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On the Hilbert-Samuel polynomial in complex analytic geometry

By

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The Hilbert-Samuel polynomials are given by the following theorem due to Ramanujam ([12], [13]).

"*Let X be an algebraic variety, $\mathscr{L}$ an invertible sheaf on X and $\mathscr{I}\subset\mathcal{O}_X$ an ideal sheaf such that* Supp $(\mathcal{O}_X/\mathscr{I})$ *is complete. Then there is a polynomial $P(n, m)$ of total degree* $\leq \dim X$ *such that for large m we have*

$$\chi(X, \mathscr{L}^n/\mathscr{I}^m\mathscr{L}^n)=P(n, m).$$

When $\mathscr{I}=0$, this is the well-known polynomial theorem of Snapper ([2], [11], [16]). These results take a central place in the intersection number theory ([11], [12]).

The main purpose of the present paper is to give an analogous theorem in complex analytic geometry. Some comments concerning intersection numbers will be also given.

I. Proof of the theorem

I.1. For a complex space X and an invertible sheaf $\mathscr{L}$, we will put $\mathscr{L}^n=\mathscr{L}\otimes\cdots\otimes\mathscr{L}$, n-times, for an integer $n\geq 0$ ($\mathscr{L}^0=\mathcal{O}$) and $\mathscr{L}^n=\mathscr{L}^{-1}\otimes\cdots\otimes\mathscr{L}^{-1}$, $-n$-times, for an integer $n<0$, where $\mathscr{L}^{-1}$ is the dual of $\mathscr{L}$. For an analytic coherent sheaf $\mathscr{F}$ on X, $\dim\mathscr{F}$ means $\dim(\operatorname{Supp}\mathscr{F})$. We will write $m\gg 0$ to mean that m is sufficiently large.

Theorem. *Let X be a complex space, Y a compact analytic subset of X, and $\mathscr{I}\subset\mathcal{O}_X$ a coherent ideal sheaf such that* $\operatorname{Supp}(\mathcal{O}_X/\mathscr{I})=Y$, *$\mathscr{L}$ an invertible sheaf on X and $\mathscr{F}$ a coherent sheaf on X. Then the function $(n, m)\mapsto\chi((\mathscr{F}\otimes\mathscr{L}^n)/\mathscr{I}^m(\mathscr{I}\otimes\mathscr{L}^n))$ is a polynomial for $m\gg 0$ and any n, and the associated polynomial $P(\mathscr{F})=P(\mathscr{F}; \mathscr{L}, \mathscr{I})$ is of total degree* $\leq\dim\mathscr{F}$.

Moreover, if X is compact and $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$ is exact, then $P(\mathscr{F})=P(\mathscr{F}')+P(\mathscr{F}'')+Q$, where degree $Q<\dim\mathscr{F}'$.

Proof. a) Suppose first $\mathscr{I}=0$, thus X is compact. We need to show that the function $n\mapsto\chi(\mathscr{F}\otimes\mathscr{L}^n)$ is a polynomial in n, of degree $\leq\dim\mathscr{F}$ (the last statement of the theorem is clear in this case). This is stated by Fujita in ([5], p. 105, foot note), at least when X is reduced and irreducible and $\mathscr{F}=\mathcal{O}$. For reader's convenience we sketch an argument for the general case by induction on the dimension of X. If $\dim X=0$, the conclusion is clear. We prove the general step of the induction. First, we will remark that if $\dim\mathscr{F}<\dim X$, then the function $n\mapsto\chi(\mathscr{F}\otimes\mathscr{L}^n)$ is a polynomial of degree $\leq\dim\mathscr{F}$, as $\mathscr{F}\otimes\mathscr{L}^n\simeq(\mathscr{F}/\mathscr{A}nn\mathscr{F})\otimes(\mathscr{L}/\mathscr{A}nn\mathscr{F}\mathscr{L})^n$. We can assume that X is reduced: indeed, if $\mathscr{N}$ is the ideal sheaf of nilpotent elements, then $\mathscr{N}^k=0$ for sufficiently large k and by additivity, using the exact sequences

$$0\longrightarrow\mathscr{N}^s\mathscr{F}/\mathscr{N}^{s+1}\mathscr{F}\longrightarrow\mathscr{F}/\mathscr{N}^{s+1}\mathscr{F}\longrightarrow\mathscr{F}/\mathscr{N}^s\mathscr{F}\longrightarrow 0$$

we conclude it.

We can assume that X is irreducible. Indeed, let $X_1,\dots, X_r$ be the irreducible components of X and $\mathscr{I}_1,\dots,\mathscr{I}_r$ the associated maximal ideal sheaves. The canonical map $\mathscr{F}\to\prod_i(\mathscr{F}/\mathscr{I}_i\mathscr{F})$ has the kernel and the cokernel both of dimension $<\dim X$, hence for these sheaves the result is true, and we obtain the desired result, using again additivity. Therefore X can be assumed to be reduced and irreducible and moreover, we can assume that $\operatorname{Supp}\mathscr{F}=X$.

By a result due to Rossi [14] there exist a compact complex manifold X^* and a modification $\pi\colon X^*\to X$ such that the sheaf $\pi^*\mathscr{F}$ is, modulo torsion, locally free. Denote $\pi^*(\mathscr{L})=\mathscr{L}^*$ and $\mathscr{F}^*=\pi^*(\mathscr{F})$. By induction, $n\mapsto\chi(t(\mathscr{F}^*)\otimes\mathscr{L}^{*n})$ is a polynomial of degree $<\dim X^*=\dim X$, where $t(\mathscr{F}^*)$ is the torsion subsheaf of $\mathscr{F}^*$. By the Riemann-Roch-Hirzebruch theorem ([10]) for compact complex manifolds (as a consequence of the Atiyah-Singer index theorem), the function $n\mapsto\chi((\mathscr{F}^*/t(\mathscr{F}^*))\otimes\mathscr{L}^{*n})$ is a polynomial of degree $\leq\dim X^*$. Hence so is the function $n\mapsto\chi(X^*,\mathscr{F}^*\otimes\mathscr{L}^{*n})$. Now, for any n, there is a spectral sequence of term $E_2^{p,q}(n)=H^p(X,R^q\pi_*(\mathscr{F}^*\otimes\mathscr{L}^{*n}))$ which converges to $H^{p+q}(X^*,\mathscr{F}^*\otimes\mathscr{L}^{*n})$. There is a canonical isomorphism $R^q\pi_*(\mathscr{F}^*\otimes\mathscr{L}^{*n})\simeq R^q\pi_*(\mathscr{F}^*)\otimes\mathscr{L}^n$. As π is a modification, $R^q\pi_*(\mathscr{F}^*)$ is zero for $q\geq 1$, outside a closed analytic subset of X of lower dimension. Using again the induction hypothesis and the invariance of the Euler-Poincaré characteristic in a spectral sequence, we conclude that the function $n\mapsto\chi(\pi_*(\mathscr{F}^*)\otimes\mathscr{L}^n)$ is a polynomial of degree $\leq\dim X=\dim\mathscr{F}$. The kernel and the cokernel of the natural map $\mathscr{F}\to\pi_*\mathscr{F}^*$ are of dimension $<\dim X$ and we obtain the desired result.

b) We prove that the function is a polynomial (for $m\gg 0$). It suffices to show that the difference function

$$(n,m)\longmapsto\chi(\mathscr{F}\otimes\mathscr{L}^n/\mathscr{I}^{m+1}(\mathscr{F}\otimes\mathscr{L}^n))-\chi(\mathscr{F}\otimes\mathscr{L}^n/\mathscr{I}^m(\mathscr{F}\otimes\mathscr{L}^n))$$

is a polynomial in (n, m) for $m\gg 0$ ([15]). Let us consider the blowing-up of X with respect to $\mathscr{I}$ ([8], [9]) and denote it by $\pi\colon X^*\to X$. X^* is the Proj of the Rees sheaf-algebra $\mathcal{O}_X\oplus\mathscr{I}\oplus\mathscr{I}^2\oplus\cdots$. Then $Y^*=\pi^{-1}(Y)$, the analytic inverse image of Y, is nothing but the Proj of the sheaf-algebra (in fact its restriction to Y) $\mathcal{O}_X/\mathscr{I}\oplus\mathscr{I}/\mathscr{I}^2\oplus\cdots$. The inverse image of $\mathscr{I}/\mathscr{I}^2$ is an invertible sheaf $\mathscr{T}$ on Y^* which is very ample relatively to the map $Y^* \to Y$ (we will denote this map also by π). $\mathscr{A} = \mathscr{O}_X/\mathscr{I} \oplus \mathscr{I}/\mathscr{I}^2 \oplus \cdots$ is a coherent sheaf of rings and $\mathscr{M} = \mathscr{F}/\mathscr{I}\mathscr{F} \oplus \mathscr{I}\mathscr{F}/\mathscr{I}^2\mathscr{F} \oplus \cdots$ has a natural structure of a graded coherent $\mathscr{A}$-module (for every compact Stein subset K, there are canonical isomorphisms $\Gamma(K, \mathscr{A}) \simeq \oplus \Gamma(K, \mathscr{I})^n/\Gamma(K, \mathscr{I})^{n+1}$, $\Gamma(K, \mathscr{M}) \simeq \oplus (\Gamma(K, \mathscr{I})^n \Gamma(K, \mathscr{F})/\Gamma(K, \mathscr{I})^{n+1}\Gamma(K, \mathscr{F}))$ and for every inclusion $K' \subset K$, isomorphisms

$$\Gamma(K, \mathscr{A}) \otimes_{\mathscr{O}(K)} \mathscr{O}(K') \simeq \Gamma(K', \mathscr{A}), \quad \Gamma(K, \mathscr{M}) \otimes_{\mathscr{O}(K)} \mathscr{O}(K') \simeq \Gamma(K', \mathscr{M})$$

and moreover the map $\mathscr{O}(K) \to \mathscr{O}(K')$ is flat; now the above coherence statements follow from these facts and by the noetherianity theorem of Frisch ([4]). As in the algebraic case ([8]), to $\mathscr{M}$, one associates a coherent sheaf $\tilde{\mathscr{M}}$ on Y^* and a natural morphism $\mathscr{M} \mapsto \bigoplus_{n=0}^{\infty} \pi_*(\tilde{\mathscr{M}} \otimes \mathscr{T}^n)$. Using the Grauert-Remmert vanishing theorem for projective maps [7], we show that this morphism is an isomorphism on the homogeneous components of sufficiently large degree, that is $\mathscr{I}^m\mathscr{F}/\mathscr{I}^{m+1}\mathscr{F} \simeq \pi_*(\tilde{\mathscr{M}} \otimes \mathscr{T}^m)$ if $m \gg 0$ (in analytic case, the details can be found in C. Bănică et O. Stănasilă, Méthodes algébriques dans la théorie globale des espaces complexes, Gauthier-Villars, p. 257–259). Using again the Grauert-Remmert theorem, we have $R^q\pi_*(\tilde{\mathscr{M}} \otimes \mathscr{T}^m) = 0$ for $q \geq 1$ and $m \gg 0$. Thus, we obtain, by the Leray spectral sequence, the isomorphisms

$$H^q(Y, \mathscr{I}^m\mathscr{F}/\mathscr{I}^{m+1}\mathscr{F}) \simeq H^q(Y^*, \tilde{\mathscr{M}} \otimes \mathscr{T}^m), \qquad \text{for} \quad q \geqslant 0,\ m \geqslant m_0.$$

The $\mathscr{A}$-graded sheaf $\mathscr{M}(n) = \oplus(\mathscr{I}^m(\mathscr{F} \otimes \mathscr{L}^n)/\mathscr{I}^{m+1}(\mathscr{F} \otimes \mathscr{L}^n))$ is nothing but $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{L}^n$ and we have the identification $\widetilde{\mathscr{M}(n)} \simeq \tilde{\mathscr{M}} \otimes \mathscr{L}^{*n}$, where $\mathscr{L}^* = \pi^*(\mathscr{L})$; indeed, if $\{V_i\}_{i \in I}$ is a covering of X which trivializes $\mathscr{L}$ and $h_{ij} \in \mathscr{O}(V_i \cap V_j)$ are transition functions of $\mathscr{L}$, then on each $\pi^{-1}(V_i)$ both sheaves are isomorphic to $\tilde{\mathscr{M}}$ and on the intersections $\pi^{-1}(V_i) \cap \pi^{-1}(V_j)$ the patching is done for both by the multiplications by h_{ij}. Thus, we have an isomorphism

$$H^q(Y, \mathscr{I}^m(\mathscr{F} \otimes \mathscr{L}^n)/\mathscr{I}^{m+1}(\mathscr{F} \otimes \mathscr{L}^n)) \simeq H^q(Y^*, \tilde{\mathscr{M}} \otimes \mathscr{L}^{*n} \otimes \mathscr{T}^m)$$

for any n, any q and $m \geq m_0$ (the integer m_0 is good for every $\mathscr{F} \otimes \mathscr{L}^n$, since the question is local on Y). Now we apply a), with the remark that the given argument can be extended when we replace $\mathscr{L}$ by a finite number of invertible sheaves.

c) To prove that degree $P \leq \dim \mathscr{F}$, we proceed by induction on $\dim \mathscr{F}$. As the initial step of the induction is clear, we prove the general step. Replacing X by $\operatorname{Supp} \mathscr{F}$, $\mathscr{L}$ by $\mathscr{L}/(\mathscr{A}nn\mathscr{F})\mathscr{L} \mid \operatorname{Supp} \mathscr{F}$ and $\mathscr{I}$ by $(\mathscr{I} + \mathscr{A}nn\mathscr{F})/\mathscr{A}nn\mathscr{F} \mid \operatorname{Supp} \mathscr{F}$, we may assume $X = \operatorname{Supp} \mathscr{F}$, hence $\dim X = \dim \mathscr{F}$. If $\dim Y < \dim X$, then $P(n, m+1) - P(n, m) = \chi(Y^*, \tilde{\mathscr{M}} \otimes \mathscr{L}^{*n} \otimes \mathscr{T}^m)$ is of total degree $\leq \dim Y^* < \dim X$, thus degree $P \leq \dim X$. Now, assume $\dim Y = \dim X$. Denote by Z the union of the irreducible components of X_{red} which are not contained in Y (if any!). One has $X = Y \cup Z$ and $\dim(Y \cap Z) < \dim X$. Let $\mathscr{J} \subset \mathscr{O}_X$ be a coherent ideal such that $\operatorname{Supp}(\mathscr{O}_X/\mathscr{J}) = Z$. Our problem is of local nature around Y, hence replacing X by a neighbourhood of Y, we can find k such that $(\mathscr{I} \cap \mathscr{J})^k = 0$. Consider the exact sequence

$$0 \longrightarrow \mathcal{J}^k\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{J}^k\mathcal{F} \longrightarrow 0$$

and denote $\mathcal{G}=\mathcal{F}/\mathcal{J}^k\mathcal{F}$. Consider the complex space $(Z, \mathcal{O}_X/\mathcal{J}^k|Z)$, $\mathcal{L}/\mathcal{J}^k\mathcal{L}|Z$, $(\mathcal{I}+\mathcal{J}^k)/\mathcal{J}^k|Z$ and $\mathcal{G}|Z$. In this case the zero set of the ideal-sheaf is $Y\cap Z$ and it follows (as $\dim Z<\dim X$, or $\dim(Y\cap Z)<\dim Z$) that the degree of $(n, m)\mapsto\chi(\mathcal{G}\otimes\mathcal{L}^n/\mathcal{I}^m(\mathcal{G}\otimes\mathcal{L}^n))$ is at most $\dim X=\dim\mathcal{F}$. On the other hand, by the Artin-Rees lemma, the coherence and the compactness, we can find an integer r such that $\mathcal{I}^m\mathcal{F}\cap\mathcal{J}^k\mathcal{F}=\mathcal{I}^{m-r}(\mathcal{I}^r\mathcal{F}\cap\mathcal{J}^k\mathcal{F})$, if $m\geq r$. It follows $\mathcal{I}^m\mathcal{F}\cap\mathcal{J}^k\mathcal{F}=0$ for $m\gg 0$ and the same is true when we replace $\mathcal{F}$ by any $\mathcal{F}\otimes\mathcal{L}^n$. Hence we obtain the exact sequences

$$0\longrightarrow\mathcal{J}^k(\mathcal{F}\otimes\mathcal{L}^n)\longrightarrow(\mathcal{F}\otimes\mathcal{L}^n)/\mathcal{I}^m(\mathcal{F}\otimes\mathcal{L}^n)\longrightarrow(\mathcal{G}\otimes\mathcal{L}^n)/\mathcal{I}^m(\mathcal{G}\otimes\mathcal{L}^n)\longrightarrow 0$$

for any n and $m\gg 0$ and the proof is finished.

d) Assume now that X is compact and let $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ be an exact sequence of coherent sheaves on X. From the exact sequences

$$0\longrightarrow\mathcal{I}^m(\mathcal{F}'\otimes\mathcal{L}^n)\longrightarrow\mathcal{F}'\otimes\mathcal{L}^n\longrightarrow\mathcal{F}'\otimes\mathcal{L}^n/\mathcal{I}^m(\mathcal{F}'\otimes\mathcal{L}^n)\longrightarrow 0,\dots,$$

it follows that it suffices to prove the required property for the polynomials associated to the functions $(n, m)\mapsto\chi(\mathcal{I}^m(\mathcal{F}'\otimes\mathcal{L}^n))$, $\chi(\mathcal{I}^m(\mathcal{F}\otimes\mathcal{L}^n))$, $\chi(\mathcal{I}^m(\mathcal{F}''\otimes\mathcal{L}^n))$. We have the exact sequences

$$0\longrightarrow(\mathcal{F}'\otimes\mathcal{L}^n)\cap\mathcal{I}^m(\mathcal{F}\otimes\mathcal{L}^n_n)\longrightarrow\mathcal{I}^m(\mathcal{F}\otimes\mathcal{L}^n)\longrightarrow\mathcal{I}^m(\mathcal{F}''\otimes\mathcal{L}^n)\longrightarrow 0.$$

By the Artin-Rees lemma, the first sheaf is isomorphic to $\mathcal{I}^{m-r}((\mathcal{F}'\cap\mathcal{I}^r\mathcal{F})\otimes\mathcal{L}^n)$, for $r\gg 0$, $m\geq r$ and any n. The proof is finished, if we show that the polynomials associated to $\mathcal{G}=\mathcal{F}'\cap\mathcal{I}^r\mathcal{F}$ and $\mathcal{F}'$ differ by a polynomial of degree $<\dim\mathcal{F}'$. We can assume $X=\operatorname{Supp}\mathcal{F}'$, hence $\dim\mathcal{F}'=\dim X$. The inclusion map $\mathcal{G}\to\mathcal{F}'$ induces a morphism between the associated graded sheaves $\mathcal{M}_{\mathcal{G}}=\oplus\mathcal{I}^m\mathcal{G}\to\mathcal{M}_{\mathcal{F}'}=\oplus\mathcal{I}^m\mathcal{F}'$ hence a morphism between the associated sheaves on $X^*=\operatorname{Proj}(\oplus\mathcal{I}^m)$, $\tilde{\mathcal{M}}_{\mathcal{G}}\to\tilde{\mathcal{M}}_{\mathcal{F}'}$. We have $\chi(\mathcal{I}^m(\mathcal{G}\otimes\mathcal{L}^n))=\chi(X^*, \tilde{\mathcal{M}}_{\mathcal{G}}\otimes\mathcal{L}^{*n}\otimes\mathcal{T}^m)$, $\chi(\mathcal{I}^m(\mathcal{F}'\otimes\mathcal{L}^n))=\chi(X^*, \tilde{\mathcal{M}}_{\mathcal{F}'}\otimes\mathcal{L}^{*n}\otimes\mathcal{T}^m)$ for any n and $m\gg 0$, where $\mathcal{L}^*=\pi^*(\mathcal{L})$ and $\mathcal{T}=\pi^*(\mathcal{I})\mathcal{O}_{X^*}$. Assume $\dim Y<\dim X$. As the map $\mathcal{G}\to\mathcal{F}'$ is an isomorphism on $X\backslash Y$, the map $\tilde{\mathcal{M}}_{\mathcal{G}}\to\tilde{\mathcal{M}}_{\mathcal{F}'}$ is an isomorphism on $X^*\backslash Y^*$, hence its kernel and its cokernel are of dimension $\leq\dim Y^*<\dim X=\dim\mathcal{F}'$ and we obtain the desired result. It remains the case $\dim Y=\dim X$. In this case, consider Z, $\mathcal{J}$ and k as in c). It follows that $\mathcal{I}^m(\mathcal{G}\otimes\mathcal{L}^n)=\mathcal{I}^m((\mathcal{G}/\mathcal{J}^k\mathcal{G})\otimes\mathcal{L}^n)$, $\mathcal{I}^m(\mathcal{F}'\otimes\mathcal{L}^n)=\mathcal{I}^m((\mathcal{F}'/\mathcal{J}^k\mathcal{F}')\otimes\mathcal{L}^n)$ for any n and $m\gg 0$, and so on. Hence the proof of the theorem is finished.

I.2. The case $\mathcal{I}=0$ (i.e. the analogue of Snapper's theorem), as we have already said, is due to Fujita ([5]). For $\dim X=2$ (and $\mathcal{L}$ associated to a divisor) this was also proved by E. Selder (München) and for a special class of Moišezon spaces by C. Horst (München)(*). When X is Moišezon or when X is arbitrary

(*) When X has a "projektiv abschließbaren Teil" (as in [3]), in any case for Moišezon manifolds (cf. Fundamental Lemma of [3]).

but has no fixed points and $\mathscr{G}$ is such that the global sections generate the fibres $\mathscr{I}_x$, $x \in Y$, the theorem is proved in [1].

D. Leistner (Regensburg) has proved the following interesting result: *if X is a compact complex space, $\mathscr{L}$ an invertible sheaf on X and $\mathscr{F} \in \mathrm{Coh}\, X$, then for every q there exists a polynomial P_q of degree $\leq \dim \mathscr{F}$ such that* $\dim H^q(X, \mathscr{F}^n \otimes \mathscr{L}^n) \leq P_q(n)$ *for any positive integer n.*

Here we give a simple proof of this statement under the assumption that X is compact Moišezon. We prove it by induction on $\dim X$. For $\dim X = 0$, this is obvious. We can assume that $\dim \mathscr{F} = \dim X$. First, let us consider the case where X is projective. There exist very ample invertible sheaves $\mathscr{L}_1$, $\mathscr{L}_2$ and effective reduced Cartier divisors D_1, D_2 such that $\mathscr{L} = \mathscr{L}_1 \otimes \mathscr{L}_2^{-1}$, $\mathscr{O}_X(D_i) \simeq \mathscr{L}_i$. From the induction hypothesis and an exact sequence

$$0 \longrightarrow \mathscr{O}_X(\mathscr{F} \otimes \mathscr{L}_1^{m-1} \otimes \mathscr{L}_2^{-m}) \longrightarrow \mathscr{O}_X(\mathscr{F} \otimes \mathscr{L}^m) \longrightarrow \mathscr{O}_{D_1}(\mathscr{F} \otimes \mathscr{L}^m) \longrightarrow 0,$$

it follows

$$|h^q(X, \mathscr{F} \otimes \mathscr{L}_1^{m-1} \otimes \mathscr{L}_2^{-m}) - h^q(X, \mathscr{F} \otimes \mathscr{L}^m)|$$

$$\leq h^{q-1}(D_1, \mathscr{F} \otimes \mathscr{L}^m) + h^q(D_1, \mathscr{F} \otimes \mathscr{L}^m) \leq P(m)$$

where $P(x)$ is a polynomial of degree $\leq \dim X - 1$.
Similarly, we have

$$|h^q(X, \mathscr{F} \otimes \mathscr{L}_1^{m-1} \otimes \mathscr{L}_2^{-m}) - h^q(X, \mathscr{F} \otimes \mathscr{L}^{m-1})| \leq Q(m-1)$$

where $Q(x)$ is a polynomial of degree $\leq \dim X - 1$.
Hence, we have

$$h^q(X, \mathscr{F} \otimes \mathscr{L}^m) - h^q(X, \mathscr{F} \otimes \mathscr{L}^{m-1}) \leq P(m) + Q(m-1)$$

This implies the above statement.

Next let us consider the general case. By Chow's lemma, there exist a projective variety $\hat{X}$ and a birational morphism $\pi: \hat{X} \to X$. There is a spectral sequence

$$E_2^{p,q} = H^p(X, R^q\pi_*(\pi^*(\mathscr{F} \otimes \mathscr{L}^m))) \Longrightarrow H^{p+q}(\hat{X}, \pi^*(\mathscr{F} \otimes \mathscr{L}^m)).$$

As $\hat{X}$ is projective, there are polynomials $P_q(x)$ of degree $\leq \dim X$ such that

$$h^q(\hat{X}, \pi^*(\mathscr{F} \otimes \mathscr{L}^m)) \leq P_q(m).$$

As we have

$$E_{k+1}^{q,0} \simeq E_k^{q,0}/\mathrm{Im}(E_k^{q-k,k-1} \longrightarrow E_k^{q,0}),$$

we obtain inequalities

$$(*) \quad h^q(X, \pi_*(\pi^*\mathscr{F}) \otimes \mathscr{L}^m) = \dim E_2^{q,0} \leq \dim E_r^{q,0} + \sum_{k=3}^{r-1} \dim E_k^{q-k,k-1}$$

$$\leq \dim E_r^{q,0} + \sum_{k=3}^{r-1} \dim E_2^{q-k,k-1}$$

$$=\dim E_r^{q,0}+\sum_{k=3}^{r-1} h^{q-k}(X, R^{k-1}\pi_*(\pi^*\mathscr{F})\otimes\mathscr{L}^m)$$

Put $r=\dim X+1$. Then, $E_r^{q,0}=E_\infty^{q,0}$. Hence, we have

$$\dim E_r^{q,0}\le h^q(\hat{X}, \pi^*(\mathscr{F}\otimes\mathscr{L}^m))\le P_q(m)\,.$$

On the other hand, as π is birational, $\dim R^p\pi_*(\pi^*\mathscr{F})\le\dim X-1$ for $p\ge 1$. Hence by the induction hypothesis and the above inequality (*), there is a polynomial $Q(x)$ of degree $\le\dim X-1$ such that

$$h^q(X, \pi_*(\pi^*\mathscr{F})\otimes\mathscr{L}^m)\le P_q(m)+Q(m)\,.$$

There is a canonical homomorphism $f\colon \mathscr{F}\to\pi_*(\pi^*\mathscr{F})$ such that $\dim\operatorname{Ker} f\le\dim X-1$, $\dim\operatorname{Coker} f\le\dim X-1$. Hence it is easy to show that

$$h^q(X, \mathscr{F}\otimes\mathscr{L}^m)\le R(m)$$

where $R(x)$ is a polynomial of degree $\le\dim X$. This is the desired result.

I.3. In the algebraic case we have the following variant of the theorem (we again neglect to consider $\mathscr{L}$)

Statement. *Let X be a non-singular algebraic variety, $\mathscr{I}$ a coherent ideal sheaf such that* $\operatorname{Supp}(\mathcal{O}_X/\mathscr{I})$ *is complete and $\mathscr{F}$ an algebraic coherent sheaf on X. Then for every q the functions $n\mapsto\chi(\mathscr{T}\!or_q^{\mathcal{O}}(\mathscr{F}, \mathcal{O}/\mathscr{I}^n))$, $n\mapsto\chi(\mathscr{E}\!xt_{\mathcal{O}}^q(\mathscr{F}, \mathcal{O}/\mathscr{I}^n))$ are polynomials for $n\gg 0$.*

Indeed, by means of a result due to Kleiman there exists a resolution $\mathscr{L}.$ of $\mathscr{F}$ by locally free coherent sheaves. Then $\mathscr{T}\!or_q^{\mathcal{O}}(\mathscr{F}, \mathcal{O}/\mathscr{I}^n)$ equals $\mathscr{H}_q(\mathscr{L}./\mathscr{I}^n\mathscr{L}.)$ and by additivity we obtain $\chi(\mathscr{H}_q(\mathscr{L}./\mathscr{I}^n\mathscr{L}.))=-\chi(\mathscr{L}_{q-1}/\mathscr{I}^n\mathscr{L}_{q-1})+\chi(\operatorname{Coker} d_q/\mathscr{I}^n \operatorname{Coker} d_q)+\chi(\operatorname{Coker} d_{q+1}/\mathscr{I}^n \operatorname{Coker} d_{q+1})$, where $d.$ are differential maps of $\mathscr{L}.$, and the conclusion for local $\mathscr{T}\!or$'s follows. Now we have a canonical isomorphism

$$\mathscr{H}\!om_{\mathcal{O}}(\mathscr{L}., \mathcal{O}/\mathscr{I}^n)\simeq\mathscr{L}^\vee_.\otimes_{\mathcal{O}}(\mathcal{O}/\mathscr{I}^n)\,,$$

where $\mathscr{L}^\vee_.$ is the dual of $\mathscr{L}.$. Thus $\mathscr{E}\!xt_{\mathcal{O}}^q(\mathscr{F}, \mathcal{O}/\mathscr{I}^n)$ is isomorphic to $\mathscr{H}_q((\mathscr{L}^\vee_./\mathscr{I}^n\mathscr{L}^\vee_.)$ and we make use again of the previous argument to finish the proof.

We do not know whether a similar statement holds when X is singular, or in the complex analytic case.

II. Comments on intersection numbers

II.1. We can generalize the theorem, using the same argument as above, taking into account finitely many invertible sheaves and ideal sheaves (cf. [11], [12] for the algebraic case). This gives the possibility to define asymptotically some numerical invariants, as in [11], [12]. We restrict ourselves to give some comments. Consider first only invertible sheaves. We have

Statement. *Let X be a compact complex space, $\mathscr{L}_1,\dots,\mathscr{L}_r$ invertible sheaves* *on X and $\mathscr{F}$ an coherent sheaf on X. Then the Euler-Poincaré characteristic $\chi(\mathscr{F}\otimes\mathscr{L}_1^{n_1}\otimes\cdots\otimes\mathscr{L}_r^{n_r})$ is a polynomial in $n_1,\ldots, n_r$ of total degree $\leq \dim \mathscr{F}$.*

When $r\geq \dim \mathscr{F}$, the coefficient of the monomial $n_1\cdots n_r$ in this polynomial is an integer (We can show this for a manifold X and a locally free sheaf $\mathscr{F}$, using the Riemann-Roch-Hirzebruch theorem. Then we reduce the general case to this case, or apply directly a well-known lemma on numerical polynomials) and we denote it by $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})$ and we call it the intersection number

In particular, when $\mathscr{L}_1\simeq\mathscr{L}_2\simeq\cdots\simeq\mathscr{L}_r\simeq\mathscr{L}$ we write $(\mathscr{L}^{\cdot r}\cdot\mathscr{F})$ and in fact $(\mathscr{L}^{\cdot r}\cdot\mathscr{F})$ is nothing but $r!$ multiplied by the coefficient of n^r in the polynomial $\chi(\mathscr{F}\otimes\mathscr{L}^n)$. When W is an analytic subspace of X, of dimension $\leq r$, we denote $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathcal{O}_W)$ by $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot W)$ and $(\mathscr{L}^{\cdot r}\cdot\mathcal{O}_W)$ by $(\mathscr{L}^{\cdot r}\cdot W)$.

We note a property of the intersection numbers which is shown in the algebraic case by dévissage [11]. This argument cannot be carry out in the analytic case. However, we can use an argument in the proof of the theorem.

For any non-singular point x, $\mathscr{F}_x\otimes_{\mathcal{O}_x}\mathscr{M}_x$ is of finite dimension over the field $\mathscr{M}_x$ of germs of meromorphic functions in x and the dimension is called the rank of $\mathscr{F}$ in x, $rk_x\mathscr{F}$. Then $rk_x\mathscr{F}$ is locally constant with respect to x. Assume that X is irreducible, then $rk_x\mathscr{F}$ is independent of the non-singular point x and we denote it $rk\mathscr{F}$. We have the formula

$$(*)\quad (\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})=(rk\mathscr{F})\cdot(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathcal{O}_X),\ r\geqslant\dim X.$$

Indeed, if $\operatorname{Supp}\mathscr{F}\neq X$, then $r>\dim\mathscr{F}$, $rk\mathscr{F}=0$ and both sides of $(*)$ are zero. Assume now $\operatorname{Supp}\mathscr{F}=X$. Let $X^*\xrightarrow{\pi}X$ be a modification such that X^* is a manifold and $\mathscr{F}^*=\pi^*(\mathscr{F})$ is, modulo torsion, locally free. The kernel and the cokernel of the map $\mathscr{F}\to\pi_*(\mathscr{F}^*)$, as well as the sheaves $R^q\pi_*(\mathscr{F}^*)$, $q\geq 1$, are zero outside a closed analytic subset of X of lower dimension. As we are looking for terms of upper degree in the polynomials, we obtain successively:

$$(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})=(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\pi_*\mathscr{F}^*)=(\mathscr{L}_1^*\cdots\mathscr{L}_r^*\cdot\mathscr{F}^*)=(\mathscr{L}_1^*\cdots\mathscr{L}_r^*\cdot\mathscr{F}^*/t(\mathscr{F}^*))$$

and also $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathcal{O}_X)=(\mathscr{L}_1^*\cdots\mathscr{L}_r^*\cdot\mathcal{O}_{X^*})$. On the other hand $rk\mathscr{F}=rk\mathscr{F}^*=rk(\mathscr{F}^*/t(\mathscr{F}^*))$ and we reduced the question to the case where X is a manifold and $\mathscr{F}$ is locally free. But in this case we can prove $(*)$, using the Riemann-Roch-Hiezebruch theorem.

Using the Riemann-Roch-Hirzebruch theorem, induction on dimension and the result of Rossi, we can prove in the analytic case (X a compact complex space, $\mathscr{L}_1,\ldots,\mathscr{L}_{r+1}$ invertible sheaves, $\mathscr{F}$ a coherent sheaf) that $\chi(\mathscr{F}\otimes(1-\mathscr{L}_1)\otimes\cdots\otimes(1-\mathscr{L}_{r+1}))=0$ when $\dim\mathscr{F}\leq r$, analogously to Cartier's algebraic result ([2]) (here χ is extended linearly). We can use this to extend the identities of Snapper and Cartier to the analytic case.

In particular, we obtain compatibility between the integers $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})$ (defined here asymptotically) and Kronecker's indices

$$[\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F}]=\sum_{t=0}^{r}(-1)^{r-t}\sum_{i_1<\cdots<i_t}\chi(\mathscr{F}\otimes\mathscr{L}_{i_1}\otimes\cdots\mathscr{L}_{i_t})$$

II.2. Concerning the uniqueness of an intersection number theory, we have a coarser result as in the algebraic case.

The numbers $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})$ satisfy the following properties.

① $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})=0$ *if* $\dim\mathscr{F}<r$, *and* $(\mathscr{F})=\dim H^0(X, \mathscr{F})$ *if* $r=\dim\mathscr{F}=0$.

② $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})$ *is a symmetric r-linear form in* $\mathscr{L}_1,\dots, \mathscr{L}_r$.

③ *If* $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$ *is exact, then*

$$(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})=(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F}')+(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F}'')\,.$$

④ *If S is the subspace of X given by* $\mathscr{A}nn\mathscr{F}$ *and* $S\subset T\subset X$ *are inclusions of subspaces, then* $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})=(\mathscr{L}_1\mid T\cdots\mathscr{L}_r\mid T\cdot\mathscr{F}\mid T)$ $(\mathscr{L}_i\mid T,\ \mathscr{F}\mid T$ *are the analytic restrictions*).

⑤ *If* $\pi\colon X^*\to X$ *is a modification of irreducible compact complex spaces and* $\mathscr{F}$ *is a torsion free coherent sheaf on X, then* $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})=(\mathscr{L}_1^*\cdots\mathscr{L}_r^*\cdot\mathscr{F}^*)$ $(\mathscr{L}_i^*=\pi^*(\mathscr{L}),\ \mathscr{F}^*=\pi^*(\mathscr{F}))$.

⑥ *If X is a manifold of dimension r and* $\mathscr{F}$ *is locally free, then* $(\mathscr{L}_1\cdots\mathscr{L}_r\cdot\mathscr{F})$ $=(rk\mathscr{F})\cdot(c_1(\mathscr{L}_1)\cdots c_1(\mathscr{L}_r))\cdot[X]$, *where* $c_1(\mathscr{L}_i)$ *is the first Chern class of* $\mathscr{L}_i$ *and* $[X]$ *is the fundamental class of X.*

Another intersection number theory which satisfies the above properties coincides with the asymptotical theory defined here.

II.3. We have an analogue of the numerical characterization of ampleness.

Statement. *Let X be a compact complex space and* $\mathscr{L}$ *an invertible sheaf on X. Then* $\mathscr{L}$ *is ample if and only if for any r-dimensional irreducible subspace W of X,* $\mathscr{L}\mid W$ *is associated to a Cartier divisor and* $(\mathscr{L}^{\cdot r}\cdot W)>0$.

The proof is the same as those of Kleiman in the algebraic case ([11]). We need the following criterion of ampleness of Grauert [6]:

"*Let X be a compact complex space (which may not be necessarily reduced) and* $\mathscr{L}$ *an invertible sheaf on X. If for any irreducible subset W of X of positive dimension, there exist an integer* $k\geq 1$ *and a non-zero section of* $\mathscr{L}^k\mid W$ *which does vanish at some point, then* $\mathscr{L}$ *is ample*".

The hypothesis that $\mathscr{L}\mid W$ is associated to a Cartier divisor is needed in the proof, to conclude that $\mathscr{L}\mid W$ is a subsheaf in the sheaf of meromorphic sections on W (see [17], Ch. II, § 4, for informations on Cartier divisors). If X is a Moišezon space, this hypothesis is verified for any invertible sheaf (Moišezon). In any case, G. Fischer ([3]) is able to prove the Nakai criterion on Moišezon manifolds.

II.4. Now a few words about the case when we deal with sheaves of ideals. We restrict ourselves to the case where we have an invertible sheaf and an ideal, that is to the situation of the theorem. For $r\geq\dim\mathscr{F}$, the coefficient in $P(n, n)$ $=\chi(\mathscr{F}\otimes\mathscr{L}^n/\mathscr{I}^n(\mathscr{F}\otimes\mathscr{L}^n))$ (for $n\gg 0$) of n^r divided by $r!$ is an integer and we denote it by $e_r(\mathscr{F})=e_r(\mathscr{F}\,;\mathscr{I},\mathscr{L})$: the r-multiplicity of $\mathscr{F}$ with respect to $\mathscr{I}$ and

$\mathscr{L}$. If X is compact, and if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, then $e_r(\mathscr{F}) = e_r(\mathscr{F}') + e_r(\mathscr{F}'')$ ($r \geq \dim \mathscr{F}$), as it follows from the theorem. It would be interesting to get rid of the hypothesis of compactness; in the algebraic case we are always able to reduced to the compact case [12].

We also mentioned the following fact (cf. [12] for the algebraic case and $\mathscr{F}$ = the structure sheaf): if X is compact and $\mathscr{L}$ and $\mathscr{I}\mathscr{L}$ are generated by global sections, then we have

$$\left|\dim H^0(X, (\mathscr{F} \otimes \mathscr{L}^n)/\mathscr{I}^n(\mathscr{F} \otimes \mathscr{L}^n)) - e_r(\mathscr{F}) \cdot \frac{n^r}{r!}\right| = O(n^{r-1}),\ r = \dim \mathscr{F}.$$

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