J. Math. Kyoto Univ. (JMKYAZ)
20-2 (1980) 327-349

On the cohomology mod p of the classifying spaces of the exceptional Lie groups, II

By

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(Received Feb. 5, 1979)

§1. Introduction

Let p be a prime, G a compact, 1-connected, simple Lie group and $\{G: p\}$ the set $\{X$: compact, associative H-space such that $H^*(X; \boldsymbol{Z}_p) \cong H^*(G; \boldsymbol{Z}_p)$ as Hopf algebras over $\mathscr{A}_p\}$.

The present paper is the second in a series studying the cohomology $H^*(BX; \boldsymbol{Z}_p)$ of the classifying space of $X \in \{G: p\}$ using the Eilenberg-Moore spectral sequence:

$$E_2 \cong \mathrm{Cotor}_A(\boldsymbol{Z}_p, \boldsymbol{Z}_p) \quad \text{with} \quad A = H^*(X; \boldsymbol{Z}_p)$$

and

$$E_\infty \cong \mathscr{G}_r H^*(BX; \boldsymbol{Z}_p).$$

Let E_8 be the compact, simple, 1-connected, exceptional Lie group of rank 8. The purpose of the paper is to determine $\mathrm{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ with $A = H^*(X_8; \boldsymbol{Z}_3)$ for $X_8 \in \{E_8: 3\}$.

The paper is organized as follows:

In Section 2, we construct an acyclic injective resolution of $\boldsymbol{Z}_3$ over $H^*(X_8; \boldsymbol{Z}_3)$ by making use of the twisted tensor product. $\mathrm{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ with $A = H^*(X_8; \boldsymbol{Z}_3)$ is shown to be isomorphic as an algebra to $H(\overline{W}: d) = \mathrm{Ker}\, d/\mathrm{Im}\, d$, where $\overline{W}$ is a differential algebra constructed in Section 2. In Section 3 we introduce two operators ∂_9 and ∂_{21} in a polynomial subalgebra V of $\overline{W}$. We determine cocycles in V using these operators in §§4 and 5. Then in §6 we study some relations among the cocycles obtained in §§4 and 5. These relations will be used in Part III, where the calculation of $\mathrm{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ will be completed and the following will be shown:

Main Theorem. $\mathrm{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$ *is commutative and is generated (as an algebra) by the following* 29 *elements*:

$$a_4, a_8, a_{20}, x_{48}, z_{52}, z_{56}, u_{56}, x_{84}, z_{88}, w_{88}, \dot{z}_{92}, w_{100}, z_{104}, x_{108}, x_{120}, w_{124},$$

$$w_{128}, w_{136}, w_{140}, x_{144}, w_{152}, v_{168}; a_9, a_{21}, y_{25}, y_{26}, y_{57}, y_{61}, y_{62},$$

where the index indicates the degree.

The references of the paper are as listed in Part I.

§2. An injective resolution of Z_3 over $H^*(X_8; Z_3)$

Let $X_8 \in \{E_8: 3\}$. First we recall from [7] the Hopf algebra structure of $H^*(X_8; Z_3)$:

$$(2.1) \qquad H^*(X_8; \boldsymbol{Z}_3) = \boldsymbol{Z}_3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}),$$

where $\deg x_i = i$;

The coalgebra structure is given by

$$(2.2) \qquad \bar{\phi} x_i = 0 \quad \textit{for} \quad i = 3, 7, 8, 19, 20,$$

$$\bar{\phi} x_{15} = x_8 \otimes x_7,$$

$$\bar{\phi} x_{39} = x_{20} \otimes x_{19},$$

$$\bar{\phi} x_{27} = x_8 \otimes x_{19} + x_{20} \otimes x_7,$$

$$\bar{\phi} x_{35} = x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_{20} x_8 \otimes x_7,$$

$$\bar{\phi} x_{47} = x_8 \otimes x_{39} + x_{20} \otimes x_{27} + x_{20} x_8 \otimes x_{19} - x_{20}^2 \otimes x_7,$$

where $\bar{\phi}$ *is the reduced diagonal map induced from the multiplication on* X_8.

Notation. $A = H^*(X_8; \boldsymbol{Z}_3)$ and $\bar{A} = \tilde{H}^*(X_8; \boldsymbol{Z}_3)$.

We shall construct an injective resolution of $\boldsymbol{Z}_3$ over A using the same construction as that in § 3 of [8].

Let L be a graded $\boldsymbol{Z}_3$-submodule of $\bar{A}$ generated by

$$\{x_3, x_7, x_8, x_{19}, x_{20}, x_8^2, x_{20}^2, x_{15}, x_{39}, x_{27}, x_{35}, x_{47}\}.$$

Let $\theta: A \to L$ be the projection and $\iota: L \to A$ the injection such that $\iota \circ \theta = 1_A$. We name the set of corresponding elements under the suspension s as

$$(2.3) \qquad sL = \{a_4, a_8, a_9, a_{20}, a_{21}, c_{17}, c_{41}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}\}.$$

Define $\bar{\theta}: A \to sL$ by $\bar{\theta} = s \circ \theta$ and $\bar{\iota}: sL \to A$ by $\bar{\iota} = \iota \circ s^{-1}$. Let $T(sL)$ be the free tensor algebra over sL with the natural product ψ. Consider the two sided ideal I of $T(sL)$ generated by $\mathrm{Im}\,(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi)(\mathrm{Ker}\,\bar{\theta})$, where ϕ is the diagonal map of A. Put $\bar{W} = T(sL)/I$, that is,

$$\bar{W} = \boldsymbol{Z}_3\{a_4, a_8, a_9, a_{20}, a_{21}, c_{17}, c_{41}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}\}$$

and I is generated by

(2.4) $[\alpha, \beta]$ *for all pairs* (α, β) *of generators of* $T(sL)$ *except*

$$(a_9, b_{16}), \quad (a_9, d_{28}), \quad (a_9, e_{36}), \quad (a_9, e_{48}), \quad (a_{21}, b_{40}), \quad (a_{21}, d_{28}),$$

$$(a_{21}, e_{36}),\quad (a_{21}, e_{48}),\quad (a_9, c_{17}),\quad (a_{21}, c_{41}),$$

$$[a_9, b_{16}]+c_{17}a_8, \qquad [a_{21}, b_{40}]+c_{41}a_{20},$$

$$[a_9, d_{28}]+c_{17}a_{20}, \qquad [a_{21}, d_{28}]+c_{41}a_8,$$

$$[a_9, e_{36}]+c_{17}d_{28}, \qquad [a_{21}, e_{36}]+c_{41}b_{16},$$

$$[a_9, e_{48}]+c_{17}b_{40}, \qquad [a_{21}, e_{48}]+c_{41}d_{28},$$

where $[\alpha, \beta]=\alpha\beta-(-1)^*\beta\alpha$ *with* $*=\deg\alpha\cdot\deg\beta$.

Note. $\overline{W}$ contains the polynomial algebra

$$V=\boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}].$$

We define a map

$$d=-\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi\circ\bar{\imath}: sL \longrightarrow T(sL)$$

and extend it naturally over $T(sL)$ as a derivation. Since $d(I)\subset I$ holds, d induces a map $\overline{W}\to\overline{W}$, which is also denoted by d by abuse of notation. It is easy to check that $d\circ d=0$ and so $\overline{W}$ is a differential algebra over $\boldsymbol{Z}_3$. By the relation

$$d\circ\bar{\theta}+\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\phi=0$$

we can construct the twisted tensor product $W=A\otimes\overline{W}$ with respect to $\bar{\theta}$ [14]. Namely, W is an A-comodule with the differential operator

$$\bar{d}=1\otimes d+(1\otimes\psi)\circ(1\otimes\bar{\theta}\otimes 1)\circ(\phi\otimes 1).$$

More explicitly, the differential operators $\bar{d}$ and d are given by

(2.5) $\bar{d}(x_i\otimes 1)=1\otimes a_{i+1}$ *for* $i=3, 7, 8, 19, 20,$

$$\bar{d}(x_8^2\otimes 1)=1\otimes c_{17}-x_8\otimes a_9,$$

$$\bar{d}(x_{20}^2\otimes 1)=1\otimes c_{41}-x_{20}\otimes a_{21},$$

$$\bar{d}(x_{15}\otimes 1)=1\otimes b_{16}+x_8\otimes a_8,$$

$$\bar{d}(x_{39}\otimes 1)=1\otimes b_{40}+x_{20}\otimes a_{20},$$

$$\bar{d}(x_{27}\otimes 1)=1\otimes d_{28}+x_8\otimes a_{20}+x_{20}\otimes a_8,$$

$$\bar{d}(x_{35}\otimes 1)=1\otimes e_{36}+x_8\otimes d_{28}-x_8^2\otimes a_{20}+x_{20}\otimes b_{16}+x_{20}x_8\otimes a_8,$$

$$\bar{d}(x_{47}\otimes 1)=1\otimes e_{48}+x_8\otimes b_{40}+x_{20}\otimes d_{28}-x_{20}^2\otimes a_8+x_{20}x_8\otimes a_{20};$$

(2.6) $da_i=0$ *for* $i=4, 8, 9, 20, 21,$

$$dc_{17}=a_9^2,$$

$$dc_{41}=a_{21}^2,$$

$$db_{16}=-a_9a_8,$$

$$db_{40} = -a_{21}a_{20},$$

$$dd_{28} = -a_9a_{20} - a_{21}a_8,$$

$$de_{36} = -a_9d_{28} + c_{17}a_{20} - a_{21}b_{16},$$

$$de_{48} = -a_9b_{40} - a_{21}d_{28} + c_{41}a_8.$$

Now we define weights in $W = A \otimes \overline{W}$ as follows:

(2.7)

A :	x_3,	x_7,	x_8,	x_{19},	x_{20},	x_8^2,	x_{20}^2,	x_{15},	x_{39},	x_{27},	x_{35},	x_{47}
sL:	a_4,	a_8,	a_9,	a_{20},	a_{21},	c_{17},	c_{41},	b_{16},	b_{40},	d_{28},	e_{36},	e_{48}
weight:	0	0	1	0	1	2	2	2	2	2	9	9

The weight of a monomial is the sum of the weights of each element.

Define a filtration

(2.8) $$F_r = \{x \mid \text{weight } x \leq r\}.$$

Put $E_0W = \sum_i F_i/F_{i-1}$. Then it is easy to see that

$$E_0W \cong \Lambda(x_3, x_7, x_{19}, x_{15}, x_{39}, x_{27}, x_{35}, x_{47})$$
$$\otimes \boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}]$$
$$\otimes C(Q(x_8)) \otimes C(Q(x_{20})),$$

where $C(Q(x_i))$ is the cobar construction of $\boldsymbol{Z}_3[x_i]/(x_i^3)$ ($i = 8, 20$). The differential formulae (2.5) and (2.6) imply that E_0W is acyclic, and hence W is acyclic.

Theorem 2.9. *W is an injective resolution of $\boldsymbol{Z}_3$ over $A = H^*(X_8; \boldsymbol{Z}_3)$.*

By the definition of Cotor we have

Corollary 2.10. $H(\overline{W}: d) = \text{Ker}\, d/\text{Im}\, d \cong \text{Cotor}_A(\boldsymbol{Z}_3, \boldsymbol{Z}_3)$.

§3. Some formulae

We define operators ∂_9 and ∂_{21} by

(3.1)

x:	a_4	a_8	a_{20}	b_{16}	b_{40}	d_{28}	e_{36}	e_{48}
$\partial_9 x$:	0	0	0	$-a_8$	0	$-a_{20}$	$-d_{28}$	$-b_{40}$
$\partial_{21} x$:	0	0	0	0	$-a_{20}$	$-a_8$	$-b_{16}$	$-d_{28}$

and extend them over $V = \boldsymbol{Z}_3[a_4, a_8, a_{20}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}]$ so that they satisfy

(3.2) $$\partial_j(P+Q) = \partial_jP + \partial_jQ \quad \textit{and} \quad \partial_j(PQ) = \partial_jP \cdot Q + P\partial_jQ$$

for any polynomials P and Q in V ($j = 9, 21$).

Then we have

Lemma 3.3. *For any polynomial P in V, we have*

(1) $\partial_9^3 P=0$, $\partial_{21}^3 P=0$ *and* $\partial_9\partial_{21}P=\partial_{21}\partial_9 P$;

(2) $[a_9, P]=c_{17}\partial_9 P$ *and* $[a_{21}, P]=c_{41}\partial_{21}P$;

(3) $dP=a_9\partial_9 P+c_{17}\partial_9^2 P+a_{21}\partial_{21}P+c_{41}\partial_{21}^2 P$.

Proof. (By induction.)

(1) Suppose that $\partial_9^3 P=0$ holds for any polynomial of degree up to l. Then for a monomial xP of degree $l+1$, we have

$$\partial_9^3(xP)=\partial_9^3 x\cdot P+x\partial_9^3 P=0.$$

Thus $\partial_9^3 P=0$ holds for any polynomial of degree $l+1$. Similarly, $\partial_{21}^3 P=0$.

Now, suppose that $\partial_9\partial_{21}P=\partial_{21}\partial_9 P$ holds for any polynomial of degree up to l. Then for a monomial xP of degree $l+1$, we have

$$\begin{aligned}\partial_9\partial_{21}(xP)&=\partial_9(\partial_{21}x\cdot P+x\partial_{21}P)\\&=\partial_9\partial_{21}x\cdot P+\partial_{21}x\cdot\partial_9 P+\partial_9 x\cdot\partial_{21}P+x\partial_9\partial_{21}P\\&=\partial_{21}\partial_9 x\cdot P+\partial_9 x\cdot\partial_{21}P+\partial_{21}x\cdot\partial_9 P+x\partial_{21}\partial_9 P\\&=\partial_{21}\partial_9(xP).\end{aligned}$$

Thus the relation $\partial_9\partial_{21}P=\partial_{21}\partial_9 P$ holds for any polynomial of degree $l+1$.

(2) Suppose that $[a_9, P]=c_{17}\partial_9 P$ holds for any polynomial of degree up to l. Then for a monomial xP of degree $l+1$, we have

$$[a_9, xP]=[a_9, x]P+x[a_9, P]=c_{17}\partial_9 x\cdot P+xc_{17}\partial_9 P=c_{17}\partial_9(xP).$$

Thus the relation $[a_9, P]=c_{17}\partial_9 P$ holds for any polynomial of degree $l+1$.

The relation $[a_{21}, P]=c_{41}\partial_{21}P$ is proved similarly.

(3) Suppose that the differential formula holds for any polynomial P of degree up to l. Then for a monomial xP of degree $l+1$, we have

$$\begin{aligned}d(xP)&=dx\cdot P+xdP\\&=(a_9\partial_9 x+c_{17}\partial_9^2 x+a_{21}\partial_{21}x+c_{41}\partial_{21}^2 x)P\\&\quad+x(a_9\partial_9 P+c_{17}\partial_9^2 P+a_{21}\partial_{21}P+c_{41}\partial_{21}^2 P)\\&=(a_9\partial_9 x+c_{17}\partial_9^2 x+a_{21}\partial_{21}x+c_{41}\partial_{21}^2 x)P\\&\quad+(a_9 x-c_{17}\partial_9 x)\partial_9 P+c_{17}x\partial_9^2 P\\&\quad+(a_{21}x-c_{41}\partial_{21}x)\partial_{21}P+c_{41}x\partial_{21}^2 P\\&=a_9\partial_9(xP)+c_{17}\partial_9^2(xP)+a_{21}\partial_{21}(xP)+c_{41}\partial_{21}^2(xP).\end{aligned}$$

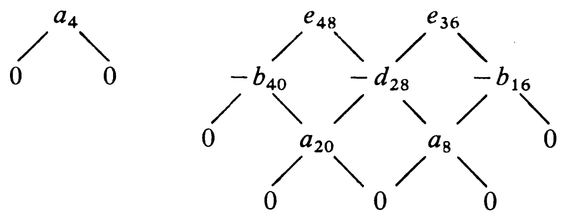
Thus the differential formula holds for any polynomial of degree $l+1$. q.e.d.

Lemma 3.4. *Let P be non-trivial in V. Then P is a non-trivial cocycle if and only if* $\partial_9 P=\partial_{21} P=0$.

Proof. If P is a cocycle, $dP=0$. Then by the differential formula (3) of Lemma 3.3, we have $\partial_9 P=0$ and $\partial_{21} P=0$.

Conversely, if $\partial_9 P=\partial_{21} P=0$, then $\partial_9^2 P=\partial_{21}^2 P=0$, hence we have $dP=0$ by the differential formula (3) of Lemma 3.3. Since P contains neither a_9 nor a_{21}, it is not in the d-image, hence it is a non-trivial cocycle. q.e.d.

We shall make much use of diagrams, in which an oblique line $/$ means ∂_9 and an oblique line $\backslash$ means ∂_{21}. The generators of V form the following diagrams:



Observe that the diagram on the right is symmetric.

Definition. We call two polynomials in V *conjugate* if one is obtained from the other by interchanging a_8 and a_{20}, b_{16} and b_{40}, and, e_{36} and e_{48}.

Then the role of ∂_9 and ∂_{21} are interchanged.

Notation. $\bar{P}$ = the conjugate of P.

We see, in particular, that P is a cocycle if and only if its conjugate $\bar{P}$ is.

We shall find cocycles in the following steps:

(0) cocycles in $Z_3[a_4]$,

(i) those in $Z_3[a_8, a_{20}, b_{16}, b_{40}, d_{28}]$,

(ii) those in $Z_3[a_8, a_{20}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}]$,

(iii) those with a_9, c_{17}, a_{21}, c_{41},

(iv) those with a_{21} and c_{41} but without a_9 or c_{17}.

(The first two steps are done in §4, (ii) in §5 and the last two steps will be in §9 and §11 of Part III.)

§4. Cocycles without elements of odd degree-I

(0) Cocycles in $Z_3[a_4]$

Clearly a_4 is a cocycle, and it is the only indecomposable one in $Z_3[a_4]$.

We see by (3.1) that a_4 is independent from the other generators in V under the operators ∂_9 and ∂_{21}. Therefore Steps (i) and (ii) are done independently from

Step (0).

(i) Cocycles in $\boldsymbol{Z}_3[a_8, a_{20}, b_{16}, b_{40}, d_{28}]$

Clearly, the elements a_8 and a_{20} are cocycles.

An element of degree 1 with respect to b_{16}, b_{40} and d_{28} is of the form

$$P=Ab_{16}+Bb_{40}+Cd_{28} \quad \text{with} \quad A,\ B,\ C\in \boldsymbol{Z}_3[a_8,\ a_{20}].$$

By Lemma 3.4, P is a cocycle if and only if

$$\partial_9 P=-a_8A-a_{20}C=0 \quad \text{and} \quad \partial_{21}P=-a_{20}B-a_8C=0.$$

Then we have a cocycle

$$u_{56}=a_{20}^2b_{16}+a_8^2b_{40}-a_8a_{20}d_{28}.$$

A cocycle of degree 2 with respect to b_{16}, b_{40} and d_{28} is of the form

$$P=Ab_{16}^2+Bb_{40}^2+Cd_{28}^2-Db_{16}b_{40}+Eb_{40}d_{28}+Fb_{16}d_{28},$$

where the coefficients $A,\dots,F$ are in $\boldsymbol{Z}_3[a_8,\ a_{20}]$. The conditions

$$\partial_9 P=(Aa_8-Fa_{20})b_{16}+(Da_8-Ea_{20})b_{40}+(Ca_{20}-Fa_8)d_{28}=0,$$

$$\partial_{21}P=(Ba_{20}-Ea_8)b_{40}+(Da_{20}-Fa_8)b_{16}+(Ca_8-Ea_{20})d_{28}=0$$

give rise to

$$Aa_8-Fa_{20}=0, \quad Da_8-Ea_{20}=0, \quad Ca_{20}-Fa_8=0,$$

$$Ba_{20}-Ea_8=0, \quad Da_{20}-Fa_8=0, \quad Ca_8-Ea_{20}=0.$$

Then we have a decomposable cocycle

$$P=a_{20}^4b_{16}^2+a_8^4b_{40}^2+a_8^2a_{20}^2d_{28}^2-a_8^2a_{20}^2b_{16}b_{40}+a_8^3a_{20}b_{40}d_{28}+a_8a_{20}^3b_{16}d_{28}=u_{56}^2.$$

A cocycle of degree 3 with respect to b_{16}, b_{40} and d_{28} is of the form

$$P=Ab_{16}^3+Bb_{40}^3+Cd_{28}^3+Db_{16}^2b_{40}+Eb_{16}b_{40}^2$$
$$+Fb_{16}^2d_{28}+Gb_{16}d_{28}^2+Hb_{40}^2d_{28}+Ib_{40}d_{28}^2.$$

Then $\partial_9 P=0$ gives rise to

$$\partial_9 A=\partial_9 B=\partial_9 C=0 \quad \text{and} \quad D=F=G=I=0$$

and $\partial_{21}P=0$ gives rise to

$$\partial_{21}A=\partial_{21}B=\partial_{21}C=0 \quad \text{and} \quad E=H=I=G=0.$$

Therefore $P=Ab_{16}^3+Bb_{40}^3+Cd_{28}^3$, where A, B and C are cocycles by Lemma 3.4. Thus $x_{48}=b_{16}^3$, $x_{120}=b_{40}^3$ and $x_{84}=d_{28}^3$ are all the indecomposable cocycles of degree 3.

It is not hard to see that there is no indecomposable cocycle of degree greater than 3. Hence

(4.1) *The following are all the indecomposable cocycles in* $\boldsymbol{Z}_3[a_8, a_{20}, b_{16}, b_{40}, d_{28}]$:

$$a_8,\quad a_{20},\quad u_{56}=a_{20}^2b_{16}+a_8^2b_{40}-a_8a_{20}d_{28},\quad x_{48}=b_{16}^3,\quad x_{120}=b_{40}^3,\quad x_{84}=d_{28}^3.$$

The following diagrams will be of use in the next step:

$$-b_{16}b_{40}+d_{28}^2$$

$$a_{20}d_{28}+a_8b_{40} \qquad a_{20}b_{16}+a_8d_{28}$$

(4.2.1)

$$-a_{20}b_{16}^2-a_8b_{16}d_{28} \qquad\qquad -a_8b_{40}^2-a_{20}b_{40}d_{28}$$

$$a_8^2d_{28} \quad a_8^2b_{16} \qquad\qquad a_{20}^2b_{40} \quad a_{20}^2d_{28}$$

(4.2.2) (4.2.3)

$$\eta \qquad \xi$$

$$u_{56}b_{40} \qquad u_{56}d_{28} \qquad u_{56}b_{16}$$

(4.2.4)

$$\lambda \qquad\qquad \mu \qquad\qquad \nu$$

$$\xi d_{28} \quad \xi b_{16} \qquad \eta b_{40} \quad \eta d_{28} \qquad \eta d_{28}+\xi b_{40} \quad \eta b_{16}+\xi d_{28}$$

(4.2.5) (4.2.6) (4.2.7)

where we put

(4.3)
$$\xi=-a_{20}b_{16}^2b_{40}-a_8b_{16}b_{40}d_{28}+a_{20}b_{16}d_{28}^2+a_8d_{28}^3,$$

$$\eta=-a_8b_{16}b_{40}^2-a_{20}b_{16}b_{40}d_{28}+a_8b_{40}d_{28}^2+a_{20}d_{28}^3,$$

$$\lambda=-b_{16}^3b_{40}^2-b_{16}^2b_{40}d_{28}^2-b_{16}d_{28}^4,$$

$$\mu=-b_{16}^2b_{40}^3-b_{16}b_{40}^2d_{28}^2-b_{40}d_{28}^4,$$

$$\nu=b_{16}^2b_{40}^2d_{28}+b_{16}b_{40}d_{28}^3+d_{28}^5.$$

Note that $\xi=\bar{\eta}$ and $\lambda=\bar{\mu}$.

§5. Cocycles without elements of odd degree-II

(ii) Cocycles in $\boldsymbol{Z_3[a_8, a_{20}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}]}$

(ii-1) A cocycle of degree 1 with respect to e_{36} and e_{48} is of the form $P=Ae_{36}$

$+Be_{48}+C$, where A, B, $C\in \boldsymbol{Z}_3[a_8, a_{20}, b_{16}, b_{40}, d_{28}]$. Then the relations

$$\partial_9 P=\partial_9 A\cdot e_{36}+\partial_9 B\cdot e_{48}-Ad_{28}-Bb_{40}+\partial_9 C=0,$$

$$\partial_{21}P=\partial_{21}A\cdot e_{36}+\partial_{21}B\cdot e_{48}-Ab_{16}-Bd_{28}+\partial_{21}C=0$$

give rise to

(5.1) $$\partial_9 A=\partial_9 B=0, \qquad \partial_{21}A=\partial_{21}B=0,$$

$$\partial_9 C=Ad_{28}+Bb_{40}, \qquad \partial_{21}C=Ab_{16}+Bd_{28}.$$

So by (4.1) A and B are cocycles in $\boldsymbol{Z}_3[a_8, a_{20}, u_{56}, x_{48}, x_{120}, x_{84}]$ and C must satisfy the following diagram:

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ Ad_{28}+Bb_{40} & & Ab_{16}+Bd_{28} \end{array}$$

Lemma 5.2. *The following are all the indecomposable cocycles of degree* 1 *with respect to* e_{36} *and* e_{48}:

$$z_{56}=a_{20}e_{36}+a_8e_{48}-(b_{16}b_{40}-d_{28}^2),$$

$$z_{52}=a_8^2e_{36}-(a_{20}b_{16}^2+a_8b_{16}d_{28}),$$

$$z_{88}=a_{20}^2e_{48}-(a_8b_{40}^2+a_{20}b_{40}d_{28}),$$

$$z_{92}=u_{56}e_{36}+\xi,$$

$$z_{104}=u_{56}e_{48}+\eta.$$

(For later use we choose z_{92} and z_{140} with the terms $a_8d_{28}^3$ and $a_{20}d_{28}^3$ respectively so that they are in the $\partial_9^2\partial_{21}^2$-image.)

Proof. First, we shall find cocycles P with $A\neq 0$ by seeking B and C satisfying $Ad_{28}=\partial_9 C-Bb_{40}$ and $Ab_{16}=\partial_{21}C-Bd_{28}$. It is sufficient to choose one such P for a cocycle A, since the difference of two cocycles with the same term Ae_{36} is a cocycle with $A=0$.

Note that the elements $x_{48}=b_{16}^3$, $x_{120}=x_{40}^3$ and $x_{84}=d_{28}^3$ are 'immobile' in seeking B and C, that is to say: when $A=A'x_{48}^ix_{120}^jx_{84}^k+A''$ with i, j and k non-negative integers, there exist B and C satisfying $Ad_{28}=\partial_9 C-Bb_{40}$ and $Ab_{16}=\partial_{21}C-Bd_{28}$ if and only if there exist B' and C' satisfying $A'd_{28}=\partial_9 C'-B'b_{40}$ and $A'b_{16}=\partial_{21}C'-B'd_{28}$, and then $B=B'x_{48}^ix_{120}^jx_{84}^k$ and $C=C'x_{48}^ix_{120}^jx_{84}^k$. In particular, there is no cocycle P for $A=x_{48}^ix_{120}^jx_{84}^k+A''$ for any cocycle A''. Thus we have only to study those A as in $\boldsymbol{Z}_3[a_8, a_{20}, u_{56}]$.

For $A=u_{56}$, we have $C=\xi$ with $B=0$ as in (4.2.4) and hence we obtain an indecomposable cocycle

$$z_{92}=u_{56}e_{36}+\xi.$$

For $A=a_8$, $Ad_{28}=a_8d_{28}=\partial_9(-b_{16}d_{28})-a_{20}b_{16}$ cannot be of the form ∂_9C-Bb_{40}. And we also see that there exist no B or C for $A=a_8+A'$ for any A'.

For $A=a_8^2$, we have $C=-a_{20}b_{16}^2-a_8b_{16}d_{28}$ with $B=0$ as in (4.2.2) and hence we obtain an indecomposable cocycle

$$z_{52}=a_8^2e_{36}-(a_{20}b_{16}^2+a_8b_{16}d_{28}).$$

For $A=a_{20}$, we have $B=a_8$ and $C=-b_{16}b_{40}+d_{28}^2$ as in (4.2.1) and hence we obtain a cocycle

$$z_{56}=a_{20}e_{36}+a_8e_{48}-(b_{16}b_{40}-d_{28}^2).$$

The element z_{56} is indecomposable, since there is no cocycle P with $A=a_{20}$ and $B=0$.

We have shown that for $A=u_{56}A'+a_8^2A''+a_{20}A'''$ we have a cocycle P with the term Ae_{36} and that $P=z_{92}A'+z_{52}A''+z_{56}A'''$. Therefore, the elements z_{92}, z_{52} and z_{56} are all the indecomposable cocycles with $A\neq 0$.

For $A=0$ we now find cocycles of the form $P=Be_{48}+C$ for cocycles B in $\boldsymbol{Z}_3[a_8, a_{20}, u_{56}]$, since the elements x_{48}, x_{120} and x_{84} are again immobile in seeking C.

Note that it suffices to choose one P with the term Be_{48}, since the difference of two such P's is a cocycle with $A=B=0$.

For $B=u_{56}$, we have the conjugate of z_{92}:

$$z_{104}=u_{56}e_{48}+\eta,$$

which is indecomposable.

For $B=a_8$, there is no cocycle P, since otherwise its conjugate $\bar{P}$ would be a cocycle with $A=a_{20}$ and $B=0$, but the existence of such a cocycle is already denied.

For $B=a_8^2$, a_{20}, a_8a_{20} or $a_8^2a_{20}$, there is no cocycle of the form $Be_{48}+C$.

For $B=a_8^3$, we have a decomposable cocycle $P=a_8^2z_{56}-a_{20}z_{52}=a_8^3e_{48}+C$.

For $B=a_{20}^2$, we have the conjugate cocycle of z_{52}:

$$z_{88}=a_{20}^2e_{48}-(a_8b_{40}^2+a_{20}b_{40}d_{28}),$$

which is indecomposable, since there is no cocycle with the term $a_{20}e_{48}$.

Thus for $B=u_{56}B'+a_8^3B''+a_{20}^2B'''$ we have a cocycle P of the form $Be_{48}+C$, and we see that the elements z_{104} and z_{88} are all the indecomposable cocycles with $A=0$.

We have shown that the elements z_{56}, z_{52}, z_{88}, z_{92} and z_{104} are all the indecomposable cocycles of degree 1 with respect to e_{36} and e_{48}. q.e.d.

(ii–2) A cocycle of degree 2 with respect to e_{36} and e_{48} is of the form

$$P=Ae_{36}^2-Be_{36}e_{48}+Ce_{48}^2-De_{36}-Ee_{48}-F$$

with A, B, C, D, E and F in $\boldsymbol{Z}_3[a_8, a_{20}, b_{16}, b_{40}, d_{28}]$. The relations

$$\partial_9P=\partial_9A\cdot e_{36}^2-\partial_9B\cdot e_{36}e_{48}+\partial_9C\cdot e_{48}^2+(Ad_{28}+Bb_{40}-\partial_9D)e_{36}$$

$$+(Bd_{28}+Cb_{40}-\partial_9 E)e_{48}+(Dd_{28}+Eb_{40}-\partial_9 F)$$
$$=0,$$

$$\partial_{21}P=\partial_{21}A\cdot e_{36}^2-\partial_{21}B\cdot e_{36}e_{48}+\partial_{21}C\cdot e_{48}^2+(Ab_{16}+Bd_{28}-\partial_{21}D)e_{36}$$
$$+(Bd_{16}+Cd_{28}-\partial_{21}E)e_{48}+(Db_{16}+Ed_{28}-\partial_{21}F)$$
$$=0$$

give rise to

(5.3) $$\partial_9 A=\partial_9 B=\partial_9 C=0, \qquad \partial_{21}A=\partial_{21}B=\partial_{21}C=0,$$
$$\partial_9 D=Ad_{28}+Bb_{40}, \qquad \partial_{21}D=Ab_{16}+Bd_{28},$$
$$\partial_9 E=Bd_{28}+Cb_{40}, \qquad \partial_{21}E=Bb_{16}+Cd_{28},$$
$$\partial_9 F=Dd_{28}+Eb_{40}, \qquad \partial_{21}F=Db_{16}+Ed_{28}.$$

Thus A, B and C are cocycles in $\boldsymbol{Z}_3[a_8, a_{20}, u_{56}, x_{48}, x_{120}, x_{84}]$ and D, E and F form the following diagrams:

```
            D                              E
          /   \                          /   \
Ad28 + Bb40    Ab16 + Bd28     Bd28 + Cb40    Bb16 + Cd28

                         F
                       /   \
             Dd28 + Eb40    Db16 + Ed28
```

Lemma 5.4. *The elements w_{88}, w_{136}, w_{100}, w_{124}, w_{128}, w_{152} and w_{140} are all the indecomposable cocycles of degree 2 with respect to e_{36} and e_{48}, where the coefficients of w_i's are as follows:*

P	A	B	C	D	E	F
w_{88}	a_8^2	0	0	$-a_{20}b_{16}^2$ $-a_8b_{16}d_{28}$	0	$b_{16}^3b_{40}$ $-b_{16}^2d_{28}^2$
w_{136}	0	0	a_{20}^2	0	$-a_8b_{40}^2$ $-a_{20}b_{40}d_{28}$	$b_{16}b_{40}^3$ $-b_{40}^2d_{28}^2$
w_{100}	a_8a_{20}	a_8^2	0	$-a_8b_{16}b_{40}$ $+a_8d_{28}^2$	$-a_{20}b_{16}^2$ $-a_8b_{16}d_{28}$	$b_{16}^2b_{40}d_{28}$ $-b_{16}d_{28}^3$
w_{124}	0	a_{20}^2	a_8a_{20}	$-a_8b_{40}^2$ $-a_{20}b_{40}d_{28}$	$-a_{20}b_{16}b_{40}$ $+a_{20}d_{28}^2$	$b_{16}b_{40}^2d_{28}$ $-b_{40}d_{28}^3$
w_{128}	u_{56}	0	0	ξ	0	λ
w_{152}	0	0	u_{56}	0	η	μ
$-w_{140}$	0	u_{56}	0	η	ξ	ν

Proof. The elements x_{48}, x_{120} and x_{84} are immobile with respect to the determination of cocycles P. Thus we have only to study cocycles A, B and C in $\boldsymbol{Z}_3[a_8, a_{20}, u_{56}]$.

Note the number of a_8's and a_{20}'s in each term of A, B, C, D, E and F. The elements D and E have one more a_8 or a_{20} in each term than F. The elements A, B and C have one more than D and E. Therefore, A, B and C must have at least one a_8^2, a_8a_{20} or a_{20}^2 in each term. (The cocycle u_{56} has a_8^2, a_8a_{20} and a_{20}^2.)

The determination of cocycles is divided into three cases: (1) $A \neq 0$; (2) $A=0$ and $C \neq 0$; (3) $A=C=0$ and $B \neq 0$.

(1) Now, let $A \neq 0$. It is sufficient to find, if any, one cocycle P for a cocycle A, since the difference of two cocycles with the same term Ae_{36}^2 is a cocycle with $A=0$.

For $A=u_{56}$, we have $D=\xi$ and $F=\lambda$ with $B=C=E=0$ as in (4.2.4) and in (4.2.5), and hence we obtain an indecomposable cocycle

$$w_{128}=u_{56}e_{36}^2-\xi e_{36}-\lambda.$$

For $A=a_8^2$ with $B=C=0$, we have a cocycle

$$w_{88}=a_8^2e_{36}^2+(a_{20}b_{16}^2+a_8b_{16}d_{28})e_{36}-(b_{16}^3b_{40}-b_{16}^2d_{28}^2),$$

which is indecomposable, since there is no cocycle beginning with a_8e_{36}.

For $A=a_8a_{20}$ with $B=a_8^2$, we have a cocycle

$$w_{100}=a_8a_{20}e_{36}^2-a_8^2e_{36}e_{48}+(a_8b_{16}b_{40}-a_8d_{28}^2)e_{36}$$
$$+(a_{20}b_{16}^2+a_8b_{16}d_{28})e_{48}-(b_{16}^2b_{40}d_{28}-b_{16}d_{28}^3),$$

which is also indecomposable, since there is neither a cocycle with $A=a_8a_{20}$ and $B=0$ nor one beginning with a_8e_{36}.

For $A=a_{20}^2$ with $B=a_8a_{20}$ and $C=a_8^2$, we have a decomposable cocycle $P=z_{56}^2 = a_{20}^2e_{36}^2-a_8a_{20}e_{36}e_{48}+a_8^2e_{48}^2+$(some other terms). It is easily seen that there is no cocycle P with $A=a_{20}^2$ and $B=0$ or $C=0$.

We have shown that for a cocycle A which has at least one u_{56}, a_8^2, a_8a_{20} or a_{20}^2 in each term, there exists a corresponding cocycle P, which is decomposable except for w_{128}, w_{88} and w_{100}.

(2) Now consider the case $A=0$ and $C \neq 0$. It is sufficient to find one P with $A=0$ for a cocycle C, since the difference of two such P's is a cocycle with $A=C=0$.

Now, taking conjugate of w_{128} and w_{88}, we obtain two indecomposable cocycles with $A=B=0$:

$$w_{152}=u_{56}e_{48}^2-\eta e_{48}-\mu,$$
$$w_{136}=a_{20}^2e_{48}^2+(a_8b_{40}^2+a_{20}b_{40}d_{28})e_{48}-(b_{16}b_{40}^3-b_{40}^2d_{28}^2).$$

We also have the conjugate of w_{100}, a cocycle for $C=a_8a_{20}$ with $B=a_{20}^2$:

$$w_{124}=-a_{20}^2e_{36}e_{48}+a_8a_{20}e_{48}^2+(a_8b_{40}^2+a_{20}b_{40}d_{28})e_{36}$$
$$+(a_{20}b_{16}b_{40}-a_{20}d_{28}^2)e_{48}-(b_{16}b_{40}^2d_{28}-b_{40}d_{28}^3),$$

which is indecomposable as is w_{100}.

There is no cocycle for $C=a_8a_{20}$ with $B=0$.

For $C=a_8^2$, there is no cocycle P with $A=0$, since otherwise its conjugate $\bar{P}$ would be a cocycle with $A=a_{20}^2$ and $C=0$, the existence of which is already denied.

For $C=a_8^3$, we have a cocycle $a_8z_{36}^2-a_{20}w_{100}$ with $B=0$.

We see that, if C has one u_{56}, a_8^3, a_8a_{20} or a_{20}^2 in each term, the corresponding cocycle P exists and is decomposable except for w_{152}, w_{136} and w_{124}.

(3) Finally, consider the case $A=C=0$ and $B\neq 0$.

For $B=u_{56}$, we have an indecomposable cocycle

$$-w_{140}=-u_{56}e_{36}e_{48}-\eta e_{36}-\xi e_{48}-\nu.$$

For $B=a_8^2$, a_8a_{20} or a_{20}^2, there is no cocycle P with $A=C=0$.

For $B=a_8^3$, we have $P=a_8w_{100}-a_{20}w_{88}$, and for $B=a_{20}^3$, we have $P=a_{20}w_{124}-a_8w_{136}$.

For $B=a_8^2a_{20}$ or $a_8a_{20}^2$, there is no cocycle with $A=C=0$.

For $B=a_8^2a_{20}^2$, we have a decomposable cocycle $P=-z_{52}z_{88}$.

Thus, for those B which have at least one u_{56}, a_8^3, a_{20}^3 or $a_8^2a_{20}^2$ in each term there exists a corresponding cocycle with $A=C=0$. Then we see that w_{140} is the only indecomposable cocycle with $A=C=0$.

We have shown that w_{88}, w_{136}, w_{100}, w_{124}, w_{128}, w_{152} and w_{140} are all the indecomposable cocycles of degree 2 with respect to e_{36} and e_{48}. q.e.d.

(ii–3) Clearly $x_{108}=e_{36}^3$ and $x_{144}=e_{48}^3$ are indecomposable cocycles. The other cocycle of degree 3 with respect to e_{36} and e_{48} is of the form

$$P=Ae_{36}^2e_{48}+Be_{36}e_{48}^2+Ce_{36}^2-De_{36}e_{48}+Ee_{48}^2-Fe_{36}-Ge_{48}-H$$

with coefficients $A,\dots, H$ in $\boldsymbol{Z}_3[a_8, a_{20}, b_{16}, b_{40}, d_{28}]$. Then the relations

$$\begin{aligned}\partial_9P=&\partial_9A\cdot e_{36}^2e_{48}+\partial_9B\cdot e_{36}e_{48}^2+(\partial_9C-Ab_{40})e_{36}^2\\&-(\partial_9D-Ad_{28}-Bd_{40})e_{36}e_{48}+(\partial_9E-Bd_{28})e_{48}^2\\&-(\partial_9F-Cd_{28}-Db_{40})e_{36}-(\partial_9G-Dd_{28}-Eb_{40})e_{48}\\&-(\partial_9H-Fd_{28}-Gb_{40})\\=&0,\end{aligned}$$

$$\begin{aligned}\partial_{21}P=&\partial_{21}A\cdot e_{36}^2e_{48}+\partial_{21}B\cdot e_{36}e_{48}^2+(\partial_{21}C-Ad_{28})e_{36}^2\\&-(\partial_{21}D-Ab_{16}-Bd_{28})e_{36}e_{48}+(\partial_{21}E-Bb_{16})e_{48}^2\\&-(\partial_{21}F-Cb_{16}-Dd_{28})e_{36}-(\partial_{21}G-Db_{16}-Ed_{28})e_{48}\\&-(\partial_{21}H-Fb_{16}-Gd_{28})\\=&0\end{aligned}$$

give rise to

(5.5)
$$\begin{aligned} &\partial_9 A=\partial_9 B=0, && \partial_{21}A=\partial_{21}B=0,\\ &\partial_9 C=Ab_{40}, && \partial_{21}C=Ad_{28},\\ &\partial_9 D=Ad_{28}+Bb_{40}, && \partial_{21}D=Ab_{16}+Bd_{28},\\ &\partial_9 E=Bd_{28}, && \partial_{21}E=Bb_{16},\\ &\partial_9 F=Cd_{28}+Db_{40}, && \partial_{21}F=Cb_{16}+Dd_{28},\\ &\partial_9 G=Dd_{28}+Eb_{40}, && \partial_{21}G=Db_{16}+Ed_{28},\\ &\partial_9 H=Fd_{28}+Gb_{40}, && \partial_{21}H=Fb_{16}+Gd_{28}. \end{aligned}$$

Thus, A and B are cocycles in $\boldsymbol{Z}_3[a_8, a_{20}, u_{56}, x_{48}, x_{120}, x_{84}]$ and $C,\dots, H$ satisfy the following diagrams:

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ Ab_{40} & & Ad_{28} \end{array} \qquad \begin{array}{ccc} & D & \\ \swarrow & & \searrow \\ Ad_{28}+Bb_{40} & & Ab_{16}+Bd_{28} \end{array}$$

$$\begin{array}{ccc} & E & \\ \swarrow & & \searrow \\ Bd_{28} & & Bb_{16} \end{array} \qquad \begin{array}{ccc} & F & \\ \swarrow & & \searrow \\ Cd_{28}+Db_{40} & & Cd_{16}+Dd_{28} \end{array}$$

$$\begin{array}{ccc} & G & \\ \swarrow & & \searrow \\ Dd_{28}+Eb_{40} & & Db_{16}+Ed_{28} \end{array} \qquad \begin{array}{ccc} & H & \\ \swarrow & & \searrow \\ Fd_{28}+Gb_{40} & & Fb_{16}+Gd_{28} \end{array}$$

Put

$$\begin{aligned} v_{168}=&\,a_8a_{20}^2e_{36}^2e_{48}+a_8^2a_{20}e_{36}e_{48}^2-a_8(a_8b_{40}^2+a_{20}b_{40}d_{28})e_{36}^2\\ &+a_8a_{20}(b_{16}b_{40}-d_{28}^2)e_{36}e_{48}-a_{20}(a_{20}b_{16}^2+a_8b_{16}d_{28})e_{48}^2\\ &-a_8(b_{16}b_{40}^2d_{28}-b_{40}d_{28}^3)e_{36}-a_{20}(b_{16}^2b_{40}d_{28}-b_{16}d_{28}^3)e_{48}\\ &-(b_{16}^2b_{40}^2d_{28}^2+b_{16}b_{40}d_{28}^4+d_{28}^6). \end{aligned}$$

Lemma 5.6. *The element v_{168} is the only indecomposable cocycle of degree* 3 *with respect to e_{36} and e_{48} other than $x_{108}=e_{36}^3$ and $x_{144}=e_{48}^3$.*

Proof. The elements x_{48}, x_{120} and x_{84} are again immobile in all the determination of cocycles P, so we have only to study A and B in $\boldsymbol{Z}_3[a_8, a_{20}, u_{56}]$. This time A and B must have at least one a_8^3, $a_8^2a_{20}$, $a_8a_{20}^2$ or a_{20}^3 in each term.

Firstly, suppose $A\neq 0$.

For $A=a_8u_{56}$ with $B=0$, we have

$$\begin{aligned} P&=z_{56}w_{128}-a_{20}u_{56}e_{36}^3\\ &=a_8u_{56}e_{36}^2e_{48}+(\text{some other terms}). \end{aligned}$$

For $A=a_{20}u_{56}$ with $B=a_8u_{56}$, we have

$$P=z_{56}w_{140}=a_{20}u_{56}e_{36}^2e_{48}+(\text{some other terms}),$$

but there is no P with $B=0$.

For $A=u_{56}^2$ with $B=0$, we have

$$P=z_{104}w_{128}=u_{56}^2e_{36}^2e_{48}+(\text{some other terms}).$$

For $A=a_{20}^3$ with $B=0$, we have

$$\begin{aligned}P&=-z_{56}w_{124}+a_8^2a_{20}e_{48}^3\\&=a_{20}^3e_{36}^2e_{48}+(\text{some other terms}).\end{aligned}$$

For $A=a_8^3$ with $B=0$, we have

$$\begin{aligned}P&=z_{56}w_{88}-a_8^2a_{20}e_{36}^3\\&=a_8^3e_{36}^2e_{48}+(\text{some other terms}).\end{aligned}$$

For $A=a_8a_{20}^2$, we have a cocycle

$$\begin{aligned}v_{168}=&a_8a_{20}^2e_{36}^2e_{48}+a_8^2a_{20}e_{36}e_{48}^2-a_8(a_8b_{40}^2+a_{20}b_{40}d_{28})e_{36}^2\\&+a_8a_{20}(b_{16}b_{40}-d_{28}^2)e_{36}e_{48}-a_{20}(a_{20}b_{16}^2+a_8b_{16}d_{28})e_{48}^2\\&-a_8(b_{16}b_{40}^2d_{28}-b_{40}d_{28}^3)e_{36}-a_{20}(b_{16}^2b_{40}d_{28}-b_{16}d_{28}^3)e_{48}\\&-(b_{16}^2b_{40}^2d_{28}^2+b_{16}b_{40}d_{28}^4+d_{28}^6),\end{aligned}$$

which is indecomposable, since there is no P with $A=a_8a_{20}^2$ and $B=0$.

For $A=a_8^2a_{20}$, there is no cocycle P.

For $A=a_8^2a_{20}^2$ with $B=0$, we have

$$P=z_{88}w_{88}=a_8^2a_{20}^2e_{36}^2e_{48}+(\text{some other terms}).$$

Thus, if A is a cocycle with at least one a_8u_{56}, $a_{20}u_{56}$, u_{56}^2, a_8^3, $a_8a_{20}^2$ or a_{20}^3 in each term, there is a cocycle P beginning with the term $Ae_{36}^2e_{48}$, which is decomposable except for v_{168}.

Now consider the case $A=0$.

We obtain cocycles with the term $Be_{36}e_{48}^2$ from those with $Ae_{36}^2e_{48}$ by taking conjugate. In fact, for each of $B=a_{20}u_{56}$, u_{56}^2, a_8^3, a_{20}^3 and $a_8^2a_{20}^2$, we have a decomposable cocycle P with $A=0$.

For $B=a_8u_{56}$, there is no cocycle P with $A=0$, since otherwise there would be a cocycle with $A=a_{20}u_{56}$ and $B=0$.

For $B=a_8^2u_{56}$, we have a cocycle

$$P=z_{52}w_{152}=a_8^2e_{36}e_{48}^2+(\text{some other terms}).$$

For $B=a_8^2a_{20}$, there is no cocycle with $A=0$, since otherwise its difference with

v_{168} would be a cocycle with $A=a_8a_{20}^2$ and $B=0$, the existence of which is already denied.

For $B=a_8a_{20}^2$, there is no cocycle P.

Thus for those B which have at least one $a_{20}u_{56}$, u_{56}^2, a_8^3, a_{20}^3, $a_8^2a_{20}^2$ or $a_8^2u_{56}$ in each term we have a cocycle P with $A=0$, all of which are decomposable. This completes the proof that v_{168} is the only indecomposable cocycle other than x_{108} and x_{144} of degree 3 with respect to e_{36} and e_{48}. q.e.d.

(ii–4) A cocycle of degree 4 with respect to e_{36} and e_{48} is the sum of $e_{36}^3\cdot$(cocycle of degree 1), $e_{48}^3\cdot$(cocycle of degree 1) and a cocycle P of the form

$$P=-Ie_{36}^2e_{48}^2+Ae_{36}^2e_{48}+Be_{36}e_{48}^2+Ce_{36}^2$$
$$-De_{36}e_{48}+Ee_{48}^2-Fe_{36}-Ge_{48}-H.$$

So we have only to study cocycles P of the above form.

Now, $\partial_9P=0$ and $\partial_{21}P=0$ give rise to relations similar to (5.5):

(5.5)′ $\partial_9I=0$, $\partial_{21}I=0$ (which are not in (5.5)),

$\partial_9A=Ib_{40}$, $\partial_{21}A=Id_{28}$ (instead of $\partial_9A=\partial_{21}A=0$),

$\partial_9B=Id_{28}$, $\partial_{21}B=Ib_{16}$ (instead of $\partial_9B=\partial_{21}B=0$),

the rest of (5.5).

Now, the cocycle I must have at least one $a_8^2u_{56}$, $a_8a_{20}u_{56}$, $a_{20}^2u_{56}$, u_{56}^2 or $a_8^ia_{20}^j$ $(i+j=4)$ in each term. Conversely, if I is such a cocycle, we have a cocycle P with the term $-Ie_{36}^2e_{48}^2$, since we have P for each I in the following:

I	P
$a_8^2u_{56}$	$-w_{88}w_{152}$
$a_8a_{20}u_{56}$	$-w_{100}w_{152}+u_{56}z_{52}e_{48}^3$
$a_{20}^2u_{56}$	$-w_{128}w_{136}$
u_{56}^2	$w_{128}w_{152}$
a_8^4	$a_8^2a_{20}z_{56}e_{36}^3-w_{100}^2$
$a_8^3a_{20}$	$-w_{88}w_{124}-a_8^2z_{88}e_{36}^3$
$a_8^2a_{20}^2$	$w_{100}w_{124}+a_8a_{20}(z_{88}e_{36}^3+z_{52}e_{48}^3)$
$a_8a_{20}^3$	$-w_{100}w_{136}-a_{20}^2z_{52}e_{48}^3$
a_{20}^4	$a_8a_{20}^2z_{56}e_{48}^3-w_{124}^2$

It is easy to see that there is no indecomposable cocycle of degree greater than 4. Thus we have obtained

Proposition 5.7. *The following 22 elements are all the indecomposable cocycles without elements of odd degree*:

$$a_4,\quad a_8,\quad a_{20},\quad u_{56}=a_{20}^2b_{16}+a_8^2b_{40}-a_8a_{20}d_{28},$$

$$x_{48}=b_{16}^3,\quad x_{120}=b_{40}^3,\quad x_{84}=d_{28}^3,\quad x_{108}=e_{36}^3,\quad x_{144}=e_{48}^3,$$

$$z_{56}=a_{20}e_{36}+a_8e_{48}-b_{16}b_{40}+d_{28}^2,$$

$$z_{52}=a_8^2e_{36}+\cdots,\quad z_{88}=a_{20}^2e_{48}+\cdots,$$

$$z_{92}=u_{56}e_{36}+\xi,\quad z_{104}=u_{56}e_{48}+\eta,$$

$$w_{88}=a_8^2e_{36}^2+\cdots,\quad w_{136}=a_{20}^2e_{48}^2+\cdots,$$

$$w_{100}=a_8a_{20}e_{36}^2-a_8^2e_{36}e_{48}+\cdots,$$

$$w_{124}=-a_{20}^2e_{36}e_{48}+a_8a_{20}e_{48}^2+\cdots,$$

$$w_{128}=u_{56}e_{36}^2+\cdots,\quad w_{152}=u_{56}e_{48}^2+\cdots,$$

$$w_{140}=u_{56}e_{36}e_{48}+\cdots,$$

$$v_{168}=a_8a_{20}^2e_{36}^2e_{48}+a_8^2a_{20}e_{36}e_{48}^2+\cdots.$$

§6. Nice elements and nice-relations

Definition. We call a cocycle *nice* if it is in the $\partial_9^2\partial_{21}^2$-image. (We call it *non-nice* otherwise.)

Of the generators in Proposition 5.7 the following elements are nice:

(6.1)
$$\begin{aligned}
u_{56}&=\partial_9^2\partial_{21}^2(d_{28}e_{36}e_{48}),\\
z_{92}&=\partial_9^2\partial_{21}^2(b_{16}e_{36}e_{48}^2),\\
z_{104}&=\partial_9^2\partial_{21}^2(b_{40}e_{36}^2e_{48}),\\
w_{152}&=\partial_9^2\partial_{21}^2(b_{40}e_{36}^2e_{48}^2),\\
w_{128}&=\partial_9^2\partial_{21}^2(b_{16}e_{36}^2e_{48}^2),\\
w_{140}&=\partial_9^2\partial_{21}^2(d_{28}e_{36}^2e_{48}^2),\\
v_{168}&=\partial_9^2\partial_{21}^2(-d_{28}^2e_{36}^2e_{48}^2);
\end{aligned}$$

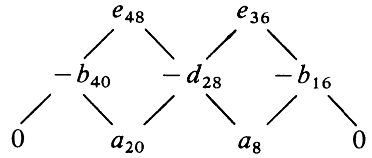
and the remainder are not.

We see that

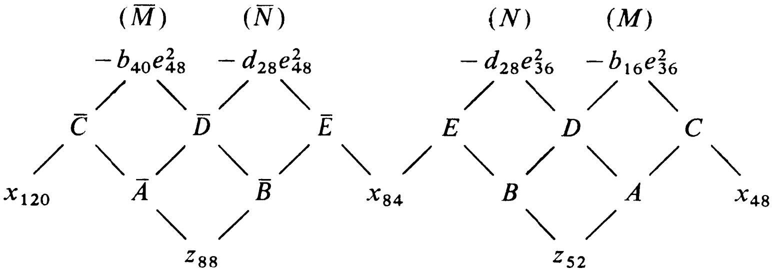
(6.1)′ a_4, z_{56}, x_{108} *and* x_{144} *are in neither the* ∂_9- *nor the* ∂_{21}-*image.*

The other non-nice generators are related in the following diagrams:

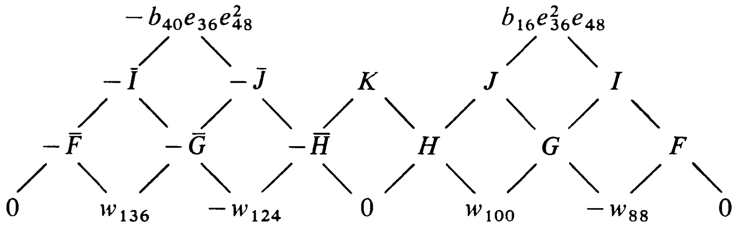
(6.2.1)



(6.2.2)



(6.2.3)



where $K=-a_{20}e_{36}^2e_{48}+a_8e_{36}e_{48}^2+b_{40}d_{28}e_{36}^2-b_{16}d_{28}e_{48}^2$, and $A,\dots, J$ are easily obtained by mere calculations of the ∂_9- and the ∂_{21}-image. ($\bar{A},\dots, \bar{J}$ are the conjugates of $A,\dots, J$ respectively.)

We use the letters M, N, $\bar{M}$ and $\bar{N}$ as above for simplicity.

Observe that an element of the form

$$\text{(nice cocycle)}\cdot\text{(any cocycle)}$$

is nice. But sometime it happens that products of non-nice generators are nice.

Definition. Two cocycles A and B are called *nicely-related* if $A-B$ is nice and denoted by $A\underset{N}{\sim}B$.

We shall show in the following three lemmas that the products in the following table are nice and that products that have the same circled number are nicely-related.

We exclude the elements a_4, x_{108} and x_{144}, since they are 'immobile' in ∂_9-∂_{21}-diagrams, that is, they are related to no elements by ∂_9-∂_{21}-diagrams and their products with other cocycles make no essential difference to diagrams.

(6.3)

	a_8	a_{20}	z_{56}	x_{48}	x_{120}	x_{84}	z_{52}	z_{88}	w_{88}	w_{100}	w_{124}	w_{136}
a_8	nice	nice	nice				nice	nice	nice	①	nice	②
a_{20}		nice	nice				nice	nice	①	nice	②	nice
z_{56}			nice				①	②	nice	nice	nice	nice
x_{48}					③			④			⑥	⑦
x_{120}							⑤		⑧	⑨		
x_{84}						③	④	⑤	⑥	⑦	⑧	⑨
z_{52}							nice	nice	nice	nice	nice	nice
z_{88}								nice	nice	nice	nice	nice
w_{88}									nice	nice	nice	nice
w_{100}										nice	nice	nice
w_{124}											nice	nice
w_{136}												nice

Lemma 6.4. *All the products in the above table denoted "nice" are nice.*

Proof. The following are checked by direct calculation:

$$a_8 z_{56} = \partial_9^2 \partial_{21}^2 (e_{36}^2 e_{48}),$$

$$a_{20} z_{56} = \partial_9^2 \partial_{21}^2 (e_{36} e_{48}^2),$$

$$z_{56}^2 = \partial_9^2 \partial_{21}^2 (e_{36}^2 e_{48}^2),$$

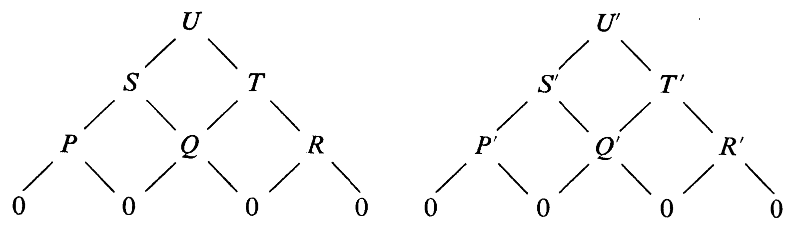
$$z_{56} w_{88} = \partial_9^2 \partial_{21}^2 (b_{16}^2 e_{36}^2 e_{48}^2 + a_{20} e_{36}^5),$$

$$z_{56} w_{100} = \partial_9^2 \partial_{21}^2 (b_{16} d_{28} e_{36}^2 e_{48}^2 + a_8 e_{36}^3 e_{48}^2),$$

$$z_{56} w_{124} = \partial_9^2 \partial_{21}^2 (b_{40} d_{28} e_{36}^2 e_{48}^2 + a_{20} e_{36}^2 e_{48}^3),$$

$$z_{56} w_{136} = \partial_9^2 \partial_{21}^2 (b_{40}^2 e_{36}^2 e_{48}^2 + a_8 e_{48}^5).$$

Now, in general, when we have two diagrams



we have $\partial_9^2 \partial_{21}^2 (UU') = PR' + QQ' + RP'$.

Thus we have the following $\partial_9^2 \partial_{21}^2$-image:

T	e_{36}^2	$-e_{36}e_{48}$	De_{36}	$\bar{D}e_{36}$	$-Ie_{36}$	$\bar{J}e_{36}$	e_{48}^2	De_{48}
$\partial_9^2 \partial_{21}^2 T$	a_8^2	$a_8 a_{20}$	$a_8 z_{52}$	$a_8 z_{88}$	$a_8 w_{88}$	$a_8 w_{124}$	a_{20}^2	$a_{20} z_{52}$

$\bar{D}e_{48}$	Je_{48}	$-\bar{I}e_{48}$	D^2	$D\bar{D}$	$-DI$	DJ
$a_{20}z_{88}$	$a_{20}w_{100}$	$a_{20}w_{136}$	z_{52}^2	$z_{52}z_{88}$	$z_{52}w_{88}$	$z_{52}w_{100}$

$D\bar{J}$	$-D\bar{I}$	$\bar{D}^2$	$-\bar{D}I$	$\bar{D}J$	$\bar{D}\bar{J}$	$-\bar{D}\bar{I}$	I^2
$z_{52}w_{124}$	$z_{52}w_{136}$	z_{88}^2	$z_{88}w_{88}$	$z_{88}w_{100}$	$z_{88}w_{124}$	$z_{88}w_{136}$	w_{88}^2

IJ	$-I\bar{J}$	J^2	$-J\bar{I}$	$\bar{J}^2$	$\bar{I}\bar{J}$	$\bar{I}^2$
$w_{88}w_{100}$	$w_{88}w_{124}$	w_{100}^2	$w_{100}w_{136}$	w_{124}^2	$w_{124}w_{136}$	w_{136}^2

We have $w_{88}w_{136}+w_{100}w_{124}=\partial_9^2\partial_{21}^2(I\bar{I})$ and $\partial_9^2\partial_{21}^2(K^2)=w_{100}w_{124}$. Thus $w_{88}w_{136}=\partial_9^2\partial_{21}^2(I\bar{I}-K^2)$. So we have shown all the required products to be nice.
q.e.d.

Using the same diagrams as in the above proof, we have also

(6.4.1) $a_8w_{100}-a_{20}w_{88}=\partial_9^2\partial_{21}^2(Je_{36})$,

(6.4.2) $a_8w_{136}-a_{20}w_{124}=\partial_9^2\partial_{21}^2(-\bar{I}e_{36})$,

whence $a_8w_{100}\underset{N}{\sim}a_{20}w_{88}$ and $a_8w_{136}\underset{N}{\sim}a_{20}w_{124}$.

It may be checked by direct calculation that

(6.4.3) *a_8w_{100} and a_8w_{136} are non-nice, hence so are their conjugates $a_{20}w_{88}$ and $a_{20}w_{124}$.*

Lemma 6.5. *We have the following nice-relations, and each term is non-nice*:

(1) $a_8w_{100}\underset{N}{\sim}a_{20}w_{88}\underset{N}{\sim}w_{56}z_{52}$, (2) $a_8w_{136}\underset{N}{\sim}a_{20}w_{124}\underset{N}{\sim}z_{56}z_{88}$,

(3) $x_{48}x_{120}\underset{N}{\sim}x_{84}^2$, (4) $z_{88}x_{48}\underset{N}{\sim}z_{52}x_{84}$, (5) $z_{52}x_{120}\underset{N}{\sim}z_{88}x_{84}$,

(6) $w_{124}x_{48}\underset{N}{\sim}w_{88}x_{84}$, (7) $w_{136}x_{48}\underset{N}{\sim}w_{100}x_{84}$, (8) $w_{88}x_{120}\underset{N}{\sim}w_{124}x_{84}$,

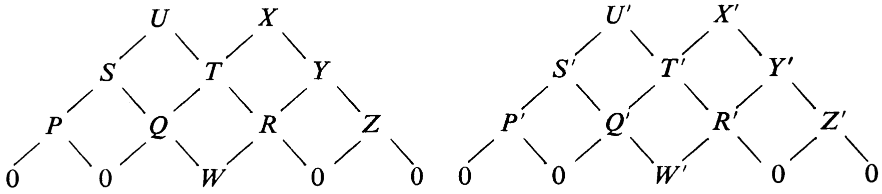
(9) $w_{100}x_{120}\underset{N}{\sim}w_{136}x_{84}$.

Proof. (1) We have $a_8w_{100}+a_{20}w_{88}+z_{56}z_{52}=0$. Since $a_{20}w_{88}=a_8w_{100}-\partial_9^2\partial_{21}^2(Je_{36})$ by (6.4.1), we have $z_{56}z_{52}=a_8w_{100}+\partial_9^2\partial_{21}^2(Je_{36})$, which shows that

$$a_8w_{100}\underset{N}{\sim}a_{20}w_{88}\underset{N}{\sim}z_{56}z_{52}.$$

(2) $a_8w_{136}\underset{N}{\sim}a_{20}w_{124}\underset{N}{\sim}z_{56}z_{88}$ is shown similarly by using the equality $a_8w_{136}+a_{20}w_{124}+z_{56}z_{88}=0$ and (6.4.2).

Now in general, if we have two diagrams



we have $\partial_9^2\partial_{21}^2(UX')=PZ'+QR'+RQ'-TW'-WT'$, and thus

$$\partial_9^2\partial_{21}^2(UX'-XU')=PZ'-ZP'.$$

So we have

(3) $x_{48}x_{120}-x_{84}^2=\partial_9^2\partial_{21}^2(M\overline{M}-N\overline{N})$,

(4) $z_{88}x_{48}-z_{52}x_{84}=\partial_9^2\partial_{21}^2(\overline{E}M-EN)$,

(5) $z_{52}x_{120}-x_{84}z_{88}=\partial_9^2\partial_{21}^2(E\overline{M}-\overline{E}\overline{N})$,

(6) $w_{124}x_{48}-x_{84}w_{88}=\partial_9^2\partial_{21}^2(-KM+NJ)$,

(7) $w_{136}x_{48}-w_{100}x_{84}=\partial_9^2\partial_{21}^2(-\overline{J}M-KN)$,

(8) $w_{88}x_{120}-w_{124}x_{84}=\partial_9^2\partial_{21}^2(-J\overline{M}+K\overline{N})$,

(9) $w_{100}x_{120}-w_{136}x_{84}=\partial_9^2\partial_{21}^2(K\overline{M}+\overline{J}\overline{N})$.

Observe that $z_{56}z_{52}$ and $z_{56}z_{88}$ in (1) and (2) are non-nice, since their nicely-related cocycles are non-nice by (6.4.3). That the terms in (3),..., (9) are non-nice may be checked by direct calculation. q.e.d.

Definition. Cocycles $A_1,\ldots, A_n$ are called *nicely-independent* if any sum of them does not belong to the $\partial_9^2\partial_{21}^2$-image.

Proposition 6.6. *The following monomials in cocycles are non-nice*:

(1) x_{84}^h, (2) $x_{48}^i x_{84}^h$ $(i\neq 0)$, (3) $x_{120}^j x_{84}^h$ $(j\neq 0)$, (4) $a_8x_{84}^h$,

(5) $a_8x_{48}^i x_{84}^h$ $(i\neq 0)$, (6) $a_8x_{120}^j x_{84}^h$ $(j\neq 0)$, (7) $a_{20}x_{84}^h$,

(8) $a_{20}x_{48}^i x_{84}^h$ $(i\neq 0)$, (9) $a_{20}x_{120}^j x_{84}^h$ $(j\neq 0)$, (10) $z_{56}x_{84}^h$,

(11) $z_{56}x_{48}^i x_{84}^h$ $(i\neq 0)$, (12) $z_{56}x_{120}^j x_{84}^h$ $(j\neq 0)$, (13) $z_{52}x_{48}^i x_{84}^h$,

(14) $z_{88}x_{120}^j x_{84}^h$, (15) $w_{88}x_{48}^i x_{84}^h$, (16) $w_{100}x_{48}^i x_{84}^h$, (17) $w_{124}x_{120}^j x_{84}^h$,

(18) $w_{136}x_{120}^j x_{84}^h$, (19) $a_8w_{100}x_{48}^i x_{84}^h \underset{N}{\sim} a_{20}w_{88}x_{48}^i x_{84}^h \underset{N}{\sim} z_{56}z_{52}x_{48}^i x_{84}^h$,

(20) $a_8w_{136}x_{120}^j x_{84}^h \underset{N}{\sim} a_{20}w_{124}x_{120}^j x_{84}^h \underset{N}{\sim} z_{56}z_{88}x_{120}^j x_{84}^h$,

where i, j and h are non-negative integers.

The monomials of the form

(*) (*any monomial in the proposition*)$\cdot a_4^r x_{108}^s x_{144}^t$

(r, s, and t are non-negative integers)
are nicely-independent and any non-nice monomial is nicely-related to one of them.

Proof. It may be shown directly that the monomials in the proposition are non-nice. In the calculations we also see that they are nicely-independent, since we see that no sum of any monomials in the proposition is nice. It follows that monomials of the form (∗) are non-nice and that they are nicely-independent.

Now, we show that any non-nice monomial without a_4, x_{108} or x_{144} is nicely-related to a monomial in the proposition. Then any non-nice monomial with $a_4^r x_{108}^s x_{144}^t$ is nicely-related to one of the form (∗). So we put aside the elements a_4, x_{108} and x_{144} until the end of the proof.

Of two monomials which are nicely-related, we shall always choose the one with the least number of x_{48} and the least number of x_{120}.

By (3) of Lemma 6.5 we have

$$x_{48}^i x_{120}^j x_{84}^h \underset{N}{\sim} \begin{cases} x_{84}^{h+2i} & \text{if } i=j, \\ x_{48}^{i-j} x_{84}^{h+2j} & \text{if } i>j, \\ x_{120}^{j-i} x_{84}^{h+2i} & \text{if } i<j, \end{cases}$$

and so we choose the right hand sides, which are of the form

(1) x_{84}^h, (2) $x_{48}^i x_{84}^h$ $(i \neq 0)$, (3) $x_{120}^j x_{84}^h$ $(j \neq 0)$.

And using this nice-relation, we see that any monomial is nicely-related to one having not both x_{48} and x_{120}.

A non-nice monomial has at most one z_k or w_l ($k=52, 88$; $l=88, 100, 124, 136$), since $z_k z_{k'}$, $w_l w_{l'}$ and $z_k w_l$ ($k, k'=52, 88$; $l, l'=88, 100, 124, 136$) are nice.

By (4)∼(9) of Lemma 6.5 we have, for $i \neq 0$ and $j \neq 0$,

$$z_{88} x_{48}^i x_{84}^h \underset{N}{\sim} z_{52} x_{48}^{i-1} x_{84}^{h+1},$$

$$z_{52} x_{120}^j x_{84}^h \underset{N}{\sim} z_{88} x_{120}^{j-1} x_{84}^{h+1},$$

$$w_{124} x_{48}^i x_{84}^h \underset{N}{\sim} w_{88} x_{48}^{i-1} x_{84}^{h+1},$$

$$w_{136} x_{48}^i x_{84}^h \underset{N}{\sim} w_{100} x_{48}^{i-1} x_{84}^{h+1},$$

$$w_{88} x_{120}^j x_{84}^h \underset{N}{\sim} w_{124} x_{120}^{j-1} x_{84}^{h+1},$$

$$w_{100} x_{120}^j x_{84}^h \underset{N}{\sim} w_{136} x_{120}^{j-1} x_{84}^{h+1}.$$

We take the right hand sides, which are of the form (13)∼(18). If $i=0$ or $j=0$, the terms in the left hand sides are also of the form (13)∼(18).

We study non-nice monomials of the form $a_8 A$. By (6.3), we see that A may contain w_{100}, w_{136}, x_{48}, x_{120} and x_{84}, and, as we have noted, it may have at most one w_{100} and w_{136}. We have non-nice monomials

$$(4)\ a_8x_{84}^h,\quad (5)\ a_8x_{48}^ix_{84}^h\ (i\neq 0),\quad (6)\ a_8x_{120}^jx_{84}^h\ (j\neq 0),$$

$$(19)\ a_8w_{100}x_{48}^ix_{84}^h\underset{N}{\sim}\cdots,\quad (20)\ a_8w_{136}x_{120}^jx_{84}^h\underset{N}{\sim}\cdots.$$

The monomial $a_8w_{100}x_{84}^h$ is of the form (19) and the monomial $a_8w_{100}x_{120}^jx_{84}^h$ ($j\neq 0$) is nicely-related to $a_8w_{126}x_{120}^{j-1}x_{84}^{h+1}$ in (20). Similarly, $a_8w_{136}x_{84}^h$ is in (20) and $a_8w_{136}x_{48}^ix_{84}^h$ ($i\neq 0$) is nicely-related to $a_8w_{100}x_{48}^{i-1}x_{84}^{h+1}$ in (19).

Therefore, any non-nice monomial of the form a_8A is nicely-related to one of (4), (5), (6), (19) and (20).

Similarly, any non-nice monomial of the form $a_{20}A$ is nicely-related to one of (7), (8), (9), (19) and (20).

Finally, we study non-nice monomials of the form $z_{56}A$. By (6.3) A may contain z_{52}, z_{88}, x_{48}, x_{120} and x_{84}, and it may contain at most one z_{52} or z_{88}.

We have non-nice monomials

$$(10)\ z_{56}x_{84}^h,\quad (11)\ z_{56}x_{48}^ix_{84}^h\ (i\neq 0),\quad (12)\ z_{56}x_{120}^jx_{84}^h\ (j\neq 0),$$

$$(19)\ \cdots\underset{N}{\sim}z_{56}z_{52}x_{48}^ix_{84}^h,\quad (20)\ \cdots\underset{N}{\sim}z_{56}z_{88}x_{120}^jx_{84}^h.$$

Here (19) includes $z_{56}z_{52}x_{84}^h$, and a monomial $z_{56}z_{52}x_{120}^jx_{84}^h$ ($j\neq 0$) is nicely-related to $z_{56}z_{88}x_{120}^{j-1}x_{84}^{h+1}$ in (20). Similarly, (20) includes $z_{56}z_{88}x_{84}^h$, and a monomial $z_{56}z_{88}x_{48}^ix_{84}^h$ is nicely-related to $z_{56}z_{52}x_{48}^{i-1}x_{84}^{h+1}$ in (19). Thus any non-nice monomial with z_{56} is nicely-related to one of (10), (11), (12), (19) and (20).

We have shown that any non-nice monomial without a_4, x_{108} or x_{144} is nicely-related to a monomial in the proposition, and then any non-nice monomial with $a_4^rx_{108}^sx_{144}^t$ is nicely-related to one of (*). q.e.d.

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