J. Math. Kyoto Univ. (JMKYAZ)
20-2 (1980) 263-289

Derivatives of Wiener functionals and absolute continuity of induced measures

By

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(Received Jan. 22, 1979)

1. Introduction

The *Wiener space*, which is a typical example of abstract Wiener spaces introduced by L. Gross [1], is a triple (B, H, μ) where
(i) B is a Banach space consisting of real-valued continuous functions $x(t)$ with $x(0)=0$ defined on the interval $T=[0, 1]$ endowed with norm $\|x\|=\sup_{0\leq t\leq 1}|x(t)|$,
(ii) H is a Hilbert space consisting of absolutely continuous functions $x(t)$ with $x(0)=0$ such that $x'(t)\in L^2(T)$ endowed with the inner product

$$\langle x, y\rangle_H=\int_0^1 x'(t)y'(t)dt$$

and
(iii) μ is the Wiener measure, i.e., the Borel probability measure on B such that

$$\int_B \exp\{i(h, x)\}\mu(dx)=\exp\left(-\frac{1}{2}\langle h, h\rangle_H\right) \tag{1.1}$$

where $h\in B^*\subset H$ and $(\ ,\ )$ is a natural pairing of B^* and B. Note that $\|x\|\leq|x|_H=\sqrt{\langle x, x\rangle_H}$ for $x\in H$, then the inclusion map $i: H\to B$ is continuous. Hence we have $B^*\subset H^*=H$ and we regard B^* as a subset of H. It is readily seen that $\{x(t); 0\leq t\leq 1\}$ is a standard Wiener process on the probability space (B, μ). A real-valued (or more generally, a Banach space-valued) measurable function defined on the probability space (B, μ) is called a *Wiener functional*. We identify two Wiener functionals $F_1(x)$ and $F_2(x)$ if $F_1(x)=F_2(x)$ a.e. (μ). Typical examples of Wiener functionals are solutions of stochastic differential equations or Ito's multiple Wiener integrals [2].

P. Malliavin introduced a notion of derivatives of Wiener functionals and applied it to the absolute continuity and the smoothness of density of the probability law induced by the solution of the stochastic differential equation at a fixed time [6], [7]. Here we define the derivatives of the Wiener functionals in a somewhat different way and rephrase a theorem of Malliavin. We shall apply it to the absolute continuity of the probability law induced by a system of multiple Wiener integrals.

The author wishes to thank Professor S. Watanabe for his valuable suggestion and encouragement.

2. Notion of derivative

Let (B, H, μ) be the Wiener space or more generally, any abstract Wiener space. Let E be a separable Banach space and F be a mapping from B into E. F is said to be *B-differentiable* (or Fréchet differentiable) *at* $x \in B$ if there exists an operator $T = T_x \in \mathscr{L}(B, E)$ (we denote the space of all bounded linear operators from B into E by $\mathscr{L}(B, E)$) such that

$$F(x+y)-F(x)=T(y)+o(\|y\|) \quad \text{as} \quad \|y\| \longrightarrow 0 \quad (y \in B). \tag{2.1}$$

The operator $T = T_x$ is called the *B-derivative* (or Fréchet derivative) *of* F *at* $x \in B$, $F'(x)$ in notation. If F is B-differentiable at every point of B, we say simply that F is *B-differentiable*. Similarly F is said to be *H-differentiable at* $x \in B$ if there exists an operator $S = S_x \in \mathscr{L}(H, E)$ such that

$$F(x+h)-F(x)=S(h)+o(|h|_H) \quad \text{as} \quad |h|_H \longrightarrow 0 \quad (h \in H). \tag{2.2}$$

The operator $S = S_x$ is called the *H-derivative of* F *at* $x \in B$, $DF(x)$ in notation. If F is H-differentiable at every point of B, we say F is *H-differentiable*. Clearly if F is B-differentiable, then F is also H-differentiable and $DF(x) = F'(x)|_H$. Inductively we can define F'', F''',... and D^2F, D^3F,.... We may regard $F^{(n)}$ as an element of $\mathscr{L}^n(B; E)$ and D^nF as an element of $\mathscr{L}^n(H, E)$ where $\mathscr{L}^n(B, E)$ is a space of continuous n-linear operators from B^n into E and $\mathscr{L}^n(H, E)$ is defined similarly. When E is a Hilbert space, $S \in \mathscr{L}^n(H, E)$ is said to be of *Hilbert-Schmidt class* if

$$\sum_{i_1, i_2, \dots, i_n = 1}^{\infty} |S(h_{i_1}, h_{i_2}, \dots, h_{i_n})|_E^2 < \infty \tag{2.3}$$

for any orthonormal system $\{h_i\}_{i=1}^{\infty}$ in H. We denote by $\mathscr{L}^n_{(2)}(H, E)$ the space of all $S \in \mathscr{L}^n(H, E)$ which are of Hilbert-Schmidt class. Then $\mathscr{L}^n_{(2)}(H, E)$ is a Hilbert space with inner product given by

$$\langle T, S \rangle_{\mathscr{L}^n_{(2)}(H,E)} = \sum_{i_1, i_2, \dots, i_n = 1}^{\infty} \langle T(h_{i_1}, \dots, h_{i_n}), S(h_{i_1}, \dots, h_{i_n}) \rangle_E \tag{2.4}$$

for $T, S \in \mathscr{L}^n_{(2)}(H, E)$. Where $\{h_i\}_{i=1}^{\infty}$ is a complete orthonormal system in H. If $F: B \to E$ is m-times B-differentiable then D^mF is of Hilbert-Schmidt class (cf. [4]).

Let K be a separable Hilbert space. For $p \geq 1$, we denote by $L^p(\mu; K)$ the set of all K-valued Wiener functionals $F: B \to K$ such that

$$\|F\|_{L^p(\mu;K)} = \left(\int_B |F(x)|_K^p \mu(dx)\right)^{\frac{1}{p}} < \infty$$

Difinition 2.1. For $p_0, p_1, \dots, p_n \geq 1$, we define $H(p_0, p_1, \dots, p_n)(K)$ to be the space of all Wiener functionals $F(x) \in L^{p_0}(\mu; K)$ such that there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of functions on B into K with the following properties;

(i) for any $k=1, 2, \dots, f_k$ is a n-times B-differentiable mapping from B into K and $f_k \in L^{p_0}(\mu; K)$,

(ii) $\lim_{k\to\infty} f_k = F$ in $L^{p_0}(\mu; K)$,

(iii) for any $m=1, 2, \dots, n$, $D^m f_k(x) \in \mathscr{L}_{(2)}^m(H, K)$ for all $x \in B$ and a sequence $\{D^m f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^{p_m}(\mu; \mathscr{L}_{(2)}^m(H, K))$,

(iv) for any $k=1, 2, \dots$, there exists a finite dimensional projection Q_k: $B \to B$ such that $Q_k|_H$ is an orthogonal projection in H and $f_k(x) = f_k(Q_k x)$.

For $F \in H(p_0, p_1, \dots, p_n)(K)$, we set $D^m F = \lim_{k\to\infty} D^m f_k$ and call it the *m-th weak H-derivative* of F.

The sequence of above definition is called an *approximating sequence.* By the following lemma our definition of weak H-derivatives are justified; they are well-defined independent of a particular choice of an approximating sequence $\{f_k\}$.

Lemma 2.1. *If $F \in H(p_0, p_1)(K)$ and $F=0$, then $DF=0$.*

Proof. In the proof we may assume $K=\mathbf{R}$. Indeed we take $l \in K^*$ and consider the functional $l(F(x))$. Then evidently $l(F(X)) \in H(p_0, p_1)(\mathbf{R})$ and $Dl(F(x)) = l \circ DF(x)$ from the chain rule of differential. Furthermore if $Dl(F(x)) = l \circ DF(x) = 0$ for all $l \in K^*$, we have $DF(x)=0$. So we shall assume $K=\mathbf{R}$. Take an approximating sequence $\{f_k\}_{k=1}^{\infty}$ of F and $h \in B^* \subset H$ such that $|h|_H = 1$. Let H_1 be a subspace of H spanned by h, H_2 be an orthogonal complement of H_1 in H and $\bar{H}_2$ be a completion of H_2 in B. Then $B = H_1 \oplus \bar{H}_2$ where $\oplus$ stands for the direct sum; indeed any $x \in B$ can be expressed as $x = (h, x)h + (x-(h, x)h)$ where $(h, x)h \in H_1$ and $x-(h, x)h \in \bar{H}_2$. Therefore $\pi(x) = x-(h, x)h$ defines a projection of B onto $\bar{H}_2$ and $(\bar{H}_2, H_2, \tilde{\mu})$ is an abstract Wiener space where $\tilde{\mu} = \mu \circ \pi^{-1}$ i.e., induced measure of μ by π. Note that if we express $x \in B$ as $x = y + th$ where $y = \pi(x)$ and $t = (h, x)$, then μ is expressed as

$$\mu(dx) = \tilde{\mu}(dy) \times \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

Since $f_k \to 0$ in $L^{p_0}(\mu)$ as $k \to \infty$,

$$\int_B |f_k(x)|^{p_0} \mu(dx) = \int_{\bar{H}_2} \tilde{\mu}(dy) \int_{-\infty}^{\infty} |f_k(y+th)|^{p_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \longrightarrow 0$$

as $k \to \infty$. Consequently if we put $g_k(y) = \int_{-\infty}^{\infty} |f_k(y+th)|^{p_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, we see that $g_k \to 0$ in $L^1(\tilde{\mu})$ as $k \to \infty$. By extracting a subsequence if necessary, we may assume $g_k \to 0$ a.e. $(\tilde{\mu})$. By a similar argument, we have

$$\int_{-\infty}^{\infty} |\langle Df_k(y+th), h\rangle_H - \langle DF(y+th), h\rangle_H|^{p_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \longrightarrow 0$$

as $k \to \infty$ a.e. $(\tilde{\mu})$. Note that for a fixed y, $f_k(y+th)$ is differentiable with respect to

t and its derivative is

$$\frac{d}{dt} f_k(y+th) = \langle Df_k(y+th), h \rangle_H .$$

Therefore we have for a.e. $y(\tilde{\mu})$

$$f_k(y+th) \longrightarrow 0 \quad \text{in} \quad L^{p_0}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt\right)$$

and

$$\frac{d}{dt} f_k(y+th) \longrightarrow \langle DF(y+th), h \rangle_H \quad \text{in} \quad L^{p_1}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt\right)$$

as $k \to \infty$. By a well-known result in one dimensional case, we have

$$\langle DF(y+th), h \rangle_H = 0 \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt\text{-}a.e.$$

Hence $\langle DF(x), h \rangle_H = 0$ a.e. (μ). Since $h \in B^*$ is arbitrary and B^* is dense in H, we have $DF(x) = 0$ a.e. (μ). Q.E.D.

Obviously $H(p_0, p_1, \ldots, p_n)(K)$ is a Banach space endowed with norm $\|F\|_{H(p_0, p_1, \ldots, p_n)(K)} = \sum_{m=0}^{n} \|D^m F\|_{L^{p_m}(\mu; \mathscr{L}^m_{(2)}(H;K))}$, especially for $p_0 = p_1 = \cdots = p_n = 2$, $H(\underbrace{2, 2, \ldots, 2}_{n+1})(K)$ is a Hilbert space endowed with norm (considering the Hilbert space structure, we modify the above norm in this case) $|F|^2_{H(2,2,\ldots,2)(K)} = \sum_{m=0}^{n} \|D^m F\|^2_{L^2(\mu; \mathscr{L}^m_{(2)}(H;K))}$. We can also characterize the space $H(p_0, p_1, \ldots, p_n)(K)$ in another way. We introduce the notion of smooth functional. Let K be a separable Hilbert space. Then a K-valued *smooth functional* $\varphi(x)$ defined on B is the mapping $\varphi: B \to K$ expressed as $\varphi(x) = f((l_1, x), (l_2, x), \ldots, (l_d, x))$ where d is a positive integer and $f: \mathbf{R}^d \to K$ is a K-valued C^∞ function with compact support and $l_1, l_2, \ldots, l_d \in B^*$. It is easy to see that a smooth functional is B-differentiable of any order and its derivatives are all smooth functionals. Then we have:

Proposition 2.1. $H(p_0, p_1, \ldots, p_n)(K)$ is the completion of all K-valued smooth functionals with norm $\|\cdot\|_{H(p_0, p_1 \ldots, p_n)(K)}$ defined above.

Proof. What we have to show is that K-valued smooth functionals are dense in $H(p_0, p_1, \ldots, p_n)(K)$. Take any $F \in H(p_0, p_1, \ldots, p_n)(K)$. For any $\varepsilon > 0$, from Definition 2.1, there exists $f \in L^{p_0}(\mu; K)$ satisfying (i) and (iv) of Definition 2.1 such that $\|f - F\|_{H(p_0, p_1, \ldots, p_n)(K)} < \varepsilon$. From (i) and (iv) of Definition 2.1, f is expressed as $f(x) = g((l_1, x), (l_2, x), \ldots, (l_d, x))$ where $l_1, l_2, \ldots, l_d \in B^*$ and g is a K-valued C^n function defined on $\mathbf{R}^d$. We may assume that $l_1, l_2, \ldots, l_d$ are orthonormal system in H. For $N = 1, 2, \ldots$, let $c_N(\xi): \mathbf{R}^d \to \mathbf{R}$ is a C^∞ function such that $0 \le c_N(\xi) \le 1$ and

$$c_N(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \le N \\ 0 & \text{if} \quad |\xi| \ge N+1. \end{cases}$$

We may assume that $\{c_N\}_{N=1}^{\infty}$ is uniformly bounded together with its derivatives. Then it is easy to show that

$$c_N((l_1, x), (l_2, x), \dots, (l_d, x))g((l_1, x), (l_2, x), \dots, (l_d, x))$$
$$\longrightarrow f(x) \quad \text{in} \quad H(p_0, p_1, \dots, p_n)(K) \quad \text{as} \quad N \longrightarrow \infty.$$

Then we may assume that g has compact support. But g can be approximated uniformly by a C^∞ function with compact support by using the molifier. Hence we can find a smooth functional φ such that $\|f-\varphi\|_{H(p_0,p_1,\dots,p_n)(K)}<\varepsilon$. This complete the proof. Q.E.D.

Proposition 2.2. *If* $F \in H(p_0, p_1)(K)$ *and* $DF=0$, *then* $F=$*constant a.e.* (μ).

Proof. We may assume $K=\mathbf{R}$ as in Lemma 2.1. It is enough to prove in case of bounded F. Indeed take a C^∞ function $c_N(\xi)$: $\mathbf{R}\to\mathbf{R}$ such that

$$c_N(\xi)=\begin{cases} \xi & \text{if } |\xi|\le N \\ \operatorname{sgn}(\xi)(N+1) & \text{if } |\xi|\ge N+1 \end{cases} \tag{2.5}$$

and consider the function $c_N\circ F$. Then clearly $c_N\circ F \in H(p_0, p_1)(\mathbf{R})$ and $Dc_N\circ F = c_N'(F)DF=0$. Note finally that if $c_N\circ F=$constant for all $N\in\mathbf{N}$, it is evident that $F=$constant. So let F be bounded. For any $h\in B^*$ such that $|h|_H=1$ we have, similarly as in Lemma 2.1 (we use the same notation)

$$B=H_1\oplus\overline{H}_2,\ \mu(dx)=\tilde{\mu}(dy)\times\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$

where $x=y+th$. In the following we fix h and $s\in\mathbf{R}$. From the assumption, for a.e. $y(\tilde{\mu})$

$$\frac{d}{dt}F(y+th)=\langle DF(y+th), h\rangle_H=0$$

where the derivative is in the distributional sense. Consequently we have for a.e. $y(\tilde{\mu})$

$$F(y+th)=\text{constant} \qquad \text{a.e. in } t$$

and hence

$$F(y+(t+s)h)=F(y+th) \qquad \text{a.e. in } t.$$

Thus we have

$$F(x+sh)=F(x) \qquad \text{a.e. } (\mu).$$

Hence

$$\int_B F(x)\mu(dx)=\int_B F(x+sh)\mu(dx)$$
$$=\int_B F(x)\exp\left\{s(h, x)-\frac{s^2}{2}|h|_H^2\right\}\mu(dx).$$

To show the second equality, we use the following fact. If we define $\mu_h(\cdot)=\mu(\cdot-h)$, then μ_h is absolutely continuous with respect to μ if and only if $h\in H$ and its Radon-Nikodym derivative is given by $\exp\left\{\langle h, x\rangle-\frac{|h|_H^2}{2}\right\}$ (see Kuo [4]). If we put $C=\int_B F(x)\mu(dx)$, we have

$$\int_B (F(x)-C)e^{(sh,x)}\mu(dx)=0$$

for any $s\in R$ and $h\in B^*$ such that $|h|_H=1$. Our assertion now follows from the following.

Lemma 2.2. *$\{e^{(h,\cdot)};\ h\in B^*\}$ spans $L^2(\mu)$.*

This fact is well known (see Lehman [5]).

Next we shall obtain a formula on the integration by parts for Wiener integrals (see also Kuo [3]). In order to state the formula we need the Hermitian polynomials defined by

$$H_n(\xi)=\frac{(-1)^n}{n!}e^{\frac{\xi^2}{2}}\frac{d^n}{d\xi^n}e^{-\frac{\xi^2}{2}} \qquad \xi\in\boldsymbol{R},\ n=0,\ 1,\dots. \tag{2.6}$$

This definition is a little different from ordinary one but it is more convenient for our purposes. We list below some properties of the Hermitian polynomials:

$$H_0(\xi)=1 \tag{2.7}$$

$$\frac{d}{d\xi}H_n(\xi)=H_{n-1}(\xi) \tag{2.8}$$

$$(n+1)H_{n+1}(\xi)-\xi H_n(\xi)+H_{n-1}(\xi)=0 \tag{2.9}$$

$$\int_{-\infty}^{\infty}H_m(\xi)H_n(\xi)\frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{2}}d\xi=\delta_{m,n}\frac{1}{n!}. \tag{2.10}$$

In (2.8) and (2.9), we set $H_{-1}(\xi)=0$ if $n=0$.

Let $\{h_i\}_{i=1}^{\infty}\subset B^*$ be a complete orthonormal system in H and fix it until Lemma 2.3. Let $a=(a_1, a_2,\dots)$ be a sequence of non negative integers such that $a_j=0$ except for finitely many j. We define the *Fourier-Hermite functionals* $H_a(x)$ on B as

$$H_a(x)=H_{a_1}((h_1, x))H_{a_2}((h_2, x))\cdots. \tag{2.11}$$

From (2.10) we have

$$\int_B H_a(x)H_b(x)\mu(dx)=\begin{cases}0 & \text{if } a\neq b\\ \frac{1}{a!} & \text{if } a=b\end{cases} \tag{2.12}$$

where $a!=a_1!a_2!\cdots$.

It is easily seen that for $F \in H(p_0, p_1, \ldots, p_n)(K)$, $D^m F(x)$ is symmetric a.e. (μ) i.e., $D^m F(x)(u_1, u_2, \ldots, u_m) = D^m F(x)(u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(m)})$ a.e. (μ) for any $u_1, u_2, \ldots, u_m \in H$ and any permutation σ of $\{1, 2, \ldots, m\}$. For a sequence $a = (a_1, a_2, \ldots)$ as above we define

$$D^{|a|}F(x)(h^a) = D^{|a|}F(x)(\underbrace{h_1, \ldots, h_1}_{a_1}, \underbrace{h_2, \ldots, h_2}_{a_2}, \ldots) \tag{2.13}$$

where $|a| = a_1 + a_2 + \cdots$.

Lemma 2.3. *Let $F \in H(p_0, p_1, \ldots, p_n)(\mathbf{R})$ such that $p_0 > 1$ and $p_1, p_2, \ldots, p_n \geq 1$. Then it holds that*

$$\begin{aligned} &\int_B D^{|a|}F(x)(h^a)\varphi(x)\mu(dx) \\ &\quad = \int_B F(x) \sum_{b \leq a} (-1)^{|b|} \frac{a!}{b!} D^{|b|}\varphi(x)(h^b) H_{a-b}(x)\mu(dx). \end{aligned} \tag{2.14}$$

for any smooth functional φ and any sequence $a = (a_j)$ such that $a_j = 0$ except for finitely many j. Here $b \leq a$ means that $b = (b_j)$ is a sequence of non negative integers such that $b_j \leq a_j$ for all j.

Proof. First we assume $p_0, p_1, \ldots, p_{n-1} > 1$ and $p_n \geq 1$. We shall prove it by induction on $|a|$. Let $|a| = 1$ i.e., $a = \delta_i = (0, 0, \ldots, 0, \overset{i}{\check{1}}, 0, \ldots)$ for some $i \in \mathbf{N}$. Take an approximating sequence $\{f_k\}_{k=1}^{\infty}$ for F. Similarly as in Lemma 2.1, we set

$$B = H_1 \oplus \bar{H}_2, \quad \mu(dx) = \tilde{\mu}(dy) \times \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

where $x = y + th_i$. Then

$$\begin{aligned} &\int_B \langle Df_k(x), h_i \rangle \varphi(x)\mu(dx) \\ &\quad = \int_{\bar{H}_2} \tilde{\mu}(dy) \int_{-\infty}^{\infty} \langle Df_k(y + th_i), h_i \rangle \varphi(y + h_i) \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \\ &\quad = \int_{\bar{H}_2} \tilde{\mu}(dy) \int_{-\infty}^{\infty} \frac{d}{dt} f_k(y + th_i) \varphi(y + th_i) \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt. \end{aligned}$$

By integration by parts in the one dimensional case, we have

$$\begin{aligned} &\int_B \langle Df_k(x), h_i \rangle \varphi(x)\mu(dx) \\ &= -\int_{\bar{H}_2} \tilde{\mu}(dy) \int_{-\infty}^{\infty} f_k(y + th_i) \frac{d}{dt} \left\{ \varphi(y + th_i) \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \right\} dt \\ &= -\int_{\bar{H}_2} \tilde{\mu}(dy) \int_{-\infty}^{\infty} f_k(y + th_i) \{ \langle D\varphi(y + th_i), h_i \rangle - \varphi(y + th_i)t \} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \\ &= \int_B f_k(x) \{ -\langle D\varphi(x), h_i \rangle + \varphi(x)(h_i, x) \} \mu(dx) \end{aligned}$$

Letting $k\to\infty$, we have

(2.15)

$$\int_B \langle DF(x), h_i\rangle \varphi(x)\mu(dx)=\int_B F(x)\{-\langle D\varphi(x), h_i\rangle+\varphi(x)(h_i, x)\}\mu(dx).$$

Thus (2.14) holds for $a=\delta_i$. Next we assume (2.14) for all a' such that $|a'|\le|a|$ and we shall show (2.14) for $a+\delta_i$. Note that $\langle DF(x), h_i\rangle\in H(p_1, p_2,\dots, p_n)(\mathbf{R})$ if $F\in H(p_0, p_1,\dots, p_n)(\mathbf{R})$. Hence by induction

$$\int_B D^{|a|}(\langle DF(x), h_i\rangle)(h^a)\varphi(x)\mu(dx)$$

$$=\int_B \langle DF(x), h_i\rangle \sum_{b\le a}(-1)^{|b|}\frac{a!}{b!}D^{|b|}\varphi(x)(h^b)H_{a-b}(x)\mu(dx).$$

Again by induction for δ_i, we have

$$\int_B D^{|a|+1}F(x)(h^{a+\delta_i})\varphi(x)\mu(dx)$$

$$=\int_B F(x)\Big[\sum_{b\le a}(-1)^{|b|+1}\frac{a!}{b!}D^{|b|+1}\varphi(x)(h^{b+\delta_i})H_{a-b}(x)$$

$$-\sum_{b\le a}(-1)^{|b|+1}\frac{a!}{b!}D^{|b|}\varphi(x)(h^b)\langle DH_{a-b}(x), h_i\rangle$$

$$+\sum_{b\le a}(-1)^{|b|}\frac{a!}{b!}D^{|b|}\varphi(x)(h^b)H_{a-b}(x)(h_i, x)\Big]\mu(dx)$$

$$=\int_B F(x)\Big[\sum_{b\le a}(-1)^{|b|+1}\frac{a!}{b!}D^{|b|+1}\varphi(x)(h^{b+\delta_i})H_{a-b}(x)$$

$$+\sum_{b\le a}(-1)^{|b|}\frac{a!}{b!}D^{|b|}\varphi(x)(h^b)\{-H_{a-b-\delta_i}(x)+(h_i, x)H_{a-b}(x)\}\Big]\mu(dx)$$

$$=\int_B F(x)\Big[\sum_{b\le a+\delta_i}(-1)^{|b|}D^{|b|}\varphi(x)(h^b)H_{a+\delta_i-b}(x)$$

$$\times\left\{\frac{a!}{(b-\delta_i)!}+\frac{a!}{b!}(a_i-b_i+1)\right\}\Big]\mu(dx)$$

$$=\int_B F(x)\sum_{b\le a+\delta_i}(-1)^{|b|}\frac{a!}{b!}D^{|b|}\varphi(x)(h^b)H_{a+\delta_i-b}(x)\mu(dx).$$

Here we used the following general formula:

$$(a_i+1)H_{a+\delta_i}(x)-(h_i, x)H_a(x)+H_{a-\delta_i}(x)=0 \tag{2.16}$$

which is a consequence of (2.9). Here we set $H_{a-\delta_i}(x)=0$ if $a_i=0$. Thus we have (2.14) for $a+\delta_i$. The restriction $p_1, p_2,\dots, p_{n-1}>1$ can be removed from Proposition 2.1. Q.E.D.

Definition 2.2. The *Ornstein-Uhlenbeck operator* L is defined, for any function $f\colon B\to\mathbf{R}$ which is twice B-differentiable, by

$$Lf(x)=\operatorname{trace} D^2 f(x)-(f'(x), x). \tag{2.17}$$

Remark. It is known that if $A\in\mathscr{L}(B, B^*)$ then $A|_H\in\mathscr{L}_{(1)}(H)$ where $\mathscr{L}_{(1)}(H)$ is a space of trace class operators on H (see Kuo [4]). Since f is twice B-differentiable, $f'(x)\in\mathscr{L}^2(B, \mathbf{R})=\mathscr{L}(B, B^*)$ and trace $D^2f(x)$ can be defined.

Now we shall extend the Ornstein-Uhlenbeck operator L as follows:

Definition 2.3. For $p_0, p_1, p_2, p_L\geq 1$, we define $H(p_0, p_1, p_2; p_L)$ to be the space of all Wiener functionals $F(x)\in H(p_0, p_1, p_2)(\mathbf{R})$ such that there exists an approximating sequence $\{f_k\}_{k=1}^\infty$ in $H(p_0, p_1, p_2)(\mathbf{R})$ for F satisfying also that $\{Lf_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^{p_L}(\mu)$. We call the limit of $\{Lf_k\}_{k=1}^\infty$ in $L^{p_L}(\mu)$ the *weak L-derivative* and denote it by LF.

This weak L-derivative is well-defined as we shall see in Lemma 2.5 below. We can take a sequence of smooth functionals as an approximating sequence.

Lemma 2.4. *If $F\in H(p_0, p_1, p_2; p_L)$, then*

$$\int_B LF(x)\varphi(x)\mu(dx)=\int_B F(x)L\varphi(x)\mu(dx) \tag{2.18}$$

for any smooth functional $\varphi(x)$.

Proof. First we prove (2.18) for a smooth functional F. Take any $h\in B^*$ such that $|h|_H=1$. Then $\langle DF(x), h\rangle_H$ is also a smooth functional and hence from Lemma 2.3,

$$\begin{aligned}\int_B &\{D^2F(x)(h, h)-(h, x)\langle DF(x), h\rangle_H\}\varphi(x)\mu(dx)\\ &=\int_B D(\langle DF(x), h\rangle_H)(h)\varphi(x)\mu(dx)\\ &\quad-\int_B (h, x)\langle DF(x), h\rangle_H\varphi(x)\mu(dx)\\ &=-\int_B \langle DF(x), h\rangle_H\langle D\varphi(x), h\rangle_H\mu(dx).\end{aligned} \tag{2.19}$$

Since F and φ are finite dimensional functions, there exists an orthonormal system $\{h_1, h_2,\dots, h_k\}$ in H and $h_i\in B^*$ $(i=1, 2,\dots, k)$ such that

$$\begin{aligned}\int_B LF(x)\varphi(x)\mu(dx)&=\int_B \{\operatorname{trace} D^2F(x)-(F'(x), x)\}\varphi(x)\mu(dx)\\ &=\int_B\sum_{i=1}^k \{D^2F(x)(h_i, h_i)-(h_i, x)\langle DF(x), h_i\rangle_H\}\varphi(x)\mu(dx)\end{aligned}$$

and

$$\int_B \langle DF(x), D\varphi(x)\rangle_H\mu(dx)=\int_B\sum_{i=1}^k \langle DF(x), h_i\rangle_H\langle D\varphi(x), h_i\rangle_H\mu(dx).$$

From (2.19) we have

$$\int_B LF(x)\varphi(x)\mu(dx) = -\int_B \langle DF(x),\ D\varphi(x)\rangle_H \mu(dx).$$

By exchanging F and φ we have

$$\int_B F(x)L\varphi(x)\mu(dx) = -\int_B \langle DF(x),\ D\varphi(x)\rangle_H \mu(dx).$$

and (2.18) is proved. The general case is easily obtained by approximating F with smooth functionals. Q. E. D.

Lemma 2.5. *If $F \in H(p_0, p_1, p_2; p_L)$ and $F=0$, then $LF=0$.*

Proof. From Lemma 2.4, we have

$$\int_B LF(x)\varphi(x)\mu(dx) = \int_B F(x)L\varphi(x)\mu(dx) = 0$$

for any smooth functional φ. Then it is easy to conclude that $LF=0$. Q. E. D.

Clearly $H(p_0, p_1, p_2; p_L)$ is a Banach space endowed with norm $\|F\|_{H(p_0,p_1,p_2;p_L)} = \|F\|_{H(p_0,p_1,p_2)(\mathbf{R})} + \|LF\|_{L^{p_L}(\mu)}$. Especially for $p_0=p_1=p_2=p_L=2$, $H(2, 2, 2; 2)$ is a Hilbert space endowed with norm $|F|^2_{H(2,2,2;2)} = |F|^2_{H(2,2,2)} + |LF|^2_{L^2(\mu)}$. We shall show later that $H(2, 2, 2; 2) = H(2, 2, 2)(\mathbf{R})$ i.e., the weak L-differentiability follows automatically. In the sequel we shall mainly consider the space $H(1, 2, 1; 1)$. This space convenient as we shall see, for instance, in the following lemmas.

Lemma 2.6. *Let $F=(F^1, F^2, \ldots, F^d)$ be an $\mathbf{R}^d$-valued Wiener functional defined on B and u be an element of $C_b^2(\mathbf{R}^d)$ (the space consisting of all twice continuously differentiable functions which are bounded together with their derivatives up to the second order). If $F^i \in H(1, 2, 1; 1)$ for $i=1, 2, \ldots, d$, then $u\circ F \in H(1, 2, 1; 1)$.*

Proof. Let $\{f_k^i\}_{k=1}^\infty$ be an approximating sequence for F^i in $H(1, 2, 1; 1)$ and put $f_k=(f_k^1, f_k^2, \ldots, f_k^d)$. By extracting a subsequence if necessary, we have $u(f_k(x)) \to u(F(x))$ a.e. (μ) as $k\to\infty$ and hence in $L^1(\mu)$. Clearly $u\circ f_k$ is twice B-differentiable. We shall show that $\{u\circ f_k\}_{k=1}^\infty$ is an approximating sequence for $u\circ F$. Since $\{f_k^i\}_{k=1}^\infty$ is an approximating sequence for $F^i(x)$ $(i=1, 2, \ldots, d)$ we have

$$\begin{aligned} D(u\circ f_k)(x) &= \sum_{i=1}^d \frac{\partial u}{\partial \xi^i}(f_k(x))Df_k^i(x) \\ &\longrightarrow \sum_{i=1}^d \frac{\partial u}{\partial \xi^i}(F(x))DF^i(x) \qquad \text{in} \quad L^2(\mu; H) \end{aligned}$$

$$\begin{aligned} D^2(u\circ f_k(x)) &= \sum_{i,j=1}^d \frac{\partial^2 u}{\partial \xi^i \partial \xi^j}(f_k(x))Df_k^i(x)\otimes Df_k^j(x) + \sum_{i=1}^d \frac{\partial u}{\partial \xi^i}(f_k(x))D^2 f_k^i(x) \\ &\longrightarrow \sum_{i,j=1}^d \frac{\partial^2 u}{\partial \xi^i \partial \xi^j}(F(x))DF^i(x)\otimes DF^j(x) + \sum_{i=1}^d \frac{\partial u}{\partial \xi^i}(F(x))D^2 F^i(x) \\ &\qquad \text{in} \quad L^1(\mu; \mathscr{L}^2_{(2)}(H; \mathbf{R})), \end{aligned}$$

$$L(u\circ f_k)(x)=\sum_{i,j=1}^{d}\frac{\partial^2 u}{\partial\xi^i\partial\xi^j}(f_k(x))\langle Df_k^i(x),\, Df_k^j(x)\rangle_H$$

$$+\sum_{i=1}^{d}\frac{\partial u}{\partial\xi^i}(f_k(x))Lf_k^i(x)$$

$$\longrightarrow\sum_{i,j=1}^{d}\frac{\partial^2 u}{\partial\xi^i\partial\xi^j}(F(x))\langle DF^i(x),\, DF^j(x)\rangle_H$$

$$+\sum_{i=1}^{d}\frac{\partial u}{\partial\xi^i}(F(x))LF^i(x)\qquad\text{in}\quad L^1(\mu)$$

as $k\to\infty$. Thus $u\circ F\in H(1, 2, 1; 1)$. Q. E. D.

Using this lemma we shall generalize Lemma 2.4 as follows:

Lemma 2.7. *If $F, G\in H(1, 2, 1; 1)$ and F, G are both bounded, then the following equality holds.*

$$\int_B LF(x)G(x)\mu(dx)=\int_B F(x)LG(x)\mu(dx). \tag{2.20}$$

Proof. From Lemma 2.4, (2.20) holds if G is a smooth functional. We can see by using Lemma 2.6 that G is approximated by smooth functionals which are uniformly bounded and hence (2.20) follows. Q. E. D.

Lemma 2.8. *If F and G belong to $H(1, 2, 1; 1)$ and they are both bounded, then $F(x)\cdot G(x)\in H(1, 2, 1; 1)$ and the following equality holds;*

$$L(F\cdot G)(x)=LF(x)G(x)+2\langle DF(x),\, DG(x)\rangle_H+F(x)LG(x). \tag{2.21}$$

Proof. Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be approximating sequences for F and G respectively. From Lemma 2.6, we may assume that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are uniformly bounded and converge a.e. (μ) as $k\to\infty$. Clearly $f_k(x)\cdot g_k(x)$ is twice B-differentiable. We shall show that $\{f_k\cdot g_k\}_{k=1}^{\infty}$ is an approximating sequence for $F\cdot G$. Since $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are the approximating sequences for F and G respectively we have

$$f_k(x)g_k(x)\longrightarrow F(x)G(x)\quad\text{in}\quad L^1(\mu),$$

$$D(f_k\cdot g_k)(x)=g_k(x)Df_k(x)+f_k(x)Dg_k(x)$$

$$\longrightarrow G(x)DF(x)+F(x)DG(x)\qquad\text{in}\quad L^2(\mu;\, H),$$

$$D^2(f_k\cdot g_k)(x)=g_k(x)D^2f_k(x)+Df_k(x)\otimes Dg_k(x)+Dg_k(x)\otimes Df_k(x)+f_k(x)D^2g_k(x)$$

$$\longrightarrow G(x)D^2F(x)+DF(x)\otimes DG(x)+DG(x)\otimes DF(x)+F(x)D^2G(x)$$

$$\text{in}\quad L^1(\mu;\, \mathscr{L}^2_{(2)}(H;\, \mathbf{R})),$$

$$L(f_k\cdot g_k)(x)=Lf_k(x)g_k(x)+2\langle Df_k(x),\, Dg_k(x)\rangle_H+f_k(x)Lg_k(x)$$

$$\longrightarrow LF(x)G(x)+2\langle DF(x),\, DG(x)\rangle_H+F(x)LG(x)\quad\text{in}\quad L^1(\mu)$$

as $k\to\infty$. Thus we have $F(x)\cdot G(x)\in H(1, 2, 1; 1)$ and (2.21) holds. Q. E. D.

3. Absolute continuity of probability laws

Let F be an $\mathbf{R}^d$-valued Wiener functional. We investigate the absolute continuity of $\mu \circ F^{-1}$ i.e., the probability law on $\mathbf{R}^d$ induced by F, with respect to Lebesgue measure. Original method is due to Malliavin using stochastic derivatives. Here we rephrase his ideas using our derivatives.

Lemma 3.1. (*Malliavin* [6]) *Let ν be a finite measure on $\mathbf{R}^d$ such that there exists a constant $C>0$ and for any $\varphi \in C_0^\infty(\mathbf{R}^d)$*

$$(3.1) \qquad \left|\int_{\mathbf{R}^d} \frac{\partial \varphi}{\partial \xi^k} \nu(d\xi)\right| \leq C\|\varphi\|_\infty \qquad for \quad k=1, 2,\dots, d.$$

Then ν is absolutely continuous i.e., there exists $u \in L^1(\mathbf{R}^d)$ such that $\nu(d\xi)=u(\xi)d\xi$.

Using this lemma we have the following main theorem.

Theorem 3.1. *Let $F=(F^1, F^2,\dots, F^d)$ be an $\mathbf{R}^d$-valued Wiener functional defined on B. We assume that F satisfies the followings:*

(i) $F^i \in H(1, 2, 1; 1) \qquad i=1, 2,\dots, d,$

(ii) $\sigma^{ij}(x)=\langle DF^i(x), DF^j(x)\rangle_H \in H(1, 2, 1; 1) \qquad i, j=1, 2,\dots, d,$

(iii) $\det(\sigma^{ij}(x)) \neq 0 \qquad a.e. \ (\mu).$

Then the probability law of F is absolutely continuous.

Proof. Take an arbitrary $\varepsilon>0$. Since $\sigma(x)=(\sigma^{ij}(x))$ is invertible a.e. (μ) i.e., $\mu \circ \sigma^{-1}$ has its full measure on $GL(n)$, there esxists $\psi_1 \in C_0^\infty(GL(n))$ such that $\|\psi(x)\mu(dx)-\mu(dx)\|<\varepsilon$ where $\psi=\psi_1 \circ \sigma$ and the norm $\|\cdot\|$ is the total variation. Moreover there exists $u\colon GL(n) \to M_n$ (M_n is a space of (n, n)-matrices) such that $u(A)=A^{-1}$ on the support of ψ_1 and $u \in C_0^\infty(GL(n))$. Let $\zeta={}^t(\zeta_1, \zeta_2,\dots, \zeta_d)$ be the first column vector of $u(\sigma)$ (for simplicity we consider only the first column vector, the other case being similar). Note that $\sum_{k=1}^d \zeta_k \sigma^{ki}=\delta_{1i}$ on the support of ψ. From Lemma 2.6 ψ, $\zeta_k \in H(1, 2, 1; 1)$, $k=1, 2,\dots, n$. Let $\varphi \in C_0^\infty(\mathbf{R}^d)$ and $\tilde{\varphi}=\varphi \circ F$. We shall show the following equality.

$$(3.2) \qquad \int_B \sum_{k=1}^d \zeta_k(x)\langle DF^k(x), D\tilde{\varphi}(x)\rangle_H \psi(x)\mu(dx)$$
$$= -\int_B \sum_{k=1}^d \{\zeta_k(x)\langle D\psi(x), DF^k(x)\rangle_H + \psi(x)\langle D\zeta_k(x), DF^k(x)\rangle_H$$
$$+ \zeta_k(x)\psi(x)LF^k(x)\}\tilde{\varphi}(x)\mu(dx).$$

First we show that (3.2) holds for $F_N^k=c_N \circ F^k$, where c_N is the function defined by (2.5). We may assume that the derivatives of c_N is uniformly bounded in N. Since F_N^k is bounded and belongs to $H(1, 2, 1; 1)$ we have, from Lemma 2.7 and Lemma 2.8, that

$$\int_B \sum_{k=1}^d \zeta_k(x)\langle DF_N^k(x),\, D\tilde{\varphi}(x)\rangle \psi(x)\mu(dx)$$

$$=\frac{1}{2}\int_B \sum_{k=1}^d \zeta_k(x)\{L(F_N^k\cdot\tilde{\varphi})(x)-F_N^k(x)L\tilde{\varphi}(x)-LF_N^k(x)\tilde{\varphi}(x)\}\psi(x)\mu(dx)$$

$$=\frac{1}{2}\int_B \sum_{k=1}^d \{L(\zeta_k\cdot\psi)(x)F_N^k(x)-L(\zeta_k\cdot F_N^k\cdot\psi)(x)-\zeta_k(x)LF_N^k(x)\psi(x)\}$$

$$\tilde{\varphi}(x)\mu(dx)$$

$$=-\int_B \sum_{k=1}^d \{\psi(x)\langle D\zeta_k(x),\, DF_N^k(x)\rangle_H+\zeta_k(x)\langle D\psi(x),\, DF_N^k(x)\rangle_H$$

$$+\zeta_k(x)\psi(x)LF_N^k(x)\}\tilde{\varphi}(x)\mu(dx)$$

Note that $DF_N^k\to DF^k$ in $L^2(\mu;\, H)$ and $LF_N^k\to LF^k$ in $L^1(\mu)$ as $N\to\infty$. Hence by letting $N\to\infty$, we have (3.2). On the other hand,

$$\int_B \sum_{k=1}^d \zeta_k(x)\langle DF^k(x),\, D\tilde{\varphi}(x)\rangle_H\psi(x)\mu(dx)$$

$$=\int_B \sum_{k,l=1}^d \zeta_k(x)\langle DF^k(x),\, \frac{\partial\varphi}{\partial\xi^l}(F(x))DF^l(x)\rangle_H\psi(x)\mu(dx)$$

$$=\int_B \sum_{k,l=1}^d \zeta_k(x)\sigma^{kl}(x)\frac{\partial\varphi}{\partial\xi^l}(F(x))\psi(x)\mu(dx)$$

$$=\int_B \frac{\partial\varphi}{\partial\xi^1}(F(x))\psi(x)\mu(dx).$$

Thus we have

$$\left|\int_B \frac{\partial\varphi}{\partial\xi^1}(F(x))\psi(x)\mu(dx)\right|\le C\|\varphi\|_\infty \tag{3.3}$$

where

$$C=\int_B\left|\sum_{k=1}^d\Big\{\zeta_k(x)\langle D\psi(x),\, DF^k(x)\rangle_H+\psi(x)\langle D\zeta_k(x),\, DF^k(x)\rangle_H\right.$$

$$\left.+\zeta_k(x)\psi(x)LF^k(x)\Big\}\right|\mu(dx)$$

Consequently $\psi(x)\mu(dx)\circ F^{-1}$ is absolutely continuous by Lemma 3.1. But $\|\psi(x)\cdot\mu(dx)-\mu(dx)\|<\varepsilon$ and ε is arbitrary. Hence $\mu\circ F^{-1}$ is absolutely continuous.

Q. E. D.

4. Derivatives of the multiple Wiener integrals.

In this section we shall investigate the differentiability of the multiple Wiener integral. Let (B, H, μ) be the Wiener space. We put $T=[0, 1]$. For $f\in L^2(T^p)$ $(p\in\mathbf{N})$, we define a multiple Wiener integral $I_p(f)$ of f as

$$I_p(f)=\sum_{\sigma\in S_p}\int_0^1 dx(t_1)\int_0^{t_1}dx(t_2)\cdots\int_0^{t_{p-1}}f(t_{\sigma(1)}, t_{\sigma(2)},\dots, t_{\sigma(p)})dx(t_p) \tag{4.1}$$

where integrals are understood in the sense of Ito's stochastic integrals and σ runs over all permutations of $\{1, 2,\dots, p\}$. We also denote $\int_0^1\cdots\int_0^1 f(t_1, t_2,\dots, t_p)dx(t_1)dx(t_2)\cdots dx(t_p)$ in place of $I_p(f)$. Then the mapping $I_p: f\mapsto I_p(f)$ is a bounded linear operator from $L^2(T^p)$ into $L^2(\mu)$ such that

$$(4,2)\qquad |I_p(f)|^2_{L^2(\mu)}=|I_p(\tilde{f})|^2_{L^2(\mu)}=p!|\tilde{f}|^2_{L^2(T^p)}\le p!|f|^2_{L^2(T^p)}$$

where $\tilde{f}$ is the symmetrization of f:

$$(4.3)\qquad \tilde{f}(t_1, t_2,\dots, t_p)=\frac{1}{p!}\sum_{\sigma\in S_p} f(t_{\sigma(1)}, t_{\sigma(2)},\dots, t_{\sigma(p)}).$$

We shall show that $I_p(f)$ has all moments and I_p is continuous mapping from $L^2(T^p)$ into $L^q(\mu)$ for any $q\ge 1$. First, we introduce some notations. For $f\in L^2(T^p)$ and $g\in L^2(T^q)$ $(p, q\ge 1)$, $f\otimes g\in L^2(T^{p+q})$ is defined by

$$f\otimes g(t_1, t_2,\dots, t_p, s_1, s_2,\dots, s_q)=f(t_1, t_2,\dots, t_p)g(s_1, s_2,\dots, s_q).$$

Let $\{i_1, i_2,\dots, i_l\}$ and $\{j_1, j_2,\dots, j_l\}$ be l different elements of $\{1, 2,\dots, p\}$ and $\{1, 2, \dots, q\}$ respectively. Then $c(i_1, i_2,\dots, i_l; j_1, j_2,\dots, j_l)f\otimes g$ is defined by

$$[c(i_1, i_2,\dots, i_l; j_1, j_2,\dots, j_l)f\otimes g](t_1,\dots, \hat{t}_{i_1},\dots, \hat{t}_{i_l},\dots, t_p, s_1,\dots, \hat{s}_{j_1},\dots, \hat{s}_{j_l},\dots, s_q)]$$

$$=\int_0^1\cdots\int_0^1 f(t_1, t_2,\dots, t_p)g(s_1,s_2,\dots, s_q)du_1\cdots du_l$$

$$\begin{array}{cc} t_{i_1}\longrightarrow u_1 & s_{j_1}\longrightarrow u_1 \\ \vdots & \vdots \\ t_{i_l}\longrightarrow u_l & s_{j_l}\longrightarrow u_l \end{array}$$

where, for example, $\hat{t}_{i_1}$ means that the variable t_{i_1} is removed and $t_{i_1}\to u_1$ means that the variable t_{i_1} is replaced by the variable u_1. By the Schwartz' inequality, it is easy to see that $c(i_1, i_2,\dots, i_l; j_1, j_2,\dots, j_l)f\otimes g\in L^2(T^{p+q-2l})$, more precisely

$$|c(i_1, i_2,\dots, i_l; j_1, j_2,\dots, j_l)f\otimes g|_{L^2(T^{p+q-2l})}\le|f|_{L^2(T^p)}|g|_{L^2(T^q)}.$$

Now we have;

Lemma 4.1. *For $f\in L^2(T^p)$, $g\in L^2(T^q)$*

$$(4.4)\quad I_p(f)I_q(g)$$

$$=\sum_{l=0}^{p\wedge q}\frac{1}{l!}\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset\{1,2,\dots,p\}\\ \{j_1,j_2,\dots,j_l\}\subset\{1,2,\dots,q\}}} I_{p+q-2l}(c(i_1, i_2,\dots, i_l; j_1, j_2,\dots, j_l)f\otimes g)$$

where $\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset\{1,2,\dots,p\}\\ \{j_1,j_2,\dots,j_l\}\subset\{1,2,\dots,q\}}}$ *denotes, for each fixed $l\le p\wedge q$, the sum over possible ways of choosing l different elements $\{i_1, i_2,\dots, i_l\}$ and $\{j_1, j_2,\dots, j_l\}$ from $\{1, 2,\dots, p\}$ and $\{1, 2,\dots, q\}$ respectively.*

Proof. Without loss of generality we may assume $q\le p$ and we prove it by induction on q. For $q=1$, it was proved by Itô ([2], Theorem 2.2). Assume that the formula is true up to q. Let $f \in L^2(T^p)$, $g \in L^2(T^q)$ and $h \in L^2(T)$. We denote $f=f(t_1, t_2,\dots, t_p)$, $g=g(s_1, s_2,\dots, s_q)$ and $h=h(s_{q+1})$. By induction for q and 1,

$$\begin{aligned} I_p(f)I_{q+1}(g\otimes h) &= I_p(f)\{I_q(g)I_1(h)-\sum_{j=1}^{q} I_{q-1}(c(j;q+1)g\otimes h)\} \\ &= I_p(f)I_q(g)I_1(h)-\sum_{j=1}^{q} I_p(f)I_{q-1}(c(j;q+1)g\otimes h). \end{aligned}$$

By induction,

$$\begin{aligned} & I_p(f)I_q(g) \\ &= \sum_{l=0}^{q}\frac{1}{l!}\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset\{1,2,\dots,p\}\\ \{j_1,j_2,\dots,j_l\}\subset\{1,2,\dots,q\}}} I_{p+q-2l}(c(i_1,i_2,\dots,i_l;j_1,j_2,\dots,j_l)f\otimes g). \end{aligned}$$

Hence we have

$$\begin{aligned} & I_p(f)I_{q+1}(g\otimes h) \\ &= \sum_{l=0}^{q}\frac{1}{l!}\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset\{1,2,\dots,p\}\\ \{j_1,j_2,\dots,j_l\}\subset\{1,2,\dots,q\}}} I_{p+q-2l}(c(i_1,i_2,\dots,i_l;j_1,j_2,\dots,j_l)f\otimes g) \\ &\quad \times I_1(h)-\sum_{j=1}^{q} I_p(f)I_{q-1}(c(j;q+1)g\otimes h). \end{aligned}$$

Again by induction for $p+q-2l$, 1 and p, $q-1$,

$$\begin{aligned} & I_p(f)I_{q+1}(g\otimes h) \\ &= \sum_{l=0}^{q}\frac{1}{l!}\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset\{1,2,\dots,p\}\\ \{j_1,j_2,\dots,j_l\}\subset\{1,2,\dots,q\}}} I_{p+q+1-2l}(\{c(i_1,i_2,\dots,i_l;j_1,j_2,\dots,j_l)f\otimes g\}\otimes h) \\ &\quad + \sum_{l=0}^{q}\frac{1}{l!}\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset\{1,2,\dots,p\}\\ \{j_1,j_2,\dots,j_l\}\subset\{1,2,\dots,q\}}}\ \sum_{i_{l+1}\neq i_1,i_2,\dots,i_l} \\ & I_{p+q-2l}(c(i_{l+1},q+1)\{c(i_1,i_2,\dots,i_l;j_1,j_2,\dots,j_l)f\otimes g\}\otimes h) \\ &\quad + \sum_{l=0}^{q-1}\frac{1}{l!}\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset\{1,2,\dots,p\}\\ \{j_1,j_2,\dots,j_l\}\subset\{1,2,\dots,q\}}}\ \sum_{j_{l+1}\neq j_1,j_2,\dots,j_l} \\ & I_{p+q-2l-1}(c(j_{l+1},q+1)\{c(i_1,i_2,\dots,i_l;j_1,j_2,\dots,j_l)f\otimes g\}\otimes h) \\ &\quad - \sum_{j=1}^{q}\sum_{l=0}^{q-1}\frac{1}{l!}\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset\{1,2,\dots,p\}\\ \{j_1,j_2,\dots,j_l\}\subset\{1,2,\dots,q\}\setminus\{j\}}} \\ & I_{p+q-2l-1}(c(i_1,i_2,\dots,i_l;j_1,j_2,\dots,j_l)f\otimes\{c(j;q+1)g\otimes h\}). \end{aligned}$$

Note that the third term and the forth term are cancelled each other. Consequently,

$$I_p(f)I_{q+1}(g\otimes h)$$
$$=\sum_{l=0}^{q+1}\frac{1}{l!}\sum_{\substack{\{i_1,i_2,\dots,i_l\}\subset\{1,2,\dots,p\}\\ \{j_1,j_2,\dots,j_l\}\subset\{1,2,\dots,q,q+1\}}} I_{p+q-2l}(c(i_1,i_2,\dots,i_l;j_1,j_2,\dots j_l)f\otimes\{g\otimes h\}).$$

Now (4.4) follows by noting that functions of the form $g\otimes h$ for $g\in L^2(T^q)$ and $h\in L^2(T)$ span $L^2(T^{q+1})$. Q.E.D.

By this lemma, we have the following estimate:

$$\int_B I_p(f)^2 I_q(g)^2\mu(dx)\le K_{p,q}(|f|^2_{L^2(T^p)}|g|^2_{L^2(T^q)})$$

for $f\in L^2(T^p)$ and $g\in L^2(T^q)$ where $K_{p,q}$ is a constant which depends only on p and q. If we use this lemma repeatedly, we have

$$\int_B I_{p_1}(f_1)^2 I_{p_2}(f_2)^2\cdots I_{p_n}(f_n)^2\mu(dx)$$
$$\le K_{p_1,p_2,\dots,p_n}|f_1|^2_{L^2(T^{p_1})}|f_2|^2_{L^2(T^{p_2})}\cdots|f_n|^2_{L^2(T^{p_n})}$$

for $f_1\in L^2(T^{p_1})$, $f_2\in L^2(T^{p_2}),\dots, f_n\in L^2(T^{p_n})$ where $K_{p_1,p_2,\dots,p_n}$ is a constant which depends only on $p_1, p_2,\dots, p_n$. Especially, for $f\in L^2(T^p)$

$$\int_B I_p(f)^{2n}\mu(dx)\le C_{p,n}|f|^{2n}_{L^2(T^p)}$$

where $C_{p,n}=K_{\underbrace{p,p,\dots,p}_{n}}$. Hence $I_p(f)$ has all moments.

Now we can investigate the weak H-derivatives of $I_p(f)$ of $f\in L^2(T^p)$. Since $I_p(f)=I_p(\tilde{f})$ where $\tilde{f}$ is the symmetrization of f, we may assume that f is symmetric. We denote the space of all symmetric functions in $L^2(T^p)$ by $\hat{L}^2(T^p)$. Let f_k, $k=1, 2,\dots$ and f be in $L^2(T^p)$ such that $|f_k-f|_{L^2(T^p)}\to 0$ as $k\to\infty$ and f_k is a special step function for every k. Here by a special step function, we mean a function of the form

$$f(t_1,t_2,\dots,t_p)\tag{4.5}$$
$$=\sum_{i_1,i_2,\dots,i_p=0}^{N-1} f_{i_1,\dots,i_p}1_{\Delta(i_1,i_2,\dots,i_p)}(t_1,t_2,\dots,t_p)$$

where $\Delta=\{0=t_0<t_1<\cdots<t_N=1\}$ is a partition of $[0, 1]$,

$$\Delta(i_1,i_2,\dots,i_p)=[t_{i_1},t_{i_1+1})\times[t_{i_2},t_{i_2+1})\times\cdots\times[t_{i_p},t_{i_p+1})$$

and $f_{i_1,i_2,\dots,i_p}$ are constants such that they are symmetric in $i_1, i_2,\dots, i_p$ and $f_{i_1,i_2,\dots,i_p}=0$ if there exist same elements among $i_1, i_2,\dots, i_p$ (cf. Ito [2]). We shall show that $\{I_p(f_k)\}_{k=1}^{\infty}$ is an approximating sequence for $I_p(f)$ in $H(p_0, p_1,\dots, p_n)(\mathbf{R})$ for any $n\in\mathbf{N}$ and $p_0, p_1,\dots, p_n\ge 1$. First we investigate $I_p(f)$ such that f is a special step symmetric function of the form (4.5). Then the multiple Wiener integral of f is expressed as

$$I_p(f)(x)=\sum_{i_1,i_2,\dots,i_p=0}^{N-1} f_{i_1,i_2,\dots,i_p}\Delta x(i_1,i_2,\dots,i_p) \tag{4.6}$$

where $\Delta x(i_1,i_2,\dots,i_p)=(x(t_{i_1+1})-x(t_{i_1}))(x(t_{i_2+1})-x(t_{i_2}))\cdots(x(t_{i_p+1})-x(t_{i_p}))$.

Clearly $I_p(f)$ is B-differentiable and its derivatives are given by

$$I'_p(f)(x)(y)=p\sum_{i_1,i_2,\dots,i_p=0}^{N-1} f_{i_1,i_2,\dots,i_p}\Delta x(i_2,\dots i_p)(y(t_{i_1+1})-y(t_{i_1})) \tag{4.7}$$

$$\begin{aligned} I''_p(f)(x)(y,z) &= p(p-1)\sum_{i_1,i_2,i_3,\dots,i_p=0}^{N-1} f_{i_1,i_2,i_3,\dots,i_p}\Delta x(i_3,\dots,i_p)(y(t_{i_1+1})-y(t_{i_1})) \\ &\quad\times(z(t_{i_2+1})-z(t_{i_2})) \end{aligned} \tag{4.8}$$

for $y, z\in B$ and so on. Finally

$$\begin{aligned} &I_p(f)^{(p)}(x)(y_1,y_2,\dots,y_p) \\ &=p!\sum_{i_1,i_2,\dots,i_p=0}^{N-1} f_{i_1,i_2,\dots,i_p}(y_1(t_{i_1+1})-y(t_{i_1}))(y_2(t_{i_2+1})-y_2(t_{i_2}))\cdots \\ &\quad\times(y_p(t_{i_p+1})-y_p(t_{i_p})) \end{aligned}$$

for $y_1, y_2,\dots, y_p\in B$ and

$$I_p(f)^{(p+1)}=0,\ I_p(f)^{(p+2)}=0,\dots .$$

Next we study the integrability of H-derivatives of $I_p(f)$.

$$\begin{aligned} &\int_B |DI_p(f)(x)|_H^2\mu(dx) \\ &=\int_B\sum_{i_1=0}^{N-1}(p\sum_{i_2,i_3,\dots,i_p=0}^{N-1} f_{i_1,i_2,i_3,\dots,i_p}\Delta x(i_2,i_3,\dots,i_p))^2|\Delta(i_1)|\mu(dx) \\ &=p^2\sum_{i_1=0}^{N-1}|\Delta(i_1)|\sum_{\substack{i_2,i_3,\dots,i_p=0\\ j_2,j_3,\dots,j_p=0}}^{N-1}\int_B f_{i_1,i_2,\dots,i_p}f_{i_1,j_2,\dots,j_p} \\ &\quad\times\Delta x(i_2,\dots,i_p)\Delta x(j_2,\dots,j_p)\mu(dx) \\ &=p^2(p-1)!\sum_{i_1,i_2,\dots,i_p=0}^{N-1} f^2_{i_1,i_2,\dots,i_p}|\Delta(i_1)||\Delta(i_2,\dots,i_p)| \\ &=p(p!)|f|^2_{L^2(T^p)} \end{aligned}$$

where $|\cdot|$ is the Lebesgue measure. Similarly, we have

$$\int_B |D^2I_p(f)|^2_{\mathscr{L}^2_{(2)}(H;\mathbf{R})}\mu(dx)=p(p-1)(p!)|f|^2_{L^2(T^p)}.$$

In general

$$\int_B |D^mI_p(f)|^2_{\mathscr{L}^m_{(2)}(H;\mathbf{R})}\mu(dx)=p(p-1)\cdots(p-m+1)(p!)|f|^2_{L^2(T^p)}.$$

Thus we obtain that $D^m I_p(f) \in L^2(\mu; \mathscr{L}^m_{(2)}(H; \mathbf{R}))$. We can obtain stronger results by using Lemma 4.1. First we note that

$$\langle DI_p(f), h\rangle_H = p\int_0^1 \cdots \int_0^1 \left\{\int_0^1 f(t_1, t_2, \ldots, t_p) h'(t_1) dt_1\right\} dx(t_2) \cdots dx(t_p)$$

for $h \in H$. Let $\{h_i\}_{i=1}^\infty$ be a complete orthonormal system in H. Then we have

$$\begin{aligned}\int_B |DI_p(f)(x)|_H^4 \mu(dx) &= \int_B (|DI_p(f)(x)|_H^2)^2 \mu(dx) \\ &= \int_B \Big(\sum_{i=1}^\infty \langle DI_p(f)(x), h_i\rangle_H^2\Big)^2 \mu(dx) \\ &= \sum_{i,j=1}^\infty \int_B \langle DI_p(f)(x), h_i\rangle_H^2 \langle DI_p(f)(x), h_j\rangle_H^2 \mu(dx).\end{aligned}$$

By the Schwartz' inequality, this is majorized by

$$\sum_{i,j=1}^\infty \left\{\int_B \langle DI_p(f)(x), h_i\rangle_H^4 \mu(dx)\right\}^{\frac{1}{2}} \left\{\int_B \langle DI_p(f)(x), h_j\rangle_H^4 \mu(dx)\right\}^{\frac{1}{2}}$$

and this is majorized by

$$\begin{aligned}&\sum_{i,j=1}^\infty \left(K\int_0^1 \cdots \int_0^1 \left\{\int_0^1 f(t_1, t_2, \ldots, t_p) h_i'(t_1) dt_1\right\}^2 dt_2 \cdots dt_p\right) \\ &\qquad \left(K\int_0^1 \cdots \int_0^1 \left\{\int_0^1 f(s_1, s_2, \ldots, s_p) h_j'(s_1) ds_1\right\}^2 ds_2 \cdots ds_p\right) \\ &= K^2 \left[\int_0^1 \cdots \int_0^1 \sum_{i=1}^\infty \left\{\int_0^1 f(t_1, t_2, \ldots, t_p) h_i'(t_1) dt_1\right\}^2 dt_2 \cdots dt_p\right]^2 \\ &= K^2 |f|_{L^2(T^p)}^4\end{aligned}$$

by the remark of Lemma 4.1 where K is a constant depending only on p. Thus we have $DI_p(f) \in L^4(\mu; H)$ and

$$\int_B |DI_p(f)|_H^4 \mu(dx) \le K_p |f|_{L^2(T^p)}^4$$

Similarly we can show that for any $p_1 \ge 1$, $DI_p(f) \in L^{p_1}(\mu; H)$ and

$$\int_B |DI_p(f)|_H^{p_1} \mu(dx) \le K_{p,p_1} |f|_{L^2(T^p)}^{p_1} \tag{4.9}$$

where K_{p,p_1} is a constant depending only on p and p_1. We can get similar results on the higher weak H-derivatives; for any $m \in \mathbf{N}$ and $p_m \ge 1$ $D^m I_p(f) \in L^{p_m}(\mu; \mathscr{L}^m_{(2)}(H; \mathbf{R}))$ and

$$\int_B |D^m I_p(f)(x)|^{p_m}_{\mathscr{L}^m_{(2)}(H;\mathbf{R})} \mu(dx) \le K_{m,p_m} |f|_{L^2(T^p)}^{p_m} \tag{4.10}$$

where K_{m,p_m} is a constant depending only on m and p_m. Now the following proposition is easily obtained.

Proposition 4.1. *Let f be an element of $\hat{L}^2(T^p)$. Then for any $n \in \mathbf{N}$ and $p_0, p_1,\dots, p_n \geq 1$, $I_p(f) \in H(p_0, p_1,\dots, p_n)(\mathbf{R})$ and its weak H-derivatives are given by*

$$(4.11)\quad \langle DI_p(f)(x), h\rangle_H = p\int_0^1\cdots\int_0^1\left\{\int_0^1 f(t_1, t_2,\dots, t_p)h'(t_1)dt_1\right\}dx(t_2)\cdots dx(t_p)$$

for $h \in H$,

$$(4.12)\quad D^2I_p(f)(x)(h, k)$$
$$= p(p-1)\int_0^1\cdots\int_0^1\left\{\int_0^1\int_0^1 f(t_1, t_2, t_3,\dots, t_p)h'(t_1)k'(t_2)dt_1dt_2\right\}dx(t_3)\cdots dx(t_p)$$

for $h, k \in H$ and so on. Moreover

$$(4.13)\qquad D^pI_p(f)(x) = p!f,\quad D^{p+1}I_p(f)(x) = 0,\quad D^{p+2}I_p(f)(x) = 0,\dots,$$

where we regard f as an element of $\mathscr{L}^p_{(2)}(H; \mathbf{R})$ in the following way;

$$(4.14)\quad f(h_1, h_2,\dots, h_p) = \int_0^1\cdots\int_0^1 f(t_1, t_2,\dots, t_p)h_1'(t_1)h_2'(t_2)\cdots h_p'(t_p)dt_1dt_2\cdots dt_p$$

for $h_1, h_2,\dots, h_p \in H$.

If $\{f_k\}_{k=1}^\infty$ is a sequence of special step functions in $\hat{L}^2(T^p)$ which converges to $f \in \hat{L}^2(T^p)$ in $\hat{L}^2(T^p)$ then $\{I_p(f_k)\}_{k=1}^\infty$ is an approximating sequence for $I_p(f)$ in $H(p_0, p_1,\dots, p_n)(\mathbf{R})$.

From this proposition we see that $p+1$-th weak H-derivative of the multiple Wiener integral $I_p(f)$ of $f \in L^2(T^p)$ is 0. The converse also holds as we shall see in the following.

Proposition 4.2. *If $F \in H(p_0, p_1,\dots, p_n)(\mathbf{R})$ and $D^mF = 0$ for $m \leq n$, then F is the linear combination of multiple Wiener integrals with degree $<m$.*

Proof. From the assumption $D^mF = 0$ and Proposition 2.2, we have that $D^{m-1}F = $constant a.e. (μ). Hence there exists $f_{m-1} \in \hat{L}^2(T^{m-1})$ such that $D^{m-1}F = (m-1)!f_{m-1}$, here we identify $L^2(T^{m-1})$ and $\mathscr{L}^{m-1}_{(2)}(H; \mathbf{R})$ as in Proposition 4.1. Then clearly $F - I_{m-1}(f_{m-1}) \in H(p_0, p_1,\dots, p_n)(\mathbf{R})$ and $D^{m-1}(F - I_{m-1}(f_{m-1})) = 0$. By repeating this procedure we have $F = \sum_{k=0}^{m-1} I_k(f_k)$ for some $f_k \in \hat{L}^2(T^k)$ $k = 1, 2,\dots, m-1$. Q.E.D.

Next we consider the weak L-derivative of the multiple Wiener integral. Let f be a special step function in $L^2(T^p)$ defined by (4.5). Then firstly we have from (4.7)

$$(I_p'(f)(x), x) = pI_p(f)(x).$$

Secondly we have

$$\text{trace } D^2I_p(f)(x) = \sum_{i=1}^\infty D^2I_p(f)(x)(h_i, h_i)$$

where $\{h_i\}_{i=1}^{\infty}$ is a complete orthonormal system in H. We choose $\{h_i\}_{i=1}^{\infty}$ as follows. Let $h_i=\bar{h}_i/|\bar{h}_i|_H$ $i=1, 2,\dots, N$, where $\bar{h}_i(t)=\int_0^t 1_{[t_{i-1},t_i)}(u)du$. By adding appropriate elements $h_{N+1}, h_{N+2},\dots$ to $\{h_i\}_{i=1}^{N}$, we get a complete orthonormal system $\{h_i\}_{i=1}^{\infty}$. Since f is a special step function

$$\begin{aligned}\operatorname{trace} D^2I_p(f)(x) &= \sum_{i=1}^{\infty} D^2I_p(f)(x)(h_i, h_i)\\ &=\sum_{i=1}^{\infty} p(p-1)\int_0^1\cdots\int_0^1\left\{\int_0^1\int_0^1 f(t_1, t_2, t_3,\dots, t_p)h_i'(t_1)h_i'(t_2)dt_1dt_2\right\}\\ &\qquad dx(t_3)\cdots dx(t_p)\\ &=0.\end{aligned}$$

Therefore

$$L(I_p(f))(x)=\operatorname{trace} D^2I_p(f)(x)-(I_p'(f)(x), x)=-pI_p(f)(x).$$

From this we can easily conclude the following.

Proposition 4.3. *If $f\in L^2(T^p)$, then $I_p(f)\in H(p_0, p_1, p_2; p_L)$ for any $p_0, p_1, p_2, p_L\geq 1$ and weak L-derivative of $I_p(f)$ is given by*

$$LI_p(f)=-pI_p(f). \tag{4.15}$$

We have defined the space $H(p_0, p_1,\dots, p_n)(\mathbf{R})$ for $p_0, p_1,\dots, p_n\geq 1$. We simply denote $H(\underbrace{2, 2,\dots, 2}_{n+1})(\mathbf{R})$ by $H^{n,2}$. We may regard $H^{n,2}$ as a subspace of $L^2(\mu)$. It is well known that $F\in L^2(\mu)$ can be expanded by the multiple Wiener integrals;

$$F=\sum_{p=0}^{\infty} I_p(f_p) \quad \text{in} \quad L^2(\mu) \tag{4.16}$$

where $f_p\in \hat{L}^2(T^p)$ for $p=1, 2,\dots$ and

$$|F|_{L^2(\mu)}^2=\sum_{p=0}^{\infty} |I_p(f_p)|_{L^2(\mu)}^2=\sum_{p=0}^{\infty} p!|f_p|_{L^2(T^p)}^2. \tag{4.17}$$

Using this expansion, we can characterize $H^{n,2}$ as follows:

Proposition 4.4. *Let $F\in L^2(\mu)$ be expanded as* (4.16). *Then $F\in H^{n,2}$ if and only if*

$$\sum_{p=n}^{\infty} p(p-1)\cdots(p-n+1)|I_p(f)|_{L^2(\mu)}^2<\infty. \tag{4.18}$$

Proof. (Necessity) Let $F\in H^{n,2}$ given by (4.16) and $\{h_i\}_{i=1}^{\infty}$ be a complete orthonormal system in H such that $h_i\in B^*$. From Lemma 2.3,

$$\begin{aligned}&\int_B \langle DF(x), h_i\rangle_H H_a(x)\mu(dx)\\ &=\int_B F(x)\{-\langle DH_a(x), h_i\rangle_H+(h_i, x)H_a(x)\}\mu(dx)\end{aligned}$$

$$= \int_B F(x)\{-H_{a-\delta_i}(x)+(h_i, x)H_a(x)\}\mu(dx)$$

$$= \int_B F(x)(a_i+1)H_{a+\delta_i}(x)\mu(dx)$$

where $H_a(x)$ is a Fourier-Hermite functional given by (2.11) and we use the formula (2.16). Here we used (2.14); in Lemma 2.3 we prove it only for smooth functional but it can be extended to cover this case. We note that $\{H_a(x)/(a!)^{\frac{1}{2}}\}_a$ is the complete orthonormal system in $L^2(\mu)$ and

$H_a(x) = \frac{1}{a!} I_{|a|}(\underbrace{h'_1\otimes\cdots\otimes h'_1}_{a_1}\otimes \underbrace{h'_2\otimes\cdots\otimes h'_2}_{a_2}\otimes\cdots)$. If $p=|a+\delta_i|$, then we have

$$\int_B \langle DF(x), h_i\rangle_H H_a(x)\mu(dx)$$

$$= \int_B I_p(f_p)(a_i+1)H_{a+\delta_i}(x)\mu(dx)$$

$$= \frac{p!(a_i+1)}{(a+\delta_i)!}\int_0^1\cdots\int_0^1 f(t_1, t_2,\ldots, t_p)h'_1(t_1)\cdots h'_1(t_{a_1})h'_2(t_{a_1+1})\cdots$$
$$h'_2(t_{a_1+a_2})\cdots dt_1 dt_2\cdots dt_p$$

$$= p\int_B I_{p-1}(c(1\,;\,1)f_p\otimes h'_i)H_a(x)\mu(dx).$$

Hence we have

$$\langle DF(x), h_i\rangle_H = \sum_{p=1}^{\infty} pI_{p-1}(c(1\,;\,1)f_p\otimes h'_i).$$

Similarly we have

$$D^m F(x)(h_{i_1}, h_{i_2},\ldots, h_{i_m})$$
$$= \sum_{p=m}^{\infty} p(p-1)\cdots(p-m+1)I_{p-m}(c(1, 2,\ldots, m\,;\, 1, 2,\ldots, m)f_p\otimes$$
$$\{h'_{i_1}\otimes\cdots\otimes h'_{i_m}\}).$$

Finally, since $F\in H^{n,2}$ we have

$$\int_B |D^n F(x)|^2_{\mathscr{L}^n_{(2)}(H;\mathbf{R})}\mu(dx)$$

$$= \int_B \sum_{i_1, i_2,\ldots, i_n=1}^{\infty} D^n F(x)(h_{i_1}, h_{i_2},\ldots, h_{i_n})^2\mu(dx)$$

$$= \sum_{i_1, i_2,\ldots, i_n=1}^{\infty}\int_B [\sum_{p=n}^{\infty} p(p-1)\cdots(p-n+1)I_{p-n}(c(1, 2,\ldots, n\,;\, 1, 2,\ldots, n)f_p$$
$$\otimes\{h'_{i_1}\otimes\cdots\otimes h'_{i_n}\})]^2\mu(dx)$$

$$= \sum_{i_1, i_2,\ldots, i_n=1}^{\infty}\sum_{p=n}^{\infty} p(p-1)\cdots(p-n+1)(p!)\int_0^1\cdots\int_0^1\{\int_0^1\cdots\int_0^1$$

$$f_p(t_1, t_2,\dots, t_p)h'_{i_1}(t_1)\cdots h'_{in}(t_n)dt_1\cdots dt_n\Big\}^2 dt_{n+1}\cdots dt_p$$

$$=\sum_{p=n}^{\infty} p(p-1)\cdots(p-n+1)(p!)|f_p|^2_{L^2(T^p)}$$

$$=\sum_{p=n}^{\infty} p(p-1)\cdots(p-n+1)|I_p(f)|^2_{L^2(\mu)}$$

Thus we have (4.18).

(Sufficiency) Take $F\in L^2(\mu)$ satisfying (4.18). Choose a complete orthonormal system in H such that $h_i\in B^*$. We put $G_N=\sum_{p=0}^{N} I_p(f_p)$ for $N=1, 2,\dots$. Then $G_N\in H^{n,2}$ from Proposition 4.1 and

$$\int_B |D^m G_N(x)|^2_{\mathscr{L}^m_{(2)}(H;\mathbf{R})}\mu(dx)$$

$$=\int_B \sum_{i_1,i_2,\dots,i_m=1}^{\infty} D^m G_N(x)(h_{i_1}, h_{i_2},\dots, h_{i_m})^2\mu(dx)$$

$$=\int_B \sum_{i_1,i_2,\dots,i_m=1}^{\infty} \sum_{p=0}^{N} D^m I_p(f_p)(x)(h_{i_1}, h_{i_2},\dots, h_{i_m})^2\mu(dx)$$

$$=\sum_{i_1,i_2,\dots,i_m=1}^{\infty}\int_B\Big[\sum_{p=m}^{N} p(p-1)\cdots(p-m+1)\int_0^1\cdots\int_0^1\Big\{\int_0^1\cdots\int_0^1$$

$$f(t_1, t_2,\dots, t_p)h'_{i_1}(t_1)h'_{i_2}(t_2)\cdots h'_{i_m}(t_m)dt_1\cdots dt_m\Big\}dx(t_{m+1})\cdots dx(t_p)\Big]^2\mu(dx)$$

$$=\sum_{i_1,i_2,\dots,i_m=1}^{\infty}\sum_{p=m}^{N}\Big[p(p-1)\cdots(p-m+1)^2(p-m)!\int_0^1\cdots\int_0^1\Big\{\int_0^1\cdots\int_0^1$$

$$f(t_1, t_2,\dots, t_p)h'_{i_1}(t_1)h'_{i_2}(t_2)\cdots h'_{i_m}(t_m)\,dt_1\cdots dt_m\Big\}^2 dt_{m+1}\cdots dt_p\Big]$$

$$=\sum_{p=m}^{N} p(p-1)\cdots(p-m+1)(p!)|f_p|^2_{L^2(T^p)}$$

$$=\sum_{p=m}^{N} p(p-1)\cdots(p-m+1)|I_p(f_p)|^2_{L^2(\mu)}.$$

Now it is easy to see that $\{G_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $H^{n,2}$ from (4.18). Since $G_N\to F$ in $L^2(\mu)$ as $N\to\infty$, the limit of $\{G_N\}_{N=1}^{\infty}$ coinside with F. Consequently $F\in H^{n,2}$. Q.E.D.

Proposition 4.5. *We have* $H^{2,2}(=H(2, 2, 2)(\mathbf{R}))=H(2, 2, 2; 2)$ *and the following equality holds;*

$$\int_B |LF(x)|^2\mu(dx)=\int_B (|DF(x)|^2_H+|D^2F(x)|^2_{\mathscr{L}^2_{(2)}(H;\mathbf{R})})\mu(dx) \tag{4.19}$$

for $F\in H(2, 2, 2; 2)$.

Proof. It is obvious that $H(2, 2, 2; 2)\subset H^{2,2}$. Let $F\in H^{2,2}$ be expanded as

(4.16). If we put $G_N=\sum_{p=0}^{N} I_p(f_p)$ for $N=1, 2,\dots$, then from Proposition 4.3, we have $G_N \in H(2, 2, 2; 2)$ and

$$LG_N=\sum_{p=1}^{N} -pI_p(f_p).$$

Hence

$$|LG_N|^2_{L^2(\mu)}=\int_B LG_N(x)^2\mu(dx)=\sum_{p=1}^{N} p^2|I_p(f_p)|^2_{L^2(\mu)}.$$

But from Proposition 4.4, we have

$$\sum_{p=2}^{\infty} p(p-1)|I_p(f_p)|^2_{L^2(\mu)}<\infty.$$

Then it is easy to see that $\{G_N\}_{N=1}^{\infty}$ is a Cauchy sequence in $H(2, 2, 2; 2)$ and hence $F\in H(2, 2, 2; 2)$. Thus $H^{2,2}=H(2, 2, 2; 2)$. Now it is clear that

$$LF=\sum_{p=1}^{\infty} -pI_p(f_p),$$

$$|LF|^2_{L^2(\mu)}=\sum_{p=1}^{\infty} p^2|I_p(f_p)|^2_{L^2(\mu)}.$$

But from Proposition 4.4, we have

$$|DF(x)|^2_{L^2(\mu;H)}=\sum_{p=1}^{\infty} p|I_p(f_p)|^2_{L^2(\mu)},$$

$$|D^2F(x)|^2_{L^2(\mu;\mathscr{L}^2_{(2)}(H;\mathbf{R}))}=\sum_{p=2}^{\infty} p(p-1)|I_p(f_p)|^2_{L^2(\mu)}.$$

Hence we have (4.19). Q.E.D.

So far we have investigated the derivatives of multiple Wiener integrals. For another example, we discuss on solutions of stochastic differential equations; we do not give a proof, however. We consider the following stochastic differential equation in one dimension (the extension to the multi-dimensional case is straightforward);

$$X(t)=\xi+\int_0^t a(X(s))dx(s)+\int_0^t b(X(s))ds. \tag{4.20}$$

We assume $a, b\in C_b^2(\mathbf{R})$. Then the solution $X(t)$ of (4.20) for fixed t belongs to $H(2, 2)(\mathbf{R})$ and its weak H-derivative $DX(t)$ is given by the solution of the stochastic differential equation

$$\begin{aligned}(4.21)\quad \langle DX(t), h\rangle_H=&\int_0^t \frac{da}{d\xi}(X(s))\langle DX(s), h\rangle_H dx(s)\\ &+\int_0^t \frac{db}{d\xi}(X(s))\langle DX(s), h\rangle_H ds+\int_0^t a(X(s))h'(s)ds\end{aligned}$$

for $h \in H$. Furthermore, if $a, b \in C_b^\infty(\mathbf{R})$, then $X(t)$ for a fixed t belongs to $H(p_0, p_1, \dots, p_n)(\mathbf{R})$ for every n and every $p_0, p_1, \dots, p_n \geq 1$.

5. Absolute continuity of probability laws of the multiple Wiener integrals

Now we investigate the absolute continuity of probability laws of the multiple Wiener integrals as an application of Theorem 3.1. First we treat the case of a single multiple Wiener integral.

Theorem 5.1. *Let F be a real valued Wiener functional given by $F = \sum_{p=0}^{N} I_p(f_p)$ for $f_p \in \hat{L}^2(T^p)$, $p=1, 2, \dots, N$. If $f_N \neq 0$, then the probability law on $\mathbf{R}$ induced by F is absolutely continuous.*

Proof. We prove this theorem by induction on the degree N. For $N=1$, the theorem is true since $I_1(f_1)$ has a normal distribution. Assume the theorem is true for N. Let $F = \sum_{p=0}^{N+1} I_p(f_p)$. The conditions (i) and (ii) of Theorem 3.1 are satisfied by Propositions 4.1 and 4.3. Therefore it suffices to establish the condition (iii), i.e.,

$$\sigma(x) = \langle DF(x), DF(x) \rangle_H = |DF(x)|_H^2 \neq 0 \quad \text{a.e. } (\mu).$$

For any $h \in H$,

$$\langle DF(x), h \rangle_H = (N+1) \int_0^1 \cdots \int_0^1 \left\{ \int_0^1 f_{N+1}(t_1, t_2, \dots, t_{N+1}) h'(t_1) dt_1 \right\} dx(t_2) \cdots$$
$$dx(t_{N+1}) + (\text{lower terms}).$$

Since $f_{N+1} \neq 0$, there exists $h \in H$ such that

$$\int_0^1 f_{N+1}(t_1, t_2, \dots, t_{N+1}) h'(t_1) dt_1 \neq 0 \qquad \text{in} \quad L^2(T^N).$$

Hence, by induction, we have $\langle DF(x), h \rangle_H \neq 0$ a.e. (μ) for such $h \in H$. Thus $\sigma(x) = |DF(x)|_H^2 \neq 0$ a.e. (μ) and the theorem is true for $N+1$. Q.E.D.

Next we discuss the case of a system of multiple Wiener integrals. This case is rather complicated. We give here one sufficient condition for the absolute continuity.

Theorem 5.2. *Let $F = (F^1, F^2, \dots, F^d)$ be an $\mathbf{R}^d$-valued Wiener functional given by*

$$F^i = \sum_{p=0}^{N_i} I_p(f_p^{(i)}), \quad i=1, 2, \dots, d, \quad f_p^{(i)} \in \hat{L}^2(T^p). \tag{5.1}$$

We assume that there exists $h \in L^2(T)$ such that the system of functions $g_1(t), g_2(t), \dots, g_d(t)$ defined by

$$g_i(t) = \int_0^1 \cdots \int_0^1 f_{N_i}^{(i)}(t, t_2, \dots, t_{N_i}) h'(t_2) \cdots h'(t_{N_i}) dt_2 \cdots dt_{N_i} \tag{5.2}$$

is linearly independent in $L^2(T)$. *Then the probability law on* $\mathbf{R}^d$ *induced by* F *is absolutely continuous.*

Proof. Put $G=I_p(g)$, $K=I_q(k)$ for $g\in L^2(T^p)$, $k\in L^2(T^q)$. We shall evaluate $\langle DG(x), DK(x)\rangle_H$ concretely. Let $\{h_i\}_{i=1}^{\infty}$ be a complete orthonormal system in H. Then by Proposition 4.1 and Lemma 4.1,

$$\langle DG(x), DK(x)\rangle_H=\sum_{i=1}^{\infty}\langle DG(x), h_i\rangle_H\langle DK(x), h_i\rangle_H$$

$$=\sum_{i=1}^{\infty} I_{p-1}\Big(\sum_{m=1}^{p} c(m;\,1)g\otimes h_i'\Big)I_{q-1}\Big(\sum_{n=1}^{q} c(n;\,1)k\otimes h_i'\Big)$$

$$=\sum_{i=1}^{\infty}\frac{1}{l!}\sum_{l=0}^{p\wedge q-1}\sum_{m=1}^{p}\sum_{n=1}^{q}\sum_{\substack{\{m_1,m_2,\dots,m_l\}\subset\{1,2,\dots,p\}\setminus\{m\}\\ \{n_1,n_2,\dots,n_l\}\subset\{1,2,\dots,q\}\setminus\{n\}}}$$

$$I_{p+q-2l-2}(c(m_1, m_2,\dots, m_l;\, n_1, n_2,\dots, n_l)(c(m;\,1)g\otimes h_i')\otimes(c(n;\,1)k\otimes h_i'))$$

$$=\sum_{l=1}^{p\wedge q}\frac{1}{l!}\sum_{\substack{\{m_1,m_2,\dots,m_l\}\subset\{1,2,\dots,p\}\\ \{n_1,n_2,\dots,n_l\}\subset\{1,2,\dots,q\}}} I_{p+q-2l}(c(m_1, m_2,\dots m_l;\, n_1, n_2,\dots, n_l)g\otimes k).$$

Hence if $g\in\hat{L}^2(T^p)$ and $k\in\hat{L}^2(T^q)$, then the highest term in the expansion of $\langle DG(x), DK(x)\rangle_H$ is the multiple Wiener integral of $pq\int_0^1 g(u, t_2,\dots, t_p)k(u, s_2,\dots, s_q)du$. Now it is clear that $\sigma^{ij}(x)=\langle DF^i(x), DF^j(x)\rangle_H\in H(1, 2, 1;\, 1)$. Thus the condition (i) and (ii) of Theorem 3.1 are clearly satisfied. It suffices to establish the condition (iii) i.e., $\det(\sigma^{ij}(x))\neq 0$ a.e. (μ). But

$$\det(\sigma^{ij}(x))=\sum_{\tau\in S_d}\operatorname{sgn}(\tau)\prod_{i=1}^{d}\sigma^{i\tau(i)}(x). \tag{5.3}$$

Since $\sigma^{ij}(x)$ is a sum of multiple Wiener integrals, so is also $\det(\sigma^{ij}(x))$. The highest term in the expansion of $\det(\sigma^{ij}(x))$ is the multiple Wiener integral $I_k(f)$ of the function $f\in L^2(T^k)$ defined by

$$f=N_1^2N_2^2\cdots N_d^2\sum_{\tau\in S_d}\operatorname{sgn}(\tau)\{c(1;\,1)f_{N_1}^{(1)}\otimes f_{N_{\tau(1)}}^{(\tau(1))}\}\otimes\cdots \otimes\{c(1;\,1)f_{N_d}^{(d)}\otimes f_{N_{\tau(d)}}^{(\tau(d))}\} \tag{5.4}$$

where $k=2(\sum_{i=1}^{d} N_i-d)$. We shall show that the symmetrization $\tilde{f}$ of f is not the zero element in $\hat{L}^2(T^k)$. It is easy to see that $\tilde{f}\neq 0$ if and only if there exists $h\in H$ such that the inner product $\langle h'\otimes h'\otimes\cdots\otimes h', \tilde{f}\rangle_{L^2(T^k)}\neq 0$. But this inner product is just a constant multiple of the Gramian of the system of functions $\{g_i(t)\}$ defined by (6.2) and consequently, $\tilde{f}\neq 0$ if and only if $\{g_i(t)\}$ defined by (6.2) is linearly independend for some $h\in H$. Thus we can conclude $\tilde{f}\neq 0$. From Theorem 5.1, $\det(\sigma^{ij}(x))$ $\neq 0$ a.e. (μ). Q.E.D.

We shall give an example.

Example. Let $f_p\in\hat{L}^2(\mathbf{R}^p)$ for $p=1, 2,\dots, N$ which has a support on $(-\infty, 0]^p$.

We define the stationary process $\{X_t;\ -\infty<t<\infty\}$ by

$$X_t=\sum_{p=0}^{N}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t}f_p(u_1-t,\ u_2-t,\dots,\ u_p-t)dx(u_1)\cdots dx(u_p)$$

where $\{x(t);\ -\infty<t<\infty\}$ is the Wiener process such that $x(0)=0$. Then any finite dimensional joint distribution is absolutely continuous.

Proof. So far we assume that the Wiener process is defined on the time interval $[0,\ 1]$. But we can easily extend above results in the case of time interval $(-\infty,\ \infty)$. We shall prove that for $-\infty<t_1<t_2<\cdots<t_n<\infty$, the probability law of $(X_{t_1},\ X_{t_2},\ \dots,\ X_{t_n})$ on $\mathbf{R}^n$ is absolutely continuous. We prove it by induction on n. We may assume that $f_0=0,\ f_1=0,\dots,f_{N-1}=0$ and the support of f_N is not included in $(-\infty,\ -\varepsilon]^N$ for any $\varepsilon>0$. The general case can be proved similarly. For $n=1$, the probability law of X_{t_1} is absolutely continuous from Theorem 5.1. Assume that the statement is true for n and consider the law of $(X_{t_1},\ X_{t_2},\dots,\ X_{t_n},\ X_{t_{n+1}})$. If we put

$$\begin{aligned}X_{t_{n+1}}&=\int_{-\infty}^{t_n}\cdots\int_{-\infty}^{t_n}f_N(u_1-t_{n+1},\dots,\ u_N-t_{n+1})dx(u_1)\cdots dx(u_N)\\&\quad+\int_{-\infty}^{t_{n+1}}\cdots\int_{-\infty}^{t_{n+1}}1_{(-\infty,t_{n+1}]^N\setminus(-\infty,t_n]^N}(u_1,\dots,\ u_N)\\&\quad\times f_N(u_1-t_{n+1},\dots,\ u_N-t_{n+1})dx(u_1)\cdots dx(u_N)\\&=Y_1+Y_2,\end{aligned}$$

then clearly $(X_{t_1},\ X_{t_2},\dots,\ X_{t_n},\ Y_1)$ and Y_2 are independent and $(X_{t_1},\dots,\ X_{t_n})$ and Y_2 have the absolutely continuous probability laws from the assumption of induction and Theorem 5.1. Hence for any Borel subset $A\subset\mathbf{R}^{n+1}$ such that $|A|=0$,

$$\begin{aligned}&\mu((X_{t_1},\ X_{t_2},\dots,\ X_{t_n},\ X_{t_{n+1}})\in A)\\&=E_\mu[1_A(X_{t_1},\ X_{t_2},\dots,\ X_{t_n},\ Y_1+Y_2)]\\&=E_\mu[E_\mu[1_A(X_{t_1},\ X_{t_2},\dots,\ X_{t_n},\ Y_1+Y_2)\,|\,X_{t_1},\ X_{t_2},\dots,\ X_{t_n},\ Y_1]]\\&=E_\mu\int_{-\infty}^{\infty}1_A(X_{t_1},\ X_{t_2},\dots,\ X_{t_n},\ Y_1+\xi)\nu_{Y_2}(d\xi)\end{aligned}$$

where ν_{Y_2} is a induced measure by Y_2. Since ν_{Y_2} is absolutely continuous and the probability law of $(X_{t_1},\ X_{t_2},\dots,\ X_{t_n})$ is absolutely continuous, we have

$$\mu((X_{t_1},\ X_{t_2},\dots,\ X_{t_n},\ X_{t_{n+1}})\in A)=0.$$

Hence the probability law of $(X_{t_1},\ X_{t_2},\dots,\ X_{t_n},\ X_{t_{n+1}})$ is absolutely continuous.

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References

[1] L. Gross, Abstract Wiener spaces. Proc. Fifth Berkeley Sympos. Math. Statist. and Prob., Univ. of California Press, vol. 1, part 1, (1965) 31–42.

[2] K. Ito, Multiple Wiener integral. J. Math. Soc. Japan, 3, (1951) 157–167.

[3] H.-H. Kuo, Integration by parts for abstract Wiener measures. Duke Math. J. **41**, (1974), 373–379.

[4] ———, Gaussian measures in Banach spaces. Lecture Notes in Math. 463, Berlin-Heiderberg-New York, Springer, (1975).

[5] E. L. Lehmann, Testing statistical hypotheses. Wiley, New York and Chapman & Hall, London, (1959).

[6] P. Malliavin, Stochastic caluculus of variation and hypoelliptic operators. Proc. Int. Sympos. on S.D.E., Kyoto, Kinokuniya, (1978) 195–263.

[7] ———, C^k-hypoellipticity with degeneracy. Stochastic Analysis. edited by A. Friedman, M. Pinsky, Academic Press, New York, (1978), 199–214, 327–340.