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On the cohomology mod p of the classifying spaces of the exceptional Lie groups, III

By

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Dedicated to Professor Tatsuji KUDO on his 60-th birthday

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§1′. Introduction

This paper is exactly a continuation of the former one, Part II, aiming to determine

$$\mathrm{Cotor}_A(Z_3, Z_3) \text{ with } A=H^*(X_8;\mathbf{Z}_3) \qquad \text{for } X_8\in\{E_8: 3\},$$

where $\mathrm{Cotor}_A(\mathbf{Z}_3, \mathbf{Z}_3)$ was shown to be isomorphic (as an algebra) to the cohomology $H(\overline{W}: d)=\mathrm{Ker}\, d/\mathrm{Im}\, d$ of the differential algebra $\overline{W}$ constructed in §2.

Our result is

Main Theorem. $\mathrm{Cotor}_A(\mathbf{Z}_3, \mathbf{Z}_3)$ *is commutative and is generated (as an algebra) by the following* 29 *elements*:

$$a_4,\ a_8,\ a_{20},\ x_{48},\ z_{52},\ z_{56},\ u_{56},\ x_{84},\ z_{88},\ w_{88},\ z_{92},\ w_{100},$$

$$z_{104},\ x_{108},\ x_{120},\ w_{124},\ w_{128},\ w_{136},\ w_{140},\ x_{144},\ w_{152},\ v_{168};$$

$$a_9,\ a_{21},\ y_{25},\ y_{26},\ y_{57},\ y_{61},\ y_{62},$$

where the index indicates the degree.

The paper is organized as follows. (The section numbers follow those of Part II to express that Part III is a continuation of Part II.)

In Section 7 some lemmas are proved for later use. In Section 8 we exhibit the form of cocycles. Then in Section 9 we determine cocycles containing a_9 and c_{17} in $\overline{W}$. In Section 10 we study necessary conditions for a cocycle to be trivial. In Section 11 we determine cocycles with neither a_9 nor c_{17} (but with a_{21} and c_{41}). We show in Section 12 that $\mathrm{Cotor}_A(\mathbf{Z}_3, \mathbf{Z}_3)$ is commutative and produce an additive basis. The last section is devoted to showing the relations used in Section 12.

§7. Some lemmas

We denote by $P(n, m)$ an element of degree n with respect to a_9 and c_{17} and of degree m with respect to a_{21} and c_{41}, where $n+m$ is called *the total degree*, *n the first degree* and *m the second degree*. Then an element Φ of degree N with respect to elements of odd degree is a sum of elements of total degree N:

$$\Phi=P(N, 0)+P(N-1, 1)+\cdots+P(0, N).$$

We define two operators d' and d'' as follows:

(7.1.1) $$d'A=a_9\partial_9 A+c_{17}\partial_9^2 A \quad \textit{for any } A \textit{ in } V,$$

$$d'a_9=0, \qquad d'c_{17}=a_9^2,$$

$$d'a_{21}=0, \qquad d'c_{41}=0;$$

(7.1.2) $$d''A=a_{21}\partial_{21} A+c_{41}\partial_{21}^2 A \quad \textit{for any } A \textit{ in } V,$$

$$d''a_9=0, \qquad d''c_{17}=0,$$

$$d''a_{21}=0, \qquad d''c_{41}=a_{21}^2;$$

and extend them as a derivation:

$$d'(xy)=d'x\cdot y+(-1)^{\deg x}x\cdot d'y,$$

$$d''(xy)=d''x\cdot y+(-1)^{\deg x}x\cdot d''y.$$

We see that $d'P(n, m)$ and $d''P(n, m)$ are of type $(n+1, m)$ and $(n, m+1)$ respectively.

Lemma 7.2. (1) $d=d'+d''$.

(2) $d'^2=0$, $d''^2=0$ *and* $d'd''=-d''d'$.

Proof. (1) Clearly $d=d'+d''$ holds for any element in V. Suppose that it holds for any element Φ of degree up to l with respect to elements of odd degree. Then,

$$d(a_9\Phi)=-a_9d\Phi=-a_9d'\Phi-a_9d''\Phi$$

$$=d'(a_9\Phi)+d''(a_9\Phi),$$

$$d(c_{17}\Phi)=a_9^2\Phi-c_{17}d\Phi=a_9^2\Phi-c_{17}d'\Phi-c_{17}d''\Phi$$

$$=d'(c_{17}\Phi)+d''(c_{17}\Phi).$$

Thus the relation $d=d'+d''$ holds for $a_9\Phi$ and $c_{17}\Phi$ and it holds similarly for $a_{21}\Phi$ and $c_{41}\Phi$. Therefore it holds for any element of degree $l+1$.

(2) Since d is a differential operator, we have

$$d^2=(d'+d'')^2=d'^2+d'd''+d''d'+d''^2=0.$$

For any element $P(n, m)$, $d'^2P(n, m)$ is of type $(n+2, m)$, $d''^2P(n, m)$ is of type $(n, m+2)$ and $d'd''P(n, m)$ and $d''d'P(n, m)$ are of type $(n+1, m+1)$. Hence, for reasons of type $d^2P(n, m)=0$ gives rise to

$$d'^2P(n, m)=0, \quad d''^2P(n, m)=0, \quad d'd''P(n, m)=-d''d'P(n, m).$$

Since the relations $d'^2=d''^2=0$ and $d'd''=-d''d'$ hold for any element $P(n, m)$ of a single type, they hold for any element. q.e.d.

We see that $d'P(n, m)$ and $d''P(n, m)$ are exactly the parts of types $(n+1, m)$ and $(n, m+1)$ respectively in $dP(n, m)$.

For technical reasons, we shall extend ∂_9 and ∂_{21} as follows:

(7.3.1) $$\partial_9 a_{21}=0, \quad \partial_9 c_{41}=0,$$
$$\partial_9(P+Q)=\partial_9 P+\partial_9 Q, \quad \partial_9(PQ)=\partial_9 P\cdot Q+P\partial_9 Q$$

for any P and Q having neither a_9 nor c_{17}, ($\partial_9 a_9$ and $\partial_9 c_{17}$ are not defined);

(7.3.2) $$\partial_{21} a_9=0, \quad \partial_{21} c_{17}=0,$$
$$\partial_{21}(P+Q)=\partial_{21} P+\partial_{21} Q, \quad \partial_{21}(PQ)=\partial_{21} P\cdot Q+P\partial_{21} Q$$

for any P and Q having neither a_{21} nor c_{41}, ($\partial_{21} a_{21}$ and $\partial_{21} c_{41}$ are not defined).

We have

Lemma 7.4. (1) $d'P(0, m)=(a_9+c_{17}\partial_9)\partial_9 P(0, m)$,

$d''P(n, 0)=(a_{21}+c_{41}\partial_{21})\partial_{21} P(n, 0)$.

(2) $\partial_9 d''P(0, m)=d''\partial_9 P(0, m)$,

$\partial_{21} d'P(n, 0)=d'\partial_{21} P(n, 0)$.

Proof. (1) Each term of $P(0, m)$ is of the form $x_1x_2\cdots x_mA$, where x_i is either a_{21} or c_{41} and $A\in V$. Recall that a_9 and c_{17} commute with x_i. Thus,

$$\begin{aligned} d'(x_1x_2\cdots x_mA) &= (-1)^m x_1x_2\cdots x_m d'A \\ &= (-1)^m x_1x_2\cdots x_m(a_9\partial_9 A+c_{17}\partial_9^2 A) \\ &= a_9x_1x_2\cdots x_m\partial_9 A+c_{17}x_1x_2\cdots x_m\partial_9^2 A \\ &= a_9\partial_9(x_1x_2\cdots x_mA)+c_{17}\partial_9^2(x_1x_2\cdots x_mA) \\ &= (a_9+c_{17}\partial_9)\partial_9(x_1x_2\cdots x_mA). \end{aligned}$$

Thus the first formula $d'P(0, m)=(a_9+c_{17}\partial_9)\partial_9 P(0, m)$ holds.

The other formula is proved similarly.

(2) For $A\in V$, we have

$$\begin{aligned} \partial_9 d''A &= \partial_9(a_{21}\partial_{21}A+c_{41}\partial_{21}^2 A) \\ &= a_{21}\partial_9\partial_{21}A+c_{41}\partial_9\partial_{21}^2 A \\ &= a_{21}\partial_{21}\partial_9 A+c_{41}\partial_{21}^2\partial_9 A \\ &= d''\partial_9 A. \end{aligned}$$

Suppose that $\partial_9 d''P = d''\partial_9 P$ for any P of type $(0, m-1)$. Then,

$$\begin{aligned}\partial_9 d''(a_{21}P) &= \partial_9(-a_{21}d''P) = -a_{21}\partial_9 d''P = -a_{21}d''\partial_9 P \\ &= d''(a_{21}\partial_9 P) = d''\partial_9(a_{21}P), \\ \partial_9 d''(c_{41}P) &= \partial_9(a_{21}^2 P - c_{41}d''P) = a_{21}^2\partial_9 P - c_{41}\partial_9 d''P \\ &= a_{21}^2\partial_9 P - c_{41}d''\partial_9 P = d''(c_{41}\partial_9 P) \\ &= d''\partial_9(c_{41}P).\end{aligned}$$

Therefore, $\partial_9 d''P = d''\partial_9 P$ holds for any P of type $(0, m)$.

The other formula is shown similarly. q.e.d.

(In other words, ∂_9 and d'' commute whenever ∂_9 is defined and ∂_{21} and d' commute whenever ∂_{21} is defined.)

Put $y_{26} = [a_9, c_{17}]$ and $y_{62} = [a_{21}, c_{41}]$. Clearly, a_9, y_{26}, a_{21} and y_{62} are cocycles, and we have

$$d'y_{26} = d''y_{26} = 0 \quad \text{and} \quad d'y_{62} = d''y_{62} = 0.$$

Notation. Throughout the calculations, we shall put $n = 2k + \varepsilon$ ($\varepsilon = 0$ or 1) and $m = 2l + \delta$ ($\delta = 0$ or 1) for the letters n and m.

The letters A, B, C and D will be used for elements of V and the letters P, Q, R, S and T will be used for elements of a single type. Thus $Q(n, m)$ or $R(n, m)$ or others means some element of type (n, m), but in calculations the type is often abbreviated and elements are written simply as Q_i, R_j and so on.

In the following lemmas, we shall study an element $P(n, m)$ which satisfies certain conditions.

Comparing type, we see that an element $P(n, m)$ contains terms of the form

$$y_{26}^k(a_9 + c_{17}\partial_9)^\varepsilon y_{62}^l(a_{21} + c_{41}\partial_{21})^\delta A \quad \text{with} \quad A \in V.$$

We shall see that terms of the above form play essential roles in all the calculations.

Lemma 7.5. *An element of type (n, m) can be written as*

$$P(n, m) = y_{26}^k(a_9 + c_{17}\partial_9)^\varepsilon P(0, m) + \sum_{i=0}^{k+\varepsilon-1} y_{26}^i c_{17}Q_i + d'R(n-1, m),$$

where Q_i is an element of type $(n-2i-1, m)$.

Proof. Since a_9 and c_{17} commute with a_{21} and c_{41}, we put a_9's and c_{17}'s before a_{21}'s and c_{41}'s in each term of $P(n, m)$. Then using the substitutions

$$\begin{aligned} y_{26}^i a_9 c_{17} Q &= y_{26}^{i+1} Q - y_{26}^i c_{17}(a_9 Q), \\ y_{26}^i a_9^2 Q &= d'(y_{26}^i c_{17} Q) + y_{26}^i c_{17}(d'Q), \end{aligned}$$

we can rewrite $P(n, m)$ as follows:

$$y_{26}^k(a_9^\varepsilon P(0, m)+\varepsilon c_{17}^\varepsilon P'(0, m))+\sum_{i=0}^{k-1} y_{26}^i c_{17} Q_i+(d'\text{-image})$$

$$=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon P(0, m)+\varepsilon y_{26}^k c_{17}^\varepsilon(P'(0, m)-\partial_9 P(0, m))$$

$$+\sum_{i=0}^{k-1} y_{26}^i c_{17} Q_i+(d'\text{-image})$$

$$=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon P(0, m)+\sum_{i=0}^{k+\varepsilon-1} y_{26}^i c_{17} Q_i+(d'\text{-image}),$$

where the type of Q_i is obvious and the last (d'-image) is $d'R(n-1, m)$ for some $R(n-1, m)$. q. e. d.

Lemma 7.6. *If $d'P(n, m)$ is of the form*

$$d'P(n, m)=y_{26}^{k+\varepsilon}(a_9+c_{17}\partial_9)^{1-\varepsilon}Q(0, m) \quad \text{for some} \quad Q(0, m),$$

then $P(n, m)$ is necessarily of the form

$$P(n, m)=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon P(0, m)+(d'\text{-image}).$$

Proof. Since an element $P(n, m)$ can be written by Lemma 7.5 as

$$P(n, m)=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon P(0, m)+\sum_{i=0}^{k+\varepsilon-1} y_{26}^i c_{17} Q_i+(d'\text{-image}),$$

we have

$$d'P(n, m)=(-1)^\varepsilon y_{26}^{k+\varepsilon}(a_9+c_{17}\partial_9)^{1-\varepsilon}\partial_9^{\varepsilon+1}P(0, m)$$

$$+\sum_{i=0}^{k+\varepsilon-1} y_{26}^i(a_9^2 Q_i-c_{17}d'Q_i).$$

Thus in order that $d'P(n, m)=y_{26}^{k+\varepsilon}(a_9+c_{17}\partial_9)^{1-\varepsilon}Q(0, m)$ for some $Q(0, m)$, it is necessary that each Q_i be 0 and hence

$$P(n, m)=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon P(0, m)+(d'\text{-image}).$$

q. e. d.

Similarly to Lemma 7.5, we have

Lemma 7.7. *An element of type* $(0, m)$ *can be written as*

$$P(0, m)=y_{62}^l(a_{21}+c_{41}\partial_{21})^\delta A+\sum_{j=0}^{l+\delta-1} y_{62}^j c_{41} R_j+d''S(0, m-1),$$

where R_j is an element of type $(0, m-2j-1)$.

Lemma 7.8. *Suppose that P of type* $(2k+\varepsilon, 2l+\delta)$ *is of the form*

$$P=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon P(0, 2l+\delta)+(d'\text{-image}).$$

Then in order that $d''P$ be of the form

$$d''P=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon y_{62}^{l+\delta}(a_{21}+c_{41}\partial_{21})^{1-\delta}B+(d'\text{-image})$$

for some $B\in V$, it is necessary that P be of the form

$$P=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}\{y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}A+d''S(0,\ m-1)\}+(d'\text{-image}).$$

Proof. The part $P(0,\ 2l+\delta)$ can be written by Lemma 7.7 as

$$P(0,\ 2l+\delta)=y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}A+\sum_{j=0}^{l+\delta-1}y_{62}^{j}c_{41}R_j+d''S(0,\ 2l+\delta-1),$$

where R_j is of type $(0,\ 2l-2j+\delta-1)$.

Thus we have

$$d''P=(-1)^{\varepsilon+\delta}y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l+\delta}(a_{21}+c_{41}\partial_{21})^{1-\delta}\partial_{21}^{\delta+1}A$$
$$+(-1)^{\varepsilon}\sum_{j=0}^{l+\delta-1}y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{j}(a_{21}^{2}R_j-c_{41}d''R_j)+d''(d'\text{-image}),$$

which must be of the form

$$y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l+\delta}(a_{21}+c_{41}\partial_{21})^{1-\delta}B+(d'\text{-image})$$

for some $B\in V$. Then the part $\sum_{j=0}^{l+\delta-1}y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{j}(a_{21}^{2}R_j-c_{41}d''R_j)$ must be in the d'-image, for which it is necessary that each $y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}R_j$ be in the d'-image, since there is neither a_9 nor c_{17} in $y_{62}^{j}a_{21}^{2}R_j$. It follows that

$$\sum_{j=0}^{l+\delta-1}y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{j}c_{41}R_j\in d'\text{-image}$$

and P is of the required form:

$$P=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}\{y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}A+d''S(0,\ m-1)\}+(d'\text{-image}).$$

q. e. d.

Lemma 7.9. *Let $U=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}c_{41}T$ with T of type $(0,\ t)$ for some t. If $d'd''U=0$, then $d''U=dU$.*

Proof. The conclusion $d''U=dU$ holds provided $d'U=0$. And

$$d'U=(-1)^{\varepsilon}y_{26}^{k+\varepsilon}(a_9+c_{17}\partial_9)^{1-\varepsilon}c_{41}\partial_9^{\varepsilon+1}T=0$$

holds provided $\partial_9^{\varepsilon+1}T=0$.

Now, recalling from (2) of Lemma 7.4 that d'' and ∂_9 commute (whenever ∂_9 is defined), we have

$$\begin{aligned}d'd''U&=-d''d'U\\&=-(-1)^{\varepsilon}(-1)^{1-\varepsilon}y_{26}^{k+\varepsilon}(a_9+c_{17}\partial_9)^{1-\varepsilon}d''(c_{41}\partial_9^{\varepsilon+1}T)\\&=y_{26}^{k+\varepsilon}(a_9+c_{17}\partial_9)^{1-\varepsilon}(a_{21}^{2}\partial_9^{\varepsilon+1}T-c_{41}\partial_9^{\varepsilon+1}d''T)\end{aligned}$$

and the relation $d'd''U=0$ gives rise to $\partial_9^{\varepsilon+1}T=0$.

Thus, if $d'd''U=0$, then $d'U=0$ and $d''U=dU$. q. e. d.

Proposition 7.10. *If an element P of type $(2k+\varepsilon, 2l+\delta)$ satisfies the conditions:*

$$d'P=0,$$
$$d''P=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l+\delta}(a_{21}+c_{41}\partial_{21})^{1-\delta}D+(d'\text{-image})$$

for some $D\in V$, then P is of the form

$$P=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}A+(d'\text{-image})+(d\text{-image})\quad \textit{with}\quad A\in V.$$

Proof. By Lemma 7.6 the condition $d'P=0$ gives rise to

$$P=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}P(0,\ 2l+\delta)+(d'\text{-image}),$$

for which the condition for $d''P$ gives rise to, by Lemma 7.8,

$$P=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}\{y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}B+d''S(0,\ 2l+\delta-1)\}+(d'\text{-image}).$$

Now we write $d''S(0, 2l+\delta-1)$ explicitly to study again the condition $d'P=0$. By Lemma 7.7, $S(0, 2l+\delta-1)$ can be written as

$$S(0,\ 2l+\delta-1)=y_{62}^{l+\delta-1}(a_{21}+c_{41}\partial_{21})^{1-\delta}C+\sum_{j=0}^{l-1}y_{62}^{j}c_{41}T_j+(d''\text{-image}),$$

where $C\in V$ and T_j is of type $(0, 2l-2j+\delta-2)$. Thus $d''S(0, 2l+\delta-1)$ is expressed as

$$d''S(0,\ 2l+\delta-1)=(-1)^{1-\delta}y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}\partial_{21}^{2-2\delta}C+\sum_{j=0}^{l-1}y_{62}^{j}d''(c_{41}T_j).$$

And P can be written explicitly as

$$\begin{aligned}P=&\,y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}\{B+(-1)^{1-\delta}\partial_{21}^{2-2\delta}C\}\\&+\sum_{j=0}^{l-1}y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{j}d''(c_{41}T_j)+(d'\text{-image}),\end{aligned}$$

or, putting $A=B+(-1)^{1-\delta}\partial_{21}^{2-2\delta}C$ and $U_j=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}c_{41}T_j$, we have

$$P=y_{62}^{l}y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}(a_{21}+c_{41}\partial_{21})^{\delta}A+(-1)^{\varepsilon}\sum_{j=0}^{l-1}y_{62}^{j}d''U_j+(d'\text{-image}).$$

And the relation

$$\begin{aligned}d'P=&\,y_{62}^{l}d'(y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}(a_{21}+c_{41}\partial_{21})^{\delta}A)+(-1)^{\varepsilon}\sum_{j=0}^{l-1}y_{62}^{j}d'd''U_j\\=&\,0\end{aligned}$$

gives rise to $d'd''U_j=0$ for each j, which yields $d''U_j=dU_j$ for each j by Lemma 7.9 and hence $\sum_{j=0}^{l-1}y_{62}^{j}d''U_j\in d$-image.

We have shown that any P satisfying the conditions:

$$d'P=0,$$

$$d''P=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon y_{62}^{l+\delta}(a_{21}+c_{41}\partial_{21})^{1-\delta}D+(d'\text{-image})$$

is necessarily of the form

$$P=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon y_{62}^l(a_{21}+c_{41}\partial_{21})^\delta A+(d'\text{-image})+(d\text{-image}).$$

q.e.d.

§8. The form of cocycles

When Φ is an element of the form

$$\Phi=P(n, m)+P(n-1, m+1)+P(n-2, m+2)+\cdots,$$

we call $P(n, m)$ *the top term*. In the following calculations any P with the first degree negative is understood to be 0.

Now the condition $d\Phi=0$ is equivalent to

(8.1) $d'P(n, m)=0,$

$d''P(n-i, m+i)+d'P(n-i-1, m+i+1)=0$ *for* $i=0, 1,\dots, n,$

which we express as a diagram

$$\begin{array}{ccccccccc} & P(n, m) & & & P(n-1, m+1) & & & P(n-2, m+2) & \cdots \\ \swarrow^{d'} & & \searrow^{d''} & \swarrow^{d'} & & \searrow^{d''} & \swarrow^{d'} & & \searrow^{d''} \\ 0 & & & \text{sum } 0 & & & \text{sum } 0 & & \cdots \end{array}$$

From now on, in any diagram, an arrow $\swarrow$ will always mean d' and $\searrow$ will mean d''.

Note that $d\Phi=0$ does *not* necessarily imply that $d'\Phi=d''\Phi=0$.

If the top term of a cocycle has a d'-image part, i.e., if $P(n, m)=\hat{P}(n, m)+d'Q$, then

$$P(n, m)+P(n-1, m+1)+P(n-2, m+2)+\cdots$$
$$=\hat{P}(n, m)+dQ+(P(n-1, m+1)-d''Q)+P(n-2, m+2)+\cdots$$
$$=\hat{P}(n, m)+\hat{P}(n-1, m+1)+P(n-2, m+2)+\cdots+dQ.$$

Thus we obtain an equivalent cocycle when we omit the d'-image part from the top term $P(n, m)$.

We shall determine cocycles such that the top term has *no* d'-image part.

Now by (8.1) the top term $P(n, m)$ of a cocycle must satisfy the conditions

(8.2) $d'P(n, m)=0$ and $d''P(n, m)\in d'$-image.

Then by Propositon 7.10, $P(n, m)$ is necessarily of the form

(8.3) $$P(n, m)=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon y_{62}^l(a_{21}+c_{41}\partial_{21})^\delta A+(d'\text{-image})+(d\text{-image}).$$

Determination of cocycles is divided into the following four cases:

Case 1. $\varepsilon=1$;
Case 2. $\varepsilon=0$, $k\neq 0$;
Case 3. $\varepsilon=k=0$, $\delta=1$;
Case 4. $\varepsilon=k=0$, $\delta=0$, $l\neq 0$.

We first determine cocycles (with a_9 and c_{17}) in Cases 1 and 2. We show that (8.2) for $n>0$ is (not only necessary but) sufficient for the top term of a cocycle. Then we find some necessary conditions for a trivial cocycle and show that the cocycles found in Cases 1 and 2 are non-trivial and linearly independent, and that they are also linearly independent of non-trivial cocycles found in Cases 3 and 4.

§9. Cocycles with elements of odd degree-I

(iii) Cocycles with a_9, c_{17}, a_{21}, c_{41}

In this section, the first degree n of $P(n, m)$ is not 0.

In order to omit further d'-image parts from $P(n, m)$, we need the following

Lemma 9.1. *Let $P(n, m)$ be of the form*

$$P(n, m)=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon y_{62}^l(a_{21}+c_{41}\partial_{21})^\delta B$$

with $B\in V$. Then it is in the d'-image if and only if $B\in\partial_9^{2-\varepsilon}$-image.

Proof. If $P(n, m)$ is in the d'-image, the term $y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon B$ must be in the d'-image. Let Q be such that $d'Q=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon B$. Then Q is of type $(2k+\varepsilon-1, 0)$ and, by Lemma 7.6, is necessarily of the form

$$Q=y_{26}^{k+\varepsilon-1}(a_9+c_{17}\partial_9)^{1-\varepsilon}C+(d'\text{-image})\qquad\text{for some}\quad C\in V.$$

Since $d'Q$ is expressed as

$$d'Q=(-1)^{1-\varepsilon}y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon\partial_9^{2-\varepsilon}C,$$

the element B must be in the $\partial_9^{2-\varepsilon}$-image.

Conversely, if $B=\partial_9^{2-\varepsilon}D$ for some D, then we see that

$$P(n, m)=d'\{(-1)^{1-\varepsilon}y_{26}^{k+\varepsilon-1}(a_9+c_{17}\partial_9)^{1-\varepsilon}y_{62}^l(a_{21}+c_{41}\partial_{21})^\delta D\}.$$

Thus $P(n, m)$ is in the d'-image if and only if $B\in\partial_9^{2-\varepsilon}$-image. q.e.d.

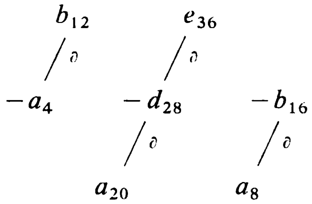
We can separate further (but not all the) d'-image part from $P(n, m)$ in (8.3), by omitting the $\partial_9^{2-\varepsilon}$-image part from A. Thus we shall study $P(n, m)$ of the following form, which will be denoted simply by P in this section:

(9.2) $$P=y_{26}^k(a_9+c_{17}\partial_9)^\varepsilon y_{62}^l(a_{21}+c_{41}\partial_{21})^\delta A$$

with A having no $\partial_9^{2-\varepsilon}$-image part.

We shall determine cocycles in Cases 1 and 2 by using some of the results in Part I.

Excluding a_4, we note that the ∂_9-structure of $\mathbf{Z}_3[a_8, a_{20}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}]$ is the same as the ∂-structure of $\mathbf{Z}_3[a_4, a_8, a_{20}, b_{12}, b_{16}, d_{28}, e_{36}]$ in the case of $X_7 \in \{E_7: 3\}$, which is shown in the diagram below:



The ∂-kernel has been completely determined for X_7. We have only to interchange a_4 and b_{12} in X_7 with b_{40} and e_{48} in X_8 respectively, and we obtain the ∂_9-kernel of X_8. (Recall that a_4 of X_8 is excluded.)

Interpretation of (3.12.1)$\sim$(3.12.3) of Part I yields:

(9.3.1) *An element of the ∂_9-kernel is in the ∂_9^2-image if and only if it has no term of the form*

$$a_8 x_{48}^i x_{108}^s x_{144}^t, \quad b_{40} x_{48}^i x_{108}^s x_{144}^t, \quad x_{48}^i x_{108}^s x_{144}^t$$

$$\textit{or} \quad (a_8 e_{48} - b_{16} b_{40}) x_{48}^i x_{108}^s x_{144}^t;$$

(9.3.2) *An element of the ∂_9-kernel is in the ∂_9-image but not in the ∂_9^2-image if it is a sum of*

$$a_8 x_{48}^i x_{108}^s x_{144}^t,\ b_{40} x_{48}^i x_{108}^s x_{144}^t \quad \textit{and} \quad \partial_9^2\text{-image};$$

(9.3.3) *An element of the ∂_9-kernel is not in the ∂_9-image if it is a sum of*

$$x_{48}^i x_{108}^s x_{144}^t,\ (a_8 e_{48} - b_{16} b_{40}) x_{48}^i x_{108}^s x_{144}^t \quad \textit{and} \quad \partial_9\text{-image}.$$

Henceforth we exclude a_4, x_{108} and x_{144}, since they are immobile with respect to the ∂_9-∂_{21}-structure and hence in d'-d''-diagrams.

Case 1. $\varepsilon = 1$: $P = y_{26}^k (a_9 + c_{17}\partial_9) y_{62}^l (a_{21} + c_{41}\partial_{21})^\delta A$ with A having no ∂_9-image part.

Now, the condition $d'P = 0$ gives rise to $\partial_9 A = 0$ or $\partial_9^2 A = 0$.

(1) If $\partial_9 A = 0$, the ∂_9-kernel A with no ∂_9-image part is, by (9.3.3), of the form

$$A = \sum_i \alpha_i x_{48}^i + \sum_i \beta_i (a_8 e_{48} - b_{16} b_{40}) x_{48}^i \quad \text{with} \quad \alpha_i, \beta_i \in \mathbf{Z}_3.$$

For $A = x_{48}^i$, $P = y_{26}^k a_9 y_{62}^l a_{21}^\delta x_{48}^i$ is a cocycle by itself.

For $A = a_8 e_{48} - b_{16} b_{40}$, recall that

$$z_{56} = a_8 e_{48} - b_{16} b_{40} + a_{20} e_{36} + d_{28}^2 = A + \partial_9^2(-e_{36}^2).$$

The element $y_{26}^k a_9 y_{62}^l a_{21}^\delta z_{56}$ is a cocycle and it can be rewritten as

$$y_{26}^k a_9 y_{62}^l a_{21}^\delta A + d'(y_{26}^k y_{62}^l a_{21}^\delta \partial_9(-e_{36}^2))$$
$$= y_{26}^k a_9 y_{62}^l a_{21}^\delta A - d''(y_{26}^k y_{62}^l a_{21}^\delta \partial_9(-e_{36}^2)) + d(y_{26}^k y_{62}^l a_{21}^\delta \partial_9(-e_{36}^2)).$$

Put $\phi_{1+\delta} = a_9 a_{21}^\delta (a_8 e_{48} - b_{16} b_{40}) - d''(a_{21}^\delta \partial_9(-e_{36}^2))$. Then $y_{26}^k y_{62}^l \phi_{1+\delta}$ is a cocycle for $A = a_8 e_{48} - b_{16} b_{40}$ and, since it is equivalent to $y_{26}^k a_9 y_{62}^l a_{21}^\delta z_{56}$, we obtain actually no new cocycle.

Finally, for $A = \sum_i \alpha_i x_{48}^i + \sum_i \beta_i (a_8 e_{48} - b_{16} b_{40}) x_{48}^i$, we have a cocycle

$$\Phi = y_{26}^k y_{62}^l \Big(\sum_i \alpha_i a_9 a_{21}^\delta x_{48}^i + \sum_i \beta_i \phi_{1+\delta} x_{48}^i\Big) \qquad \text{for} \quad \delta = 0, 1.$$

(2) Let $\partial_9 A \neq 0$ and $\partial_9^2 A = 0$. As we consider A with no ∂_9-image part, the ∂_9-kernel of the form $\partial_9 A$ has no ∂_9^2-image part. By (9.3.2), $\partial_9 A$ is of the form

$$\partial_9 A = \sum_i \gamma_i a_8 x_{48}^i + \sum_i \delta_i b_{40} x_{48}^i \quad \text{with} \quad \gamma_i, \delta_i \in \mathbf{Z}_3,$$

for which we may choose $A = -\sum_i \gamma_i b_{16} x_{48}^i - \sum_i \delta_i e_{48} x_{84}^i$. In fact, for A and A' with $\partial_9 A = \partial_9 A'$, we have P and P' the difference of which is $y_{26}^k a_9 y_{62}^l (a_{21} + c_{41}\partial_{21})^\delta (A - A')$ and $\partial_9(A - A') = 0$. This is the case studied in (1).

For $A = b_{16}$ (and $\partial_9 A = -a_8$), we have a cocycle

$$\Phi = P = y_{26}^k y_{62}^l (a_9 b_{16} - c_{17} a_8) a_{21}^\delta.$$

We put

$$y_{25} = a_9 b_{16} - c_{17} a_8$$

and $\Phi = y_{26}^k y_{62}^l y_{25} a_{21}^\delta$ is a cocycle for $A = b_{16}$.

For $A = e_{48}$ (and $\partial_9 A = -b_{40}$), we study the cases $\delta = 0$ and $\delta = 1$ separately.

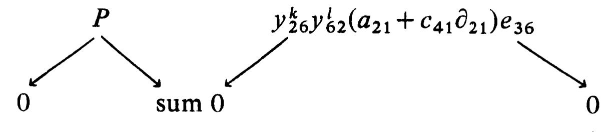
First, let $\delta = 0$: $P = y_{26}^k y_{62}^l (a_9 + c_{17}\partial_9) e_{48}$. Then

$$d''P = y_{26}^k y_{62}^l (a_9 + c_{17}\partial_9)(a_{21} + c_{41}\partial_{21}) d_{28}$$
$$= -d'(y_{26}^k y_{62}^l (a_{21} + c_{41}\partial_{21}) e_{36})$$

with

$$d''(y_{26}^k y_{62}^l (a_{21} + c_{41}\partial_{21}) e_{36}) = 0,$$

which is expressed as



Therefore, we have a cocycle

$$\Phi = P + y_{26}^k y_{62}^l (a_{21} + c_{41}\partial_{21}) e_{36}$$
$$= y_{26}^k y_{62}^l (a_9 e_{48} - c_{17} b_{40} + a_{21} e_{36} - c_{41} b_{16}).$$

We put

$$y_{57}=a_9e_{48}-c_{17}b_{40}+a_{21}e_{36}-c_{41}b_{16}$$

and $\Phi=y_{26}^ky_{62}^ly_{57}$ is a cocycle for $A=e_{48}$ and $\delta=0$.

Now let $\delta=1$. Then,

$$P=y_{26}^ky_{62}^l(a_9+c_{17}\partial_9)(a_{21}+c_{41}\partial_{21})e_{48}$$
$$=y_{26}^ky_{62}^l(a_9+c_{17}\partial_9)a_{21}e_{48}-d'(y_{26}^ky_{62}^lc_{41}e_{36}).$$

Thus omitting the d'-image part from P, we rechoose

$$P=y_{26}^ky_{62}^l(a_9+c_{17}\partial_9)a_{21}e_{48}=-y_{26}^ky_{62}^la_{21}(a_9+c_{17}\partial_9)e_{48}.$$

Then, as in the case of $\delta=0$, we obtain a cocycle

$$\Phi=-y_{26}^ky_{62}^la_{21}y_{57} \quad \text{for} \quad A=e_{48} \quad \text{and} \quad \delta=1.$$

Finally, for $A=\sum_i \gamma_ib_{16}x_{48}^i+\sum_i \delta_ie_{48}x_{48}^i$, we have a cocycle

$$\Phi=y_{26}^ky_{62}^l\{\sum_i \gamma_iy_{25}a_{21}^\delta x_{48}^i+\sum_i(-1)^\delta\delta_ia_{21}^\delta y_{57}x_{48}^i\} \qquad \text{for} \quad \delta=0,\ 1.$$

Summing up Case 1, we have studied P of the form (9.2) for $\varepsilon=1$ by studying A with no ∂_9-image part such that $\partial_9^2A=0$ (including the case $\partial_9A=0$). We have seen that A satisfying these conditions is of the form

$$\sum_i \alpha_ix_{48}^i+\sum_i \beta_i(a_8e_{48}-b_{16}b_{40})x_{48}^i+\sum_i \gamma_ib_{16}x_{48}^i+\sum_i \delta_ie_{48}x_{48}^i$$

and that for any such A there is a cocycle with top term P having no d'-image part:

(9.4) $$\Phi_{2k+1,2l+\delta}=y_{26}^ky_{62}^l\{\sum_i \alpha_ia_9a_{21}^\delta x_{48}^i+\sum_i \beta_i\phi_{1+\delta}x_{48}^i$$
$$+\sum_i \gamma_iy_{25}a_{21}^\delta x_{48}^i+\sum_i(-1)^\delta\delta_ia_{21}^\delta y_{57}x_{48}^i\}.$$

Now if an element $P(2k+1, m)$ satisfies (8.2), then it is of the form $P+(d'\text{-image})+(d\text{-image})$.

Thus,

(9.5) *If $P(2k+1, m)$ satisfies* (8.2), *there is a cocycle with top term $P(2k+1, m)$.*

Case 2. $\varepsilon=0$, $k\neq0$: $P=y_{26}^ky_{62}^l(a_{21}+c_{41}\partial_{21})^\delta A$ with A having no ∂_9^2-image part.

The condition $d'P=0$ gives rise to $\partial_9A=0$.

By (9.3.2) and (9.3.3) we have

$$A=\sum_i \alpha_ix_{48}^i+\sum_i \beta_i(a_8e_{48}-b_{16}b_{40})x_{48}^i+\sum_i \gamma_ia_8x_{48}^i+\sum_i \delta_ib_{40}x_{48}^i,$$

where α_i, β_i, γ_i, $\delta_i\in\mathbf{Z}_3$.

For $A=\sum_i \alpha_ix_{48}^i+\sum_i \gamma_ia_8x_{48}^i$, P is a cocycle by itself.

For $A=a_8e_{48}-b_{16}b_{40}$, we consider a cocycle with z_{56}:

$$\begin{aligned} y_{26}^k y_{62}^l a_{21}^\delta z_{56} &= y_{26}^k y_{62}^l a_{21}^\delta A + d'(y_{26}^{k-1}(a_9+c_{17}\partial_9) y_{62}^l a_{21}^\delta e_{36}^2) \\ &= y_{26}^k y_{62}^l a_{21}^\delta A - d''(y_{26}^{k-1}(a_9+c_{17}\partial_9) y_{62}^l a_{21}^\delta e_{36}^2) \\ &\qquad + d(y_{26}^{k-1}(a_9+c_{17}\partial_9) y_{62}^l a_{21}^\delta e_{36}^2). \end{aligned}$$

Thus putting

$$\phi_{3+\delta}=y_{26}a_{21}^\delta(a_8e_{48}-b_{16}b_{40})-d''((a_9+c_{17}\partial_9)a_{21}^\delta e_{36}^2),$$

we have a cocycle $y_{26}^{k-1}y_{62}^l\phi_{3+\delta}$ for $A=a_8e_{48}-b_{16}b_{40}$. Since this is equivalent to $y_{26}^k y_{62}^l a_{21}^\delta z_{56}$, we obtain actually no new cocycle.

For $A=b_{40}$, we study the cases $\delta=0$ and $\delta=1$ separately.

First, let $\delta=0$, then $P=y_{26}^k y_{62}^l b_{40}$. We have

$$\begin{aligned} d''P &= -y_{26}^k y_{62}^l a_{21}a_{20} = -y_{26}^k y_{62}^l (a_{21}+c_{41}\partial_{21})a_{20} \\ &= d'(y_{26}^{k-1}y_{62}^l(a_9+c_{17}\partial_9)(a_{21}+c_{41}\partial_{21})e_{36}) \end{aligned}$$

with

$$d''(y_{26}^{k-1}y_{62}^l(a_9+c_{17}\partial_9)(a_{21}+c_{41}\partial_{21})e_{36})=0,$$

which is expressed as

P $-y_{26}^{k-1}y_{62}^l(a_9+c_{17}\partial_9)(a_{21}+c_{41}\partial_{21})e_{36}$

↙ ↘ ↙ ↘

0   sum 0   0

Putting

$$\phi_5=y_{26}b_{40}-(a_9+c_{17}\partial_9)(a_{21}+c_{41}\partial_{21})e_{36},$$

the cocycle $\Phi=y_{26}^{k-1}y_{62}^l\phi_5$ is a cocycle for $A=b_{40}$ and $\delta=0$.

However, the cocycle ϕ_5 can be rewritten as

$$\begin{aligned} \phi_5 &= a_9c_{17}b_{40}+c_{17}a_9b_{40}-a_9a_{21}e_{36}+a_9c_{41}b_{16}+c_{17}a_{21}d_{28}-c_{17}c_{41}a_8 \\ &= a_9(-a_9e_{48}+c_{17}b_{40}-a_{21}e_{36}+c_{41}b_{16})+(a_9^2e_{48}+c_{17}a_9b_{40}+c_{17}a_{21}d_{28}-c_{17}c_{41}a_8) \\ &= -a_9y_{57}+d(c_{17}e_{48}). \end{aligned}$$

Therefore, the cocycle Φ is equivalent to $-y_{26}^{k-1}y_{62}^l a_9y_{57}$ and hence we obtain no new cocycle.

Now, let $\delta=1$, then $P=y_{26}^k y_{62}^l(a_{21}b_{40}-c_{41}a_{20})$, which is a cocycle. We put

$$y_{61}=a_{21}b_{40}-c_{41}a_{20}.$$

Thus, for

$$A=\sum_i \alpha_i x_{48}^i+\sum_i \beta_i(a_8e_{48}-b_{16}b_{40})x_{48}^i+\sum_i \gamma_i a_8x_{48}^i+\sum_i \delta_i b_{40}x_{48}^i$$

with α_i, β_i, γ_i, δ_i of $\mathbf{Z}_3$, we have cocycles

(9.6.1) $$\Phi_{2k,2l}=\sum_i \alpha_i y_{26}^k y_{62}^l x_{48}^i+\sum_i \beta_i y_{26}^{k-1} y_{62}^l \phi_3 x_{48}^i + \sum_i \gamma_i y_{26}^k y_{62}^l a_8 x_{48}^i+\sum_i \delta_i y_{26}^{k-1} y_{62}^l \phi_5 x_{48}^i$$

and

(9.6.2) $$\Phi_{2k,2l+1}=\sum_i \alpha_i y_{26}^k y_{62}^l a_{21} x_{48}^i+\sum_i \beta_i y_{26}^{k-1} y_{62}^l a_{21} \phi_4 x_{48}^i + \sum_i \gamma_i y_{26}^k y_{62}^l a_{21} a_8 x_{48}^i+\sum_i \delta_i y_{26}^k y_{62}^l y_{61} x_{48}^i$$

for $\delta=0$ and $\delta=1$, respectively.

Summing up Case 2, we have studied P of the form (9.2) for $\varepsilon=0$ but $k\neq 0$ by studying ∂_9-kernel A with no ∂_9^2-image part. For any such A, we have obtained a cocycle of the form $\Phi_{2k,2l+\delta}$. That is, for P with no d'-image part, there exists a cocycle and since any element $P(2k, m)$ with $k\neq 0$ satisfying (8.2) is of the form

$$P+(d'\text{-image})+(d\text{-image}),$$

we have

(9.7) *If an element* $P(2k, m)$ $(k\neq 0)$ *satisfies* (8.2), *there exists a cocycle with top term* $P(2k, m)$.

§10. Necessary conditions for a cocycle to be trivial

For ease of calculation, we consider a cocycle Φ, for a while, with top term of type $(n, m+1)=(2k+\varepsilon, 2l+\delta+1)$:

$$\Phi=P(n, m+1)+P(n-1, m+2)+\cdots,$$

or, in short,

$$\Phi=P_0+P_{-1}+\cdots \qquad (P_{-i}=P(n-i, m+i+1)).$$

Suppose that $\Phi=d\Psi$ for some Ψ. Then Ψ is of degree $n+m$ and, in general, of the form

$$\Psi=Q(n+h, m-h)+Q(n+h-1, m-h+1)+\cdots+Q(n, m)+\cdots,$$

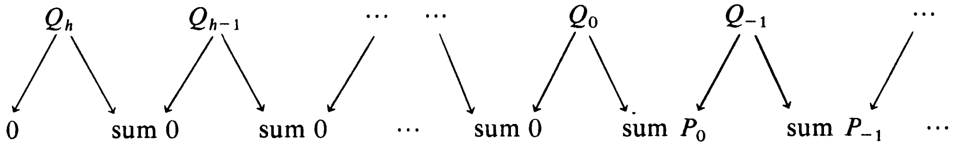
or, in short,

$$\Psi=Q_h+Q_{h-1}+\cdots+Q_0+\cdots \qquad (Q_j=Q(n+j, m-j)).$$

The relation $d\Psi=\Phi$ gives rise to

$$d'Q_h=0,\quad d''Q_h+d'Q_{h-1}=0,\ldots,\quad d''Q_1+d'Q_0=0,\quad d''Q_0+d'Q_{-1}=P_0,\ldots,$$

which are expressed as



If $h \geqq 1$, then the top term Q_h must satisfy

$$d'Q_h = 0 \quad \text{and} \quad d''Q_h \in d'\text{-image},$$

which is the condition (8.2) and, by (9.5) and (9.7) there exists a cocycle $\hat{\Psi} = Q_h + \hat{Q}_{h-1} + \cdots$ with top term Q_h. Then

$$\Psi - \hat{\Psi} = (Q_{h-1} - \hat{Q}_{h-1}) + \cdots,$$
$$d\Psi = d(\Psi - \hat{\Psi}),$$

since $d\hat{\Psi} = 0$. Therefore, if a cocycle Φ is $d\Psi$ for some Ψ, we can take Ψ with top term Q_0:

$$\Psi = Q_0 + Q_{-1} + \cdots$$

with the conditions $d'Q_0 = 0$ and $d''Q_0 + d'Q_{-1} = P_0$.

Now, suppose that $P_0 = P$ has no d'-image part. Then $Q_0 = Q$ is such that

$$d'Q = 0 \quad \text{and} \quad d''Q = P.$$

Moreover, Q has no d'-image part. For, if $Q = \hat{Q} + d'R$, then $P = d''Q = d''\hat{Q} + d''d'R = d''\hat{Q} + d'(-d''R)$.

The argument required to find the form of such Q is parallel to that for the top term of a cocycle. Recall that Q is of type (n, m) and P of type $(n, m+1)$. Thus P is of the form

$$P = y_{26}^k (a_9 + c_{17}\partial_9)^\varepsilon y_{62}^{l+\delta} (a_{21} + c_{41}\partial_{21})^{1-\delta} A.$$

The condition $d'Q = 0$ gives rise to, by Lemma 7.6,

$$Q = y_{26}^k (a_9 + c_{17}\partial_9)^\varepsilon Q(0, 2l + \delta) + (d'\text{-image}),$$

where the d'-image may be omitted. By Lemma 7.8 the condition

$$d''Q = P = y_{26}^k (a_9 + c_{17}\partial_9)^\varepsilon y_{62}^{l+\delta} (a_{21} + c_{41}\partial_{21})^{1-\delta} A$$

gives rise to

$$Q = y_{26}^k (a_9 + c_{17}\partial_9)^\varepsilon \{ y_{62}^l (a_{21} + c_{41}\partial_{21})^\delta C + d''S(0, 2l + \delta - 1) \} + (d'\text{-image}),$$

where the d'-image may be omitted. For such Q, exactly as in the proof of Proposition 7.10, the condition $d'Q = 0$ again gives rise to

$$Q = y_{26}^k (a_9 + c_{17}\partial_9)^\varepsilon y_{62}^l (a_{21} + c_{41}\partial_{21})^\delta B$$
$$+ (d'\text{-image}) + (d\text{-image}) \qquad \text{for some} \quad B \in V.$$

Omitting the d'-image and the d-image from Q, Q must be of the form

$$Q=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}B.$$

Now to return to our original convention, we change the second degree of P as $P(n, m)$:

$$P=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}A$$

with no d'-image part. Then as we have shown, if $d'Q=0$ and $d''Q=P$ for such P, Q must be of the form

$$Q=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l+\delta-1}(a_{21}+c_{41}\partial_{21})^{1-\delta}B.$$

Finally, we study the conditions for B. The condition $d''Q=P$ can be written as

$$(-1)^{1+\varepsilon-\delta}y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}\partial_{21}^{2-\delta}B$$
$$=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}A,$$

which gives rise to $A=(-1)^{1+\varepsilon-\delta}\partial_{21}^{2-\delta}B$. The condition

$$d'Q=(-1)^{\varepsilon}y_{26}^{k+\varepsilon}(a_9+c_{17}\partial_9)^{1-\varepsilon}y_{62}^{l+\delta-1}(a_{21}+c_{41}\partial_{21})^{1-\delta}\partial_9^{\varepsilon+1}B$$
$$=0$$

gives rise to $\partial_9^{\varepsilon+1}B=0$.

Thus we have

Proposition 10.1. *Let Φ be a cocycle with top term*

$$P=y_{26}^{k}(a_9+c_{17}\partial_9)^{\varepsilon}y_{62}^{l}(a_{21}+c_{41}\partial_{21})^{\delta}A \quad \textit{having no } d'\textit{-image part.}$$

Then, if Φ is trivial, A has some B such that $\partial_{21}^{2-\delta}B=A$ and $\partial_9^{1+\varepsilon}B=0$.

For the cocycle $\Phi_{2k+1,2l+\delta}$,

$$A=\sum_i \alpha_i x_{48}^{i}+\sum_i \beta_i(a_8e_{48}-b_{16}b_{40})x_{48}^{i}+\sum_i \gamma_i b_{16}x_{48}^{i}+\sum_i \delta_i e_{48}x_{48}^{i}$$

with α_i, β_i, γ_i, $\delta_i\in \mathbf{Z}_3$, for which we see by direct calculation that there is no B with $\partial_{21}^{2-\delta}B=A$ and $\partial_9^{2}B=0$. Thus the top term of $\Phi_{2k+1,2l+\delta}$ does not satisfy the necessary conditions for a trivial cocycle.

Similarly, for the cocycle $\Phi_{2k,2l+\delta}$,

$$A=\sum_i \alpha_i x_{48}^{i}+\sum_i \beta_i(a_8e_{48}-b_{16}b_{40})x_{48}^{i}+\sum_i \gamma_i a_8x_{48}^{i}+\sum_i \delta_i b_{40}x_{48}^{i}$$

for which there is no B with $\partial_{21}^{2-\delta}B=A$ and $\partial_9B=0$. Thus the top term of $\Phi_{2k+1,2l+\delta}$ does not satisfy the necessary conditions for a trivial cocycle.

Note that we have only to consider a sum of cocycles of the same total degree, when we check the non-triviality of a sum of cocycles. We consider an arbitrary sum (within the same total degree) of the cocycles in the $\Phi_{2k+\varepsilon,2l+\delta}$'s and cocycles that will be found in Cases 3 and 4. If the top term of the sum is of first degree positive, then it is the top term of one of the $\Phi_{2k+\varepsilon,2l+\delta}$'s, which does not satisfy the necessary conditions for a trivial cocycle.

Therefore, the following cocycles are non-trivial and linearly independent and they are linearly independent of non-trivial cocycles containing neither a_9 nor c_{17}:

(10.2.1) (Cocycles in $\Phi_{2k+1,2l}$)

$y_{26}^k y_{62}^l a_9 x_{48}^i$, $y_{26}^k y_{62}^l \phi_1 x_{48}^i$ (which is equivalent to $y_{26}^k y_{62}^l a_9 z_{56}$),

$y_{26}^k y_{62}^l y_{25} x_{48}^i$, $y_{26}^k y_{62}^l y_{57} x_{48}^i$;

(10.2.2) (Those in $\Phi_{2k+1,2l+1}$)

$y_{26}^k y_{62}^l a_9 a_{21} x_{48}^i$, $y_{26}^k y_{62}^l \phi_2 x_{48}^i$ (which is equivalent to $y_{26}^k y_{62}^l a_9 a_{21} z_{56}$),

$y_{26}^k y_{62}^l y_{25} a_{21} x_{48}^i$, $y_{26}^k y_{62}^l a_{21} y_{57} x_{48}^i$;

(10.2.3) (Those in $\Phi_{2k,2l}$)

$y_{26}^k y_{62}^l x_{48}^i$, $y_{26}^{k-1} y_{62}^l \phi_3 x_{48}^i$ (which is equivalent to $y_{26}^k y_{62}^l z_{56}$),

$y_{26}^k y_{62}^l a_8 x_{48}^i$, $y_{26}^{k-1} y_{62}^l \phi_5 x_{48}^i$ (which is equivalent to $-y_{26}^{k-1} y_{62}^l a_9 y_{57}$);

(10.2.4) (Those in $\Phi_{2k,2l+1}$)

$y_{26}^k y_{62}^l a_{21} x_{48}^i$, $y_{26}^{k-1} y_{62}^l a_{21} \phi_4 x_{48}^i$ (which is equivalent to $y_{26}^k y_{62}^l a_{21} z_{56}$),

$y_{26}^k y_{62}^l a_{21} a_8 x_{48}^i$, $y_{26}^k y_{62}^l y_{61} x_{48}^i$.

§11. Cocycles with elements of odd degree-II

(iv) Cocycles with a_{21} and c_{41} but without a_9 or c_{17}

In this section we shall study cocycles with top term $P(0, m)$. Since $P(0, m)$ has neither a_9 nor c_{17}, $d''P(0, m)$ cannot be in the d'-image and the condition (8.2) for $n=0$ reads

$$d'P(0, m)=0 \quad \text{and} \quad d''P(0, m)=0$$

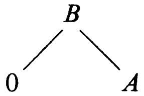
and (8.3) reads $P(0, m)=P+(d\text{-image})$ with $P=y_{62}^l(a_{21}+c_{41}\partial_{21})^\delta A$. Thus we determine cocycles P of the above form.

Case 3. $k=\varepsilon=0$, $\delta=1$: $P=y_{62}^l(a_{21}+c_{41}\partial_{21})A$.

The condition $d'P=d''P=0$ gives rise to

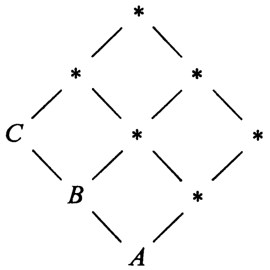
$$\partial_9 A=0 \quad \text{and} \quad \partial_{21}A \quad \text{or} \quad \partial_{21}^2 A=0.$$

(1) If $\partial_9 A=0$ and $\partial_{21}A=0$, A is a cocycle and $P=y_{62}^l a_{21}A$ is a cocycle. By Proposition 10.1 the cocycle P is trivial provided there exists B such that $\partial_{21}B=A$ and $\partial_9 B=0$, which is expressed as



(Recall that an oblique line ／ means ∂_9 and an oblique line ＼ means ∂_{21}.)

If A is nice, there exists such a B, and P is trivial. Thus we have to study non-nice cocycles A.



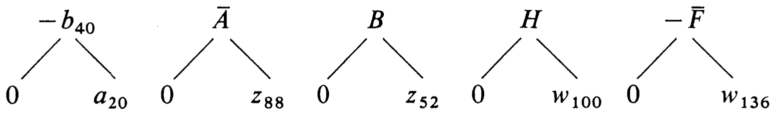
(nice)

If $A \underset{N}{\sim} A'$, then $A-A'$ being nice, the difference

$$P-P'=y^l_{62}a_{21}A-y^l_{62}a_{21}A'=y^l_{62}a_{21}(A-A')$$

is trivial. Thus we can choose one of two nicely-related monomials A and A'.

By (6.2.1)~(6.2.3) we have diagrams



Thus $P=y^l_{62}a_{21}A$ is trivial for A a monomial of the form (7), (8), (9), (13), (14), (16), (18), (19) and (20) in Proposition 6.6.

On the other hand, there is no B for any sum of cocycles of the form (1), (2), (3), (4), (5), (6), (10), (11), (12), (15) and (17) and hence any sum of the following are non-trivial:

(11.1) $y^l_{62}a_{21}x^h_{84}$, $y^l_{62}a_{21}x^i_{48}x^h_{84}$ $(i\neq 0)$,

$y^l_{62}a_{21}x^j_{120}x^h_{84}$ $(j\neq 0)$, $y^l_{62}a_{21}a_8x^h_{84}$,

$y^l_{62}a_{21}a_8x^i_{48}x^h_{84}$ $(i\neq 0)$, $y^l_{62}a_{21}a_8x^j_{120}x^h_{84}$ $(j\neq 0)$,

$y^l_{62}a_{21}z_{56}x^h_{84}$, $y^l_{62}a_{21}z_{56}x^i_{48}x^h_{84}$ $(i\neq 0)$,

$y^l_{62}a_{21}z_{56}x^j_{120}x^h_{84}$ $(j\neq 0)$, $y^l_{62}a_{21}w_{88}x^i_{48}x^h_{84}$,

$y^l_{62}a_{21}w_{124}x^j_{120}x^h_{84}$.

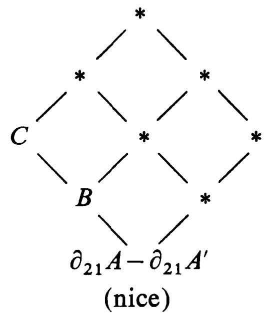
(2) $\partial_9A=0$, $\partial_{21}A\neq 0$ and $\partial^2_{21}A=0$: $P=y^l_{62}(a_{21}+c_{41}\partial_{21})A$. Now, $\partial_{21}A$ is a cocycle, for which it is sufficient to choose one A. In fact, for A and A' with $\partial_9A'=\partial_9A=0$ and $\partial_{21}A'=\partial_{21}A$, the difference between the P's is

$$P'-P=y_{62}^{l}(a_{21}+c_{41}\partial_{21})(A'-A)=y_{62}^{l}a_{21}(A'-A)$$

with $\partial_9(A'-A)=\partial_{21}(A'-A)=0$, which is the case studied in (1).

If $\partial_{21}A=\partial_{21}^2C$ for some C with $\partial_9C=0$, then $P=y_{62}^{l}(a_{21}+c_{41}\partial_{21})A=d(y_{62}^{l}C)$ is trivial by choosing $A=\partial_{21}C$. In particular, if the cocycle $\partial_{21}A$ is nice, there exists C for a suitable choice of A, and P is trivial.

If two cocycles $\partial_{21}A$ and $\partial_{21}A'$ are nicely-related, then we have a diagram:



Let B and C be as in the diagram above: $\partial_{21}C=B$, $\partial_{21}B=\partial_{21}A-\partial_{21}A'$ and $\partial_9B=\partial_9C=0$. Then

$$\begin{aligned}P-P'&=y_{62}^{l}(a_{21}+c_{41}\partial_{21})A-y_{62}^{l}(a_{21}+c_{41}\partial_{21})A'\\&=y_{62}^{l}(a_{21}+c_{41}\partial_{21})B+y_{62}^{l}a_{21}(A-A'-B)\\&=d(y_{62}^{l}C)+y_{62}^{l}a_{21}(A-A'-B)\end{aligned}$$

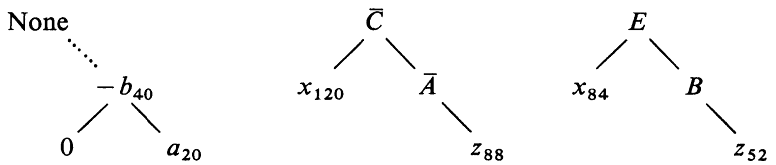
with $\partial_9(A-A'-B)=0$ and $\partial_{21}(A-A'-B)=0$. Thus the difference is equivalent to a cocycle studied in (1). Hence we may choose one of two nicely-related cocycles $\partial_{21}A$ and $\partial_{21}A'$.

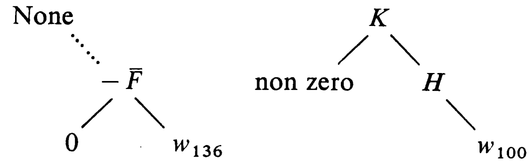
For any sum D of non-nice monomials in Proposition 6.6 (whether it is of the form $\partial_{21}A$ or not), we see by direct calculation that there is no C satisfying $\partial_{21}^2C=D$ and $\partial_9C=0$. Thus any cocycle found in the following determination is non-trivial.

Of the monomials in Proposition 6.6, all but $z_{56}x_{120}^{j}$ ((12) with $h=0$) are of the form $\partial_{21}A$.

Of the non-nice generators, a_8, x_{48}, x_{84}, w_{88} and w_{124} are of the form $\partial_{21}A$ but with $\partial_9A\neq0$ and so is any sum of monomials of the form (12) with $h\neq0$, (1), (2), (3), (4), (5), (6), (10), (11), (15) and (17). Hence, $P=y_{62}^{l}(a_{21}A+c_{41}\partial_{21}A)$ are not cocycles for any such sum $\partial_{21}A$.

The other non-nice generators, namely, a_{20}, z_{88}, z_{52}, w_{100} and w_{136}, form the following diagram (cf. (6.2.1)$\sim$(6.2.3)):





and we have non-trivial cocycles

$$y_{61}=a_{21}b_{40}-c_{41}a_{20},$$

$$P_{93}=a_{21}B+c_{41}z_{52}, \qquad P_{129}=a_{21}\bar{A}+c_{41}z_{88},$$

$$P_{141}=a_{21}H+c_{41}w_{100}, \quad P_{177}=-a_{21}\bar{F}+c_{41}w_{136}.$$

However we have

$$P_{93}=dE-a_9x_{84}, \quad P_{129}=d\bar{C}-a_9x_{120},$$

$$P_{141}=d(-a_{20}e_{36}^2e_{48}+b_{40}d_{28}e_{36}^2-d_{28}^2e_{36}e_{48})+y_{57}x_{84},$$

$$P_{177}=-a_{21}\bar{F}+c_{41}w_{136}=d(-b_{40}^2e_{48}^2)+y_{57}x_{120}.$$

Thus, they are equivalent to a decomposable cocycle and we obtain no new indecomposable cocycle. (y_{61} was already found in Case 2.)

For $\partial_{21}A$ any sum of monomials of the form (7), (8), (9), (13), (14), (16), (18), (19) and (20), we have a sum of the following non-trivial and linearly independent cocycles:

(11.2) $-y_{62}^l y_{61}x_{84}^h, \quad -y_{62}^l y_{61}x_{48}^i x_{84}^h\ (i\neq 0), \quad -y_{62}^l y_{61}x_{120}^j x_{84}^h\ (j\neq 0),$

$y_{62}^l P_{93}x_{48}^i x_{84}^h$ (which is equivalent to $-y_{62}^l a_9 x_{48}^i x_{84}^{h+1}$),

$y_{62}^l P_{129}x_{120}^j x_{84}^h$ (which is equivalent to $-y_{62}^l a_9 x_{120}^{j+1} x_{84}^h$),

$y_{62}^l P_{141}x_{48}^i x_{84}^h$ (which is equivalent to $y_{62}^l y_{57}x_{48}^i x_{84}^{h+1}$),

$y_{62}^l P_{177}x_{120}^j x_{84}^h$ (which is equivalent to $y_{62}^l y_{57}x_{120}^{j+1} x_{84}^h$),

$-y_{62}^l y_{61}w_{88}x_{48}^i x_{84}^h, \quad -y_{62}^l y_{61}w_{124}x_{120}^j x_{84}^h.$

Cocycles in (11.1) and (11.2) are all the non-trivial cocycles in Case 3 and they are linearly independent.

Case 4. $\varepsilon=k=0$, $\delta=0$, $l\neq 0$: $P=y_{62}^l A$.

In this case, A must be a cocycle. By Proposition 10.1 P is trivial provided there exists C such that $\partial_{21}^2 C=A$ and $\partial_9 C=0$. In particular, P is trivial for a nice cocycle A.

For two non-nice cocycles A and A' such that $A \underset{N}{\sim} A'$, the difference $y_{62}^l A-y_{62}^l A'$ $=y_{62}^l(A-A')$ is trivial and $y_{62}^l A$ and $y_{26}^l A'$ are equivalent cocycles. Thus we may choose one of two nicely-related cocycles A and A'.

As we have checked in Case 3, there is no C for any sum D of monomials in Proposition 6.6, and thus

(11.3) *The $y_{62}^l A$'s for all monomials A in Proposition 6.6 are non-trivial and linearly independent.*

More explicitly, (11.3) is stated as:

(11.3)′ *The following are non-trivial and linearly independent:*

$$y_{62}^l x_{84}^h,\quad y_{62}^l x_{48}^i x_{84}^h\ (i\neq 0),\quad y_{62}^l x_{120}^j x_{84}^h\ (j\neq 0),$$
$$y_{62}^l a_8 x_{84}^h,\quad y_{62}^l a_8 x_{48}^i x_{84}^h\ (i\neq 0),$$
$$y_{62}^l a_8 x_{120}^j x_{84}^h\ (j\neq 0),\quad y_{62}^l a_{20} x_{84}^h,$$
$$y_{62}^l a_{20} x_{48}^i x_{84}^h\ (i\neq 0),\quad y_{62}^l a_{20} x_{120}^j x_{84}^h\ (j\neq 0),$$
$$y_{62}^l z_{56} x_{84}^h,\quad y_{62}^l z_{56} x_{48}^i x_{84}^h\ (i\neq 0),$$
$$y_{62}^l z_{56} x_{120}^j x_{84}^h\ (j\neq 0),\quad y_{62}^l z_{52} x_{48}^i x_{84}^h,$$
$$y_{62}^l z_{88} x_{120}^j x_{84}^h,\quad y_{62}^l w_{88} x_{48}^i x_{84}^h,\quad y_{62}^l w_{100} x_{48}^i x_{84}^h,$$
$$y_{62}^l w_{124} x_{120}^j x_{84}^h,\quad y_{62}^l w_{136} x_{120}^j x_{84}^h,$$
$$y_{62}^l z_{56} z_{52} x_{48}^i x_{84}^h,\quad y_{62}^l z_{56} z_{88} x_{120}^j x_{84}^h.$$

§ 12. Structure of $H(\overline{W}: d)$

We have shown

Proposition 12.1. *$H(\overline{W}: d)$ is generated (as an algebra) by the 22 elements in Proposition 5.7 and by the following 7 elements: a_9, a_{21}, y_{26}, y_{62}, y_{25}, y_{61} and y_{57}.*

Recall that

$$y_{26}=[a_9, c_{17}],\quad y_{62}=[a_{21}, c_{41}],$$
$$y_{25}=a_9 b_{16}-c_{17} a_8,\quad y_{61}=a_{21} b_{40}-c_{41} a_{20},$$
$$y_{57}=a_9 e_{48}-c_{17} b_{40}+a_{21} e_{36}-c_{41} b_{16}.$$

Proposition 12.2. *Monomials in the 22 cocycles in Proposition 5.7, $y_{26}^k a_9^\varepsilon y_{62}^l a_{21}^\delta a_4^r x_{108}^s x_{144}^t$ and cocycles of the form $\Phi\cdot a_4^r x_{108}^s x_{144}^t$ with Φ an element in (10.2.1)~(10.2.4) and (11.1)~(11.3) form an additive basis of $H(\overline{W}: d)$.*

Proposition 12.3. *$H(\overline{W}: d)$ is commutative.*

Proof. Since the cocycles in V (including a_4, x_{108} and x_{144}) satisfy $\partial_9 A=\partial_{21} A=0$, they commute with a_9, a_{21}, c_{17} and c_{41}, and hence with a_9, a_{21}, y_{26}, y_{62}, y_{25}, y_{61} and y_{57}.

We have that

$[a_9, a_{21}]=0,$

$[a_9, y_{25}]=d(-c_{17}b_{16}),$ $[a_{21}, y_{25}]=0,$

$[a_9, y_{61}]=0,$ $[a_{21}, y_{61}]=d(-c_{41}b_{40}),$

$[a_9, y_{57}]=d(-c_{17}e_{48}),$ $[a_{21}, y_{57}]=d(-c_{41}e_{36}),$

$[a_9, y_{26}]=d(c_{17}^2),$ $[a_{21}, y_{26}]=0,$

$[a_9, y_{62}]=0,$ $[a_{21}, y_{62}]=d(c_{41}^2),$

$[y_{25}, y_{61}]=0,$

$[y_{25}, y_{57}]=d(-c_{17}b_{16}e_{48}),$ $[y_{61}, y_{57}]=d(-c_{41}b_{40}e_{36}),$

$[y_{25}, y_{26}]=d(c_{17}^2b_{16}),$ $[y_{61}, y_{26}]=0,$

$[y_{25}, y_{62}]=0,$ $[y_{61}, y_{62}]=d(c_{41}^2b_{40}),$

$[y_{57}, y_{26}]=d(c_{17}^2e_{48}),$

$[y_{57}, y_{62}]=d(c_{41}^2e_{36}),$

$[y_{26}, y_{62}]=0.$

Thus commutativity holds in $H(\overline{W}: d)$. q.e.d.

Proposition 12.4. *The generators in $H(\overline{W}: d)$ satisfy the following equivalences*:

(12.5)

left \ right	a_9	a_{21}	y_{25}	y_{61}	y_{57}
a_9	0	remains	$-y_{26}a_8$	$-y_{62}a_8$	remains
a_{21}		0	$-y_{26}a_{20}$	$-y_{62}a_{20}$	remains
y_{25}			0	$a_9a_{21}z_{56}$	$y_{26}z_{56}$
y_{61}				0	$y_{62}z_{56}$
y_{57}					0

where the table reads, for example: a_9y_{25} *is equivalent to* $-y_{26}a_8,\dots, y_{61}y_{57}$ *is equivalent* $y_{62}z_{56}$.

(12.6) $a_9a_{21}y_{57}$ *is equivalent to* $y_{26}y_{61}-y_{62}y_{25}$.

Proof. The equivalences in (12.5) and (12.6) are shown directly as follows:

$a_9^2=dc_{17},\quad a_9y_{25}=-y_{26}a_8+d(c_{17}b_{16}),$

$a_9y_{61}=-y_{62}a_8+d(-a_{21}e_{48}+c_{41}d_{28}),\quad a_{21}^2=dc_{41},$

$a_{21}y_{25}=-y_{26}a_{20}+d(-a_9e_{36}+c_{17}d_{28}),$

$a_{21}y_{61}=-y_{62}a_{20}+d(c_{41}b_{40}),\quad y_{25}^2=d(c_{17}b_{16}^2),$

$$y_{25}y_{61}=a_9a_{21}z_{56}+d(-a_9(d_{28}e_{48}+b_{40}e_{36})+c_{17}(a_{20}e_{48}-b_{40}d_{28})),$$
$$y_{25}y_{57}=y_{26}z_{56}+d(-a_9e_{36}^2-c_{17}d_{28}e_{36}+c_{17}b_{16}e_{48}),$$
$$y_{61}^2=d(c_{41}b_{40}^2),$$
$$y_{61}y_{57}=y_{62}z_{56}+d(-a_{21}e_{48}^2-c_{41}d_{28}e_{48}+c_{41}b_{40}e_{36}),$$
$$y_{57}^2=d(c_{17}e_{48}^2+c_{41}e_{36}^2),$$
$$a_9a_{21}y_{57}=y_{26}y_{61}-y_{62}y_{25}+d(-a_9c_{41}e_{36}+a_{21}c_{17}e_{48}+c_{17}c_{41}d_{28}).$$

q.e.d.

Note that by (12.5) and (12.6) any cocycle is equivalent to a cocycle each term of which has at most one of a_9, a_{21}, y_{25}, y_{61}, y_{57}, a_9a_{21}, a_9y_{57} and $a_{21}y_{57}$.

We have also the following equivalences, a proof of which will be given in § 13, since it is rather tedious.

Proposition 12.7. *The products a_9A, $a_{21}A$, $y_{25}A$, $y_{61}A$, $y_{57}A$, $y_{26}A$ and $y_{62}A$ are trivial for a nice cocycle A. For a non-nice cocycle $A\in V$, we have the following equivalences*:

(12.8)

left \ right	a_8	a_{20}	z_{56}	x_{48}	x_{120}	z_{52}	z_{88}
a_9	0					0	0
a_{21}	$-a_9a_{20}$	0				0	0
y_{25}	0	0	0			$a_9a_{20}x_{48}$	$a_9a_{20}x_{84}$
y_{61}	0	0	0			$-a_9a_{20}x_{84}$	$-a_9a_{20}x_{120}$
y_{57}	$-a_9z_{56}$	$-a_{21}z_{56}$				$-y_{25}x_{84}$ $-y_{61}x_{48}$	$-y_{25}x_{120}$ $-y_{61}x_{84}$
y_{26}					$-y_{62}x_{84}$	$-a_9a_{21}x_{48}$	$-a_9a_{21}x_{84}$
y_{62}				$-y_{26}x_{84}$		$a_9a_{21}x_{84}$	$a_9a_{21}x_{120}$
a_9a_{21}	0	0				0	0
a_9y_{57}	0	$-a_9a_{21}z_{56}$	0			0	0
$a_{21}y_{57}$	$-a_9a_{21}z_{56}$	0	0			0	0

left \ right	w_{88}	w_{100}	w_{124}	w_{136}
a_9	0	$-y_{25}x_{84}$ $-y_{61}x_{48}$	0	$-y_{25}x_{120}$ $-y_{61}x_{84}$
a_{21}	$-y_{25}x_{84}$ $-y_{61}x_{48}$	0	$-y_{25}x_{120}$ $-y_{61}x_{84}$	0
y_{25}	$-a_9z_{56}x_{48}$	$a_{21}z_{56}x_{48}$	$-a_9z_{56}x_{84}$	$a_{21}z_{56}x_{84}$
y_{61}	$a_9z_{56}x_{84}$	$-a_{21}z_{56}x_{84}$	$a_9z_{56}x_{120}$	$-a_{21}z_{56}x_{120}$
y_{57}	0	0	0	0
y_{26}	$a_9y_{57}x_{48}$	$-a_{21}y_{57}x_{48}$	$a_9y_{57}x_{84}$	$-a_{21}y_{57}x_{84}$
y_{62}	$-a_9y_{57}x_{84}$	$a_{21}y_{57}x_{84}$	$-a_9y_{57}x_{120}$	$a_{21}y_{57}x_{120}$
a_9a_{21}	0	0	0	0
a_9y_{57}	0	0	0	0
$a_{21}y_{57}$	0	0	0	0

(The last three lines are added for convenience of calculations.)

The table reads similarly as the table (12.5).

Propositions 12.1 (*generators*), 12.3 (*commutativity*) and the relations in Propositions 12.4 and 12.7 are sufficient to know the structure of $H(\overline{W}: d)$. (The sufficiency of Propositions 12.4 and 12.7 will be assured later.) We have an additive basis in Proposition 12.2, although the set of elements is not so convenient (for instance, the conjugation between cocycles in Proposition 5.7 and between a_9 and a_{21}, y_{25} and y_{61}, y_{26} and y_{62} are not observable). So we would rather have another additive basis. This is done by taking Φ in Proposition 12.2 to be an element in the following:

(12.9.1) $y_{26}^k y_{62}^l x_{84}^h$, $y_{26}^k y_{62}^l a_8 x_{84}^h$, $y_{26}^k y_{62}^l a_{20} x_{84}^h$,

$y_{26}^k y_{62}^l z_{56} x_{84}^h$, $y_{26}^k y_{62}^l a_9 x_{84}^h$, $y_{26}^k y_{62}^l a_9 a_{20} x_{84}^h$,

$y_{26}^k y_{62}^l a_9 z_{56} x_{84}^h$, $y_{26}^k y_{62}^l a_{21} x_{84}^h$, $y_{26}^k y_{62}^l a_{21} z_{56} x_{84}^h$,

$y_{26}^k y_{62}^l y_{25} x_{84}^h$, $y_{26}^k y_{62}^l y_{61} x_{84}^h$, $y_{26}^k y_{62}^l y_{57} x_{84}^h$,

$y_{26}^k y_{62}^l a_9 a_{21} x_{84}^h$, $y_{26}^k y_{62}^l a_9 a_{21} z_{56} x_{84}^h$,

$y_{26}^k y_{62}^l a_9 y_{57} x_{84}^h$, $y_{26}^k y_{62}^l a_{21} y_{57} x_{84}^h$.

(12.9.2) (In the following, $i \neq 0$)

$y_{26}^k x_{48}^i x_{84}^h$, $y_{26}^k a_8 x_{48}^i x_{84}^h$, $y_{26}^k a_{20} x_{48}^i x_{84}^h$,

$y_{26}^k z_{56} x_{48}^i x_{84}^h$, $y_{26}^k a_9 x_{48}^i x_{84}^h$, $y_{26}^k a_9 a_{20} x_{48}^i x_{84}^h$,

$y_{26}^k a_9 z_{56} x_{48}^i x_{84}^h$, $y_{26}^k a_{21} x_{48}^i x_{84}^h$, $y_{26}^k a_{21} z_{56} x_{48}^i x_{84}^h$,

$y_{26}^k y_{25} x_{48}^i x_{84}^h$, $y_{26}^k y_{61} x_{48}^i x_{84}^h$, $y_{26}^k y_{57} x_{48}^i x_{84}^h$,

$y_{26}^k a_9 a_{21} x_{48}^i x_{84}^h$, $y_{26}^k a_9 a_{21} z_{56} x_{48}^i x_{84}^h$,

$y_{26}^k a_9 y_{57} x_{48}^i x_{84}^h$, $y_{26}^k a_{21} y_{57} x_{48}^i x_{84}^h$.

(12.9.3) (In the following, $j \neq 0$)

$y_{62}^l x_{120}^j x_{84}^h$, $y_{62}^l a_8 x_{120}^j x_{84}^h$, $y_{62}^l a_{20} x_{120}^j x_{84}^h$,

$y_{62}^l z_{56} x_{120}^j x_{84}^h$, $y_{62}^l a_9 x_{120}^j x_{84}^h$, $y_{62}^l a_9 a_{20} x_{120}^j x_{84}^h$,

$y_{62}^l a_9 z_{56} x_{120}^j x_{84}^h$, $y_{62}^l a_{21} x_{120}^j x_{84}^h$, $y_{62}^l a_{21} z_{56} x_{120}^j x_{84}^h$,

$y_{62}^l y_{25} x_{120}^j x_{84}^h$, $y_{62}^l y_{61} x_{120}^j x_{84}^h$, $y_{62}^l y_{57} x_{120}^j x_{84}^h$,

$y_{62}^l a_9 a_{21} x_{120}^j x_{84}^h$, $y_{62}^l a_9 a_{21} z_{56} x_{120}^j x_{84}^h$,

$y_{62}^l a_9 y_{57} x_{120}^j x_{84}^h$, $y_{62}^l a_{21} y_{57} x_{120}^j x_{84}^h$.

We shall call Set I the set of elements in (10.2.1)~(10.2.4) and (11.1)~(11.3), and Set II the set of elements in (12.9.1)~(12.9.3). Then

Lemma 12.10. *The two sets, Set I and Set II, are equivalent in the sense that one is expressible as the other and the elements are linearly independent in each set.*

Proof. We can make Set II by eliminating products which are trivial by Propositions 12.4 and 12.7.

We can easily check using equivalences in Propositions 12.4 and 12.7 that each element in Set I is equivalent to a unique element which is a sum of monomials in Set II and conversely that each element in Set II is uniquely equivalent to a sum of monomials in Set I. Since Set I is linearly independent, so is Set II. Therefore, the two sets of elements are equivalent in the sense stated. q.e.d.

Note that we have assured that Propositions 12.4 and 12.7 contain all the equivalences necessary to determine the structure of $H(\overline{W}: d)$.

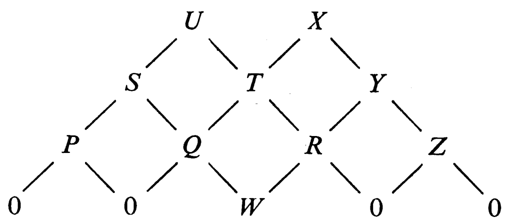
Proposition 12.2′. *Monomials in the* 22 *cocycles in Proposition* 5.7, $y_{26}^k a_9^\varepsilon y_{62}^l a_{21}^\delta a_4^r x_{108}^s x_{144}^t$ *and the cocycles of the form* $\Phi \cdot a_4^r x_{108}^s x_{144}^t$ *with* Φ *an element in* (12.9.1)~(12.9.3) *form an additive basis of* $H(\overline{W}: d)$.

§13. A proof of Proposition 12.7

In this section, the letters $A, \dots, N$ and $\bar{A}, \dots, \bar{N}$ are as in the diagrams (6.2.1)~(6.2.3).

(1) *Products with* y_{57}.

If we have a diagram of the form



we have

$$d(Qe_{36}+Re_{48}+Td_{28}+Sb_{16}+Yb_{40}+Ua_8+Xa_{20})=y_{57}W+y_{25}P+y_{61}Z.$$

Thus, we have

$$y_{57}\cdot(\text{nice cocycle}) \in d\text{-image},$$

$$y_{57}z_{52}+y_{25}x_{84}+y_{61}x_{48} \in d\text{-image},$$

$$y_{57}z_{88}+y_{25}x_{120}+y_{61}x_{84} \in d\text{-image}.$$

On the other hand, by direct calculation we have

$$y_{57}z_{52}-a_9w_{100}$$
$$=d(-a_8d_{28}e_{36}^2-a_{20}b_{16}e_{36}^2+a_8b_{16}e_{36}e_{48}+b_{16}d_{28}^2e_{36}-b_{16}^2d_{28}e_{48}),$$

whence

$$a_9w_{100}+y_{25}x_{84}+y_{61}x_{48}\in d\text{-image}.$$

Its conjugate yields

$$a_{21}w_{124}+y_{25}x_{120}+y_{61}x_{84}\in d\text{-image}.$$

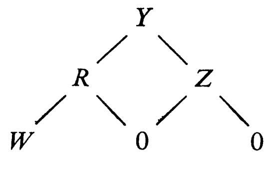
By direct calculation we have

$$y_{57}a_8+a_9z_{56}=d(b_{16}e_{48}-d_{28}e_{36}),$$
$$y_{57}a_{20}+a_{21}z_{56}=d(b_{40}e_{36}-d_{28}e_{48}),$$
$$y_{57}z_{56}=d(-b_{40}e_{36}^2-d_{28}e_{36}e_{48}-b_{16}e_{48}^2),$$
$$y_{57}w_{88}=d(a_{20}b_{16}e_{36}^3-a_8d_{28}e_{36}^3-a_8b_{16}e_{36}^2e_{48}$$
$$-b_{16}^2b_{40}e_{36}^2-b_{16}^2d_{28}e_{36}e_{48}-b_{16}^3e_{48}^2),$$
$$y_{57}w_{100}=d(-a_8b_{40}e_{36}^3+a_8d_{28}e_{36}^2e_{48}+a_{20}b_{16}e_{36}^2e_{48}+a_8b_{16}e_{36}e_{48}^2$$
$$-b_{16}b_{40}d_{28}e_{36}^2-b_{16}d_{28}^2e_{36}e_{48}-b_{16}^2d_{28}e_{48}^2),$$
$$y_{57}w_{124}=d(-a_{20}b_{16}e_{48}^3+a_{20}d_{28}e_{36}e_{48}^2+a_8b_{40}e_{36}e_{48}^2+a_{20}b_{40}e_{36}^2e_{48}$$
$$-b_{16}b_{40}d_{28}e_{48}^2-b_{40}d_{28}^2e_{36}e_{48}-b_{40}^2d_{28}e_{36}^2),$$
$$y_{57}w_{136}=d(a_8b_{40}d_{48}^3-a_{20}d_{28}e_{48}^3-a_{20}b_{40}e_{36}e_{48}^2$$
$$-b_{16}b_{40}^2e_{48}^2-b_{40}^2d_{28}e_{36}e_{48}-b_{40}^3e_{36}^2).$$

We fill in the table the line of y_{57} and the boxes a_9w_{100} and $a_{21}w_{124}$. The monomials $y_{57}x_{48}$, $y_{57}x_{120}$ and $y_{57}x_{84}$ remain as they are.

(2) *Products with* a_{21}.

If we have a diagram of the form below, then $dY=a_{21}Z+a_9R+c_{17}W$.



Thus we have

$$a_{21}\cdot(\text{nice cocycle})\in d\text{-image},\quad a_{21}a_8+a_9a_{20}=d(-d_{28}),$$
$$a_{21}a_{20}=d(-b_{40}),\quad a_{21}z_{52}=dB,\quad a_{21}z_{88}=d\bar{A},$$

$$a_{21}w_{88}-a_9w_{100}=d(-G),\quad a_{21}w_{100}=dH,$$

$$a_{21}w_{124}-a_9w_{136}=d\bar{G},\qquad a_{21}w_{136}=d(-\bar{F}).$$

Therefore we see that

$a_{21}\cdot$(nice), $a_{21}a_{20}$, $a_{21}z_{52}$, $a_{21}z_{88}$, $a_{21}w_{100}$ and $a_{21}w_{136}$ are trivial;

$a_{21}a_8$ is equivalent to and will be replaced by $-a_9a_{20}$;

$a_{21}w_{88}$ is equivalent to a_9w_{100} which has been shown in (1) to be equivalent to $-y_{25}x_{84}-y_{61}x_{48}$;

$a_{21}w_{124}$ has been shown to be equivalent to $-y_{25}x_{120}-y_{61}x_{84}$.

The line of a_{21} in the table is thus filled. The monomials $a_{21}z_{56}$, $a_{21}x_{48}$, $a_{21}x_{120}$ and $a_{21}x_{84}$ remain as they are.

(3) *Products with* a_9.
Taking conjugates, we have that

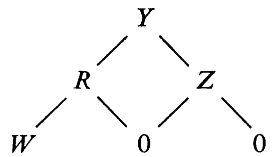
$a_9\cdot$(nice), a_9a_8, a_9z_{52}, a_9z_{88}, a_9w_{88} and a_9w_{124} are trivial;

a_9w_{100} is equivalent to $-y_{25}x_{84}-y_{61}x_{48}$;

a_9w_{136} is equivalent to $-y_{25}x_{120}-y_{61}x_{84}$.

The line of a_9 in the table is filled except a_9a_{20}, a_9z_{56}, a_9x_{48}, a_9x_{120} and a_9x_{84}.

(4) *Products with* y_{25}.
If we have a diagram of the form below,



then $d(a_8Y+b_{16}R)=y_{25}W+a_{21}a_8Z$. And, since $d(d_{28})=-a_9a_{20}-a_{21}a_8$, we have $d(a_8Y+b_{16}R+d_{28}Z)=y_{25}W-a_9a_{20}Z$. Thus we have

$$y_{25}\cdot(\text{nice cocycle})\in d\text{-image},$$

$$y_{25}z_{52}-a_9a_{20}x_{48}\in d\text{-image},$$

$$y_{25}z_{88}-a_9a_{20}x_{84}\in d\text{-image}.$$

By direct calculation we have

$$y_{25}a_8=d(b_{16}^2),$$

$$y_{25}a_{20}=d(-b_{16}d_{28}+a_8e_{36}),$$

$$y_{25}z_{56}=d(a_8e_{36}^2-b_{16}^2e_{48}+b_{16}d_{28}e_{36}),$$

$$y_{25}w_{88}+a_9z_{56}x_{48}=d(b_{16}F),$$

$$y_{25}w_{100}-a_{21}z_{56}x_{48}=d(a_8I+b_{16}G).$$

We have also

$$y_{25}w_{124}+a_9z_{56}x_{84}=d(-b_{16}\bar{H})+a_9w_{140}\in d\text{-image},$$

$$y_{25}w_{136}-a_{21}z_{56}x_{84}=d(-a_8\bar{J}-b_{16}\bar{G})-a_{21}w_{140}\in d\text{-image},$$

since w_{140} is a nice cocycle.

The line of y_{25} in the table is filled except $y_{25}x_{48}$, $y_{25}x_{120}$ and $y_{25}x_{84}$.

(5) *Products with* y_{61}.

Taking conjugate and using the equivalence of a_9a_{20} and $-a_{21}a_8$, we have the following products in the d-image:

$$y_{61}\cdot(\text{nice}),\quad y_{61}z_{52}+a_9a_{20}x_{84},\quad y_{61}z_{88}+a_9a_{20}x_{120},$$

$$y_{61}a_8,\quad y_{61}a_{20},\quad y_{61}z_{56},$$

$$y_{61}w_{88}-a_9z_{56}x_{84},\quad y_{61}w_{100}+a_{21}z_{56}x_{84},$$

$$y_{61}w_{124}-a_9z_{56}x_{120},\quad y_{61}w_{136}+a_{21}z_{56}x_{120}.$$

The line of y_{61} is filled except for $y_{61}x_{48}$, $y_{61}x_{120}$ and $y_{61}x_{84}$.

(6) *Products with* y_{26} *and* y_{62}.

The product $y_{62}A$ for a nice cocycle A has been shown to be trivial in Case 4 of § 11. Taking conjugate, we see that $y_{26}A$ is also trivial.

The products $-a_9y_{25}$, $-a_{21}y_{25}$, $-a_9y_{61}$ and $-a_{21}y_{61}$ are equivalent by (12.5) to $y_{26}a_8$, $y_{26}a_{20}$, $y_{62}a_8$ and $y_{62}a_{20}$ respectively. The latter (with y_{26} and y_{62}) will be used in the following.

We have that

$$y_{62}z_{52}-a_9a_{21}x_{84}=d(-a_{21}E-c_{41}B),$$

$$y_{62}z_{88}-a_9a_{21}x_{120}=d(-a_{21}\bar{C}-c_{41}\bar{A}),$$

$$y_{62}w_{88}+a_9(a_{21}H+c_{41}w_{100})=d(a_{21}J+c_{41}G),$$

$$y_{62}w_{100}-a_{21}(a_{21}H+c_{41}w_{100})=d(-c_{41}H),$$

$$y_{62}w_{124}+a_9(-a_{21}\bar{F}+c_{41}w_{136})=d(-a_{21}\bar{I}-c_{41}\bar{G}),$$

$$y_{62}w_{136}-a_{21}(-a_{21}\bar{F}+c_{41}w_{136})=d(c_{41}\bar{F}).$$

Recall that $a_{21}H+c_{41}w_{100}=P_{141}=y_{57}x_{84}+(d\text{-image})$ and $-a_{21}\bar{F}+c_{41}w_{136}=P_{177}=y_{57}x_{120}+(d\text{-image})$. Therefore, replacing P_{141} and P_{177} by the right hand sides, we have

$$y_{62}z_{52}-a_9a_{21}x_{84},\quad y_{62}z_{88}-a_9a_{21}x_{120},\quad y_{62}w_{88}+a_9y_{57}x_{84},$$

$$y_{62}w_{100}-a_{21}y_{57}x_{84},\quad y_{62}w_{124}+a_9y_{57}x_{120},\quad y_{62}w_{136}-a_{21}y_{57}x_{120}$$

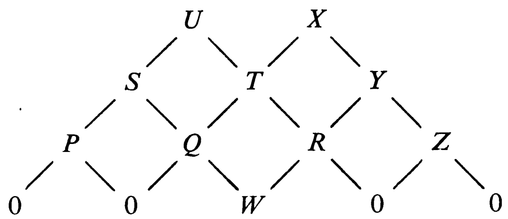
in the d-image.

Taking conjugate, we have the following in the d-image:

$$y_{26}z_{52}+a_9a_{21}x_{48},\quad y_{26}z_{88}+a_9a_{21}x_{84},\quad y_{26}w_{88}-a_9y_{57}x_{48},$$

$$y_{26}w_{100}+a_{21}y_{57}x_{48},\quad y_{26}w_{124}-a_9y_{57}x_{84},\quad y_{26}w_{136}+a_{21}y_{57}x_{84}.$$

We have some more relations. If we have a diagram below,



then

$$d((a_{21}+c_{41}\partial_{21})X)=(a_9+c_{17}\partial_9)(a_{21}+c_{41}\partial_{21})T-y_{62}Z,$$

$$d((a_9+c_{17}\partial_9)U)=-y_{26}P+(a_{21}+c_{41}\partial_{21})(a_9+c_{17}\partial_9)T,$$

from which $y_{26}P+y_{62}Z=d(-(a_9+c_{17}\partial_9)U-(a_{21}+c_{41}\partial_{21})X)$. Thus the following are in the d-image:

$$y_{26}z_{88}+y_{62}z_{52},\quad y_{26}w_{124}+y_{62}w_{88},\quad y_{26}w_{136}+y_{62}w_{100},$$

$$y_{26}x_{120}+y_{62}x_{84},\quad y_{26}x_{84}+y_{62}x_{48}.$$

The first three relations are not used, since each term in them may be replaced by another cocycle. The monomials $y_{26}x_{120}$ and $y_{62}x_{48}$ may be replaced by $-y_{62}x_{84}$ and $-y_{26}x_{84}$ respectively by the last two relations.

The lines of y_{26} and y_{62} are thus filled. q.e.d.

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