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Sample path properties of stochastic processes

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§1. Introduction. We will concern ourselves mainly with some integral tests which guarantee sample path continuity or integrability of the supremum of sample paths for all stochastic processes having the same moment condition. In this direction perhaps the first theorem is due to Kolmogorov [34]: All separable stochastic proosses $\{X(t);\, 0\leq t\leq 1\}$ such that $\exists a>0,\ b>0,\ \varepsilon>0,\ 0\leq\forall s,\ t\leq 1$,

$$E[|X(s)-X(t)|^{a}]\leq b|s-t|^{1+\varepsilon}$$

have continuous sample paths with probability 1.

After Kolmogorov, many people have developed his theorem. (A part of them are listed in References.)

In this paper, we will treat a more general class of stochastic processes than preceding people's with much more elementary proofs and give a new sufficient condition of sample path continuity for L_p-processes with stationary increments.

§2. Formulation. Let (T, d) be a separable pseudo-metric space, i.e. $d(s, t)=0$ does not necessarily mean $s=t$, and $N_T(d, \varepsilon)$ be the minimal cardinal number of ε-nets. A subset S of T is called an ε-net if for any $t\in T$, there exists $s\in S$ such that $d(s, t)\leq\varepsilon$, and an ε-net is called minimal if its cardinal number equals $N_T(d, \varepsilon)$. To describe our class of stochastic processes, we introduce the following three functions:

$\Phi(x)$: a non-negative, non-decreasing convex function defined on $[0, +\infty)$,
$\phi(x)$: the inverse function of Φ defined on $[\Phi(0), +\infty)$ with $\phi(\Phi(0))=\lim_{x\downarrow\Phi(0)}\phi(x)$,
$\sigma(x)$: a non-decreasing continuous function defined on $[0, +\infty)$ with $\sigma(0)=0$ and $\sigma(x)>0$ for $x>0$.

Then we denote by $\mathfrak{X}\{\Phi, \sigma, (T, d)\}$ a collection of all real valued stochastic processes $\{X(t, \omega);\, t\in T, \omega\in\Omega\}$ defined on a probability space $(\Omega, \mathfrak{A}, P)$, which may be different for each process, satisfying the following three conditions:

(i) $$P(X(s, \omega)=X(t, \omega))=1,\quad\text{if}\quad d(s, t)=0, \tag{1}$$

(ii) there exists a constant $M_X>0$, which may depend on the process, such that

$$E[\Phi(X^*(s, t, \omega))]\leq M_X<+\infty \tag{2}$$

holds for any $s, t \in T$, where $X^*(s, t, \omega)=|X(s, \omega)-X(t, \omega)|/\sigma(d(s, t))$ if $d(s, t) \neq 0$ and $=0$, otherwise,

(iii) there exists a point $t_0 \in T$ such that

$$E[|X(t_0)|]<+\infty. \tag{3}$$

Remark 1. Boulicaut [3], and Nanopoulos-Nobelis [29] have investigated essentially the same class when Φ generates an Orlicz space, i.e. Φ is a N-function in the sense of [25].

Remark 2. Set $f_X(t)=E[|X(t)|]$. Then $f_X(t)$ is a continuous function defined on (T, d) satisfying $|f_X(s)-f_X(t)| \leq \sigma(d(s, t))\phi(M_X)$ by Jensen's inequality, and it is a bounded function if $N_T(d, \varepsilon)$ is finite ($\varepsilon>0$).

Remark 3. It is easy to verify all our theorems for stochastic processes taking their value in a separable locally compact metric space.

Example 1. (Generalized Gaussian process).
We will call a stochastic process belonging to the class $\mathfrak{X}\{\exp cx^2, \sigma, (T, d)\}$ ($c>0$), a generalized Gaussian process. It is easily checked that a sub-Gaussian process in the sense of Jain-Marcus [22] or Heinkel [20], ($\exists A>0$, $\forall \alpha>0$

$$E[\exp\{\alpha(X(s)-X(t))\}] \leq A \exp\{\alpha^2\sigma^2(d(s, t))/2\}$$

holds for any $s, t \in T$), is a generalized Gaussian process in our sense, but the converse is not true.

Example 2. (L_p-process, $p \geq 1$) We will call a stochastic process belonging to the class $\mathfrak{X}\{x^p, \sigma, (T, d)\}$ ($p \geq 1$), a L_p-process.

Example 3. (Skorohod's example [33]) We will cite below as Skorohod's example any stochastic process belonging to the class $\mathfrak{X}\{\exp cx^\alpha, \sigma, (T, d)\}$ (c, $\alpha>0$), where (T, d) is a compact subset of a separable Hilbert space H such that

$$T=\{t \in H;\ \|A^{-1}t\| \leq 1\}$$

associated with a positive completely continuous operator A. His original process is the following:

$$\exists \alpha, \alpha', \delta, \gamma>0,\ \forall s, t \in T,\ \forall x>0$$

$$P(|X(s)-X(t)| \geq x) \leq \gamma \exp\{-\delta x^\alpha \|s-t\|^{-\alpha'}\}.$$

Remark 4. Since any stochastic process of our classes is continuous in probability, we can choose a modification which is separable with respect to any countable dense subset D of T, and measurable with respect to $\mathscr{B}(d) \times \mathfrak{A}$ by the theorem due to D. L. Cohn [6], where $\mathscr{B}(d)$ is the topological Borel field of (T, d). He has proved it when (T, d) is a separable metric space, but there is nothing to change for a separable pseudo-metric space.

In case that T has two pseudo-metrics d_1 and d_2, we can establish the following:

Proposition 1. *Assume that (T, d_1) is compact and the identity map from (T, d_1) to (T, d_2) is continuous. If a real function f on T is continuous with respect to d_1 and if $d_2(s, t)=0$ implies $f(s)=f(t)$, then f is also continuous with respect to d_2.*

Proof. Set

$$\Lambda_t=\{s\in T,\ d_2(s, t)=0\},$$

$$\Lambda_t^\varepsilon=\{s;\ d_2(s, t)\leq\varepsilon\},$$

$$\mu_t(s)=d_1(s, \Lambda_t)=\inf\{d_1(s, u);\ u\in\Lambda_t\},$$

and

$$\nu_t(\varepsilon)=\sup\{\mu_t(s);\ s\in\Lambda_t^\varepsilon\}.$$

Since (T, d_1) is compact, we have $\lim_{\varepsilon\downarrow 0}\nu_t(\varepsilon)=0$. Combining this with continuity with respect to d_1, it follows for any $\eta>0$ that there exist $\delta>0$ and $\varepsilon>0$ such that $\nu_t(\varepsilon)<\delta$ and $|f(s)-f(t)|<\eta$ for any s, t with $d_1(s, t)<\delta$. This implies that for any $s\in\Lambda_t^\varepsilon$, there exists $u\in\Lambda_t$ such that $d_1(s, u)<\delta$ and $|f(s)-f(t)|\leq|f(s)-f(u)|+|f(u)-f(t)|=|f(s)-f(u)|<\eta$. Q.E.D.

Proposition 2. *Under the same assumptions for (T, d_1) and (T, d_2) as Proposition* 1, *if a stochostic process on (T, d_1) having continuous sample paths with respect to d_1 satisfies the condition* (1), *then the sample paths are continuous with respect to d_2 with probability* 1.

This proposition has been pointed out by Fernique [15, p 704] without proof in case of Gaussian random fields.

Proof. Define a pseudo-metric $\bar{d}_1$ on $D_2=\{(s, t)\in T\times T;\ d_2(s, t)=0\}$ as follows:

$$\bar{d}_1((s, t), (s', t'))=d_1(s, s')+d_1(t, t').$$

Since $(D_2, \bar{d}_1)$ is compact, there exists a countable dense subset S of D_2. By continuity with respect to d_1, it follows that

$$\sup_{(s,t)\in D_2}|X(s, \omega)-X(t, \omega)|=\sup_{(s,t)\in S}|X(s, \omega)-X(t, \omega)|.$$

Setting

$$N_{s,t}=\{\omega;\ X(s, \omega)\neq X(t, \omega)\},$$

and

$$N=\bigcup_{(s,t)\in S}N_{s,t},$$

we have $P(N)=0$ from the condition (1), and $X(s, \omega)=X(t, \omega)$ for $\omega\notin N$, if $d_2(s, t)=0$. Therefore $X(t, \omega)$, $\omega\notin N$, is continuous with respect to d_2 by Proposition 1. Q.E.D.

§3. Integrability. In this section, we will consider integrability of the supremum of sample paths.

Theorem 1. *If there exists a positive decreasing sequence* $\{\varepsilon_n\}_{n=0}^{\infty}$ *with* $N_T(d, \varepsilon_1) > \Phi(0)$ *such that*

$$\sum_{n=1}^{\infty} \sigma(\varepsilon_{n-1})\phi(N_T(d, \varepsilon_n)) < +\infty, \tag{4}$$

then for any separable process $\{X(t); t \in T\}$ *belonging to* $\mathfrak{X}\{\Phi, \sigma, (T, d)\}$, *we have*

$$E[\sup_{t\in T}|X(t, \omega)|] \leq \sum_{t\in D_0} E[|X(t, \omega)|] + C_X \sum_{n=1}^{\infty} \sigma(\varepsilon_{n-1})\phi(N_T(d, \varepsilon_n)),$$

where

$$D_0 = \text{a minimal } \varepsilon_0\text{-net},$$

$$C_X = \sup_{x \geqq N_T(d,\varepsilon_1)} \phi(M_X x)/\phi(x) (< +\infty),$$

and

$$M_X = \sup_{s,t\in T} E[\Phi(X^*(s, t, \omega))] (< +\infty).$$

Proof. Let D_n be a minimal ε_n-net and set $D = \bigcup_{n=0}^{\infty} D_n$. It is sufficient to prove the inequality for a separable process with the separant D according as Remark 4. From the definition of a net, for each $t \in T$ there exists $\tau_n(t) \in D_n$ such that $d(t, \tau_n(t)) \leq \varepsilon_n (\tau_n(t) = t$ if $t \in D_n)$. Then setting $N = \{\omega; \exists(s, t) \in D \times D$ with $d(s, t) = 0$ such that $X(s, \omega) \neq X(t, \omega)\}$, and $A_n(\omega) = \max_{t\in D_n} |X(t, \omega) - X(\tau_{n-1}(t), \omega)|$, $n = 1, 2, \ldots$, we have $P(N) = 0$ by (1) and

$$|X(t, \omega)| \leq \sum_{t\in D_0} |X(t, \omega)| + \sum_{n=1}^{\infty} A_n(\omega)$$

$$\leq \sum_{t\in D_0} |X(t, \omega)| + \sum_{n=1}^{\infty} \sigma(\varepsilon_{n-1}) \max_{t\in D_n} X^*(t, \tau_{n-1}(t), \omega)$$

$$\leq \sum_{t\in D_0} |X(t, \omega)| + \sum_{n=1}^{\infty} \sigma(\varepsilon_{n-1})\phi(\sum_{t\in D_n} \Phi(X^*(t, \tau_{n-1}(t), \omega)))$$

for $t \in D$ and $\omega \notin N$. Since the right had side does not depend on $t \in D$, it follows by Jensen's inequality that

$$E[\sup_{t\in D} |X(t, \omega)|]$$

$$\leq \sum_{t\in D_0} E[|X(t, \omega)|] + \sum_{n=1}^{\infty} \sigma(\varepsilon_{n-1})\phi(M_X N_T(d, \varepsilon_n))$$

$$\leq \sum_{t\in D_0} E[|(t, \omega)|] + C_X \sum_{n=1}^{\infty} \sigma(\varepsilon_{n-1})\phi(N_T(d, \varepsilon_n)).$$

Q. E. D.

Corollary 1. *Choosing* $\varepsilon_n = 2^{-n} T_d$ *in the Theorem* 1, *we have*

$$E[\sup_{t\in T} |X(t, \omega)|]$$

$$\leq E[|X(t_0, \omega)|] + 2C_X \int_{+0}^{T_d/2} \phi(N_T(d, u))\tau(4u)/u du,$$

where T_d is the diameter of T and t_0 is defined in (3).

Corollary 2. *In case of $T=[0, 1]^N$ with the usual Euclidean metric $\| \|$, choosing $2\varepsilon_n = 1/\Phi(2^n)^{1/N}$, we have*

$$E[\sup_{t\in T} |X(t, \omega)|]$$

$$\leq \sum_{t\in D_0} E[|X(t, \omega)|] + 4C_X^2 \int_{1/2}^{+\infty} \sigma(1/\Phi(x)^{1/N})dx.$$

In case of a special class which includes Example 1 and 3 or "exponential type" of [29], we have another sufficient condition obtained by the method due to Fernique [13], [15] or Heinkel [19], who have applied it to Gaussian or sub-Gaussian case.

Theorem 2. *Assume that $\Phi(0) \leq 1$ and there exists a constant $c \geq 1$ such that*

$$2 \log \Phi(x) \leq \log \Phi(cx), \quad (x>0) \tag{5}$$

or equivalently,

$$\phi(x^2) \leq c\phi(x), \quad (x > \Phi(0)).$$

We concern ourselves in case of $\sigma(x) = x$.

(i) *If there exists a probability measure $(T, \mathscr{B}(d), \mu)$ such that*

$$I(\mu, \varepsilon) = \sup_{t\in T} \int_0^{\varepsilon} \phi(1/\mu(B(t, u))du < +\infty, \ (\varepsilon>0), \tag{6}$$

where $B(t, u) = \{s;\ d(s, t) < u\}$, then

$$E[\sup_{t\in T} |X(t, \omega)|] \leq \int_T E[|X(s, \omega)|]d\mu(s)$$

$$+ 3c\ \phi(M_X) + 6c^2 I(\mu, T_d/2) < +\infty$$

holds for any separable and measurable stochastic process $\{X(t);\ t\in T\}$ belonging to $\mathfrak{X}\{\Phi, x, (T, d)\}$.

(ii) *Denote by $\mathfrak{M}(T)$ a collection of all probability measures $(T, \mathscr{B}(d), \mu)$. If*

$$J(\varepsilon) = \sup_{\mu\in \mathfrak{m}(T)} \int_T \int_{+0}^{\varepsilon} \phi(1/\mu(B(t, u)))du\, d\mu(t) < +\infty, \ (\varepsilon>0) \tag{7}$$

then

$$E[\sup_{t\in T} |X(t, \omega)] \leq K + 3c\ \phi(M_X) + 6c^2 J(T_d/2) < +\infty$$

holds for any separable and measurable stochastic process $\{X(t);\ t\in T\}$ belonging to $\mathfrak{X}\{\Phi, x, (T, d)\}$, where $K = \sup_{t\in T} E[|X(t)|] < +\infty$ (Remark 2).

Proof. We remark that if $\{X(t, \omega); t \in T\}$ is a measurable function with respect to $\mathscr{B}(d)$, then $d(s, t)=0$ implies $X(s, \omega)=X(t, \omega)$.

Set

$$B(\omega)=\int_{T\times T}\Phi(X^*(s, t, \omega))d\mu(s)d\mu(t),$$

$$N_1=\{\omega;\ B(\omega)=+\infty\}.$$

and

$$X_n(t, \omega)=\int_{B_n(t)}X(s, \omega)d\mu(s)/\mu_n(t),$$

where $B_n(t)=\{s;\ d(s, t)<2^{-n}T_d\}$, and $\mu_n(t)=\mu(B_n(t))(>0)$.
Then by Fubini argument, we have $P(N_1)=0$ and as the same technique as that of [20], we have

$$\begin{aligned}&\sum_{n=0}^{\infty}E[|X(t, \omega)-X_n(t, \omega)|]\\&\le\sum_{n=0}^{\infty}2^{-n}T_d\phi\Big(\int_{B_n(t)}E[\Phi(X^*(t, s, \omega))]d\mu(s)/\mu_n(t)\Big)\\&\le\sum_{n=0}^{\infty}2^{-n}T_d\phi(M_X)<+\infty.\end{aligned}$$

Therefore, setting

$$N_2=\bigcup_{t\in D}\{\omega;\ \sum_{n=0}^{\infty}|X(t, \omega)-X_n(t, \omega)|=+\infty\},$$

(D is a separant)

if $\omega\notin N_1\cup N_2$, then it follows for $t\in D$ that

$$\begin{aligned}|X(t, \omega)|&\le|X_0(t, \omega)|+\sum_{n=0}^{\infty}|X_{n+1}(t, \omega)-X_n(t, \omega)|\\&\le\Big|\int_T X(s, \omega)d\mu(s)\Big|\\&\quad+\sum_{n=0}^{\infty}3\cdot2^{-n-1}T_d\phi\Big(\int_{B_{n+1}(t)\times B_n(t)}\Phi(X^*(s, s', \omega))\\&\qquad d\mu(s)d\mu(s')/(\mu_{n+1}(t)\mu_n(t))\Big)\\&\le\Big|\int_T X(s, \omega)d\mu(s)\Big|+3\sum_{n=0}^{\infty}2^{-n-1}T_d\phi(B(\omega)/\mu_{n+1}^2(t))\\&\le\Big|\int_T X(s, \omega)d\mu(s)\Big|+3cT_d\phi(B(\omega))\\&\quad+3c^2T_d\sum_{n=0}^{\infty}2^{-n-1}\phi(1/\mu_{n+1}(t))\end{aligned}\tag{8}$$

$$\leq \left|\int_T X(s, \omega) d\mu(s)\right| + 3cT_d\phi(B(\omega))$$
$$+6c^2 I(\mu, T_d/2).$$

Since the last term of the above estimate does not depend on t, it follows by Remark 2 and Jensen's inequality that

$$E[\sup_{t\in D} |X(t, \omega)|]$$
$$\leq \int_T E[|X(s, \omega)|] d\mu(s) + 3cT_d\phi(M_X)$$
$$+6c^2 I(\mu, T_d/2) < +\infty.$$

The proof of (ii) is essentially the same as that of Gaussian case proved by Fernique [15].

Let S be a finite subset of D, say $S=\{t_1,\dots, t_p\}$, and define a random variable τ_S by

$$\tau_S(\omega) = t_i \text{ if and only if } \max_{t\in S} |X(t, \omega)| = |X(t_i, \omega)|$$

and $|X(t_j, \omega)| < |X(t_i, \omega)|$, $1 \leq j \leq i-1$. Choosing as a probability space $(T, \mathscr{B}(d), \mu)$ the distribution of τ_S, it follows by (8) that

$$\max_{t\in S} |X(t, \omega)| = |X(\tau_S(\omega), \omega)|$$
$$\leq \left|\int_T X(s, \omega) d\mu(s)\right| + 3cT_d\phi(B(\omega))$$
$$+3c^2 T_d \sum_{n=0}^{\infty} 2^{-n-1}\phi(1/\mu_{n+1}(\tau_S(\omega))),$$

and

$$E[\max_{t\in S} |X(t, \omega)|]$$
$$\leq K + 3cT_d\phi(M_X)$$
$$+6c^2 J(T_d/2) < +\infty.$$

This yields the proof of (ii) since the right hand side does not depend on S. Q.E.D.

As is well known in case of Gaussian processes [13, p 77], if

$$\int_{+0} \phi(N_T(d, u)) du < +\infty,$$

then there exists a probability measure μ such that $I(\mu, \varepsilon) < +\infty (\varepsilon > 0)$. In fact, let D_n be a minimal 2^{-n}-net and define a probability μ such as

$$\mu(E) = \sum_{n=1}^{\infty} 2^{-n}\left(\sum_{t\in D_n} \delta_t(E)/N_T(d, 2^{-n})\right),$$

where $\delta_t(E)=1$ if $t\in E$ and $=0$, otherwise. By the definition of a net, we have $B(t, 2^{-n})\cap D_{n+1}\neq\phi$, and

$$\mu(B(t, 2^{-n}))\geq 2^{-n-1}/N_T(d, 2^{-n-1}).$$

Since we have

$$\begin{aligned}
&\phi(1/\mu(B(t, 2^{-n}))\\
&\leq\phi(2^{n+1}N_T(d, 2^{-n-1}))\\
&\leq c\phi(2^{n+1})+c\phi(N_T(d, 2^{-n-1}))\\
&\leq c\phi(2)(n+1)\log_2 c+c\phi(N_T(d, 2^{-n-1})),
\end{aligned}$$

it follows that

$$\begin{aligned}
I(\mu, \varepsilon)&\leq\int_{+0}^{\varepsilon}\sup_{t\in T}\phi(1/\mu(B(t, u))du\\
&\leq\text{const.}+\text{const.}\int_{+0}^{\varepsilon}\phi(N_T(d, u))du<+\infty.
\end{aligned}$$

§4. Sample path continuity. In this direction many people have obtained various sufficient conditions for various classes. Here, we will give another theorem, from which the known results are obtained as corollaries and, in addition, the proof of the theorem leads to a new sufficient condition for sample path continiuity of L_p-processes with stationary inerements.

Theorem 3. *For a positive decreasing sequence* $\{\varepsilon_n\}$ *such that* $\sum_{n=1}^{\infty}\varepsilon_n<+\infty$, *set*

$$\delta_n=\sum_{k=n}^{\infty}\varepsilon_k,$$

$$D_n=\text{a minimal } \varepsilon_n\text{-net of } (T, d),$$

$$B_n=\{(s, t);\ s, t\in D_n \text{ with } d(s, t)\leq 5\delta_n\},$$

and

$$\# B_n=\text{the cardinal number of } B_n.$$

If

$$\sum_{n}^{\infty}\sigma(5\delta_n)\phi(\# B_n)<+\infty, \tag{9}$$

and

$$\sum_{n}^{\infty}\sigma(\varepsilon_n)\phi(N_T(d, \varepsilon_{n+1}))<+\infty, \tag{10}$$

then any separable process $\{X(t);\ t\in T\}$ *belonging to* $\mathfrak{X}\{\Phi, \sigma, (T, d)\}$ *has, with probability* 1, *continuous sample paths, and*

$$E[\sup_{d(s,t)\le\delta_n}|X(s,\omega)-X(t,\omega)|]$$
$$\le 2C_X\sum_{k=n}^{\infty}\sigma(\varepsilon_k)\phi(N_T(d,\varepsilon_{k+1}))$$
$$+C_X\sum_{k=n}^{\infty}\sigma(5\delta_k)\phi(\sharp B_k)$$

holds for any $n\ge n_0$, *where* n_0 *is the first* n *such that* $N_T(d,\varepsilon_n)>\Phi(0)$ *and* $C_X=\sup_{x>N_T(d,\varepsilon_{n_0})}\phi(M_X x)/\phi(x)$.

Proof. Let $\{X(t,\omega);\, t\in T\}$ be a separable and measurable process with a separant $D=\bigcup_{n=1}^{\infty}D_n$. Using the same terminologies as the proof of Theorem 1, we have

$$|X(t,\omega)-X(\tau_n(t),\omega)|$$
$$\le\sigma(\varepsilon_n)X^*(t,\tau_n(t),\omega)$$
$$\le\sigma(\varepsilon_n)\phi(\Phi(X^*(t,\tau_n(t),\omega))),\ (\omega\notin N,\ t\in D)$$

and by Jensen's inequality, we have

$$\sum_n^{\infty}E[|X(t,\omega)-X(\tau_n(t),\omega)|]$$
$$\le\sum_n^{\infty}\sigma(\varepsilon_n)\phi(M_X)$$
$$\le\text{const. }\phi(M_X)\sum_n^{\infty}\sigma(\varepsilon_n)\phi(N_T(d,\varepsilon_{n+1}))<+\infty.$$

It follows that $P(N_1)=0$, where

$$N_1=\bigcup_{t\in D}\{\omega;\ \sum_n^{\infty}|X(t,\omega)-X(\tau_n(t),\omega)|=+\infty\}.$$

Set for $\omega\notin N\cup N_1$,

$$F_n(\omega)=\sup_{t\in D_{n+1}}|X(t,\omega)-X(\tau_n(t),\omega)|,$$
$$G_n(\omega)=\sup_{(s,t)\in B_n}|X(s,\omega)-X(t,\omega)|,$$
$$N_2=\{\omega\notin N\cup N_1;\ \sum_n^{\infty}F_n(\omega)=+\infty\},$$

and

$$N_3=\{\omega\notin N\cup N_1;\ \sum_n^{\infty}G_n(\omega)=+\infty\}.$$

By the habitual calculus, we have

$$F_n(\omega)\leq\sigma(\varepsilon_n)\phi(\Phi(\sup_{t\in D_{n+1}} X^*(t,\tau_n(t),\omega)))$$

$$\leq\sigma(\varepsilon_n)\phi(\sup_{t\in D_{n+1}} \Phi(X^*(t,\tau_n(t),\omega)))$$

$$\leq\sigma(\varepsilon_n)\phi(\sum_{t\in D_{n+1}} \Phi(X^*(t,\tau_n(t),\omega)))\,,$$

and

$$\sum_n^{\infty} E[F_n(\omega)]\leq\sum_n^{\infty}\sigma(\varepsilon_n)\phi(M_X N_T(d,\varepsilon_{n+1}))$$

$$\leq C_X\sum_n^{\infty}\sigma(\varepsilon_n)\phi(N_T(d,\varepsilon_{n+1}))<+\infty.$$

Analogouly, we have

$$G_n(\omega)\leq\sigma(5\delta_n)\phi(\sum_{(s,t)\in B_n}\Phi(X^*(s,t,\omega)))\,,$$

and

$$\sum_n^{\infty} E[G_n(\omega)]\leq C_X\sum_n^{\infty}\sigma(5\delta_n)\phi(\# B_n)<+\infty.$$

It follows that $P(N_2\cup N_3)=0$.

Now we take two points $t, t'\in D$ with $d(t,t')\leq\delta_m$, then for any $n(>m)$ there exist two sequenses $\{s_k(t)\}_{k=m}^n$ and $\{s_k(t')\}_{k=m}^n$ such that $s_n(t)=\tau_n(t)$, $\tau_k(s_{k+1}(t))=s_k(t)$, $k=m,\dots, n-1$ and $s_n(t')=\tau_n(t')$, $\tau_k(s_{k+1}(t'))=s_k(t')$, $k=m,\dots, n-1$. Since we have

$$d(s_m(t),s_m(t'))\leq\sum_{k=m}^{n-1} d(s_{k+1}(t),s_k(t))+d(s_n(t),t)$$

$$+d(t,t')+d(t',s_n(t'))+\sum_{k=m}^{n-1} d(s_{k+1}(t'),s_k(t'))$$

$$\leq 2\sum_{k=m}^{n-1}\varepsilon_k+2\varepsilon_n+\delta_m<5\delta_m,$$

it follows for $\omega\notin N\cup N_1\cup N_2\cup N_3$ that

$$|X(t,\omega)-X(t',\omega)|\leq|X(t,\omega)-X(s_n(t),\omega)|$$

$$+|X(t',\omega)-x(s_n(t'),\omega)|+2\sum_{k=m}^{n-1}F_k(\omega)+G_m(\omega)$$

$$\to 2\sum_{k=m}^{\infty}F_k(\omega)+G_m(\omega)\quad\text{as}\quad n\to+\infty.$$

Therefore $X(t,\omega)$, $\omega\notin N\cup N_1\cup N_2\cup N_3$, is uniformly continuous on D and moreover, we have

$$E[\sup_{0\leq d(s,t)\leq\delta_n}|X(s,\omega)-X(t,w)|]$$

$$\leq 2C_X\sum_{k=n}^{\infty}\sigma(\varepsilon_k)\phi(N_T(d,\varepsilon_{k+1}))$$

$$+C_X \sum_{k=n}^{\infty} \sigma(5\delta_k)\phi(\sharp B_k) < +\infty. \qquad \text{Q.E.D.}$$

Remark 5. We notice that the above proof is also valid even if the sum of a sub-sequence of $\{\sigma(5\delta_n)\phi(\sharp B_n)\}$ is convergent and this idea is useful for the proof of Theorem 5.

Corollary 1. *In case that* $T=[0, 1]^N$ *with the usual Euclidean metric* $d=\| \ \|$, *if*

$$\int^{+\infty} \sigma(1/\Phi(x)^{1/N})dx < +\infty, \tag{11}$$

then any separable stochastic process belonging to $\mathfrak{X}\{\Phi, \sigma, ([0, 1]^N, \| \ \|)\}$ *has continuous sample paths with probability* 1.

This integral test is the best possible in a sense when $\Phi(x)=x^p$, $p\geq 2$, and $N=1$. ([18], [23])

Proof. To avoid a technical complication, we assume that $\Phi(0)=0$. Set

$$\varepsilon_n = C_N \Phi(2^n)^{-1/N}, \quad \text{and} \quad C_N=(1-2^{-1/N})/5$$

in Theorem 3, then we have

$$N_T(\| \ \|, \varepsilon_n) \leq N^{N/2} 2^{-N} C_N^{-N} \Phi(2^n),$$

and by convexity of $\Phi(x)$, we have

$$\delta_n = \sum_{k=n}^{\infty} \varepsilon_k \leq C_N \Phi(2^n)^{-1/N} \sum_{k=n}^{\infty} (\Phi(2^n)/\Phi(2^k))^{1/N}$$
$$\leq 5^{-1}\Phi(2^n)^{-1/N}.$$

and

$$\sharp B_n \leq a_N N_T(\| \ \|, \varepsilon_n),$$

where a_N is a constant dependent only on N. Therefore, it follows that

$$\sum_{n=k}^{\infty} \sigma(5\delta_n)\phi(\sharp B_n)$$
$$\leq b_N \sum_{n=k}^{\infty} \sigma(1/\Phi(2^n)^{1/N})\phi(\Phi(2^n))$$
$$\leq 2b_N \int_{2^{k-1}}^{\infty} \sigma(1/\Phi(x)^{1/N})dx < +\infty,$$

and

$$\sum_{n=k}^{\infty} \sigma(\varepsilon_n)\phi(N_T(\| \ \|, \varepsilon_{n+1}))$$
$$\leq b_N \sum_{n=k}^{\infty} \sigma(1/\Phi(2^n)^{1/N})\phi(\Phi(2^{n+1}))$$

$$\leq 4b_N \int_{2^{k-1}}^{+\infty} \sigma(1/\Phi(x)^{1/N})dx < +\infty,$$

where

$$b_N = \sup_{x\geq 1} \phi(a_N N^{N/2} 2^{-N} C_N^{-N} \Phi(x))/\phi(\Phi(x)) \quad (< +\infty).$$

Remark 5. In case that $\Phi(x)=x$, Corollary 1 is nonsence because if

$$\int^{+\infty} \sigma(x^{-1/N})dx < +\infty, \tag{11$'$}$$

then $\mathfrak{X}\{x, \sigma, ([0, 1]^N, \| \ \|)\}$ contains only constant random variables as their separable modifications. In fact, set

$$Q_n(\omega) = \sum |X(k_1 2^{-n}, \dots, k_N 2^{-n}), \omega) - X(k'_1 2^{-n}, \dots, k'_N 2^{-n}), \omega)|$$

where $\sum$ sums up all $0 \leq k'_i \leq k_i \leq 2^n$, $i=1,\dots, N$ such that $\sum_{i=1}^{N} (k_i - k'_i) = 1$. Since $Q_n(\omega)$ is non-decreasing, it follows by (11)$'$ that

$$E[\lim_{n\to\infty} Q_n(\omega)] = \lim_{n\to\infty} E[Q_n(\omega)] \leq \lim_{n\to\infty} N 2^{nN} M_X \sigma(2^{-n}) = 0.$$

This yields that $Q_n(\omega) \equiv 0$ for all n with probability 1. Q. E. D.

Corollary 2. *Assume the condition* (5) *of Theorem* 2. *If*

$$\int_{+0} \phi(N_T(d, u))du < +\infty, \tag{12}$$

then any separable stochastic process belonging to $\mathfrak{X}\{\Phi, x, (T, d)\}$ *has continuous sample paths with probability* 1.

Proof. Set $\varepsilon_n = 2^{-n}$ in Theorem 3. Since we have $\delta_n = 2^{-n+1}$ and $\sharp B_n \leq N_T(d, 2^{-n})^2$, it follows that

$$\sum_{n=k}^{\infty} 5\cdot 2^{-n+1}\phi(\sharp B_n) \leq 5c \sum_{n=k}^{\infty} 2^{-n+1}\phi(N_T(d, 2^{-n})) \leq 20c \int_{+0}^{2^{-k}} \phi(N_T(d, u))du < +\infty,$$

and

$$\sum_{n=k}^{\infty} 2^{-n}\phi(N_T(d, 2^{-n-1})) \leq 4\int_{+0}^{2^{-k-1}} \phi(N_T(d, u))du < +\infty.$$

Q. E. D.

This Corollary 2 is applicable with a slight modification to Skorohod's example

(Example 3). In fact, if we set $\sigma(x)=x^{\alpha'/\alpha}(\alpha'>0)$, and assume that Trace $A^{\beta}<+\infty$ for some $\beta>0$, then we have

$$N_T(d, 2^{-n/\bar{\beta}})\leq a_1^n 2^{n2^{n/\theta}},$$

where $\bar{\beta}=\max(1, \beta)$, $\theta=\max(1, 1/\beta-1/2)$ and a_1 is a constant. In place of (12), convergence of the following integral guarantees sample path continuity:

$$\begin{aligned}
&\int_{+0}\phi(N_T(d, u))\sigma(20u)/u\, du \\
&\leq \text{const.} \int_{+0}(\log N_T(d, u))^{1/\alpha}u^{\alpha/\alpha'-1}du \\
&\leq \text{const.} \sum_{n}^{\infty}(n\log a_1+n2^{n/\theta}\log 2)^{1/\alpha}\sigma(2^{-n/\bar{\beta}}) \\
&<+\infty, \quad \text{if} \quad \alpha'\theta>\bar{\beta}.
\end{aligned}$$

The last condition is satisfied when one of the following three conditions is fulfilled:

(i) $0<\beta<2/3$ and $1<\alpha'(1/\beta-1/2)$,

(ii) $2/3\leq\beta\leq 1$ and $1<\alpha'$, or

(iii) $1\leq\beta\leq 2$ and $\beta<\alpha'$.

This is nothing but Skorohod's result.

Corresponding to Theorem 2, we have other sufficient conditions for sample path continuity.

Theorem 4. *Assume the condition* (5) *of Theorem* 2.
(i) *If there exists a probability measure* $(T, \mathscr{B}(d), \mu)$ *such that*

$$\lim_{\varepsilon\downarrow 0} I(\mu, \varepsilon)=0, \tag{13}$$

then any separable and measurable process $\{X(t); t\in T\}$ *belonging to* $\mathfrak{X}\{\Phi, x, (T, d)\}$ *has continuous sample paths with probability* 1, *and*

$$E[\sup_{d(s,t)\leq 2^{-k}T_d}|X(s, \omega)-X(t, \omega)|]\leq 9\cdot 2^{-k}cT_d\phi(M_X)+12c^2 I(\mu, 2^{-k}T_d).$$

(ii) *If*

$$\lim_{\varepsilon\downarrow 0} J(\varepsilon)=0, \tag{14}$$

then the same conclusion as (i) *is valid and*

$$E[\sup_{d(s,t)\leq 2^{-k}T_d}|X(s, \omega)-X(t, \omega)|]\leq 9\cdot 2^{-k}cT_d\phi(M_X)+12c^2 J(2^{-k}T_d).$$

Proof. (i) Using the same notations as in the proof of Theorem 2, it follows for $\omega\notin N_1$ that

$$|X_{n+1}(t, \omega)-X_n(t, \omega)|$$
$$\leq 3\cdot 2^{-n-1}T_d\phi\left(\int_{B_{n+1}(t)\times B_n(t)}\Phi(X^*(s, s', \omega))d\mu(s)d\mu(s')/(\mu_{n+1}(t)\mu_n(t))\right)$$
$$\leq 3\cdot 2^{-n-1}cT_d(\phi(B(\omega))+c\phi(1/\mu_{n+1}(t))),$$

and

$$\sum_{n=k}^{\infty}|X_{n+1}(t, \omega)-X_n(t, \omega)|$$
$$\leq 3\cdot 2^{-k}cT_d\phi(B(\omega))+6c^2I(\mu, 2^{-k-1}T_d).$$

Analogously, if $d(t, t')\leq 2^{-k}T_d$ and $\mu_k(t)\leq\mu_k(t')$, we have

$$|X_k(t, \omega)-X_k(t', \omega)|$$
$$\leq 3\cdot 2^{-k}cT_d(\phi(B(\omega))+c\phi(1/\mu_k(t))).$$

Therefore, it follows for $\omega\notin N_1\cup N_2$ and $t, t'\in D$ with $d(t, t')\leq 2^{-k}T_d$ that

$$|X(t, \omega)-X(t', \omega)|\leq 9\cdot 2^{-k}cT_d\phi(B(\omega))$$
$$+12c^2I(\mu, 2^{-k}T_d),$$

which means that $X(t, \omega)$, $\omega\notin N_1\cup N_2$ is uniformly continuous on D and moreover we have

$$E[\sup_{d(t,t')\leq 2^{-k}T_d}|X(t, \omega)-X(t', \omega)|]$$
$$\leq 9\cdot 2^{-k}cT_d\phi(M_X)+12c^2I(\mu, 2^{-k}T_d).$$

(ii) Let S be any finite subset of D, say $\{t_1,\dots, t_r\}$, and define a couple of random variables (τ_1, τ_2) as follows:

$$\tau_1(\omega)=t_i \quad\text{and}\quad \tau_2(\omega)=t_j \quad\text{if and only if}$$
$$\max_{\substack{(t,t')\in S\times S\\ d(t,t')\leq 2^{-k}T_d}}(X(t, \omega)-X(t', \omega))=X(t_i, \omega)-X(t_j, \omega)$$
$$>X(t_p, \omega)-X(t_q, \omega)$$

holds for $p<i$, or $p=i$ and $q<j$.

Denote by $(T, \mathscr{B}(d), \mu_1)$ and $(T, \mathscr{B}(d), \mu_2)$ the distributions of τ_1 and τ_2 respectively and set

$$D_i(\omega)=\int_{T\times T}\Phi(X^*(s, t, \omega))d\mu_i(s)d\mu_i(t), \quad i=1, 2,$$

and

$$D_{1,2}(\omega)=\int_{T\times T}\Phi(X^*(s, t, \omega))d\mu_1(s)d\mu_2(t).$$

Analogously as in (i), it follows for $\omega\notin N_1\cup N_2$ that

$$\max_{\substack{(t,t')\in S\times S\\ d(t,t')\leq 2^{-k}T_d}} |X(t,\omega)-X(t',\omega)| = X(\tau_1(\omega),\omega)-X(\tau_2(\omega),\omega)$$

$$\leq \sum_{n=k}^{\infty} |X_{n+1}(\tau_1(\omega),\omega)-X_n(\tau_1(\omega),\omega)|$$

$$+|X_k(\tau_1(\omega),\omega)-X_k^1(\tau_2(\omega),\omega)|$$

$$+\sum_{n=k}^{\infty} |X_{n+1}^1(\tau_2(\omega),\omega)-X_n^1(\tau_2(\omega),\omega)|$$

$$\leq 3c2^{-k}T_d(\phi(D_1(\omega))+\phi(D_2(\omega))+\phi(D_{1,2}(\omega)))$$

$$+6c^2\int_{+0}^{2^{-k}T_d} \phi(1/\mu_1(B(\tau_1(\omega),u)))du$$

$$+6c^2\int_{+0}^{2^{-k}T_d} \phi(1/\mu_2(B(\tau_2(\omega)),u)du,$$

where X_k and X_k^1 are defined by the distributions μ_1 and μ_2 respectively such as in the proof of Theorem 2. Therefore, we have

$$E[\max_{\substack{(t,t')\in S\times S\\ d(t,t')\leq 2^{-k}T_d}} |X(t,\omega)-X(t',\omega)|]$$

$$=E[X(\tau_1(\omega),\omega)-X(\tau_2(\omega),\omega))]$$

$$\leq 9c2^{-k}T_d\phi(M_X)+12c^2J(2^{-k}T_d).$$

Since S is arbitrary, we conclude (ii).

Now we establish a new sufficient condition for sample path continuity of a class of stochastic processes including L_p-processes with stationary increments. This condition is just the analogy of that for Gaussian case due to Jain-Marcus [22]. Let $T=[0,1]^N$ with the usual Euclidean norm $\|\ \|$, and assume that there exists a pseudo-metric d on T which depends only on the Euclidean norm; that is, $d(s,t)=\psi(\|s-t\|)$ with a continuous function ψ. Denote by $\bar{\psi}$ the non-decreasing rearrangement of ψ; i.e.

$$\bar{\psi}(x)=\sup\{y;\ \mu(\{t\in T;\ \psi(\|t\|)<y\})<x\},$$

where μ is the Lebesgue measure on T.

Theorem 5. *Assume that* $\Phi(x)\leq K_1x^p+K_2$, $p>1$. *If*

$$\int^{+\infty} \bar{\psi}(1/\Phi(x))dx<+\infty, \tag{15}$$

or equivalently,

$$\int_{+0} \phi(N_T(d,u))du<+\infty, \tag{16}$$

then any separable and measurable process $\{X(t);\ t\in T=[0,1]^N\}$ *belonging to*

$\mathfrak{X}\{\Phi(x), x, (T, d)\}$ has continuous sample paths with probability 1.

By Proposition 2, the continuity with respect to the pseudometric d is equivalent to the continuity with respect to the Euclidean norm $\| \ \|$.

Proof. We will prove that sample paths are continuous with respect to the pseudo-metric d with probability 1. Let D_n be a minimal 2^{-n}-net on (T, d), and τ_n be a mapping from T to D_n such that $d(\tau_n(t), t) \leq 2^{-n}$, and $\tau_n(t)=t$ if $t \in D_n$. Set $B_n(t)=\{s;\ d(s, t)=\psi(\|s-t\|)<2^{-n}\}$ and $\mu_n(t)=\mu(B_n(t))$, the Lebesgue measure of $B_n(t)$. By the analogous way as that of the proof of Theorem 2, we have $\sum_{n=0}^{\infty} E[|X(t, \omega)-X_n(\tau_n(t), \omega)|] \leq \sum_{n=0}^{\infty} 2^{-n+1}\phi(M_X)<+\infty$. Set

$$F_n(\omega)=\sup_{t \in D_{n+1}} |X_{n+1}(t, \omega)-X_n(\tau_n(t), \omega)|,$$

and

$$G_n(\omega)=\sup_{\substack{t,t' \in D_n \\ d(t,t') \leq 6.2^{-n}}} |X_n(t, \omega)-X_n(t', \omega)|,$$

then, we have

$$\begin{aligned}
&\sum_{n}^{\infty} E[F_n(\omega)] \\
&\leq 5 \sum_{n} 2^{-n-1} E\Big[\phi\Big(\sum_{t \in D_{n+1}} \int_{B_{n+1}(t) \times B_n(\tau_n(t))} \Phi(X^*(s, s', \omega)) \\
&\qquad d\mu(s)\, d\mu(s')/\mu_{n+1}(t)\mu_n(\tau_n(t))\Big)\Big] \\
&\leq 5 \sum_{n} 2^{-n-1} \phi(M_X N_T(d, 2^{-n-1})) \\
&\leq 5 C_X \int_{+0} \phi(N_T(d, u))\, du < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
E[G_n(w)] &\leq 2^{-n+3} E\Big[\Big(\sup_{\substack{t,t' \in D_n \\ d(t,t') \leq 6.2^{-n}}} \int_{d(s,s') \leq 2^{-n+3}} \Phi(X^*(s, s', \omega)) \\
&\qquad d\mu(s)\, d\mu(s')/\mu_n(t)\mu_n(t')\Big)\Big] \\
&\leq C_X 2^{-n+3} \phi(N_T(d, 2^{-n})^2/N_T(d, 2^{-n+5})),
\end{aligned}$$

here we use Lemma 2, 1 of [22, p 120].

Now let us estimate the last term. If there exists n_0 such that

$$\phi(N_T(d, 2^{-n})^2/N_T(d, 2^{-n+5})) \geq 2^{6p} \phi(N_T(d, 2^{-n}))$$

holds for any $n \geq n_0$, then we have

$$N_T(d, 2^{-n})/N_T(d, 2^{-n+5})$$

$$\geq \Phi(2^{6p}\phi(N_T(d, 2^{-n})))/\Phi(\phi(N_T(d, 2^{-n})))$$
$$\geq 2^{6p}-(2^{6p}-1)\Phi(0)/N_T(d, 2^{-n}).$$

Therefore, there exists n_1 such that

$$N_T(d, 2^{-n})/N_T(d, 2^{-n+5}) \geq 2^{6p-1}$$

holds for any $n \geq n_1$. By induction, we have

$$N_T(d, 2^{-n_1-5m}) \geq 2^{(6p-1)m} N_T(d, 2^{-n_1}),$$

and

$$\sum_{m=0}^{\infty} 2^{-n_1-5m}\phi(N_T(d, 2^{-n_1-5m}))$$
$$\geq \sum_{m=0}^{\infty} 2^{-n_1-5m}\{(2^{(6p-1)m}N_T(d, 2^{-n_1})-K_2)/K_1\}^{1/p}$$
$$=+\infty,$$

which contradicts to (16). This shows that there exists a subsequence $\{n_k\}$ such that

$$\sum_k^{\infty} E[G_{n_k}(\omega)] \leq C_X^2 \sum_k^{\infty} 2^{-n_k+3}\phi(N_T(d, 2^{-n_k})) < +\infty.$$

Combining the above estimates, the ω-sets

$$N_3 = \{\omega ; \sum_n^{\infty} F_n(\omega) = +\infty\},$$

and

$$N_4 = \{\omega ; \sum_k^{\infty} G_{n_k}(\omega) = +\infty\}$$

have probability 0. Therefore, it follows for $t, t' \in D$ with $d(t, t') \leq 2^{-n_k}$ that

$$|X(t, \omega) - X(t', \omega)|$$
$$\leq |X(t, \omega) - X_n(\tau_n(t), \omega)| + 2\sum_{m=n_k}^{n} F_m(\omega)$$
$$+ G_{n_k}(\omega) + |X(t', \omega) - X_n(\tau_n(t'), \omega)|$$
$$\to 2\sum_{m=n_k}^{\infty} F_m(\omega) + G_{n_k}(\omega) \quad \text{as} \quad n \to +\infty,$$

and so $\{X(t, \omega); t \in D\}$ is uniformly continuous on D with probability 1. Q.E.D.

Taking account of the Gaussian case, I have a conjecture that Theorem 5 is the best possible in the following sense: in order that any separable L_p-process with stationary increments, $E[|X(s)-X(t)|^p]^{1/p} = \psi(\|s-t\|)$, has continuous sample paths with probability 1, it is necessary and sufficient that an integral test $\int^{+\infty}\bar{\psi}(x^{-p})dx$ converges.

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