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Eikonal equations and spectral representations for long-range Schrödinger Hamiltonians

By

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§1. Introduction

We shall investigate in this paper spectral representations for the Schrödinger operator defined as the self-adjoint realization in $L_2(\mathbf{R}^n)$ of $H=-\Delta+V(x)$, where Δ denotes the Laplacian in $L_2(\mathbf{R}^n)$ and the *potential* $V(x)$ satisfies the following

Assumption: *$V(x)$ is a real-valued $C^3(\mathbf{R}^n)$-function such that for some $\delta>0$*

$$D_x^\alpha V(x)=O(r^{-|\alpha|-\delta}) \quad as \quad r=|x|\to\infty \quad (0\leq|\alpha|\leq 3)$$

where $D_x=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)$, α is a multi-index.

As has been noted by Ikebe [2], the usual Fourier transform (precisely speaking, its restriction to the sphere) is obtained from the asymptotic expansion of the solution of the Helmholtz equation (for instance in $\mathbf{R}^3$, and if $f\in C_0^\infty(\mathbf{R}^3)$,

$$(1.1)\qquad \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}f(y)dy=\frac{e^{i\sqrt{\lambda}r}}{4\pi r}\int_{\mathbf{R}^3}e^{-i\sqrt{\lambda}\omega y}f(y)dy+O(r^{-2})$$

as $r=|x|\to\infty$, where $\omega=x/r$).

Suggested by the above observation, Ikebe [2] and Saitō [10] have obtained the spectral representation theorems for Schrödinger operators with long-range potentials by considering the following limit

$$(1.2)\qquad \lim_{r\to\infty} r^{(n-1)/2}e^{-iK(x,\lambda)}(R(\lambda+i0)f)(r\cdot)$$

in $L_2(S^{n-1})$, where $R(z)=(H-z)^{-1}$, and it has been observed that $K(x,\lambda)$ should be chosen as an (approximate) solution of the *eikonal equation*

$$(1.3)\qquad |\nabla_x K(x,\lambda)|^2+V(x)=\lambda$$

(see Ikebe-Isozaki [3]). This procedure has also been adopted by Mochizuki-

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Uchiyama [9] in a slightly modified form in the case that the potential has an oscillation at infinity.

Here we remark the following: First in the work of some previous authors (see e.g. Saitō [10] and Isozaki [5]) it seems that too much smoothness has been required of $V(x)$ in order to solve (approximately) the eikonal equation (1.3). Second, the existence of the limit (1.2) has been guaranteed only along a certain sequence $\{r_m\}_{m=1}^{\infty}$ diverging to infinity.

The purpose of the present paper is two-fold. First, we shall construct the exact (asymptotic) solution of the eikonal equation under the above mentioned assumption on the potential $V(x)$. (This solution is utilized in a *stationary* proof of the completeness of the *time-dependent* modified wave operators (see Ikebe-Isozaki [4].) To solve (1.3) we follow the standard line of Hamilton-Jacobi's theory of solving first order partial differential equations. However, the attempt to find a *local* solution at *infinity* of (1.3) will complicate our arguments considerably. Second, we shall prove the existence of the limit (1.2) without taking a sequence $\{r_m\}_{m=1}^{\infty}$, which will remove the inconvenience that has so far occurred whenever we have dealt with limits like (1.2) in discussions connected to spectral representations for the Schrödinger operators. Here we should mention the work of Saitō [10], which has also shown the existence of the limit (1.2) without taking a sequence $\{r_m\}_{m=1}^{\infty}$. However, since he transforms the Schrödinger operators $-\Delta+V(x)$ into the ordinary differential operators with operator valued coefficients by passing to the spherical coordinates, his theory cannot be applied directly to the case $\mathbf{R}^2$. Whereas, our arguments hold good in the case $\mathbf{R}^n$, $n\geq 2$.

The contents of this paper are as follows. In § 2, we shall prove the existence of the limit (1.2) and the spectral representation theorem for H assuming the existence and certain asymptotic properties of the solution of the eikonal equation (1.3). Some technical lemmas will be proved in § 3. The solution of the eikonal equation (1.3) will be constructed in § 4.

§ 2. Spectral representation

Let us begin with the following lemma.

Lemma 2.1. *Let* $\mathbf{R}_+=(0, \infty)$. *There exists a real-valued* $C^3(\mathbf{R}^n\times\mathbf{R}_+)$-*function* $Y(x, \lambda)$ *having the following properties*:

(1) *Let* Λ *be an arbitrary compact set in* $\mathbf{R}_+$. *Then there exists a constant* $R_0=R_0(\Lambda)>0$ *such that*

$$2\sqrt{\lambda}\frac{\partial Y}{\partial r}(x, \lambda)=|\nabla_x Y(x, \lambda)|^2+V(x)$$

for $r=|x|>R_0$, $\lambda\in\Lambda$.

(2) *For any compact set* Λ *in* $\mathbf{R}_+$, *there exists a constant* $C=C(\Lambda)$ *such that*

$$|D_x^\alpha Y(x, \lambda)|\leq C(1+r)^{1-|\alpha|-\delta} \qquad (0\leq|\alpha|\leq 3).$$

We shall prove this lemma in § 4. Note that if we put $K(x, \lambda)=\sqrt{\lambda}r-Y(x, \lambda)$, it must satisfy the eikonal equation $|\nabla_x K(x, \lambda)|^2 + V(x) = \lambda$ for sufficiently large $|x|$.

We introduce several notations.

For a domain G in $\mathbf{R}^n$ and a real constant β, let $L_{2,\beta}(G)$ denote the Hilbert space of all measurable functions f over G such that $\|f\|^2_{\beta,G} = \int_G (1+|x|)^{2\beta}|f(x)|^2 dx < \infty$. If $\beta = 0$ or $G = \mathbf{R}^n$, we often omit the subscript.

Let $K(x, \lambda) = \sqrt{\lambda} r - Y(x, \lambda)$ as above, and the differential operators $\mathscr{D}^{\pm}_j$, $\mathscr{D}^{\pm}$, $\mathscr{D}^{\pm}_r$, $\mathscr{D}^{\pm}_\omega$ be defined by

$$\mathscr{D}^{\pm}_j = \mathscr{D}^{\pm}_j(\lambda) = \frac{\partial}{\partial x_j} + \frac{n-1}{2r}\tilde{x}_j \mp i\frac{\partial K}{\partial x_j}(x, \lambda) \qquad (\tilde{x}_j = x_j/r,\ j = 1, \dots, n),$$

$$\mathscr{D}^{\pm} = (\mathscr{D}^{\pm}_1, \cdots, \mathscr{D}^{\pm}_n),$$

$$\mathscr{D}^{\pm}_r = \sum_j \tilde{x}_j \mathscr{D}^{\pm}_j = \frac{\partial}{\partial r} + \frac{n-1}{2r} \mp i\frac{\partial K}{\partial r}(x, \lambda),$$

$$\mathscr{D}^{\pm}_\omega = \widetilde{grad} \mp i\widetilde{grad} K(x, \lambda), \qquad (\widetilde{grad} = grad - \tilde{x}\frac{\partial}{\partial r},\ \tilde{x} = x/r).$$

Let $E = \{x \in \mathbf{R}^n : |x| > 1\}$. In general $\mathbf{B}(H_1 : H_2)$ denotes the totality of bounded linear operators from a Banach space H_1 to a Banach space H_2.

The following theorem due to Ikebe-Saitō [6] asserts the existence of the boundary values $R(\lambda \pm i0)$ of the resolvent $R(z) = (H - z)^{-1}$ $(z \in \mathbf{C} - \mathbf{R})$.

Theorem 2.2. *Let ε_0 be a constant such that $0 < \varepsilon_0 \le \delta/2$.*

(1) *For $\lambda > 0$, there exists a strong limit*

$$\operatorname*{s-lim}_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon) \equiv R(\lambda \pm i0) \in B(L_{2,(1+\varepsilon_0)/2} : L_{2,-(1+\varepsilon_0)/2}).$$

Moreover, for $f \in L_{2,(1+\varepsilon_0)/2}$, $R(\lambda \pm i0)f$ is continuous for $\lambda > 0$ in $L_{2,-(1+\varepsilon_0)/2}$.

(2) *For $f \in L_{2,(1+\varepsilon_0)/2}$, $R(\lambda \pm i0)f$ satisfies the radiation condition*

$$\mathscr{D}^{\pm}(\lambda)R(\lambda \pm i0)f \in L_{2,-(1-\varepsilon_0)/2}(E).$$

(3) *$R(\lambda \pm i0)f$ is the unique solution of the following problem*

$$\begin{cases} (-\Delta + V - \lambda)u = f, \quad f \in L_{2,(1+\varepsilon_0)/2}, \\ u \in L_{2,-(1+\varepsilon_0)/2}, \\ \mathscr{D}^{\pm}(\lambda)u \in L_{2,-(1-\varepsilon_0)/2}(E). \end{cases}$$

(4) *The part of H in $\mathbf{R}_+$ is absolutely continuous.*

We also need the following lemma whose proof has been given in Isozaki [5].

Lemma 2.3. *Let $f \in L_{2,(3-\varepsilon_0)/2}$. Then we have*

$$\|\mathscr{D}^{\pm}(\lambda)R(\lambda \pm i0)f\|_{(1-\varepsilon_0)/2,E} \le C\|f\|_{(3-\varepsilon_0)/2},$$

where the constant C is independent of λ in a compact set in $\mathbf{R}_+$.

In the following we refer only to the *outgoing* radiation operator $\mathscr{D}^+(\lambda)$, and write it $\mathscr{D}(\lambda)$. Now, let $u=R(\lambda+i0)f$ $(f\in L_{2,(1+\varepsilon_0)/2})$. Then u satisfies

$$(2.1)\qquad (-\Delta+V-\lambda)u=f.$$

Let $\mathscr{D}_j^*$ be the formal adjoint of $\mathscr{D}_j$, and Λ be the Laplace-Beltrami operator on S^{n-1} (the unit sphere in $\mathbf{R}^n$). Then by a direct calculation we can rewrite the equation (2.1) as follows

$$(2,2)\qquad \sum_j \mathscr{D}_j^*\mathscr{D}_j u-2i\sum_j \frac{\partial K}{\partial x_j}\mathscr{D}_j u$$
$$=f-\Big(\tilde{V}+|\nabla Y|^2-2\sqrt{\lambda}\frac{\partial Y}{\partial r}\Big)u-i\Big(\frac{\partial^2 Y}{\partial r^2}+\frac{\Lambda Y}{r^2}\Big)u,$$

where $\tilde{V}(x)=V(x)+(n-1)(n-3)/(4r^2)$.

First we investigate some properties of the surface integral.

Lemma 2.4. *Let* $f\in L_{2,(3-\varepsilon_0)/2}$, $u=R(\lambda+i0)f$. *Let* $v\in H^2_{loc}$[2] *be such that* $v\in L_{2,-(1+\varepsilon_0)/2}$ *and* $\mathscr{D}(\lambda)v\in L_{2,-(1-\varepsilon_0)/2}(E)$. *Then we have the following equality*

$$\frac{d}{dr}\int_{|x|=r}(\mathscr{D}_r u)\bar{v}\,dS=-2i\sqrt{\lambda}\int_{|x|=r}(\mathscr{D}_r u)\bar{v}\,dS+F(r),$$

where $\int_1^\infty |F(r)|dr<\infty$.

Proof. Noting that $\mathscr{D}_r^*=-\frac{\partial}{\partial r}-\frac{n-1}{2r}+i\frac{\partial K}{\partial r}$, we have by a straightforward calculation

$$(2.3)\qquad \frac{d}{dr}\int_{|x|=r}(\mathscr{D}_r u)\bar{v}\,dS=-\int_{|x|=r}(\mathscr{D}_r^*\mathscr{D}_r u)\bar{v}\,dS+\int_{|x|=r}(\mathscr{D}_r u)(\overline{\mathscr{D}_r v})\,dS.$$

The second term of the right hand side of (2.3) is easily seen to belong to $L_1((1,\infty))$ by our assumption and Lemma 2.3. Since $\sum_j \mathscr{D}_j^*\mathscr{D}_j=\mathscr{D}_r^*\mathscr{D}_r+\mathscr{D}_\omega^*\mathscr{D}_\omega$, we have in view of (2.2)

$$-\mathscr{D}_r^*\mathscr{D}_r u=\mathscr{D}_\omega^*\mathscr{D}_\omega u-2i\frac{\partial K}{\partial r}\mathscr{D}_r u-2i\widetilde{\mathrm{grad}K}\cdot\mathscr{D}_\omega u+A(x),$$

where $A(x)=-f+\Big(\tilde{V}+|\nabla Y|^2-2\sqrt{\lambda}\frac{\partial Y}{\partial r}\Big)u+i\Big(\frac{\partial^2 Y}{\partial r^2}+\frac{\Lambda Y}{r^2}\Big)u$. By Lemma 2.1 and Theorem 2.2 we see that

$$\int_{|x|=r}A(x)\bar{v}\,dS\in L_1((1,\infty)).$$

Since $\widetilde{\mathrm{grad}K}=-\widetilde{\mathrm{grad}Y}=O(r^{-\delta})$ by Lemma 2.1, we have by the use of Lemma 2.3

$$\int_{|x|=r}(\widetilde{\mathrm{grad}K}\cdot\mathscr{D}_\omega u)\bar{v}\,dS\in L_1((1,\infty)).$$

2) H^m = the Sobolev space of order m.

We can also see that by integration by parts

$$\int_{|x|=r}(\mathscr{D}_\omega^*\mathscr{D}_\omega u)\bar{v}\,dS=\int_{|x|=r}(\mathscr{D}_\omega u)(\overline{\mathscr{D}_\omega v})\,dS,$$

which implies by Lemma 2.3 that

$$\int_{|x|=r}(\mathscr{D}_\omega^*\mathscr{D}_\omega u)\bar{v}\,dS\in L_1((1,\infty)).$$

Now we have only to evaluate $\frac{\partial K}{\partial r}\mathscr{D}_r u\bar{v}$. Since $\frac{\partial K}{\partial r}=\sqrt{\lambda}-\frac{\partial Y}{\partial r}$ and $\frac{\partial Y}{\partial r}=O(r^{-\delta})$ by Lemma 2.1 we have by using Lemma 2.3 that

$$\int_{|x|=r}\frac{\partial Y}{\partial r}\mathscr{D}_r u\bar{v}\,dS\in L_1((1,\infty)).$$

We have thus

$$\frac{d}{dr}\int_{|x|=r}(\mathscr{D}_r u)\bar{v}\,dS=-2i\sqrt{\lambda}\int_{|x|=r}(\mathscr{D}_r u)\bar{v}\,dS+F(r),$$

where

$$\int_1^\infty |F(r)|\,dr<\infty. \qquad \textit{Q.E.D.}$$

Lemma 2.5. *Under the same assumption as in Lemma 2.4 we have*

$$\int_{|x|=r}\mathscr{D}_r u\bar{v}\,dS\to 0 \quad \textit{as} \quad r\to\infty.$$

Proof. Put $\phi(r)=\int_{|x|=r}\mathscr{D}_r u\bar{v}\,dS$. Then we have by Lemma 2.4

$$\frac{d}{dr}\phi(r)=-2i\sqrt{\lambda}\phi(r)+F(r), \quad \int_1^\infty |F(r)|\,dr<\infty.$$

Letting $\psi(r)=e^{2i\sqrt{\lambda}r}\phi(r)$ we have $\frac{d}{dr}\psi(r)=e^{2i\sqrt{\lambda}r}F(r)$, which implies $\psi(r)=\psi(1)+\int_1^r e^{2i\sqrt{\lambda}s}F(s)\,ds$. Since $F(s)\in L_1((1,\infty))$, we see that there exists a limit $\lim_{r\to\infty}\psi(r)$. By our assumption and Lemma 2.3 we have $\int_1^\infty r^{-\varepsilon_0}|\phi(r)|\,dr<\infty$, from which we see that $\liminf_{r\to\infty}|\psi(r)|=0$. This implies, since there exists a limit $\lim_{r\to\infty}\psi(r)$, that $|\psi(r)|=|\phi(r)|\to 0$ as $r\to\infty$. Q.E.D.

Lemma 2.6. *Let* $f\in L_{2,(3-\varepsilon_0)/2}$, $u=R(\lambda+i0)f$. *Then we have*

$$(R(\lambda+i0)f-R(\lambda-i0)f,f)=\lim_{r\to\infty}2i\sqrt{\lambda}\int_{|x|=r}|u|^2\,dS.$$

Proof. Since $(-\Delta+V-\lambda)u=f$, we have by Green's formula

$$\int_{|x|<r}(u\bar{f}-f\bar{u})\,dx=\int_{|x|=r}\left(\frac{\partial u}{\partial r}\bar{u}-u\frac{\partial\bar{u}}{\partial r}\right)dS$$

$$=\int_{|x|=r}(\mathscr{D}_r u\bar{u}-u\overline{\mathscr{D}_r u})\,dS+2i\int_{|x|=r}\frac{\partial K}{\partial r}|u|^2\,dS.$$

In view of Lemma 2.5, we see that $\int_{|x|=r} \mathscr{D}_r u\bar{u}dS \to 0$ as $r\to\infty$. We also see that the left hand side of the above equality tends to $(R(\lambda+i0)f-R(\lambda-i0)f, f)$ as $r\to\infty$. Thus to complete the proof of the lemma, we have only to note that $\frac{\partial K}{\partial r}=\sqrt{\lambda}-\frac{\partial Y}{\partial r}$ and $\frac{\partial Y}{\partial r}=O(r^{-\delta})$. Q. E. D.

Definition 2.7. For $f\in L_{2,(3-\varepsilon_0)/2}$, let $\mathscr{F}(\lambda, r)f\in L_2(S^{n-1})$ be deaned by

$$(\mathscr{F}(\lambda, r)f)(\omega)=C(\lambda)r^{(n-1)/2}e^{-iK(r\omega,\lambda)}(R(\lambda+i0)f)(r\omega),$$

$$\omega\in S^{n-1},\quad C(\lambda)=e^{(n-3)\pi i/4}\pi^{-1/2}\lambda^{1/4}.$$

In view of Lemma 2.6, we have for $f\in L_{2,(3-\varepsilon_0)/2}$

$$\frac{1}{2\pi i}(R(\lambda+i0)f-R(\lambda-i0)f, f)=\lim_{r\to\infty}\|\mathscr{F}(\lambda, r)f\|^2_{L_2(S^{n-1})}. \tag{2.4}$$

Lemma 2.8. *Let $f\in L_{2,(3-\varepsilon_0)/2}$. Then there exists a weak limit*

$$\text{w-}\lim_{r\to\infty}\mathscr{F}(\lambda, r)f\equiv\mathscr{F}(\lambda)f \textit{ in } L_2(S^{n-1}).$$

Proof. Since $\|\mathscr{F}(\lambda, r)f\|^2_{L_2(S^{n-1})}$ is uniformly bounded in r (by (2.4)), we have only to show the existence of the limit $\lim_{r\to\infty}(\phi, \mathscr{F}(\lambda, r)f)_{L_2(S^{n-1})}$ for $\phi\in C^\infty(S^{n-1})$. Now let $\phi(\omega)\in C^\infty(S^{n-1})$ and v be defined by

$$v=\rho(r)r^{-(n-1)/2}e^{iK(x,\lambda)}\phi(\omega)\quad (r=|x|,\ \omega=x/r), \tag{2.5}$$

where $\rho(r)\in C^\infty(\mathbf{R}_+)$ such that $\rho(r)=0$ $(r<1)$, $\rho(r)=1$ $(r>2)$. We let

$$g=(-\Delta+V-\lambda)v. \tag{2.6}$$

Then we have by a straightforward calculation if $r>2$

$$g(x, \lambda)=e^{iK(x,\lambda)}\{(-2\sqrt{\lambda}\frac{\partial Y}{\partial r}+|\nabla_x Y|^2+\tilde{V})\phi(\omega)+i\Big(\frac{\partial^2 Y}{\partial r^2}+\frac{\Lambda Y}{r^2}\Big)\phi(\omega) \tag{2.7}$$
$$-\frac{\Lambda}{r^2}\phi(\omega)+2i\widetilde{grad}Y\cdot\widetilde{grad}\phi(\omega)\}r^{-(n-1)/2}.$$

In view of Lemma 2.1, we see that g satisfies $|g|\le C(1+r)^{-(n+1)/2-\delta}$, in particular $g(\cdot, \lambda)\in L_{2,(1+\varepsilon_0)/2}$. We can also see that $v\in L_{2,-(1+\varepsilon_0)/2}$, $\mathscr{D}(\lambda)v\in L_{2,-(1-\varepsilon_0)/2}(E)$ and $\mathscr{D}_r v=0$ if $r>2$. Now letting $u=R(\lambda+i0)f$, we have by using Green's formula

$$\int_{|x|<r}\{(\Delta v)\bar{u}-v(\Delta\bar{u})\}dx=\int_{|x|=r}\{(\overline{\mathscr{D}_r v})\bar{u}-v(\mathscr{D}_r u)\}dS \tag{2.8}$$
$$+2i\int_{|x|=r}\frac{\partial K}{\partial r}v\bar{u}dS.$$

As r tends to infinity, the left hand side of (2.8) tends to $(v, f)-(g, R(\lambda+i0)f)$. Since $\mathscr{D}_r v=0$ for $r>2$ and $\lim_{r\to\infty}\int_{|x|=r} v\overline{\mathscr{D}_r u}dS=0$ by Lemma 2.5, we see that the first term of the right hand side of (2.8) tends to 0 as $r\to\infty$. We can also see that, since $\dfrac{\partial Y}{\partial r}=O(r^{-\delta})$,

$$\lim_{r\to\infty} 2i\int_{|x|=r}\frac{\partial K}{\partial r}v\bar{u}\,dS=\lim_{r\to\infty} 2i\sqrt{\lambda}\int_{|x|=r}v\bar{u}\,dS$$

$$=\lim_{r\to\infty} C(\lambda)(\phi,\mathscr{F}(\lambda,r)f)_{L_2(S^{n-1})},$$

where $C(\lambda)=2e^{(n-1)\pi i/4}\pi^{1/2}\lambda^{1/4}$.
Thus we have by letting r tend to infinity in (2.8),

$$(v,f)-(g,R(\lambda+i0)f)=\lim_{r\to\infty} C(\lambda)(\phi,\mathscr{F}(\lambda,r)f)_{L_2(S^{n-1})}, \tag{2.9}$$

which proves the lemma.

We also need the following lemma whose proof will be given in § 3.

Lemma 2.9. *Let $f\in L_{2,(3-\varepsilon_0)/2}$. Then there exists a sequence $\{r_m\}_{m=1}^{\infty}$ tending to infinity such that $\mathscr{F}(\lambda,r_m)f$ converges strongly to $\mathscr{F}(\lambda)f$ in $L_2(S^{n-1})$ as $m\to\infty$.*

Now we can obtain the following theorem.

Theorem 2.10. *Let $f\in L_{2,(3-\varepsilon_0)/2}$. Then $\mathscr{F}(\lambda,r)f$ converges strongly to $\mathscr{F}(\lambda)f$ in $L_2(S^{n-1})$ as $r\to\infty$. Moreover we have*

$$\frac{1}{2\pi i}(R(\lambda+i0)f-R(\lambda-i0f),f)=\|\mathscr{F}(\lambda)f\|^2_{L_2(S^{n-1})}. \tag{2.10}$$

Proof. Since $\mathscr{F}(\lambda)f$ is the weak limit of $\mathscr{F}(\lambda,r)f$ (Lemma 2.8), we have

$$\|\mathscr{F}(\lambda)f\|_{L_2(S^{n-1})}\le\liminf_{r\to\infty}\|\mathscr{F}(\lambda,r)f\|_{L_2(S^{n-1})}. \tag{2.11}$$

By the use of the sequence $\{r_m\}_{m=1}^{\infty}$ given in Lemma 2.9 we have

$$\liminf_{r\to\infty}\|\mathscr{F}(\lambda,r)f\|_{L_2(S^{n-1})}\le\lim_{m\to\infty}\|\mathscr{F}(\lambda,r_m)f\|_{L_2(S^{n-1})}. \tag{2.12}$$

$$=\|\mathscr{F}(\lambda)f\|_{L_2(S^{n-1})}.$$

In view of (2.11) and (2.12) we have

$$\|\mathscr{F}(\lambda)f\|_{L_2(S^{n-1})}=\liminf_{r\to\infty}\|\mathscr{F}(\lambda,r)f\|_{L_2(S^{n-1})}$$

$$=\lim_{r\to\infty}\|\mathscr{F}(\lambda,r)f\|_{L_2(S^{n-1})}\qquad\text{(by (2.4))},$$

which shows the existence of the strong limit $\operatorname{s-lim}_{r\to\infty}\mathscr{F}(\lambda,r)f=\mathscr{F}(\lambda)f$ in $L_2(S^{n-1})$. The equality (2.10) will be derived from (2.4). Q.E.D.

Lemma 2.11. *$\mathscr{F}(\lambda)\in B(L_{2,(3-\varepsilon_0)/2}: L_2(S^{n-1}))$. $\mathscr{F}(\lambda)f$ is jointly continuous for $\lambda>0$ and $f\in L_{2,(3-\varepsilon_0)/2}$.*

Proof. It follows from Theorems 2.2 and 2.10 that

$$\|\mathscr{F}(\lambda)f\|_{L_2(S^{n-1})} \le C\|f\|_{(1+\varepsilon_0)/2}$$
$$\le C\|f\|_{(3-\varepsilon_0)/2},$$

which shows the first assertion, moreover the operator norm of $\mathscr{F}(\lambda)$ is locally bounded for $\lambda>0$. Also by Theorems 2.2 and 2.10 we see that $\|\mathscr{F}(\lambda)f\|_{L_2(S^{n-1})}$ is continuous in $\lambda>0$. Thus to show the continuity of $\mathscr{F}(\lambda)f$ in λ we have only to show that for $\phi\in C^\infty(S^{n-1})$ $(\phi, \mathscr{F}(\lambda)f)_{L_2(S^{n-1})}$ is continuous for $\lambda>0$. However, this follows from Theorem 2.2 and (2.9). Q.E.D.

Now we are in a position of stating the spectral representation theorem for H. Let $\hat{\mathscr{H}}=L_2(\mathbf{R}_+: L_2(S^{n-1}))$ be the Hilbert space of $L_2(S^{n-1})$-valued square integrable functions over $\mathbf{R}_+$. Let the operator $\mathscr{F}$ be defined by $(\mathscr{F}f)(\lambda)\equiv\mathscr{F}(\lambda)f$. Then

Theorem 2.12. (*Spectral Representation for H*).
(1) *$\mathscr{F}$, defined above, is uniquely extended to a partial isometry on $\mathscr{H}=L_2(\mathbf{R}^n)$ with initial set $\mathscr{H}_{ac}(H)$ (the absolutely continuous subspace for H) and final set $\hat{\mathscr{H}}$, which will be denoted by $\mathscr{F}$ also.*
(2) *For a bounded Borel function $\alpha(\lambda)$ defined on $\mathbf{R}$ we have for $f\in\mathscr{H}$*

$$(\mathscr{F}\alpha(H)f)(\lambda)=\alpha(\lambda)(\mathscr{F}f)(\lambda) \quad a.e.\ \lambda>0.$$

(3) *For $f\in\mathscr{H}_{ac}(H)$ the following inversion formula holds:*

$$f=\text{s-}\lim_{N\to\infty}\int_{1/N}^{N}\mathscr{F}(\lambda)^*(\mathscr{F}f)(\lambda)d\lambda.$$

(4) *$\mathscr{F}(\lambda)^*\in\mathbf{B}(L_2(S^{n-1}): L_{2,-(3-\varepsilon_0)/2})$ is an eigenoperator of H with eigenvalue λ in the sense that for any $\phi\in L_2(S^{n-1})$*

$$(-\Delta+V)\mathscr{F}(\lambda)^*\phi=\lambda\mathscr{F}(\lambda)^*\phi$$

holds.

Since this theorem is proved in the same way as Theorem 2.8 of Ikebe [2], we omit the proof.

§3. Proof of Lemma 2.9

As in §2, we let $u=R(\lambda+i0)f$, where $f\in L_{2,(3-\varepsilon_0)/2}$.

Proposition 3.1. *There exists a sequence $\{r_m\}_{m=1}^{\infty}$ tending to infinity such that*

$$r_m^{2-\varepsilon_0}\int_{|x|=r_m}|\mathscr{D}u|^2dS\to 0 \quad \text{as} \quad m\to\infty.$$

Proof. We have by Lemma 2.3

$$\int_1^\infty r^{1-\varepsilon_0}\Big(\int_{|x|=r}|\mathscr{D}u|^2dS\Big)dr<\infty.$$

Thus the lemma readily follows from this inequality.

Proposition 3.2. *Let $\{r_m\}_{m=1}^{\infty}$ be a sequence specified in Proposition* 3.1, *and $\phi \in C^1(S^{n-1})$. Then we have if $r_m < r_n$*

$$|(\mathscr{F}(\lambda, r_m)f - \mathscr{F}(\lambda, r_n)f, \phi)_{L_2(S^{n-1})}|$$
$$\leq C(r_m)(\|\phi\|_{L_2(S^{n-1})} + r_m^{-(2-\varepsilon_0)/2}\|\Lambda^{1/2}\phi\|_{L_2(S^{n-1})}),$$

where the constant $C(r_m)$ is independent of ϕ, and $C(r_m)$ tends to 0 as $m \to \infty$.

Proof. We have only to show the proposition when $\phi \in C^\infty(S^{n-1})$. Now let $\phi \in C^\infty(S^{n-1})$, v and g be defined by (2.5) and (2.6), respectively. By the use of (2.8) we have

$$\int_{r_m<|x|<r_n} (v\bar{f} - g\bar{u})dx + \left(\int_{|x|=r_n} - \int_{|x|=r_m}\right) v\overline{\mathscr{D}_r u}\, dS \tag{3.1}$$
$$= 2i\left(\int_{|x|=r_m} - \int_{|x|=r_n}\right)\frac{\partial K}{\partial r} v\bar{u}\, dS.$$

Since $\|\mathscr{F}(\lambda, r)\|_{L_2(S^{n-1})}$ is uniformly bounded in r (by (2.4)), by using Lemma 2.1 we can estimate the right hand side of (3.1) as follows

$$\left|2i\left(\int_{|x|=r_n} - \int_{|x|=r_m}\right)\frac{\partial K}{\partial r} v\bar{u}\, dS\right| \tag{3.2}$$
$$\geq Const.|(\mathscr{F}(\lambda, r_n)f - \mathscr{F}(\lambda, r_m)f, \phi)_{L_2(S^{n-1})}|$$
$$- Const.r_m^{-\delta}(\sup_{n\geq m}\|\mathscr{F}(\lambda, r_n)f\|_{L_2(S^{n-1})})\|\phi\|_{L_2(S^{n-1})}.$$

The second term of the left hand side of (3.1) is estimated from above as follows

$$\left|\left(\int_{|x|=r_n} - \int_{|x|=r_m}\right) v\overline{\mathscr{D}_r u}\, dS\right| \tag{3.3}$$
$$\leq Const.\left(\sup_{n\geq m}\int_{|x|=r_n}|\mathscr{D}_r u|^2 dS\right)^{1/2}\|\phi\|_{L_2(S^{n-1})}.$$

We shall evaluate the first term of the left hand side of (3.1). By Schwarz' inequality we have

$$\left|\int_{r_m<|x|<r_n} v\bar{f}\, dx\right| \tag{3.4}$$
$$\leq \left(\int_{r_m}^{\infty} r^{-1-\varepsilon_0}\|\phi\|^2_{L_2(S^{n-1})}dr\right)^{1/2}\left(\int_{r_m}^{\infty} r^{n+\varepsilon_0}\|f(r\cdot)\|^2_{L_2(S^{n-1})}dr\right)^{1/2}$$
$$\leq Const.r_m^{-\varepsilon_0/2}\left(\int_{|x|>r_m}(1+r)^{1+\varepsilon_0}|f(x)|^2 dx\right)^{1/2}\|\phi\|_{L_2(S^{n-1})}.$$

To evaluate $\int_{r_m<|x|<r_n} g\bar{u}\, dx$, using (2.7), we split the integral into two parts I_1 and I_2, where

$$I_1=\int_{r_m<|x|<r_n} e^{iK(x,\lambda)}\Big\{\Big(-2\sqrt{\lambda}\frac{\partial Y}{\partial r}+|\nabla_x Y|^2+\tilde{V}\Big)$$
$$+i\Big(\frac{\partial^2 Y}{\partial r^2}+\frac{\Lambda Y}{r^2}\Big)\Big\}\phi(\omega)\bar{u}r^{-(n-1)/2}dx,$$
$$I_2=\int_{r_m<|x|<r_n} e^{iK(x,\lambda)}\Big(-\frac{\Lambda\phi(\omega)}{r^2}$$
$$+2i\widetilde{grad}Y\cdot\widetilde{grad}\phi(\omega)\Big)\bar{u}r^{-(n-1)/2}dx.$$

By Lemma 2.1, we have

$$|I_1|\le Const.\int_{r_m}^{\infty} r^{-1-\delta}\|\mathscr{F}(\lambda, r)f\|_{L_2(S^{n-1})}dr\|\phi\|_{L_2(S^{n-1})} \tag{3.5}$$
$$\le Const.r_m^{-\delta}(\sup_{r\ge r_m}\|\mathscr{F}(\lambda, r)f\|_{L_2(S^{n-1})})\|\phi\|_{L_2(S^{n-1})}.$$

Since

$$\frac{1}{r}\|\Lambda^{1/2}(e^{-iK(r\cdot,\lambda)}u)(r\cdot)\|_{L_2(S^{n-1})}=\|(\mathscr{D}_\omega u)(r\cdot)\|_{L_2(S^{n-1})},$$
$$\|(\widetilde{grad}e^{-iK(x,\lambda)}u)(r\cdot)\|_{L_2(S^{n-1})}=\|(\mathscr{D}_\omega u)(r\cdot)\|_{L_2(S^{n-1})},$$

we have by integration by parts

$$|I_2|\le Const.\int_{r_m}^{\infty} r^{(n-3)/2}\|\Lambda^{1/2}\phi\|_{L_2(S^{n-1})}\|(\mathscr{D}_\omega u)(r\cdot)\|_{L_2(S^{n-1})}dr \tag{3.6}$$
$$+Const.\int_{r_m}^{\infty} r^{-1-\delta}\|\phi\|_{L_2(S^{n-1})}\|\mathscr{F}(\lambda, r)f\|_{L_2(S^{n-1})}dr$$
$$+Const.\int_{r_m}^{\infty} r^{(n-1)/2-\delta}\|\phi\|_{L_2(S^{n-1})}\|(\mathscr{D}_\omega u)(r\cdot)\|_{L_2(S^{n-1})}dr$$
$$\le Const.r_m^{-(2-\varepsilon_0)/2}\Big(\int_{|x|>r_m}(1+r)^{1-\varepsilon_0}|\mathscr{D}u|^2dx\Big)^{1/2}\|\Lambda^{1/2}\phi\|_{L_2(S^{n-1})}$$
$$+Const.r_m^{-\delta}(\sup_{r\ge r_m}\|\mathscr{F}(\lambda, r)f\|_{L_2(S^{n-1})})\|\phi\|_{L_2(S^{n-1})}$$
$$+Const.r_m^{-\varepsilon_0/2}\Big(\int_{r>r_m}(1+r)^{1-\varepsilon_0}|\mathscr{D}u|^2dx\Big)^{1/2}\|\phi\|_{L_2(S^{n-1})}.$$

In view of (3.1) to (3.6), we get the present lemma.

Proof of Lemma 2.9.

Let $\{r_m\}_{m=1}^{\infty}$ be a sequence specified in Proposition 3.1. Then we have

$$\|\Lambda^{1/2}\mathscr{F}(\lambda, r_m)f\|_{L_2(S^{n-1})}=r_m^{(n+1)/2}\|(\mathscr{D}_\omega u)(r\cdot)\|_{L_2(S^{n-1})} \tag{3.7}$$
$$=o(r_m^{\varepsilon_0/2})$$

as $m\to\infty$.

Using Proposition 3.2 we have by choosing $\phi=\mathscr{F}(\lambda, r_m)f$

$$(3.8)\quad |(\mathscr{F}(\lambda, r_m)f-\mathscr{F}(\lambda, r_n)f, \mathscr{F}(\lambda, r_m)f)_{L_2(S^{n-1})}|$$
$$\leq C(r_m)(\|\mathscr{F}(\lambda, r_m)f\|_{L_2(S^{n-1})}+r_m^{-(2-\varepsilon_0)/2}\|\Lambda^{1/2}\mathscr{F}(\lambda, r_m)f\|_{L_2(S^{n-1})}.$$

Since $\mathscr{F}(\lambda, r_n)f$ converges weakly to $\mathscr{F}(\lambda)f$ by Lemma 2.8, we have by letting n tend to infinity in (3.8)

$$(3.9)\quad |\|\mathscr{F}(\lambda, r_m)f\|^2_{L_2(S^{n-1})}-(\mathscr{F}(\lambda)f, \mathscr{F}(\lambda, r_m)f)_{L_2(S^{n-1})}|$$
$$\leq C(r_m)(\|\mathscr{F}(\lambda, r_m)f\|_{L_2(S^{n-1})}+r_m^{-(2-\varepsilon_0)/2}\|\Lambda^{1/2}\mathscr{F}(\lambda, r_m)f\|_{L_2(S^{n-1})}).$$

In view of (3.7), we have by letting m tend to infinity in (3.8)

$$\lim_{m\to\infty}\|\mathscr{F}(\lambda, r_m)f\|_{L_2(S^{n-1})}=\|\mathscr{F}(\lambda)f\|_{L_2(S^{n-1})},$$

from which we can conclude the strong convergence of $\mathscr{F}(\lambda, r_m)f$, since it converges weakly to $\mathscr{F}(\lambda)f$.

§4. Solution of the eikonal equation

In this section we shall find a solution of the eikonal (Hamilton-Jacobi) equation

$$(4.1)\qquad |\nabla_x K(x, \lambda)|^2+V(x)=\lambda.$$

Here we assume on the potential the following condition:

$V(x)$ is a real-valued C^∞-function of $x\in\mathbf{R}^n$ such that $D_x^\alpha V(x)=O(|x|^{-m(|\alpha|)})$ as $|x|\to\infty$, where

$$m(k)=\begin{cases} k+\delta & (0\leq k\leq 3)\\ 3+\delta+2(k-3)/3 & (k\geq 4).\end{cases}$$

Theorem 4.1. *There exists a real-valued function $Y(x, \lambda)\in C^\infty(\mathbf{R}^n\times\mathbf{R}_+)$ satisfying*

(1) *For any compact set $\Lambda\subset\mathbf{R}_+$, there exists a constant $R=R(\Lambda)$ such that for $\lambda\in\Lambda$, $|x|>R$*

$$2\sqrt{\lambda}\frac{\partial Y}{\partial r}(x, \lambda)=V(x)+|\nabla_x Y(x, \lambda)|^2, \qquad (r=|x|).$$

(2)
$$|D_x^\alpha D_\lambda^k Y(x, \lambda)|\leq C_{\alpha,k}(1+r)^{1-|\alpha|-\delta} \qquad (|\alpha|+k\leq 3,\ k\leq 2),$$
$$|D_\lambda^k Y(x, \lambda)|\leq C_k(1+r)^{\mu(k)+1} \qquad (k\geq 3),$$
$$|D_x^\alpha D_\lambda^k Y(x, \lambda)|\leq C_{\alpha,k}(1+r)^{\mu(|\alpha|-1+k)-|\alpha|+1} \qquad (|\alpha|+k\geq 4,\ |\alpha|\geq 1),$$

where $\mu(k)=\max\{0, k+1-m(k+1)\}$, and the constants C_k, $C_{\alpha,k}$ are independent of λ in a compact set in $\mathbf{R}_+$.

Before proving Theorem 4.1, we shall give some remarks on the assumption of the potential $V(x)$. The above theorem is the same as Lemma 2.1 except for the differentiability of $Y(x, \lambda)$ and the estimates for its higher order derivatives. As for the spectral representations, it is sufficient to assume that $Y(x, \lambda)$ is a C^3-function, since Lemma 2.3 which is crucial to the spectral representations can be proved under the above *weak* condition (see Appendix of Isozaki [5]). However, as has been stated in the introduction, the results of this paper is utilized in the proof of the completeness of modified wave operators (see [4]), in the case of which we need the more differentiability of $Y(x, \lambda)$. By this reason we shall prove here the *stronger* version of Lemma 2.1 (which is really Theorem 4.1). Our assumption on V in this section is stronger than that of § 1. However, the proof of Lemma 2.1, which is based on the Assumption in § 1, will be obtained in the course of the proof of Theorem 4.1.[3)]

First let us begin with the following Hamilton's canonical equation of motion

$$\text{(4.2)} \qquad \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial \xi}, \\ \dfrac{d\xi}{dt} = -\dfrac{\partial H}{\partial x}, \end{cases}$$

where $H = H(x, \xi) = |\xi|^2 + V(x)$. The following lemma is essentially due to Hörmander [1].

Lemma 4.2. *Let K be a compact set in $\mathbf{R}^n - \{0\}$. Then if $1/T$ and ε are sufficiently small, the equation* (4.2) *has a solution $x = x(t, \zeta, p)$, $\xi = \xi(t, \zeta, p)$ for all $t > T$ with an arbitrary Cauchy data $x(T, \zeta, p) = T\zeta$, $\xi(T, \zeta, p) = p$ such that $\zeta \in K$, $|\zeta - 2p| < \varepsilon$*[4)] *having the following properties*:

(1) $|x(t, \zeta, p)| \geq \varepsilon t$.

(2) $|D^\alpha_{\zeta,p}(\xi(t, \zeta, p) - p)| \leq CT^{-\delta}$ $\quad (|\alpha| \leq 1)$.

(3) $|D^\alpha_{\zeta,p}\xi(t, \zeta, p)| \leq Ct^{\mu(|\alpha|)}$ $\quad$ *for any α, where* $\mu(k) = \max\{0, k+1-m(k+1)\}$.

(4) $|D^k_t D^\alpha_{\zeta,p}\xi(t, \zeta, p)| \leq Ct^{-m(k+|\alpha|)+|\alpha|}$ $\quad$ *for any α and k $(k \geq 1)$.*

Moreover, we also have the more refined estimates:

(5) *There exists $\xi_\infty(\zeta, p)$ such that it is twice continuously differentiable with respect to ζ and p, and $\xi(t, \zeta, p) \to \xi_\infty(\zeta, p)$ as $t \to \infty$.*

(6) $|D^k_t D^\alpha_{\zeta,p}(x(t, \zeta, p)/t - 2\xi_\infty(\zeta, p))| \leq Ct^{-k-\delta}$ $\quad (k + |\alpha| \leq 2)$.

(7) $|D^k_t D^\alpha_{\zeta,p}(x(t, \zeta, p)/t - 2\xi(t, \zeta, p))| \leq Ct^{-k-\delta}$ $\quad (k + |\alpha| \leq 2)$.

Proof. The assertions (1), (2) and (3) have been shown in Lemma 3.7 of Hörmander [1]. Let us prove (4) by induction. By (4.2) we have

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3) The relavance of our two assumptions on $V(x)$ is seen in Lemma 3.3 of Hörmander [1].

4) In this condition, the number 2 is not essential at all. The crucial fact is that the directions of ζ and p are very close.

$$D_t D^{\alpha}_{\zeta,p}\xi = -\sum_{j=1}^{|\alpha|}\left(D^j_x \frac{\partial V}{\partial x}\right)\phi_j, \tag{4.3}$$ [5]

where ϕ_j consists of a finite sum of the following terms

$$D^{k_1}_{\zeta,p}x\cdots D^{k_j}_{\zeta,p}x, \quad k_i\geq 1, \quad k_1+\cdots+k_j=|\alpha|.$$

(When $|\alpha|=0$, we set $j=0$ and $\phi_j=1$.) Now, suppose $\left(D^j_k \frac{\partial V}{\partial x}\right)\phi_j$ is estimated by a constant times t to the power $|\alpha|-m(1+|\alpha|)$. (This is true when $|\alpha|=0$.) Then by the condition of the potential V and the estimate (3), when we differentiate the above equality (4.3) with respect to ζ or p, the greatest contributions arise from the terms $D_{\zeta,p}\left(D^j_x \frac{\partial V}{\partial x}\right)\phi_j$ $(j=|\alpha|)$, and they are estimated by a constant times t to the power

$$\begin{aligned}&-m(1+|\alpha|)+|\alpha|+\{-m(2+|\alpha|)+m(1+|\alpha|)\}+1\\ &=-m(1+|\alpha|+1)+|\alpha|+1.\end{aligned}$$

Thus we see that (4) is true for any α and $k=1$. Again by the use of (4.2)

$$D^{N+1}_t\xi = -\sum_{i=1}^{N}\left(D^j_x \frac{\partial V}{\partial x}\right)\psi_j, \tag{4.4}$$

Where ψ_j is a finite sum of the following terms

$$D^{k_1}_t x\cdots D^{k_j}_t x, \quad k_i\geq 1, \quad k_1+\cdots+k_j=N.$$

(When $N=0$, we set $j=0$ and $\psi_j=1$). Now, let us suppose that $\left(D^j_x \frac{\partial V}{\partial x}\right)\psi_j$ is bounded by t raised to the power $-m(N+1)$. (This is true when $N=0$.) Then when we differentiate (4.1) with respect to t, the greatest contributions arise from the terms $D_t\left(D^N_x \frac{\partial V}{\partial x}\right)\psi_N$, which will be estimated from above by a constant times t to the power $+m(N+1)+\{m(N+1)-m(N+2)\}=-m(N+2)$. Thus we see that (4) is true for any $k\geq 1$ and $|\alpha|=0$. The proof of (4) for any α is obtained in a similar way.

Now, by (4.2) we have

$$\xi(t,\zeta,p)-\xi(r,\zeta,p) = -\int_r^t \frac{\partial V}{\partial x}(x(s,\zeta,p))ds,$$

from which in view of (1), we see that there exists $\xi_\infty(\zeta,p)$ such that $\xi(t,\zeta,p)\to\xi_\infty(\zeta,p)$ as $t\to\infty$, and

$$\xi(t,\zeta,p)-\xi_\infty(\zeta,p) = \int_t^\infty \frac{\partial V}{\partial x}(x(s,\zeta,p))ds.$$

Using this equality and (3) we see $\xi_\infty(\zeta,p)$ is twice continuously differentiable and satisfies

$$|D^k_t D^{\alpha}_{\zeta,p}(\xi(t,\zeta,p)-\xi_\infty(\zeta,p))|\leq Ct^{-k-\delta} \qquad (k+|\alpha|\leq 2). \tag{4.5}$$

5) We denote by D^j_x the differential operators $D^{\beta}_x(|\beta|=j)$.

Since $x(t, \zeta, p)-T\zeta=2\int_T^t \xi(s, \zeta, p)ds$, we have

$$x(t, \zeta, p)/t-2\xi_\infty(\zeta, p)=T\zeta/t-2T\xi_\infty/t+\int_T^t (\xi-\xi_\infty)ds/t. \tag{4.6}$$

In view of (4.5) and (4.6), we get (6). The assertion (7) is obtained from (4.5) and (6). Q.E.D.

Now, for $\omega\in S^{n-1}$ let $p(\omega:\lambda)=\sqrt{\lambda-V(T\omega)}\omega$, and consider the equation

$$\begin{cases} \dfrac{dx}{dt}=\dfrac{\partial H}{\partial \xi}, & t\geq T, \\ \dfrac{d\xi}{dt}=-\dfrac{\partial H}{\partial x}, & t\geq T, \\ x(T)=T\omega, \\ \xi(T)=p(\omega:\lambda), \end{cases} \tag{4.7}$$

where we restrict λ on a compact set $\Lambda\subset\mathbf{R}_+$. In view of Lemma 4.2, by taking T sufficiently large, we see that the solution $x=x(t, \omega:\lambda)$, $\xi=\xi(t, \omega:\lambda)$ of (4.5) exists for $t\geq T$. Let $U(t, \omega:\lambda)=x(t, \omega:\lambda)$. We show by choosing R sufficiently large that the map $(t, \omega)\mapsto U(t, \omega:\lambda)$ gives a diffeomorphism for $t\geq R$. For this purpose we first show the following lemma. Recall that by Lemma 4.2, $\xi(t, \omega:\lambda)\to\xi_\infty=\xi_\infty(\omega:\lambda)$ as $t\to\infty$.

Lemma 4.3. *For sufficiently large T and t, the sets $\{p(\omega:\lambda): \omega\in S^{n-1}\}$, $\{x(t, \omega:\lambda): \omega\in S^{n-1}\}$ and $\{\xi_\infty(\omega:\lambda): \omega\in S^{n-1}\}$ are compact hypersurfaces of codimension 1, each diffeomorphic to S^{n-1}. Moreover $\{\xi_\infty(\omega:\lambda): \omega\in S^{n-1}\}$ is the sphere of the radius $\sqrt{\lambda}$ with the center at the origin.*

Proof. We know in general the following fact: Suppose K and K_1 are open sets in $\mathbf{R}^n$ and Φ is a diffeomorphism from K to K_1. Then if S is a compact hypersurface of codimension 1 in K, the image of S by Φ is also an $n-1$ dimensional compact hypersurface in K_1. Now, since $D_\omega(p(\omega:\lambda)-\sqrt{\lambda}\omega)=O(T^{-\delta})$, we see taking T sufficiently large that $p(\omega:\lambda)$ gives a diffeomorphism from an open set containing the unit sphere to an open set containing the sphere of the radius $\sqrt{\lambda}$. Thus we see that $\{p(\omega:\lambda): \omega\in S^{n-1}\}$ is a compact hypersurface of codimension 1 diffeomorphic to S^{n-1}. By Lemma 4.2 (2) we have

$$D_\omega(\xi_\infty(\omega:\lambda)-p(\omega:\lambda))=O(T^{-\delta}),$$

from which we see that $\{\xi_\infty(\omega:\lambda): \omega\in S^{n-1}\}$ is diffeomorphic to $\{p(\omega:\lambda): \omega\in S^{n-1}\}$ by choosing T sufficiently large. By Lemma 4.2 (7), by taking t sufficiently large, we see that $\{x(t, \omega:\lambda): \omega\in S^{n-1}\}$ is diffeomorphic to $\{\xi_\infty(\omega:\lambda): \omega\in S^{n-1}\}$. We show the final assertion. Since *the energy conservation law* holds;

$$|\xi(t, \omega:\lambda)|^2+V(x(t, \omega:\lambda))=\lambda, \quad \omega\in S^{n-1},$$

taking account of Lemma 4.2 (1), we see by letting t tend to infinity that $|\xi_\infty(\omega: \lambda)|^2 = \lambda$. We also see that $\{\xi_\infty(\omega: \lambda): \omega \in S^{n-1}\}$ is open and closed in the sphere of radius $\sqrt{\lambda}$. Thus we have only to use the connectedness of the sphere to get the result. Q.E.D.

Now, let

$$F(t, \omega: \lambda) = T\omega - 2T\xi_\infty(\omega: \lambda) + \int_T^t (\xi(s, \omega: \lambda) - \xi_\infty(\omega: \lambda))ds.$$

By (4.5) and (4.6) we have

$$(4.8) \quad \begin{cases} x(t, \omega: \lambda) = 2\xi_\infty(\omega: \lambda)t + F(t, \omega: \lambda), \\ |D_\lambda^j D_t^k D_\omega^\alpha F(t, \omega: \lambda)| \le Ct^{1-k-\delta} \qquad (j+k+|\alpha| \le 2). \end{cases}$$

Since the map $\omega \mapsto \xi_\infty(\omega: \lambda)$ is a diffeomorphism, we can express ω as a function of ξ_∞, $\omega = \omega(\xi_\infty: \lambda)$. Let $f(t, \xi_\infty: \lambda) = F(t, \omega(\xi_\infty: \lambda): \lambda)$. We have by (4.8)

$$(4.9) \quad \begin{cases} x(t, \omega(\xi_\infty: \lambda): \lambda) = 2\xi_\infty t + f(t, \xi_\infty: \lambda), \\ |D_\lambda^j D_t^k D_{\xi_\infty}^\alpha f(t, \xi_\infty: \lambda)| \le Ct^{1-k-\delta} \qquad (j+k+|\alpha| \le 2). \end{cases}$$

Now let us enter into a heuristic argument. If there exists a function $y(x, \lambda)$ such that $x = U(|y|, y/|y|: \lambda)$, it must satisfy

$$x = 2\xi_\infty(y/|y|: \lambda)|y| + f(|y|, \xi_\infty(y/|y|: \lambda): \lambda).$$

Let $|x| = r$, $x/r = z$ and $\zeta = 2\xi_\infty(y/|y|:\lambda)|y|/r$. Then we have

$$(4.10) \quad \begin{cases} z = \zeta + f(r|\zeta|/(2\sqrt{\lambda}), \sqrt{\lambda}\zeta/|\zeta|: \lambda)/r, \\ |D_\lambda^j D_\zeta^\alpha D_r^k (f(r|\zeta|/(2\sqrt{\lambda}), \sqrt{\lambda}\zeta/|\zeta|: \lambda)/r)| \le Cr^{-k-\delta} \qquad (j+k+|\alpha| \le 2). \end{cases}$$

Lemma 4.4. *Let K be an open set containing the unit sphere. Suppose $g(\zeta: r, \lambda)$ be a C^2-function for $\zeta \in K$, $r > 0$, $\lambda \in \Lambda$ (Λ is a compact set in $\mathbf{R}_+$), such that*

$$D_\lambda^j D_\zeta^\alpha D_r^k g(\zeta: r, \lambda) = O(r^{-k-\delta}) \quad \text{as} \quad r \to \infty \quad (j+k+|\alpha| \le 2).$$

Then for any compact set K_1 in K, taking R_1 sufficiently large, there exists a C^2-function $\zeta(z: r, \lambda)$ defined for $z \in K_1$, $r > R_1$, $\lambda \in \Lambda$ such that

$$z = \zeta(z: r, \lambda) + g(\zeta(z: r, \lambda): r, \lambda), \quad z \in K_1, \quad r > R_1, \lambda \in \Lambda,$$

$$D_\lambda^j D_z^\alpha D_r^k (\zeta(z: r, \lambda) - z) = O(r^{-k-\delta}) \quad \text{as} \quad r \to \infty \qquad (j+k+|\alpha| \le 2).$$

Proof. This lemma will be proved by the method of successive approximation. For the details see Lemma 4.1 of Ikebe-Isozaki [3].

Now, let us return to the equation (4.10). Let K_1 be a compact set containing the unit sphere excluding the origin. By lemma 4.4, taking R_1 sufficiently large, we see that there exists a C^2-function $\zeta(z: r, \lambda)$, ($z \in K_1$, $r > R_1$, $\lambda \in \Lambda$) such that

$$(4.11)\quad \begin{cases} z=\zeta(z:r,\lambda)+f(r|\zeta|/(2\sqrt{\lambda}),\ \sqrt{\lambda}\zeta/|\zeta|:\lambda), & z\in K_1,\quad r>R_1,\ \lambda\in\Lambda, \\ |D_\lambda^j D_z^\alpha D_r^k(\zeta(z:r,\lambda)-z)|\le Cr^{-k-\delta} & (j+k+|\alpha|\le 2). \end{cases}$$

Let $t(x, \lambda)=|x|\,|\zeta(x/|x|:|x|, \lambda)|/(2\sqrt{\lambda})$, and $\omega(x, \lambda)$ be the image of $\sqrt{\lambda}\zeta(x/|x|:|x|, \lambda)/|\zeta(x/|x|:|x|, \lambda)|$ by the map $\xi_\infty(\omega:\lambda)|\to\omega$. Define $y(x, \lambda)=t(x, \lambda)\omega(x. \lambda)$. Then we see by (4.11) that for $|x|>R_1$ and $\lambda\in\Lambda$ $y(x, \lambda)$ satisfies

$$(4.12)\quad \begin{cases} x=U(|y|, y/|y|:\lambda), \\ D_\lambda^k D_x^\alpha(t(x,\lambda)-r/(2\sqrt{\lambda}))=O(r^{1-|\alpha|-\delta}), \quad k+|\alpha|\le 2, \\ D_\lambda^k D_x^\alpha\omega(x,\lambda)=O(r^{-|\alpha|}), \quad k+|\alpha|\le 2, \end{cases}$$

where $r=|x|$.

We have shown that $y(x, \lambda)$ is a C^2-function of x and λ. However, since $U(t, \omega:\lambda)$ is a C^∞-function, we see that $y(x, \lambda)$ is a C^∞-function of x and λ. We must get the estimate of $D_x^\alpha D_\lambda^k y(x, \lambda)$. The following Lemma 4.5 will be proved by induction.

Lemma 4.5. *Let $\Phi(y, \lambda)=U(|y|, y/|y|:\lambda)$. Then we have $|D_y^\alpha D_\lambda^k\Phi(y, \lambda)|\le C|y|^{\mu(|\alpha|+k)-|\alpha|+1}$ for every α and k.*

Lemma 4.6. *$|D_x^\alpha D_\lambda^k y(x, \lambda)|\le Cr^{\mu(|\alpha|+k)-|\alpha|+1}$ for any α and k $(r=|x|)$.*

Proof. We differentiate $\Phi(y(x, \lambda):\lambda)=x$ by x to obtain $(D_y\Phi)\cdot D_x y=1$. Since we know by (4.12) $D_x y(x, \lambda)$ is bounded, we see $((D_y\Phi)(y(x, \lambda):\lambda))^{-1}$ is bounded. By a further differentiation we have

$$(4.13)\quad \sum_{j=2}^{|\alpha|}(D_y^j\Phi)\phi_j+(D_y\Phi)\cdot D_x^\alpha y=0, \quad (|\alpha|\ge 2),$$

where ϕ_j is a finite sum of the following terms

$$D_x^{k_1}y\cdots D_x^{k_j}y, \quad k_i\ge 1, \quad k_1+\cdots+k_j=|\alpha|.$$

Now, suppose $(D_y^j\Phi)\phi_j$ $(j\ge 2)$ is estimated by a constant times r to the power $\mu(|\alpha|)-|\alpha|+1$. (This is true when $|\alpha|=2$ by a direct calculation.) When we differentiate (4.13) with respect to x, by Lemma 4.5 and our assumption, we see that the greatest contributions arise from the terms $D_x(D_y^j\Phi)\phi_j$ $(j=|\alpha|)$, which are estimated from above by a constant times r raised to the power

$$\mu(|\alpha|)-|\alpha|+1+[\{\mu(|\alpha|+1)-(|\alpha|+1)+1\}-\{\mu(|\alpha|)-|\alpha|+1\}]$$
$$=\mu(|\alpha|+1)-(|\alpha|+1)+1.$$

Thus $D_x^{|\alpha|+1}y=O(r^{\mu(|\alpha|+1)-(|\alpha|+1)+1})$. We have thus shown that the present lemma is true for any α and $k=0$. The assertion for any k is proved in a similar way.

Summing up, we have obtained the following lemma.

Lemma 4.7. *For sufficiently large R_1, there exists a C^∞-function $y(x, \lambda)$ $(|x|\ge R_1, \lambda\in\Lambda)$ such that*

$$x=U(|y|, y/|y|: \lambda) \quad for \quad |x|\geq R_1, \lambda\in\Lambda,$$

$$D_x^\alpha D_\lambda^k y(x, \lambda)=O(r^{\mu(|\alpha|+k)-|\alpha|+1}) \quad for\ any\ \alpha\ and\ k,$$

$$D_x^\alpha D_\lambda^k(|y(x, \lambda)|-r/(2\sqrt{\lambda}))=O(r^{1-|\alpha|-\delta}), \quad k+|\alpha|\leq 2,$$

$$D_x^\alpha D_\lambda^k(y(x, \lambda)/|y(x, \lambda)|)=O(r^{-|\alpha|}), \quad k+|\alpha|\leq 2,$$

for $r=|x|\geq R_1$, $\lambda\in\Lambda$.

Now, let $x_0=x(R, \omega: \lambda)$, $p_0=\xi(R, \omega: \lambda)$, where $x(t, \omega: \lambda)$ and $\xi(t, \omega: \lambda)$ is the solution of (4.7). Let $M_R=\{x_0=x(R, \omega: \lambda): \omega\in S^{n-1}\}$, which is a compact hypersurface of codimension 1 by Lemma 4.3, and consider the equation

$$(4.14)\qquad \begin{cases} \dfrac{dx}{dt}=\dfrac{\partial H}{\partial\xi} & (t\geq R),\\[2ex] \dfrac{d\xi}{dt}=-\dfrac{\partial H}{\partial x} & (t\geq R),\\[2ex] x(R)=x_0, & \\ \xi(R)=p_0. & \end{cases}$$

By Lemma 4.7, taking R sufficiently large, the solution $x(t)$, $\xi(t)$ of (4.14) exists for $t\geq R$, and the map $(t, \omega)|\rightarrow x(t)$ defines a diffeomorphism from $\{(t, \omega): t\geq R, \omega\in S^{n-1}\}$ to $\{x: x$ is outside of $M_R\}$. Let

$$u_0=\int_T^R \xi(t, \omega: \lambda)\frac{\partial x}{\partial t}(t, \omega: \lambda)dt+\sqrt{\lambda}T,$$

where $x(t, \omega: \lambda)$, $\xi(t, \omega: \lambda)$ is the solution of (4.7), and consider the Cauchy problem

$$(4.15)\qquad \begin{cases} |\nabla_x K|^2+V(x)=\lambda & \text{outside}\quad M_R,\\ K(x, \lambda)=u_0 & \text{on}\quad M_R. \end{cases}$$

The corresponding characteristic equation is given by (4.14). By what we have proved (Lemma 4.7), we see that the solution of (4.15) really exists and is given by

$$K(x, \lambda)=\int_R^t \xi(s)\frac{\partial x}{\partial s}(s)ds+u_0.$$

Let $Y(x, \lambda)=\sqrt{\lambda}|x|-K(x, \lambda)$. We show that this function $Y(x, \lambda)$ has the properties enumerated in Theorem 4.1. Since $\nabla_x K(x, \lambda)=\xi(t)$ by our construction, we have

$$\nabla_x Y(x, \lambda)=\sqrt{\lambda}x/|x|-\xi(t).$$

By Lemma 4.2 (7) and Lemma 4.7, we have by a direct calculation

$$D_\lambda^k D_x^\alpha \nabla_x Y(x, \lambda)=O(r^{-|\alpha|-\delta}) \qquad (k+|\alpha|\leq 2),$$

which shows the first assertion of Theorem 4.1 (2). Let us show the remainder assertions. By the notation of Lemma 4.2, $\xi(t)=\xi(t, \omega, p)$, where

$p=\sqrt{\lambda-V(Ty/|y|)}y/|y|$. Let $\Psi(y, \lambda)=\xi(t, \omega, p)$, where $t=|y|$, $\omega=y/|y|$, $p=\sqrt{\lambda-V(Ty/|y|)}y/|y|$. Then by induction we see that

(4.16) $$|D_y^\alpha D_\lambda^k \psi(y, \lambda)| \leq C|y|^{\mu(|\alpha|+k)-|\alpha|} \quad \text{for any } \alpha \text{ and k.}$$

Let $\Phi(x, \lambda)=\Psi(y(x, \lambda), \lambda)$, where $y(x, \lambda)$ is inverse function given in Lemma 4.7. Then we have by Lemma 4.7 and (4.16)

$$|D_x^\alpha D_\lambda^k \Phi(x, \lambda)| \leq Cr^{\mu(|\alpha|+k)-|\alpha|} \qquad (r=|x|).$$

Clearly $D_x^\alpha D_\lambda^k(\sqrt{\lambda}x/|x|)=O(r^{-|\alpha|})$. Thus we see that

$$|D_x^\alpha D_\lambda^k \nabla_x Y(x, \lambda)| \leq Cr^{\mu(|\alpha|+k)-|\alpha|},$$

which proves the remainder assertions of Theorem 4.1 (2) when $|\alpha|\geq 1$. The case for $|\alpha|=0$ will be proved by integrating $D_x D_\lambda^k Y(x, \lambda)$.

Now, let Λ_j $(j=1, 2,...)$ be open sets in $\mathbf{R}_+$ such that $\Lambda_1 \subset \Lambda_2 \subset \cdots \mathbf{R}_+$ and the closure of Λ_j is also contained in the following one. By the above results there exists a solution $K_1(x, \lambda)$ of eikonal equation (4.1) for $|x|\geq R_1$ (R_1 sufficiently large), $\lambda \in \Lambda_1$. Let $\tilde{\Lambda}_1$ be an open set in Λ_1 such that its closure is also contained in Λ_1. Let $\phi_1(\lambda)\in C_0^\infty(\Lambda_1)$ such that $\phi_1(\lambda)=1$ for $\lambda \in \tilde{\Lambda}_1$, and $S_T=\{x\in \mathbf{R}^n: |x|=T\}$. We consider the function $v(\omega, \lambda)$ on S_T defined by

$$v(\omega, \lambda)=\phi_1(\lambda)K_1(T\omega, \lambda)+(1-\phi_1(\lambda))\sqrt{\lambda}T \qquad (T\geq R_1).$$

We shall find a function $p=(p_1,..., p_n)$ on S_T such that

$$p_1^2+\cdots+p_n^2+V(x)=\lambda \qquad (\lambda\in\Lambda_2),$$

and as an 1-form on S_T

$$dv=\sum_{j=1}^n p_j dx_j.$$

Choosing T sufficiently large we see that such a function p really exists and has the property

$$p=p(\omega, \lambda)=\sqrt{\lambda}\omega+O(T^{-\delta}) \qquad (\omega\in S^{n-1}).$$

Let us consider the equation

(4.17) $$\begin{cases} \dfrac{dx}{dt}=\dfrac{\partial H}{\partial \xi} & t\geq T, \\ \dfrac{d\xi}{dt}=-\dfrac{\partial H}{\partial x} & t\geq T, \\ x(T)=T\omega & \omega\in S^{n-1}, \\ \xi(T)=p(\omega, \lambda) & \omega\in S^{n-1}, \end{cases}$$

where $p(\omega, \lambda)$ is the function we have just constructed. By Lemma 4.2, the solution $x=x(t, \omega, p)$, $\xi=\xi(t, \omega, p)$ of (4.17) really exists for $t\geq T$ and by the similar arguments we have given in the proof of Lemma 4.3, we see that the set

$M_R=\{x(R, \omega, p)\colon \omega\in S^{n-1}\}$ is a compact hypersurface of codimension 1 by choosing R sufficiently large. Let

$$u_1=\int_T^R \xi(t, \omega, p)\frac{\partial x}{\partial t}(t, \omega, p)dt+v(\omega, \lambda),$$

and consider the Cauchy problem

$$(4.18)\qquad \begin{cases} |\nabla_x K|^2+V(x)=\lambda & \text{outside } M_R,\\ K(x, \lambda)=u_1 & \text{on } M_R,\end{cases}$$

where $\lambda\in\Lambda_2$. By the similar arguments we have just given above, by taking R sufficiently large, we see that the solution $K_2(x, \lambda)$ of (4.18) exists, moreover $K_2(x, \lambda) =K_1(x, \lambda)$ for $\lambda\in\tilde{\Lambda}_1$, $|x|\geq R_2$ (R_2 sufficiently large). Repeating these arguments we see that there exist such constants R_j that there exists a solution $K(x, \lambda)$ of (4.1) in the region $G=\bigcup_i\{x\colon |x|\geq R_j\}\times\tilde{\Lambda}_j$. Outside G we continue $K(x, \lambda)$ in a C^∞-fashion. We have thus completed the proof of Theorem 4.1.

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