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Pseudoconvex domains over Grassmann manifolds

By

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Introduction. The Levi problem, or the inverse problem of Hartogs, for domains over a complex manifold, is stated as follows: Let D be a pseudoconvex unramified domain over a complex manifold X. Is D a Stein manifold? Oka [8] solved affirmatively this problem in the original and fundamental case, i.e., for domains over an affine space $\mathbf{C}^n$. Since then this result has been generalized for domains over various complex manifold X, for example over a Stein manifold by Docquier-Grauert [1] and over a projective space by Fujita [2] and Takeuchi [9]. Hirschowitz [4], [5], [6] investigated the case where X is an infinitesimally homogeneous manifold and especially showed that the problem is affirmatively answered if X is an irreducible compact rational homogeneous manifold, for example a Grassmann manifold, and if the projection of D to X is of finite fiber.

In this note we shall show that the problem is solved in the case where X is a Grassmann manifold without the finiteness condition of the fibers, in a way different from that of Hirschowitz, reducing it to the problem over an affine space. We remark that the proof becomes simple if X is a projective space.

1. By an unramified domain over a complex manifold X we mean a connected Hausdorff space E together with a locally homeomorphic map Φ of E to X, which we call the projection. We denote such an unramified domain by the triple $\mathscr{E}=(E, \Phi, X)$ and call simply a domain. For a domain $\mathscr{E}$, a structure of complex manifold is induced on E so that the projection Φ is a holomorphic map. For the definition of the boundary points of a domain we refer to Grauert-Remmert [3] (Definition 4). The set of all boundary points of the domain is denoted by ∂E. We can define a structure of Hausdorff space on $\check{E}=E\cup\partial E$ and a continuous map $\check{\Phi}$ of $\check{E}$ to X such that $\check{\Phi}\,|\,E=\Phi$. The domain $\mathscr{E}$ is called pseudoconvex at a boundary point q, if there exists a neighborhood U of q such that $U\cap E$ is a Stein manifold. When $\mathscr{E}$ is pseudoconvex at every boundary point, $\mathscr{E}$ is called pseudoconvex.

2. The Stiefel manifold $V_{n,r}$ is the set of all $n\times r$ matrices of rank r. We can regard $V_{n,r}$ as a Zariski open set in the affine space $\mathbf{C}^{nr}$. The Grassmann manifold $G_{n,r}$ is the quotient space $V_{n,r}/GL(r, \mathbf{C})$ of $V_{n,r}$ by the operations of the general linear group $GL(r, \mathbf{C})$ defined by

$$V_{n,r} \times GL(r, \mathbf{C}) \ni (A, Z) \longrightarrow AZ \in V_{n,r}.$$

The canonical projection π of $V_{n,r}$ onto $G_{n,r}$ defines a holomorphic principal bundle with the structure group $GL(r, \mathbf{C})$. We note that, for x in $G_{n,r}$, the closure $\overline{\pi^{-1}(x)}$ in $\mathbf{C}^{nr}$ of the fiber $\pi^{-1}(x)$ is a vector subspace of $\mathbf{C}^{nr}$ of dimension r^2.

3. Our purpose is to prove the following

Theorem. *Let $\mathscr{D}=(D, \Phi, G_{n,r})$ be a pseudoconvex unramified domain over a Grassmann manifold $G_{n,r}$. If there exists at least one boundary point, i.e., unless D is homeomorphic to $G_{n,r}$ by the projection Φ, then D is a Stein manifold.*

To prove the theorem, we construct the fiber product $\tilde{D}$ of the bundle $V_{n,r} \to G_{n,r}$ and the domain $D \to G_{n,r}$, namely,

$$\tilde{D} = \{(A, p) \in V_{n,r} \times D | \quad \pi(A) = \Phi(p)\}.$$

We have the commutative diagram

$$\begin{array}{ccc} \tilde{D} & \xrightarrow{\tilde{\pi}} & D \\ {\scriptstyle \tilde{\Phi}}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathbf{C}^{nr} \supset V_{n,r} & \xrightarrow[\pi]{} & G_{n,r} \end{array} .$$

The map $\tilde{\pi}$ of $\tilde{D}$ onto D defines a holomorphic principal bundle with the structure group $GL(r, \mathbf{C})$. The map $\tilde{\Phi}$ of $\tilde{D}$ to $V_{n,r}$ defines a domain $(\tilde{D}, \tilde{\Phi}, V_{n,r})$ over $V_{n,r}$ and consequently a domain $\tilde{\mathscr{D}}=(\tilde{D}, \tilde{\Phi}, \mathbf{C}^{nr})$ over $\mathbf{C}^{nr}$. Clearly the domain $(\tilde{D}, \tilde{\Phi}, V_{n,r})$ is pseudoconvex. Therefore $\tilde{\mathscr{D}}$ is pseudoconvex at each boundary point which lies over $V_{n,r}$. We shall prove later that the domain $\tilde{\mathscr{D}}$ is pseudoconvex (at every boundary point). Let us assume this for some time. Then, by Oka's fundamental result, $\tilde{D}$ is a Stein manifold. From this we infer that D is also a Stein manifold, by virtue of the following theorem of Matsushima-Morimoto [7] (Théorème 5):

Let $P \to B$ a holomorphic principal bundle over a complex manifold B whose structure group G is the complexification of a maximal compact subgroup of G. If the total space P is a Stein manifold, then the base B is also a Stein manifold.

Indeed, in our problem, the structure group $GL(r, \mathbf{C})$ of the principal bundle $\tilde{D} \to D$ is the complexification of the unitary group $U(r)$, which is a maximal compact subgroup of $GL(r, \mathbf{C})$. Thus the proof of the theorem will be completed, if we show the pseudoconvexity of $\tilde{\mathscr{D}}$. The rest of this note is devoted to a proof of this fact.

4. Let us first recall some definitions of Grauert-Remmert [3]. Let $\mathscr{E}=(E, \Phi, X)$ be a domain. A boundary point q is called removable (hebbar), if there exists a neighborhood U of q such that $(U, \check{\Phi} | U, X)$ is a "schlicht" domain and that $U \cap \partial E$ is contained in an analytic set of positive codimension in U (Definition 5). A subset T of the boundary ∂E is called thin (dünn), if for each point q in T there exist a neighborhood U of q and a holomorphic function f on $U \cap E$ which does not identically vanish, with the following property: for any point q' in $U \cap T$ there exists a sequence p_ν, $\nu=1, 2,\dots$, of points in $U \cap E$ tending to q', such that $\lim_{\nu\to\infty} f(p_\nu)=0$ (Definition 6). One of the main results of [3] is the following (Satz 4):

Let $\mathscr{E}=(E, \Phi, X)$ be a domain. If there exists a thin subset T of the boundary ∂E such that no point in T is removable and that $\mathscr{E}$ is pseudoconvex at each point in $\partial E-T$, then $\mathscr{E}$ is pseudoconvex.

Now let S be an analytic set of positive codimension in a complex manifold X and let $\mathscr{E}=(E, \Phi, X)$ be again a domain. A boundary point q of $\mathscr{E}$ is called removable along S, if there exists a neighborhood U of q such that $(U, \check{\Phi} \mid U, X)$ is a "schlicht" domain and that $U \cap \partial E$ is contained in $\check{\Phi}^{-1}(S)$. Let R denote the set of all boundary points that are removable along S. Then, setting $E^*=E \cup R$ and $\Phi^*=\check{\Phi} \mid E^*$, we obtain a domain $\mathscr{E}^*=(E^*, \Phi^*, X)$, which we call the extension of the domain $\mathscr{E}$ along S. We have $\partial E^*=\partial E-R$ by means of the natural identification of boundary points.

Lemma. *Let S be an analytic set in X of positive codimension and let $\mathscr{E}=(E, \Phi, X)$ be a domain. Assume that $\mathscr{E}$ is pseudoconvex at every boundary point lying over $X-S$.*
(1) *If there exists no boundary point which is removable along S, then $\mathscr{E}$ is pseudoconvex.*
(2) *Let $\mathscr{E}^*=(E^*, \Phi^*, X)$ be the extension of $\mathscr{E}$ along S. Then $\mathscr{E}^*$ is pseudoconvex.*

Remark. The first assertion is considered to be a generalization of Satz 6 in [3].

Proof. The second assertion follows immediately from the first, which we prove now. (cf. the proof of Satz 6.) Clearly $T=\check{\Phi}^{-1}(S) \cap \partial E$ is a thin subset. So, in view of the above theorem of Grauert-Remmert, it suffices to show that $\mathscr{E}$ is pseudoconvex at each removable point in T. Let q be such a point. Then there exists a neighborhood U of q such that $(U, \check{\Phi} \mid U, X)$ is a "schicht" domain and that $U \cap \partial E$ is contained in an analytic set M in U. We set $N=(U-\check{\Phi}^{-1}(S)) \cap \partial E$. The domain $\mathscr{E}$ is pseudoconvex at every point in N, and N is contained in $(U-\check{\Phi}^{-1}(S)) \cap M$. Therefore, by Hartogs' continuity theorem, N is an analytic set in $U-\check{\Phi}^{-1}(S)$ of pure codimension 1, composed of some of the irreducible components of $(U-\check{\Phi}^{-1}(S)) \cap M$. The closure $\bar{N}$ of N in U is an analytic set in U. We have $\bar{N} \subseteq U \cap \partial E$ since ∂E is closed. We assert that $\bar{N}=U \cap \partial E$. In fact, otherwise, $(U \cap \partial E)-\bar{N}$ would be non-empty and contained in $\check{\Phi}^{-1}(S)$; hence the points in $(U \cap \partial E)-\bar{N}$ would be removable along S, which would contradict the assumption. Thus we see that $\bar{N}=U \cap \partial E$ and that $\mathscr{E}$ is pseudoconvex at q. q.e.d.

5. Now let us show that the domain $\tilde{\mathscr{D}}=(\tilde{D}, \tilde{\Phi}, \mathbf{C}^{nr})$ is pseudoconvex. We write $S=\mathbf{C}^{nr}-V_{n,r}$. By the lemma, it suffices to prove that there exists no boundary point removable along S. To prove this, let us assume the contrary. Adding to $\tilde{D}$ the non-empty set R of all boundary points removable along S, we get the extension $\mathscr{D}^*=(\tilde{D}^*, \tilde{\Phi}^*, \mathbf{C}^{nr})$ of $\mathscr{D}$ along S. The domain $\mathscr{D}^*$ is pseudoconvex by the lemma. Let q be a point in R. There exists a point x in $G_{n,r}$ such that the closure $\overline{\pi^{-1}(x)}$ in $\mathbf{C}^{nr}$ of the fiber $\pi^{-1}(x)$ contains the point $\tilde{\Phi}^*(q)$. We set $F^*=\tilde{\Phi}^{*-1}(\overline{\pi^{-1}(x)})$ and $F=\tilde{\Phi}^{-1}(\overline{\pi^{-1}(x)})=\tilde{\Phi}^{-1}(\pi^{-1}(x))=\tilde{\pi}^{-1}(\Phi^{-1}(x))$. Each connected component of F corresponds to a point in D which lies over x, and is homeomorphic to $\pi^{-1}(x)$ by the projection. Let F_0^* be the connected component of F^* which contains the point q, and consider the domain $\mathscr{F}_0^*=(F_0^*, \tilde{\Phi}^* \mid F_0^*, \overline{\pi^{-1}(x)})$. The domain $\mathscr{F}_0^*$ is pseudoconvex, since $\mathscr{D}^*$ is pseudoconvex. The restriction of F_0^* over $\pi^{-1}(x)$ is a connected component of F. Since F_0^* contains the point q lying over a point in $\overline{\pi^{-1}(x)}-\pi^{-1}(x)$, which is an irreducible analytic set in $\overline{\pi^{-1}(x)}$, the component F_0^* is homeomorphic to $\overline{\pi^{-1}(x)}$ by the projection $\tilde{\Phi}^* \mid F_0^*$, by Hartogs' continuity theorem. This implies that there exists a point in R over every point in $\overline{\pi^{-1}(x)}-\pi^{-1}(x)$, in particular over the origin of $\mathbf{C}^{nr}$. Let q_0 be such a point. Then there exists a neighborhood U of q_0 which is homeomorphic to a neighborhood of the origin of $\mathbf{C}^{nr}$ by the projection. Hence $\tilde{\pi}(U \cap \tilde{D})$ is homeomorphic to $G_{n,r}$ by the projection Φ. Since D is connected, we have $\tilde{\pi}(U \cap \tilde{D})=D$. But this case was excluded by the assumption. Thus we have proved the pseudoconvexity of the domain $\mathscr{D}$.

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