$$
$$\leqq 4(r+2)^2 2^{r+3/p}\int_{+0}^{4|s-t|}Q_p^{(r+1)}(\delta, f)\delta^{-(1+r+2/p)}d\delta.$$

Finally, we have to show that $f_{\infty}^{(r)}(t)$ is the r-th derivative of an $\bar{f}(t)$ which coincides with $f(t)$ almost everywhere. Let $\rho(s)$ be a non-negative c^{∞}-function on $(-1, 1)$ such that $\int_{-1}^{1}\rho(s)ds=1$, and set

$$f^{(\varepsilon)}(t)=\int_{-\varepsilon}^{\varepsilon}f(t+\varepsilon-s)\rho(s/\varepsilon)ds/\varepsilon\,, \qquad 0\leqq t\leqq 1-2\varepsilon_0 \quad 0<\varepsilon<\varepsilon_0\,,$$

for arbitrarily small $\varepsilon_0>0$.

Then we have

$$Q_p'^{(r+1)}(\delta, f^{(\varepsilon)})\equiv\Big(\int_0^{\delta}\int_0^{1-(r+1)h-2\varepsilon_0}|\Delta_h^{(r+1)}f^{(\varepsilon)}(u)|^p dudh\Big)^{1/p}$$

$$\leqq\Big(\int_0^{\delta}\int_0^{1-2\varepsilon_0-(r+1)h}\int_{-\varepsilon}^{\varepsilon}|\Delta_h^{(r+1)}f(u+\varepsilon-s)|^p\rho(s/\varepsilon)\varepsilon^{-1}dsdudh\Big)^{1/p}$$

$$\leqq\Big(\int_0^{\delta}\int_0^{1-(r+1)h}|\Delta_h^{(r+1)}f(u)|^p dudh\Big)^{1/p}=Q_p^{(r+1)}(\delta, f)\,.$$

Since the convergence of f_n to zero in $L_p([0, 1-2\varepsilon_0], dt)$ implies that $Q_p'^{(r+1)}(\delta, f_n)$ tends to zero, we have the convergence of $Q_p'^{(r+1)}(\delta, f^{(\varepsilon)}-f)$ to zero as ε goes to zero. On the other hand, we have

$$Q_p'^{(r+1)}(\delta, f^{(\varepsilon)}-f)\leqq 2Q_p^{(r+1)}(\delta, f)\,,$$

and

$$\Big|\frac{d^r f^{(\varepsilon)}}{dt^r}-f_{\infty}^{(r)}(t)\Big|$$

$$\leqq 4(r+2)^2 2^{r+3/p}\int_{+0}^{2^{-q}}Q_q'^{(r+1)}(\delta, f^{(\varepsilon)}-f)\delta^{-(1+r+2/p)}d\delta+|f_q^{(\varepsilon)(r)}(t)-f_q^{(r)}(t)|\,.$$

The first term tends to zero uniformly on $[0, 1-2\varepsilon_0]$ as $\varepsilon\downarrow 0$ by Lebesgue's convergence theorem. The second term is estimated by

$$|f_q^{(\varepsilon)(r)}(t)-f_q^{(r)}(t)|$$

$$=(r+2)2^{(r+2)q}\Big|\int_0^{2^{-q}}\int_{i2^{-q}}^{i2^{-q}+h}\Delta_h^{(r)}(f^{(\varepsilon)}(s)-f(s))dsdh\Big|$$

$$\leqq(r+2)2^{(r+2)q+r}\int_0^{2^{-q}}\int_0^{1-2\varepsilon_0}|f(s)-f^{(\varepsilon)}(s)|dsdh\longrightarrow 0,$$

as $\varepsilon\downarrow 0$ uniformly on $[0, 1-2\varepsilon_0]$.

Therefore $\dfrac{d^r f^{(\varepsilon)}}{dt^r}$ converges uniformly on $[0, 1-2\varepsilon_0]$ to $f_{\infty}^{(r)}(t)$. By taking account of $f^{(\varepsilon)}$ tending to f in L_p $([0, 1-2\varepsilon_0], dt)$, $f^{(\varepsilon)}$ conveges to an $\bar{f}$ uniformly on $[0, 1-2\varepsilon_0]$ which coincides with f almost everywhere, where ε_0 is arbitrarily small and $f_{\infty}^{(r)}(t)$ is continuous on $[0, 1]$. This implies that $f_{\infty}^{(r)}(t)$ is the r-th derivative of $\bar{f}(t)$ on $[0, 1]$ which coincides with f almost everywhere.

Q. E. D.

§ 4. Remarks.

Let σ be a non-negative continuous (not necessarily non-decreasing) function defined on $[0, 1]$, and set

$$Q_p(\delta)=\left(\int_0^\delta \sigma^p(h)dh\right)^{1/p}, \qquad (p\geqq 1).$$

Then we have

Lemma 4. *If*

$$\int_{+0} Q_p(\delta)\delta^{-(1+2/p)}d\delta<+\infty,$$

then

$$\int_{+0} \sigma(h)h^{-(1+1/p)}dh<+\infty.$$

Proof. Since we have

$$\int_{+0}\left(\int_0^\delta \sigma(h)dh\right)\delta^{-(2+1/p)}d\delta\leqq\int_{+0} Q(\delta)\delta^{-(1+2/p)}d\delta<+\infty,$$

it follows that

$$\frac{1-2^{-(1+1/p)}}{1+1/p}\cdot 2^{(1+1/p)n}\int_0^{2^{-n}} \sigma(h)dh$$

$$\leqq\int_{2^{-n}}^{2^{-n+1}}\left(\int_0^\delta \sigma(h)dh\right)\delta^{-(2+1/p)}d\delta \longrightarrow 0, \qquad \text{as} \quad n\longrightarrow +\infty.$$

This implies that

$$\lim_{\delta\downarrow 0}\delta^{-(1+1/p)}\int_0^\delta \sigma(h)dh=0.$$

Therefore, we have from integration by parts,

$$+\infty>\int_{+0}\left(\int_0^\delta \sigma(h)dh\right)\delta^{-(2+1/p)}d\delta=-\frac{1}{1+1/p}\delta^{-(1+1/p)}\int_0^\delta \sigma(h)dh\bigg|_{+0}$$

$$+\frac{1}{1+1/p}\int_{+0}\sigma(h)h^{-(1+1/p)}dh. \quad \text{Q.E.D.}$$

Lemma 5. *In addition, if σ is sub-additive, i.e. $\sigma(s+t)\leqq\sigma(s)+\sigma(t)$, and $1\leqq p<\log 6/\log 2=2.58\cdots$, then*

$$\sum_n 2^{n/p}\sigma(2^{-n})<+\infty$$

implies

$$\int_{+0} Q_p(\delta)\delta^{-(1+2/p)}d\delta<+\infty.$$

Proof. First we have

$$\int_{+0} Q_p(\delta)\delta^{-(1+2/p)}d\delta=\sum_n^\infty\int_{2^{-n-1}}^{2^{-n}} Q_p(\delta)\delta^{-(1+2/p)}d\delta$$

$$\leqq p(2^{2/p}-1)/2\sum_{n}^{\infty}2^{2n/p}Q_p(2^{-n}).$$

By sub-additivity of σ and convexity of x^p,

$$\sigma^p(h)\leqq 2^{p-1}(\sigma^p(h-2^{-n-1})+\sigma^p(2^{-n-1}))$$

holds for $2^{-n-1}<h\leqq 2^{-n}$. Therefore integrating this by h, we have

$$\int_{2^{-n-1}}^{2^{-n}}\sigma^p(h)dh\leqq 2^{p-1}\int_0^{2^{-n-1}}\sigma^p(h)dh+2^{p-n-2}\sigma^p(2^{-n-1}),$$

and

$$\int_0^{2^{-n}}\sigma^p(h)dh\leqq(2^{p-1}+1)\int_0^{2^{-n-1}}\sigma^p(h)dh+2^{p-n-2}\sigma^p(2^{-n-1}).$$

This yields

$$2^{2n/p}Q_p(2^{-n})\leqq(2^{p-1}+1)^{1/p}2^{-2/p+2(n+1)/p}Q_p(2^{-n-1})+2^{1-2/p+n/p}\sigma(2^{-n-1}),$$

and

$$\sum_{n}^{\infty}2^{2n/p}Q_p(2^{-n})<+\infty \qquad \text{if} \quad p<\log 6/\log 2. \qquad \text{Q.E.D.}$$

If the above σ is a majorant of an L_p-process $\{X(t,\omega);0\leqq t\leqq 1,\omega\in\Omega\}$, i.e. $(E[|X(t+h)-X(t)|^p])^{1/p}\leqq\sigma(|h|)$, then

$$\sum_{n}^{\infty}2^{n/p}\sigma(2^{-n})<+\infty$$

is a sufficient condition for sample continuity of $\{X(t,\omega)\}$ (Theorem 1 of [7]). On the other hand,

$$\int_{+0}Q_p(\delta)\delta^{-(1+2/p)}d\delta<+\infty$$

is another sufficient condition for sample continuity of $\{X(t,\omega)\}$ from our Corollary 2.

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