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Irreducible Banach representations of locally compact groups of a certain type

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Introduction

Let G be a locally compact unimodular group, and $\{\mathfrak{H}, T_x\}$ a topologically irreducible representation of G on a complete locally convex topological vector space $\mathfrak{H}$. If there exists a compact subgroup K of G and an equivalence class δ of irreducible representations of K such that $\{\mathfrak{H}, T_x\}$ contains δ finitely many times, then $\{\mathfrak{H}, T_x\}$ is called "nice". Let χ_δ be the normalized trace of δ and du the normalized Haar measure on K, and put $\mathfrak{H}(\delta)=E(\delta)\mathfrak{H}$ where $E(\delta)=\int_K T_u\overline{\chi_\delta}(u)du$. If the multiplicity of δ in $\{\mathfrak{H}, T_x\}$ is p, then the function

$$\phi_\delta(x)=\mathrm{trace}\,[E(\delta)T_x]$$

on G is called a spherical function of type δ of height p. Two topologically irreducible nice representations $\{\mathfrak{H}, T_x\}$ and $\{\mathfrak{H}', T'_x\}$ of G are called spherical-function-equivalent (or SF-equivalent) if there exists a common spherical function corresponding to both representations. In fact, this is an equivalence relation (see Theorem 3). If G is a connected unimodular Lie group and if $\{\mathfrak{H}, T_x\}$, $\{\mathfrak{H}', T'_x\}$ are K-finite Banach representations for a compact analytic subgroup K, then they are SF-equivalent if and only if infinitesimally equivalent (Theorem 13 in [4]).

Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible nice representation of G on a Banach space $\mathfrak{H}$. Put $\rho(x)=\|T_x\|$ and denote by dx a Haar measure on G. Then the set $L_\rho(G)$ of all functions f on G satisfying

$$\|f\|_\rho=\int_G |f(x)|\rho(x)dx<+\infty$$

is an algebra with the convolution product. By the assumption, there exists a compact subgroup K and an equivalence class δ of irreducible representations of K such that $0<\dim\mathfrak{H}(\delta)<+\infty$. For a non-zero vector $a\in\mathfrak{H}(\delta)$, put

$$\mathfrak{H}_\rho=\{T_f a\,;\, f\in L_\rho(G)\}$$

where $T_f=\int_G T_x f(x)dx$. This is a G-invariant dense subspace of $\mathfrak{H}$, and independent of the choice of such K, δ, and a (cf. Lemma 4 in [2]).

Assume that $G=S\cdot K$, $S\cap K=\{1\}$, where S is a closed subgroup and K a compact subgroup of G, and that the decomposition $x=su$ ($s\in S$, $u\in K$) is continuous. Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible nice representation of G on a Banach space $\mathfrak{H}$ which contains an equivalence class δ of irreducible representations of K finitely many times. Then our main theorem consists of two assertions. The first is that there exists a topologically irreducible representation Λ of S on a Banach space such that $\{\mathfrak{H}, T_x\}$ is SF-equivalent to a constituent of the induced representation of G from Λ. The other is that one of such representations Λ of S is obtained as follows; we take a non-trivial maximal $L_\rho(S)$-invariant subspace $\mathscr{K}$ of $\mathfrak{H}_\rho$ with $\rho(x)=\|T_x\|$ (the existence of such $\mathscr{K}$ will be proved in this paper), and introduce a suitable topology into $\mathfrak{H}_\rho/\mathscr{K}$ with respect to which it becomes a Banach space. Since it is proved that $\mathscr{K}$ is S-invariant, we obtain the naturally defined representation of S on $\mathfrak{H}_\rho/\mathscr{K}$. This representation is one of those we want.

§1. Representations of the algebra $L^\circ(\delta)$ corresponding to those of G

Let G be a locally compact unimodular group, K a compact subgroup of G, and $\hat{K}$ the set of all equivalence classes of irreducible representations of K. Let δ be an element of $\hat{K}$ with degree d. Fix an irreducible unitary matricial representation $D(u)$ of K belonging to δ, and denote by $d_{ij}(u)$ its (i, j)-matrix element. Put $\chi_\delta(u)=d\cdot\mathrm{trace}\,D(u)$. We shall denote by $L(G)$ the algebra of all continuous functions on G with compact supports, and, for every function $f\in L(G)$, define

$$f^\circ(x)=\int_K f(uxu^{-1})du, \qquad f*\overline{\chi_\delta}(x)=\int_K f(xu^{-1})\overline{\chi_\delta}(u)du,$$

$$\overline{\chi_\delta}*f(x)=\int_K f(u^{-1}x)\overline{\chi_\delta}(u)du,$$

where du is the normalized Haar measure on K. We shall regard the algebra $L(G)$ to be endowed with the usual inductive topology generated by Banach spaces $L_C(G)$ of all continuous functions with supports in compact subsets $C\subset G$ with supremum norm. Then the sets $L^\circ(G)=\{f^\circ; f\in L(G)\}$, $L(\delta)=\{\overline{\chi_\delta}*f*\overline{\chi_\delta}; f\in L(G)\}$, and $L^\circ(\delta)=L^\circ(G)\cap L(\delta)$ are closed subalgebras of $L(G)$.

Let $\mathfrak{H}$ be a complete locally convex topological vector space, and $\{\mathfrak{H}, T_x\}$ a representation of G on $\mathfrak{H}$. The operators

$$E(\delta)=\int_K T_u\overline{\chi_\delta}(u)du \quad\text{and}\quad E_{ij}(\delta)=d\int_K T_u\overline{d_{ij}}(u)du,$$

where $i, j=1,\dots, d$, are continuous and satisfy

$$E(\delta)=\sum_{i=1}^{d} E_{ii}(\delta)\,, \qquad E_{ij}(\delta)E_{kl}(\delta)=\delta_{jk}E_{il}(\delta)\,,$$

denoting by δ_{jk} the Kronecker's delta. Put

$$\mathfrak{H}(\delta)=E(\delta)\mathfrak{H}, \quad \mathfrak{H}_i(\delta)=E_{ii}(\delta)\mathfrak{H} \qquad (i=1,\dots, d).$$

Then $\mathfrak{H}(\delta)$ is invariant under the operators T_u $(u \in K)$ and $T_f = \int_G T_x f(x) dx$ $(f \in L(\delta))$ where dx is a Haar measure on G. For simplicity, we say that $\mathfrak{H}(\delta)$ is K-invariant and $L(\delta)$-invariant. The subspaces $\mathfrak{H}_i(\delta)$ are $L^\circ(\delta)$-invariant and the representations $T_f \mid \mathfrak{H}_i(\delta)$ of the algebra $L^\circ(\delta)$ are mutually equivalent since $E_{ij}(\delta)T_f = T_f E_{ij}(\delta)$ for all $f \in L^\circ(\delta)$. Therefore the representation $T_f \mid \mathfrak{H}(\delta)$ of $L^\circ(\delta)$ is equivalent to the direct sum of d copies of a certain representation $U(f)$.

Theorem 1. *Let G be a locally compact unimodular group, K a compact subgroup of G, and δ an element of $\hat{K}$. If a representation $\{\mathfrak{H}, T_x\}$ of G is topologically irreducible, then the corresponding representation $U(f)$ of the algebra $L^\circ(\delta)$ is also topologically irreducible.*

Proof. Let W be a $L^\circ(\delta)$-invariant subspace of $\mathfrak{H}_1(\delta)$, then the subspace $V = \sum_{i=1}^{d} \oplus E_{i1}(\delta)W$ is $L^\circ(\delta)$-invariant. And for all $u \in K$, we have

$$T_u V = \sum_{i=1}^{d} \oplus T_u E_{i1}(\delta) W = \sum_{i=1}^{d} \sum_{j=1}^{d} d_{ji}(u) E_{j1}(\delta) W$$
$$= \sum_{j=1}^{d} E_{j1}(\delta) W = V,$$

i.e., V is also K-invariant. Therefore $\bar{V}$ is $L(\delta)$-invariant (Lemma 14 in [2]). This means $\bar{V} = \{0\}$ or $\bar{V} = \mathfrak{H}(\delta)$ since the representation $T_f \mid \mathfrak{H}(\delta)$ of the algebra $L(\delta)$ is topologically irreducible (Lemma 2 in [2]). Then it follows that

$$W = \{0\} \quad \text{or} \quad \bar{W} = E_{11}\bar{V} = \mathfrak{H}_1(\delta)$$

respectively. Thus the theorem is proved. Q. E. D.

Lemma 1. *Let $\{\mathfrak{H}, T_x\}$ and $\{\mathfrak{H}', T'_x\}$ be two representations of G. If the corresponding representations $U(f)$ and $U'(f)$ of the algebra $L^\circ(\delta)$ are equivalent, then the representations $T_f \mid \mathfrak{H}(\delta)$ and $T'_f \mid \mathfrak{H}'(\delta)$ of the algebra $L(\delta)$ are also equivalent.*

Proof. From the assumption it follows that there exists a linear isomorphism ϕ of $\mathfrak{H}(\delta)$ onto $\mathfrak{H}'(\delta)$ such that

$$\phi T_f = T'_f \phi, \quad \phi E_{ij}(\delta) = E'_{ij}(\delta)\phi$$

for any $f \in L^\circ(\delta)$ and $i, j = 1, \ldots, d$. For every $u \in K$, we have

$$\phi^{-1} T'_u \phi = E(\delta)\phi^{-1} T'_u \phi = \sum_{i=1}^{d} E_{ii}(\delta)\phi^{-1} T'_u \phi = \phi^{-1}\Big(\sum_{i=1}^{d} E'_{ii}(\delta) T'_u\Big)\phi$$
$$= \phi^{-1}\Big(\sum_{i,j=1}^{d} d_{ij}(u) E'_{ij}(\delta)\Big)\phi = \sum_{i,j=1}^{d} d_{ij}(u) E_{ij}(\delta)$$
$$= E(\delta) T_u.$$

Namely $T'_u \phi = \phi T_u$ for all $u \in K$ on $\mathfrak{H}(\delta)$. Thus $\phi T_f = T'_f \phi$ for all $f \in L(\delta)$ (Lemma

14 in [2]). Q.E.D.

Theorem 2. *Let G be a locally compact unimodular group, K a compact subgroup of G, and δ an element of $\hat{K}$. Let $\{\mathfrak{H}, T_x\}$, $\{\mathfrak{H}', T'_x\}$ be two topologically irreducible representations of G which contain δ, and $U(f)$, $U'(f)$ corresponding topologically irreducible representations of $L^\circ(\delta)$ respectively. Then U is equivalent to U' if and only if there exists a linear mapping ψ of $\mathfrak{H}$ into $\mathfrak{H}'$ which satisfies the following conditions;*

(a) *ψ is defined on a G-invariant dense subspace $\mathscr{D}(\psi)$ of $\mathfrak{H}$, and injective,*

(b) *$T'_x\psi=\psi T_x$ on $\mathscr{D}(\psi)$ for all $x\in G$,*

(c) *$E(\gamma)\mathscr{D}(\psi)\subset\mathscr{D}(\psi)$ for every $\gamma\in\hat{K}$, and $E'(\gamma)\psi=\psi E(\gamma)$,*

(d) *$\psi\mid\mathscr{D}(\psi)\cap\mathfrak{H}(\delta)$ can be extended to a bijective and bicontinuous linear mapping of $\mathfrak{H}(\delta)$ onto $\mathfrak{H}'(\delta)$.*

Proof. Assume that U is equivalent to U'. Then, by Lemma 1, there exists a bijective and bicontinuous linear mapping ϕ of $\mathfrak{H}(\delta)$ onto $\mathfrak{H}'(\delta)$ satisfying $\phi T_f=T'_f\phi$ for any $f\in L(\delta)$. Fix a non-zero vector $a_0\in\mathfrak{H}(\delta)$ and put

$$\mathscr{D}(\psi)=\{T_fa_0;\, f\in L(G)\}.$$

For arbitrary f, $g\in L(G)$, we have

$$E'(\delta)T'_gT'_fa'_0=T'_{\overline{\chi_\delta}*g*f*\overline{\chi_\delta}}\phi(a_0)=\phi(T_{\overline{\chi_\delta}*g*f*\overline{\chi_\delta}}a_0)=\phi E(\delta)T_gT_fa_0$$

where $a'_0=\phi(a_0)$. This means that $T_fa_0=0$ implies $T'_fa'_0=0$. Therefore we may define a linear mapping ψ of $\mathscr{D}(\psi)$ to $\mathfrak{H}'$ by $\psi(T_fa_0)=T'_fa'_0$. The injectivity of ψ follows from the above equality. Now it is clear that ψ satisfies the conditions (a), (b), and (c). To prove that ψ satisfies the condition (d), we have only to show $\phi\mid\mathscr{D}(\psi)\cap\mathfrak{H}(\delta)=\psi\mid\mathscr{D}(\psi)\cap\mathfrak{H}(\delta)$, but this is easy.

Conversely, we assume that a linear mapping ψ satisfies the above four conditions. By the condition (c), we obtain $\mathscr{D}(\psi)\cap\mathfrak{H}(\delta)=E(\delta)\mathscr{D}(\psi)$ and therefore $\mathscr{D}(\psi)\cap\mathfrak{H}(\delta)$ is dense in $\mathfrak{H}(\delta)$. Denote by ϕ a bijective and bicontinuous linear mapping of $\mathfrak{H}(\delta)$ onto $\mathfrak{H}'(\delta)$ which is an extension of the mapping $\psi\mid\mathscr{D}(\psi)\cap\mathfrak{H}(\delta)$. Since $E'(\delta)T'_xE'(\delta)\phi=\phi E(\delta)T_xE(\delta)$ for all $x\in G$ and ϕ is continuous, we obtain $E'(\delta)T'_fE'(\delta)\phi=\phi E(\delta)T_fE(\delta)$ for all $f\in L(G)$, i.e., $T'_f\phi=\phi T_f$ for all $f\in L(\delta)$. Now it is clear that U is equivalent to U' by Lemma 1. Q.E.D.

Definition. A representation $\{\mathfrak{H}, T_x\}$ of G is called "*nice*" if there exists a pair (K, δ) of a compact subgroup K of G and $\delta\in\hat{K}$ which satisfies $0<\dim\mathfrak{H}(\delta)<+\infty$.

Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible nice representation of G. Then we can find a pair (K, δ) which satisfies $0<\dim\mathfrak{H}(\delta)<+\infty$. Now we take an arbitrary non-zero vector $a\in\mathfrak{H}(\delta)$ and put

$$\mathfrak{H}_0=\{T_fa;\, f\in L(G)\}.$$

This is a G-invariant dense subspace of $\mathfrak{H}$, and an important fact is that $\mathfrak{H}_0$ is independent of the choice of such (K, δ) and a (cf. Lemma 4 in [2]). The following theorem, which is the same as Theorem 9 in [4], is a corollary to Theorem 2.

Theorem 3. *Let G be a locally compact unimodular group, and $\{\mathfrak{H}, T_x\}$, $\{\mathfrak{H}', T'_x\}$ two topologically irreducible nice representations of G. Let $\mathfrak{H}_0$, $\mathfrak{H}'_0$ be the corresponding subspaces of $\mathfrak{H}$, $\mathfrak{H}'$ as above. Then the following conditions are equivalent.*

(i) *For a pair (K, δ) of a compact subgroup K of G and $\delta \in \hat{K}$ satisfying $0<\dim \mathfrak{H}(\delta)<+\infty$, the corresponding two irreducible representations $U(f)$, $U'(f)$ of the algebra $L^\circ(\delta)$ are equivalent.*

(ii) *For every pair (K, δ) of a compact subgroup K of G and $\delta \in \hat{K}$ satisfying $0<\dim \mathfrak{H}(\delta)<+\infty$, the corresponding two irreducible representations $U(f)$, $U'(f)$ of the algebra $L^\circ(\delta)$ are equivalent.*

(iii) *There exists a bijective linear mapping ψ of $\mathfrak{H}_0$ onto $\mathfrak{H}'_0$ satisfying $\psi T_x = T'_x \psi$ for all $x \in G$ and $\psi E(\delta) = E'(\delta)\psi$ for all pairs (K, δ).*

Definition. Two topologically irreducible nice representations $\{\mathfrak{H}, T_x\}$ and $\{\mathfrak{H}', T'_x\}$ are called *spherical-function-equivalent* (or *SF-equivalent*) if the conditions in Theorem 3 are satisfied.

Let $U(a)$ be a representation of an algebra A on a topological vector space $\mathfrak{H}$. If there exists a closed invariant subspace $\mathscr{L}$ of $\mathfrak{H}$, then the representation $U(a) \mid \mathscr{L}$ on $\mathscr{L}$ is called a *subrepresentation* of $U(a)$.

Theorem 4. *Let G be a locally compact unimodular group, K a compact subgroup of G. Let $\{\mathfrak{H}, T_x\}$ be a representation of G on a complete locally convex topological vector space $\mathfrak{H}$. Assume $\{\mathfrak{H}, T_x\}$ contains $\delta \in \hat{K}$, and denote by $U(f)$ the corresponding representation of the algebra $L^\circ(\delta)$. If we can find a topologically irreducible subrepresentation $U_0(f)$ of $U(f)$, then there exist closed G-invariant subspaces $\mathscr{H}_1$, $\mathscr{H}_2$ of $\mathfrak{H}$ satisfying the following conditions;*

(a) $\mathscr{H}_1 \supset \mathscr{H}_2$, $E(\delta)\mathscr{H}_2 = \{0\}$,

(b) *the naturally defined representation τ of G on $\mathscr{H}_1/\mathscr{H}_2$ is topologically irreducible, and the corresponding topologically irreducible representation of the algebra $L^\circ(\delta)$ is equivalent to $U_0(f)$.*

Remark. The author does not know whether $\mathscr{H}_1/\mathscr{H}_2$ is complete or not. But the integrals $\int_G \tau_x f(x)dx$ $(f \in L(G))$ and $\int_K \tau_u \overline{\chi_\delta}(u)du$ converge in $\mathscr{H}_1/\mathscr{H}_2$, and therefore we can make the same arguments as in the case of representations on complete topological vector spaces.

Proof of Theorem 4. By the assumption, there exists a closed $L^\circ(\delta)$-invariant subspace $\mathscr{L}_1$ of $\mathfrak{H}_1(\delta) = E_{11}(\delta)\mathfrak{H}$ such that the representation $T_f \mid \mathscr{L}_1$ of $L^\circ(\delta)$ is equivalent to $U_0(f)$. Then the closed subspace $\mathscr{L} = \mathscr{L}_1 + E_{21}(\delta)\mathscr{L}_1 + \cdots + E_{d1}(\delta)\mathscr{L}_1$, where d is the degree of δ, is $L^\circ(\delta)$-invariant and K-invariant, and therefore $L(\delta)$-invariant. For every $a \in \mathscr{L}$, put $\mathscr{L}_a = \overline{\{T_f a ; f \in L(\delta)\}}$. Since $\mathscr{L}_a$ is invariant under $E_{ii}(\delta)$, we have

$$\mathscr{L}_a = E_{11}(\delta)\mathscr{L}_a + \cdots + E_{dd}(\delta)\mathscr{L}_a.$$

Clearly $E_{11}(\delta)\mathscr{L}_a = \mathscr{L}_a \cap \mathfrak{H}_1(\delta) \subset \mathscr{L} \cap \mathfrak{H}_1(\delta) = \mathscr{L}_1$ and $E_{11}(\delta)\mathscr{L}_a$ is $L^\circ(\delta)$-invariant, thus $E_{11}(\delta)\mathscr{L}_a = \mathscr{L}_1$ or $=\{0\}$. On the other hand, we have $E_{i1}(\delta)E_{11}(\delta)\mathscr{L}_a = E_{ii}(\delta)\mathscr{L}_a$. Therefore we obtain $\mathscr{L}_a = \mathscr{L}$ or $=\{0\}$, and this means that $\mathscr{L}$ is topologically irreducible under $T_f \mid \mathscr{L}$ $(f \in L(\delta))$.

Now the closed subspace

$$\mathscr{H}_1 = \bigcap_{\substack{a \in \mathscr{L} \\ a \neq 0}} \overline{\{T_f a; f \in L(G)\}}$$

is G-invariant, and $E(\delta)\mathscr{H}_1 = \mathscr{L}$. Denote by $\mathscr{H}_2$ the maximal G-invariant subspace of $\mathscr{H}_1$ satisfying $E(\delta)\mathscr{H}_2 = \{0\}$. Then these subspaces $\mathscr{H}_1$, $\mathscr{H}_2$ satisfy the conditions (a) and (b). Q.E.D.

§2. Irreducible Banach representations of $G = SK$

Let G be a locally compact unimodular group, and K a compact subgroup of G. We assume that there exists a closed subgroup S of G such that all $x \in G$ are uniquely and continuously decomposed into the products $x = su$ where $s \in S$ and $u \in K$. Let $d\mu(s)$ be a left Haar measure on S and du the normalized Haar measure on K, then $dx = d\mu(s)du$ $(x = su)$ is a Haar measure on G.

In the following, we shall denote by $\{\mathfrak{H}, T_x\}$ a fixed topologically irreducible representation of G on a Banach space $\mathfrak{H}$. We assume $\dim \mathfrak{H}(\delta) = pd$ for a fixed equivalence class $\delta \in \hat{K}$, where d is the degree of δ and p a natural number. If we denote by $\rho(x)$ the operator norm of T_x, then $\rho(x)$ is a semi-norm on G (cf. [1]). Let $L_\rho(G)$ be the algebra of all measurable functions f on G which satisfy

$$\|f\|_\rho = \int_G |f(x)|\rho(x)dx < +\infty.$$

Then $L_\rho(G) * \overline{\chi_\delta}$ and $L_\rho^\circ(\delta) = \{f^\circ; f \in L_\rho(G) * \overline{\chi_\delta}\}$ are closed subalgebras of $L_\rho(G)$.

On the other hand, we shall denote by A_ρ the space of all $d \times d$-matrix valued measurable functions F on S which satisfy

$$\|F\|_\rho = d. \operatorname*{Max}_{1 \leq i, j \leq d} \int_S |f_{ij}(s)|\rho(s)d\mu(s) < +\infty,$$

where $f_{ij}(s)$ are (i, j)-matrix elements of $F(s)$. A_ρ is a Banach algebra with the convolution product

$$F * G(s) = \int_S F(t)G(t^{-1}s)d\mu(t).$$

Fix an irreducible unitary matricial representation $D(u)$ of K belonging to δ, and define a transformation

$$\Phi(f)(s) = \int_K \overline{D(u)} f(su^{-1})du$$

of $L_\rho(G)*\overline{\chi_\delta}$ into A_ρ. This is continuous, bijective, and linear. The inverse transformation Φ^{-1} is also continuous, and given by

$$\Phi^{-1}(F)(x)=d\cdot\text{trace}\,[F(s)\overline{D(u)}]$$

where $x=su$. For every element $F=\Phi(f)\in A_\rho$ we put $F^\circ=\Phi(f^\circ)$. Then $F\to F^\circ$ is a continuous projection, and we easily have an equality

$$\Phi(f*g^\circ)=\Phi(f)*\Phi(g^\circ)=\Phi(f)*\Phi(g)^\circ.$$

Therefore $A_\rho^\circ=\{F^\circ;\ F\in A_\rho\}$ is a closed subalgebra of A_ρ, and isomorphic to the Banach algebra $L_\rho^\circ(\delta)$.

Put

$$\mathfrak{p}=\{f\in L_\rho^\circ(\delta);\ T_f=0\},$$

then this is a regular closed two-sided ideal in $L_\rho^\circ(\delta)$, and an element $\mathfrak{e}\in L_\rho^\circ(\delta)$ is a right identity modulo $\mathfrak{p}$ if and only if $T_e\,|\,\mathfrak{H}(\delta)$ is the identity operator on $\mathfrak{H}(\delta)$. A non-trivial subspace V of $\mathfrak{H}(\delta)$ is called K-irreducible if V is invariant and irreducible under T_u $(u\in K)$. For a K-irreducible subspace V of $\mathfrak{H}(\delta)$, we put

$$\mathfrak{a}_V=\{f\in L_\rho^\circ(\delta);\ T_f\,|\,V=0\}.$$

Lemma 2. *The mapping $V\to\mathfrak{a}_V$ of the set of all K-irreducible subspaces of $\mathfrak{H}(\delta)$ to the set of all maximal left ideals in $L_\rho^\circ(\delta)$ containing $\mathfrak{p}$ is bijective.*

Proof. Let V be a K-irreducible subspace of $\mathfrak{H}(\delta)$, and $\mathfrak{a}$ a left ideal in $L_\rho^\circ(\delta)$ such that $\mathfrak{a}_V\subsetneqq\mathfrak{a}$. Then $\sum_{f\in\mathfrak{a}}T_fV$ is invariant under all operators T_f $(f\in L_\rho^\circ(\delta))$ and T_u $(u\in K)$. Therefore $\sum_{f\in\mathfrak{a}}T_fV$ is invariant under all operators T_f $(f\in\overline{\chi_\delta}*L_\rho(G)*\overline{\chi_\delta})$ by Lemma 11 in [1], and this means $\mathfrak{H}(\delta)=\sum_{f\in\mathfrak{a}}T_fV$. Since $\dim\mathfrak{H}(\delta)=pd$, there exist p functions $f_1,\dots,f_p\in\mathfrak{a}$ such that

$$\mathfrak{H}(\delta)=T_{f_1}V\oplus\cdots\oplus T_{f_p}V\qquad\text{(direct sum)}.$$

Thus every vector $a\in\mathfrak{H}(\delta)$ is uniquely written in the form $a=T_{f_1}a_1+\cdots+T_{f_p}a_p$ where $a_1,\dots,a_p\in V$. Since the linear transformation $a\to a_i$ on V commutes with all operators T_u $(u\in K)$, we have $a_i=\lambda_ia$ for some $\lambda_i\in\mathbf{C}$, i.e.,

$$a=(\lambda_1T_{f_1}+\cdots+\lambda_pT_{f_p})a.$$

Therefore, for every function $f\in L_\rho^\circ(\delta)$, we can find a function $g\in\mathfrak{a}$ such that $T_fa=T_ga$ for all $a\in V$. This means $L_\rho^\circ(\delta)=\mathfrak{a}$. Now we have proved that $\mathfrak{a}_V$, which clearly contains $\mathfrak{p}$, is a maximal left ideal in $L_\rho^\circ(\delta)$.

Conversely let $\mathfrak{a}$ be a maximal left ideal in $L_\rho^\circ(\delta)$ containing $\mathfrak{p}$. Suppose $\mathfrak{a}\not\subset\mathfrak{a}_V$ for every K-irreducible subspace V of $\mathfrak{H}(\delta)$. Take an arbitrarily chosen non-zero vector $a\in\mathfrak{H}(\delta)$. Then there exist $u_1,\dots,u_r\in K$ and $\lambda_1,\dots,\lambda_r\in\mathbf{C}$ such that $b=\sum_{i=1}^r\lambda_iT_{u_i}a$ is a non-zero vector in a K-irreducible subspace V of $\mathfrak{H}(\delta)$. By our assumption, at least one function $f\in\mathfrak{a}$ satisfies $T_fV\neq\{0\}$, or equivalently, $\dim T_fV=\dim V=d$. For such a function $f\in\mathfrak{a}$, we have $T_fb\neq0$. On the other hand, we put

$$f*\varepsilon_u(x)=f(xu^{-1})=f(u^{-1}x)=\varepsilon_u*f(x)$$

for all $f\in\mathfrak{a}$ and $u\in K$. Let A be the algebra generated by $\{f*\varepsilon_u; f\in\mathfrak{a}, u\in K\}$. Then above consideration shows that $\{T_f a; f\in A\}\neq\{0\}$. This subspace $\{T_f a; f\in A\}$ is $L_\rho^\circ(\delta)$- and K-invariant, and therefore $\overline{\chi_\delta}*L_\rho(G)*\overline{\chi_\delta}$-invariant. Consequently $\{T_f a; f\in A\}=\mathfrak{H}(\delta)$. Namely, $f\to T_f\,|\,\mathfrak{H}(\delta)$ is a p-dimensional irreducible representation of A. By the Burnside's theorem, we can find a function $g\in A$ satisfying $T_g\,|\,\mathfrak{H}(\delta)=1$. If we take $f_1,\dots, f_t\in\mathfrak{a}$ and $u_1,\dots, u_t\in K$ such that $g=\sum_{i=1}^t f_i*\varepsilon_{u_i}$, then

$$1=T_g|\mathfrak{H}(\delta)=T_{g^\circ}|\mathfrak{H}(\delta)=\sum_{i=1}^t \xi_i T_{f_i}|\mathfrak{H}(\delta)$$

where $\xi_1,\dots,\xi_t$ are certain constants. Thus we have $T_h\,|\,\mathfrak{H}(\delta)=1$ for $h=\sum_{i=1}^t \xi_i f_i\in\mathfrak{a}$, and it follows that $f*h-f\in\mathfrak{p}\subset\mathfrak{a}$ for all $f\in L_\rho^\circ(\delta)$. This is a contradiction since $\mathfrak{a}\subsetneqq L_\rho^\circ(\delta)$. Therefore we have proved that there exists a K-irreducible subspace V of $\mathfrak{H}(\delta)$ satisfying $\mathfrak{a}=\mathfrak{a}_V$.

At last, we show that the mapping $V\to\mathfrak{a}_V$ is injective. Let V and V' be two distinct K-irreducible subspaces of $\mathfrak{H}(\delta)$. Since $V\cap V'=\{0\}$, there exists a linear operator L on $\mathfrak{H}(\delta)$ such that $LT_u=T_uL$ $(u\in K)$, $LV=V$, and $LV'=\{0\}$. A function $f\in L_\rho^\circ(\delta)$ which satisfies $T_f\,|\,\mathfrak{H}(\delta)=L$ belongs to $\mathfrak{a}_{V'}$ but does not to $\mathfrak{a}_V$. Therefore $\mathfrak{a}_V\neq\mathfrak{a}_{V'}$. Q.E.D.

Since our topologically irreducible representation $\{\mathfrak{H}, T_x\}$ contains δ p times, the irreducible representation $U(f)$ of the algebra $L^\circ(\delta)$ corresponding to $\{\mathfrak{H}, T_x\}$ in the sense of § 1 is p-dimensional. If we denote by $U=U(x)$ the spherical matrix function of degree p of type δ defined from $\{\mathfrak{H}, T_x\}$ (see [3]), then we have

$$U(f)=\int_G U(x)f(x)dx \qquad (f\in L^\circ(\delta))$$

up to equivalence. The right hand side converges for $f\in L_\rho^\circ(\delta)$, therefore we can extend $U(f)$ to a representation of $L_\rho^\circ(\delta)$. We shall denote this by the same notation $U(f)$. On the other hand, for every K-irreducible subspace V of $\mathfrak{H}(\delta)$, we have a naturally defined irreducible representation of $L_\rho^\circ(\delta)$ on $L_\rho^\circ(\delta)/\mathfrak{a}_V$. It is easily seen that this representation is equivalent to $U(f)$.

For every K-irreducible subspace V of $\mathfrak{H}(\delta)$, we put

$$\mathfrak{A}_V=\Phi(\mathfrak{a}_V)\,.$$

Since Φ maps $L_\rho^\circ(\delta)$ isomorphically onto A_ρ°, $\mathfrak{A}_V$ is a closed regular maximal left ideal in A_ρ°, and an element $\mathfrak{E}=\Phi(\mathfrak{e})$, where $\mathfrak{e}$ is a function in $L_\rho^\circ(\delta)$ satisfying $T_\mathfrak{e}\,|\,\mathfrak{H}(\delta)=1$, is a right identity modulo $\mathfrak{A}_V$. Moreover

$$\mathfrak{M}_V=\{F\in A_\rho;\ (G*F)^\circ\in\mathfrak{A}_V \text{ for all } G\in A_\rho\}$$

is a closed regular left ideal in A_ρ, and $\mathfrak{E}$ is a right identity modulo $\mathfrak{M}_V$.

Definition. Let V be a K-irreducible subspace of $\mathfrak{H}(\delta)$. We shall denote by

$\{e_1^V,\dots, e_d^V\}$ a base of V with respect to which the operators $T_u | V$ are represented by our fixed unitary matricies $D(u)$.

Lemma 3. *Let V be a K-irreducible subspace of $\mathfrak{H}(\delta)$. For every $F=\Phi(f)\in A_\rho$ $(f\in L_\rho(G)*\overline{\chi_\delta})$ whose (i, j)-matrix coefficient is denoted by f_{ij}, we have*

$$T_f e_i^V = \sum_{j=1}^{d} T_{f_{ij}} e_j^V \qquad (i=1,\dots, d)$$

where $T_{f_{ij}}=\int_S T_s f_{ij}(s) d\mu(s)$.

Proof. Denoting by $d_{\alpha\beta}(u)$ the (α, β)-matrix coefficient of $D(u)$, we have

$$f(su)=d\cdot \text{trace}\,[F(s)\overline{D(u)}]=d\sum_{\alpha,\beta=1}^{d} f_{\beta\alpha}(s)\overline{d_{\alpha\beta}(u)}\,.$$

Therefore

$$\begin{aligned}
T_f e_i^V &= \int_{S\times K} T_s T_u f(su) e_i^V d\mu(s) du \\
&= d\sum_{\alpha,\beta=1}^{d}\int_{S\times K} T_s T_u f_{\beta\alpha}(s)\overline{d_{\alpha\beta}(u)} e_i^V d\mu(s) du \\
&= d\sum_{\alpha,\beta=1}^{d}\sum_{j=1}^{d}\int_{S\times K} T_s f_{\beta\alpha}(s)\overline{d_{\alpha\beta}(u)} d_{ji}(u) e_j^V d\mu(s) du \\
&= \sum_{j=1}^{d} T_{f_{ij}} e_j^V.
\end{aligned}$$

Q.E.D.

Corollary. *Let f be a function in $L_\rho(G)*\overline{\chi_\delta}$ and put $F=\Phi(f)\in A_\rho$. If we denote by E_{ii} the $d\times d$-matrix whose (i, i)-matrix coefficient is 1 and the others are 0, then the functions $f_i=\Phi^{-1}(E_{ii}F)\in L_\rho(G)*\overline{\chi_\delta}$ satisfy $f=f_1+\cdots+f_d$ and*

$$T_{f_i} e_i^V = T_f e_i^V, \quad T_{f_i} e_j^V = 0 \quad (i\neq j).$$

Lemma 4. *Let V be a K-irreducible subspace of $\mathfrak{H}(\delta)$, and $F=\Phi(f)$ an element in A_ρ where $f\in L_\rho(G)*\overline{\chi_\delta}$. Then $F\in\mathfrak{M}_V$ if and only if $T_f V=\{0\}$.*

Proof. We shall denote by $L_\rho(S)$ the algebra of all functions ϕ on S satisfying $\int_S |\phi(s)|\rho(s) d\mu(s)<+\infty$. Let $E_{\alpha\beta}$ be the $d\times d$-matrix whose (α, β)-matrix coefficient is 1 and the others are 0. For every $\phi\in L_\rho(S)$ and $E_{\alpha\beta}$, the $d\times d$-matrix valued function $\phi E_{\alpha\beta}$, whose (α, β)-coefficient is ϕ and the others are 0, belongs to A_ρ. If we denote by f_{ij} the (i, j)-matrix coefficient of $F=\Phi(f)$ and by g_{mn} the (m, n)-matrix coefficient of $G=(\phi E_{\alpha\beta})*F$, then $g_{mn}=\delta_{m\alpha}(\phi * f_{\beta n})$ where $\delta_{m\alpha}$ is the Kronecker's delta. Putting $g=\Phi^{-1}(G)$ and using Lemma 3,

$$\begin{aligned}
T_{g^\circ} e_i^V &= \int_K T_u T_g T_{u^{-1}} e_i^V du \\
&= \sum_{m=1}^{d}\int_K T_u d_{mi}(u^{-1}) T_g e_m^v du
\end{aligned}$$

$$=\sum_{m=1}^{d}\int_K T_u\overline{d_{im}(u)}\,[\sum_{n=1}^{d} T_{g_{mn}}e_n^V]\,du$$

$$=\int_K T_u\overline{d_{i\alpha}(u)}\,[\sum_{n=1}^{d} T_\phi T_{f_{\beta n}}e_n^V]\,du$$

$$=\int_K T_u\overline{d_{i\alpha}(u)}\,[T_\phi T_f e_\beta^V]\,du.$$

Therefore we have

$$F\in\mathfrak{M}_V \Longleftrightarrow [(\phi E_{\alpha\beta})*F]^\circ\in\mathfrak{A}_V \quad \text{for all} \quad \alpha,\beta\in\{1,\dots,d\} \quad \text{and} \quad \phi\in L_\rho(S),$$

$$\Longleftrightarrow [\Phi^{-1}(\phi E_{\alpha\beta}*F)]^\circ\in\mathfrak{a}_V \quad \text{for all} \quad \alpha,\beta\in\{1,\dots,d\} \quad \text{and} \quad \phi\in L_\rho(S),$$

$$\Longleftrightarrow \int_K T_u\overline{d_{i\alpha}(u)}\,[T_\phi T_f e_\beta^V]du=0 \quad \text{for all} \quad i,\alpha,\beta\in\{1,\dots,d\} \quad \text{and} \quad \phi\in L_\rho(S),$$

$$\Longleftrightarrow E(\delta)T_\phi T_f V=\{0\} \quad \text{for all} \quad \phi\in L_\rho(S),$$

$$\Longleftrightarrow E(\delta)T_s T_f V=\{0\} \quad \text{for all} \quad s\in S,$$

$$\Longleftrightarrow E(\delta)T_u T_s T_f V=\{0\} \quad \text{for all} \quad u\in K \quad \text{and} \quad s\in S,$$

$$\Longleftrightarrow E(\delta)T_x T_f V=\{0\} \quad \text{for all} \quad x\in G,$$

$$\Longleftrightarrow T_f V=\{0\}.$$

Q. E. D.

For every $d\times d$-matrix M and every element $F\in A_\rho$, we put $(MF)(s)=M\times F(s)$ where the right hand side is the product of two matrices M and $F(s)$. MF is obviously an element in A_ρ.

Lemma 5. *Let V be a K-irreducible subspace of $\mathfrak{H}(\delta)$, and $\mathfrak{M}$ a left ideal in A_ρ such that $\mathfrak{M}\supset\mathfrak{M}_V$ and that $M\mathfrak{M}\subset\mathfrak{M}$ for every $d\times d$-matrix M. Then the subspace $\{T_f a;\, f\in L_\rho(G)*\overline{\chi_\delta},\ \Phi(f)\in\mathfrak{M}\}$ of $\mathfrak{H}$, where $a\in V-\{0\}$, is independent of the choice of nonzero vector $a\in V$.*

Proof. Let a, $b\in V$ and $a\neq 0$, $b\neq 0$. We can find a continuous function $\xi(u)$ on K such that $\xi*\overline{\chi_\delta}=\xi$ and that

$$T_\xi b=\int_K T_u\xi(u)b\,du=a.$$

For every function $f\in L_\rho(G)*\overline{\chi_\delta}$ satisfying $\Phi(f)\in\mathfrak{M}$, we see $f*\xi\in L_\rho(G)*\overline{\chi_\delta}$ and

$$\Phi(f*\xi)(s)=\int_K \overline{D(u)}f*\xi(su^{-1})\,du$$

$$=\int_{K\times K}\overline{D(u)}f(su^{-1}v^{-1})\xi(v)\,du\,dv$$

$$=\int_{K\times K}\overline{D(v^{-1}u)}f(su^{-1})\xi(v)dudv$$

$$=\left[\int_K\overline{D(v^{-1})}\xi(v)dv\right]\Phi(f)(s)\in\mathfrak{M}$$

by our assumption on $\mathfrak{M}$. Since $T_f a=T_{f*\xi}b$, we clearly have $\{T_f a; f\in L_\rho(G)*\overline{\chi_\delta}, \Phi(f)\in\mathfrak{M}\}\subset\{T_f b; f\in L_\rho(G)*\overline{\chi_\delta}, \Phi(f)\in\mathfrak{M}\}$. Q.E.D.

Definition. For a K-irreducible subspace of $\mathfrak{H}(\delta)$ and a left ideal $\mathfrak{M}$ in A_ρ such that $\mathfrak{M}\supset\mathfrak{M}_V$ and that $M\mathfrak{M}\subset\mathfrak{M}$ for every $d\times d$-matrix M, we put

$$\mathfrak{H}_V(\mathfrak{M})=\{T_f a; f\in L_\rho(G)*\overline{\chi_\delta}, \Phi(f)\in\mathfrak{M}\}$$

where $a\in V$, $a\neq 0$.

Since our representation $\{\mathfrak{H}, T_x\}$ is topologically irreducible and nice, we can define the subspace $\mathfrak{H}_0$ of $\mathfrak{H}$ as in §1. But $\{\mathfrak{H}, T_x\}$ is a Banach representation, so let's define another subspace $\mathfrak{H}_\rho$ of $\mathfrak{H}$ which is a natural extension of $\mathfrak{H}_0$, i.e., taking a non-zero vector a in $\mathfrak{H}(\delta)$, we put

$$\mathfrak{H}_\rho=\{T_f a; f\in L_\rho(G)\}.$$

As in the case of $\mathfrak{H}_0$, this subspace $\mathfrak{H}_\rho$ is independent of K, δ, and a. Namely, if a pair (K', δ') of a compact subgroup K' and $\delta'\in\hat{K}'$ satisfies $0<\dim\mathfrak{H}(\delta')<+\infty$, then, for every nonzero vector $a'\in\mathfrak{H}(\delta')$, we have $\mathfrak{H}_\rho=\{T_f a'; f\in L_\rho(G)\}$. Our subspace $\mathfrak{H}_V(\mathfrak{M})$ in the above definition is a subspace of $\mathfrak{H}_\rho$ and $L_\rho(S)$-invariant, i.e., invariant under all operators $T_\phi=\int_S T_s\phi(s)d\mu(s)$ for $\phi\in L_\rho(S)$.

Lemma 6. *Let V be a K-irreducible subspace of $\mathfrak{H}(\delta)$, and $\mathscr{K}$ a $L_\rho(S)$-invariant subspace of $\mathfrak{H}_\rho$. Then there exists a left ideal $\mathfrak{M}$ in A_ρ such that $\mathfrak{M}\supset\mathfrak{M}_V$, $M\mathfrak{M}\subset\mathfrak{M}$ for all $d\times d$-matrices M, and that $\mathscr{K}=\mathfrak{H}_V(\mathfrak{M})$.*

Proof. Put $\mathfrak{M}=\{F\in A_\rho;\ T_f V\subset\mathscr{K}$ for $f=\Phi^{-1}(F)\}$. Let F be any element of $\mathfrak{M}$, and denote by $g_{ij}\in L_\rho(S)$ the (i, j)-matrix coefficient of an arbitrary element $G\in A_\rho$. The function $h=\Phi^{-1}(G*F)$ is given as follows;

$$h(su)=d\cdot\text{trace}\,[G*F(s)\overline{D(u)}]$$

$$=\sum_{i,j=1}^{d} d\cdot\text{trace}\,[g_{ij}*(E_{ij}F)(s)\overline{D(u)}]$$

$$=\sum_{i,j=1}^{d} g_{ij}*\Phi^{-1}(E_{ij}F)(su)$$

where E_{ij} is the $d\times d$-matrix whose (i, j)-matrix coefficient is 1 and the others are 0. Choose a continuous function ξ_{ij} on K such that $\xi_{ij}*\overline{\chi_\delta}=\xi_{ij}$ and that $\int_K\overline{D(u^{-1})}\xi_{ij}\cdot(u)du=E_{ij}$, then we have $E_{ij}F=\Phi(f*\xi_{ij})$ and

$$T_{\Phi^{-1}(E_{ij}F)}a=T_f(T_{\xi_{ij}}a)\in T_f V\subset\mathscr{K}\qquad(a\in V)$$

i.e., $E_{ij}F\in\mathfrak{M}$ $(1\leqq i, j\leqq d)$. From this, we know two facts; the one is that $M\mathfrak{M}\subset\mathfrak{M}$ for all $d\times d$-matrices M and the other is that $T_h a\in\mathscr{K}$ for all $a\in V$, namely, $G*F\in\mathfrak{M}$. Therefore $\mathfrak{M}$ is a left ideal in A_ρ. The inclusion $\mathfrak{M}\supset\mathfrak{M}_V$ is clear. At last let's prove $\mathscr{K}=\mathfrak{H}_V(\mathfrak{M})$. Let $\{e_1^V,\dots, e_d^V\}$ be, as was already defined, a base of V with respect to which the operator $T_u|V$ is represented by the matrix $D(u)$. For every vector $a\in\mathscr{K}\subset\mathfrak{H}_\rho=\{T_f e_1^V; f\in L_\rho(G)*\overline{\chi_\delta}\}$, there exists a function $f\in L_\rho(G)*\overline{\chi_\delta}$ such that $T_f e_1^V=a$. From Corollary to Lemma 3, we may assume $T_f e_i^V=0$ $(i=2,\dots, d)$ without loss of generality. Then $T_f V\subset\mathscr{K}$ or, by definition, $\Phi(f)\in\mathfrak{M}$. Therefore $a=T_f e_1^V\in\mathfrak{H}_V(\mathfrak{M})$. Thus we obtain $\mathscr{K}\subset\mathfrak{H}_V(\mathfrak{M})$. Since $\mathfrak{H}_V(\mathfrak{M})\subset\mathscr{K}$ is clear, we have proved the equality $\mathscr{K}=\mathfrak{H}_V(\mathfrak{M})$. Q.E.D.

Lemma 7. *Let V be a K-irreducible subspace of $\mathfrak{H}(\delta)$. The mapping $\mathfrak{M}\to\mathscr{K}=\mathfrak{H}_V(\mathfrak{M})$ is a bijection of the set of all left ideals $\mathfrak{M}$ in A_ρ which satisfy $\mathfrak{M}\supset\mathfrak{M}_V$ and $M\mathfrak{M}\subset\mathfrak{M}$ for all $d\times d$-matrices M onto the set of all $L_\rho(S)$-invariant subspace $\mathscr{K}$ of $\mathfrak{H}_\rho$.*

Proof. We have only to prove the injectivity. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be distinct left ideals in A_ρ which satisfy the above conditions. We may assume that there exists an element $F\in\mathfrak{M}_1$ such that $F\notin\mathfrak{M}_2$. Since $F=E_{11}F+\cdots+E_{dd}F$, one of the d terms of the right hand side, say $E_{11}F$, does not belong to $\mathfrak{M}_2$. We put $f_1=\Phi^{-1}(E_{11}F)$. Suppose $\mathfrak{H}_V(\mathfrak{M}_1)=\mathfrak{H}_V(\mathfrak{M}_2)$, then $T_{f_1}e_1^V\in\mathfrak{H}_V(\mathfrak{M}_1)$ has another expression of the form $T_{f_1}e_1^V=T_g e_1^V$ with a suitable function $g\in\Phi^{-1}(\mathfrak{M}_2)$. Since $E_{11}G\in\mathfrak{M}_2$ where $G=\Phi(g)$, the function $g_1=\Phi^{-1}(E_{11}G)$ satisfies $T_{g_1}a=T_{f_1}a$ for every $a\in V$ by Corollary to Lemma 3. Therefore, by Lemma 4, we obtain $E_{11}F-E_{11}G\in\mathfrak{M}_V\subset\mathfrak{M}_2$. This means $E_{11}F\in\mathfrak{M}_2$, but this is a contradiction. Q.E.D.

Since $\mathfrak{M}_V$, where V is a K-irreducible subspace of $\mathfrak{H}(\delta)$, is a regular left ideal in the Banach algebra A_ρ, a maximal left ideal $\mathfrak{M}$ in A_ρ which contains $\mathfrak{M}_V$ is closed in A_ρ. Therefore $M\mathfrak{M}\subset\mathfrak{M}$ for every $d\times d$-matrix M, and $\mathfrak{M}$ is invariant under left translations ε_s $(s\in S)$ where $(\varepsilon_s F)(t)=F(s^{-1}t)$. Now we naturally define the left multiplication by $d\times d$-matrix and the left translations ε_s on the quotient space $A_\rho/\mathfrak{M}$ which is a Banach space with the usual norm. Put

$$H_i=E_{ii}(A_\rho/\mathfrak{M})\qquad (i=1,\dots, d)$$

where E_{ii} denotes, as before, the $d\times d$-matrix whose (i, i)-matrix coefficient is 1 and the others are 0. We shall denote by $\pi_i(s)$ the left translation by an element $s\in S$ on the Banach space H_i, then $\{H_i, \pi_i(s)\}$ are mutually equivalent topologically irreducible representations of S.

On the other hand, for a maximal left ideal $\mathfrak{M}$ in A_ρ which contains $\mathfrak{M}_V$, $\mathscr{K}=\mathfrak{H}_V(\mathfrak{M})$ is a maximal $L_\rho(S)$-invariant subspace of $\mathfrak{H}_\rho$ by Lemma 7. Since $\mathfrak{M}$ is invariant under ε_s $(s\in S)$, the subspace $\mathscr{K}$ is obviously S-invariant, i.e., $T_s\mathscr{K}\subset\mathscr{K}$ for all $s\in S$. Thus the operator T_s naturally induces a linear operator, which is denoted by $\Lambda(s)$, on the vector space $\mathfrak{H}_\rho/\mathscr{K}$. $\Lambda(s)$ is a representation of S on the vector space $\mathfrak{H}_\rho/\mathscr{K}$ in a purely algebraic sense.

Lemma 8. *Let $\mathfrak{M}$ be a maximal left ideal in A_ρ which contains $\mathfrak{M}_V$. The representations $\pi_i(s)$ and $\Lambda(s)$ of S, which are defined for $\mathfrak{M}$ as above, are algebraically equivalent. In other words, there exists a linear bijection I_i of H_i onto $\mathfrak{H}_\rho/\mathscr{K}$, where $\mathscr{K}=\mathfrak{H}_V(\mathfrak{M})$, such that $I_i\circ\pi_i(s)=\Lambda(s)\circ I_i$ for $s\in S$.*

Proof. For $F\in A_\rho$ we define $I'_i(E_{ii}F)=T_{f_i}e_i^V$ where $f_i=\Phi^{-1}(E_{ii}F)$. If $E_{ii}F\in\mathfrak{M}$, then $T_{f_i}e_i^V\in\mathfrak{H}_V(\mathfrak{M})$ by the definition of $\mathfrak{H}_V(\mathfrak{M})$. Conversely, if $T_{f_i}e_i^V\in\mathfrak{H}_V(\mathfrak{M})$, then $T_{f_i}a\in\mathfrak{H}_V(\mathfrak{M})$ for all $a\in V$ by Corollary to Lemma 3. Therefore $E_{ii}F=\Phi(f_i)\in\mathfrak{M}$ by Lemma 7. These facts mean that I'_i induces naturally a linear bijection I_i of H_i onto $\mathfrak{H}_\rho/\mathfrak{H}_V(\mathfrak{M})$. The equality $I_i\circ\pi_i(s)=\Lambda(s)\circ I_i$ is clear. Q.E.D.

Let $\mathscr{H}$ be a non-trivial maximal $L_\rho(S)$-invariant subspace of $\mathfrak{H}_\rho$. For a K-irreducible subspace V of $\mathfrak{H}(\delta)$, there exists a maximal left ideal $\mathfrak{M}$ in A_ρ which contains $\mathfrak{M}_V$ such that $\mathscr{H}=\mathfrak{H}_V(\mathfrak{M})$ (Lemma 7). For this maximal left ideal $\mathfrak{M}$, we can define topologically irreducible representations $\{H_i, \pi_i(s)\}$ of S as above. If we introduce a structure of Banach space into $\mathfrak{H}_\rho/\mathscr{K}$ with respect to which the linear bijection I_i of H_i onto $\mathfrak{H}_\rho/\mathscr{K}$ is an isomorphism, then we obtain a topologically irreducible representation $\Lambda(s)$ of S on the Banach space $\mathfrak{H}_\rho/\mathscr{H}$.

§ 3. Main theorem

Let $G=S\cdot K$ be the same locally compact group as in § 2. Let $\{H, \Lambda(s)\}$ be a topologically irreducible representation of S on a Banach space H. We shall denote by $\mathfrak{H}^\Lambda$ the Banach space of all H-valued continuous functions ξ on K with a norm $|||\xi|||=\sup\|\xi(u)\|_H$, where $\|\cdot\|_H$ is the norm in H. For every pair $(x, y)\in G\times G$, we define $\kappa(x, y)\in K$ and $\sigma(x, y)\in S$ by

$$xy=\kappa(x, y)\sigma(x, y).$$

With this notations, we define a bounded linear operator T_x^Λ on $\mathfrak{H}^\Lambda$ for every $x\in G$ by

$$(T_x^\Lambda\xi)(u)=\Lambda(\sigma(x^{-1}, u)^{-1})\xi(\kappa(x^{-1}, u)) \qquad (u\in K).$$

Then $\{\mathfrak{H}^\Lambda, T_x^\Lambda\}$ is a representation of G.

Let δ be an equivalence class of irreducible representations of K. As in § 1, we choose an irreducible unitary matricial representation $D(u)$ of K belonging to δ, and denote by $d_{ij}(u)$ its (i, j)-matrix coefficient. Put

$$E^\Lambda(\delta)=\int_K T_u^\Lambda\overline{\chi_\delta(u)}du, \quad E_{ij}^\Lambda(\delta)=d\int_K T_u^\Lambda\overline{d_{ij}(u)}du \qquad (1\leqq i, j\leqq d)$$

where d is the degree of δ. By the arguments in § 1, mutually equivalent d representations of the algebra $L^\circ(\delta)$ are defined on subspaces

$$\mathfrak{H}_i^\Lambda(\delta)=E_{ii}^\Lambda(\delta)\mathfrak{H}^\Lambda=\{\xi(u)=\sum_{j=1}^d \overline{d_{ij}(u)}a_j\,;\,a_j\in H\} \qquad (1\leqq i\leqq d).$$

Denote by e_j a d-dimensional column vector whose j-th component is 1 and the others are 0, then the mapping P defined by

$$P(\sum_{j=1}^{d} \overline{d_{ij}} a_j) = \sum_{j=1}^{d} e_j \otimes a_j$$

is a linear isomorphism of $\mathfrak{H}_i^A(\delta)$ onto $\mathbf{C}^d \otimes H$. If we adopt $\sum_{j=1}^{d} \|a_j\|_H$ as a norm of $\sum_{j=1}^{d} e_j \otimes a_j$, then $\mathbf{C}^d \otimes H$ is a Banach space and P gives an isomorphism of the Banach space $\mathfrak{H}_i^A(\delta)$ onto the Banach space $\mathbf{C}^d \otimes H$.

For every function $f \in L^\circ(\delta)$ we obtain

$$\begin{aligned}
(T_f^A \overline{d_{ij}} a)(u) &= \int_G \overline{d_{ij}(\kappa(x^{-1}, u))} A(\sigma(x^{-1}, u)^{-1}) a\, f(x) dx \\
&= \int_{G \times K} \overline{d_{ij}(\kappa(x^{-1}, u))} A(\sigma(x^{-1}, u)^{-1}) a\, f(vxv^{-1}) dx dv \\
&= \int_{G \times K} \overline{d_{ij}(v \cdot \kappa(x^{-1}, u))} A(\sigma(x^{-1}, v^{-1}u)^{-1}) a\, f(x) dx dv \\
&= \int_{G \times K} \overline{d_{ij}(uv \cdot \kappa(x^{-1}, v^{-1}))} A(\sigma(x^{-1}, v^{-1})^{-1}) a\, f(x) dx dv \\
&= \sum_{n=1}^{d} \overline{d_{ij}(u)} \left[\int_{G \times K} \overline{d_{nj}(v \cdot \kappa(x^{-1}, v^{-1}))} A(\sigma(x^{-1}, v^{-1})^{-1}) a\, f(x) dx dv \right].
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&P \circ (T_f^A | \mathfrak{H}_i^A(\delta)) \circ P^{-1}(e_j \otimes a) \\
&= \sum_{n=1}^{d} e_n \otimes \left[\int_{G \times K} \overline{d_{nj}(v \cdot \kappa(x^{-1}, v^{-1}))} A(\sigma(x^{-1}, v^{-1})^{-1}) a\, f(x) dx dv \right] \\
&= \int_G \left[\int_K (\sum_{n=1}^{d} \overline{d_{nj}(v \cdot \kappa(x^{-1}, v^{-1}))} e_n) \otimes A(\sigma(x^{-1}, v^{-1})^{-1}) a\, dv \right] f(x) dx.
\end{aligned}$$

Now put

$$W^A(x) = \int_K \tilde{W}^A(vx^{-1}v^{-1}) dv$$

where $\tilde{w}^A(x) = \overline{D(u)} \otimes A(s^{-1})$ with $x = us$, then it follows that

$$P \circ (T_f^A | \mathfrak{H}_i^A(\delta)) \circ P^{-1} = W^A(f) = \int_G W^A(x) f(x) dx$$

for $f \in L^\circ(\delta)$.

Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible representation of G on a Banach space $\mathfrak{H}$ which contains δ p times $(0 < p < +\infty)$, i.e., $\dim \mathfrak{H}(\delta) = pd$. As is proved in § 2, there exists a maximal $L_\rho(S)$-invariant subspace $\mathscr{K}$, which is S-invariant at the same time, of $\mathfrak{H}_\rho$ where $\rho(x) = \|T_x\|$, and we introduce the Banach space structure into $H = \mathfrak{H}_\rho / \mathscr{K}$ defined in the last paragraph of § 2. $A(s)$ denotes the topologically irreducible representation of S naturally defined on H. For this representation

$\{H, \Lambda(s)\}$ we consider the induced representation $\{\mathfrak{H}^{\Lambda}, T_x^{\Lambda}\}$ of G. Let $U_0(f)$ be a p-dimensional irreducible representation of the algebra $L^{\circ}(\delta)$ which is equivalent to $T_f \mid \mathfrak{H}_i(\delta)$ on $\mathfrak{H}_i(\delta)$, then this is naturally extended to a representation of the algebra $L_{\rho}^{\circ}(\delta)$, denoted by the same notation $U_0(f)$. In [5] it is proved that there exists a p-dimensional subspace $\mathscr{L}$ of $\mathbf{C}^d \otimes H$ which is invariant for all $W^{\Lambda}(f)$ $(f \in L_{\rho}^{\circ}(\delta))$ such that $U_0(f)$ is equivalent to $W^{\Lambda}(f) \mid \mathscr{L}$. Of course the representation $L^{\circ}(\delta) \ni f \rightarrow W^{\Lambda}(f) \mid \mathscr{L}$ of the algebra $L^{\circ}(\delta)$ is irreducible and equivalent to $U_0(f)$. On the other hand, $W^{\Lambda}(f)$ is equivalent to the representation $T_f^{\Lambda} \mid \mathfrak{H}_i^{\Lambda}(\delta)$ of the algebra $L^{\circ}(\delta)$, therefore $U_0(f)$ is equivalent to a subrepresentation of $T_f^{\Lambda} \mid \mathfrak{H}_i^{\Lambda}(\delta)$. Now, by Theorem 4, we can find closed G-invariant subspaces $\mathscr{H}_1$, $\mathscr{H}_2$ of $\mathfrak{H}^{\Lambda}$ satisfying the following conditions;

(a) $\mathscr{H}_1 \supset \mathscr{H}_2$, $E^{\Lambda}(\delta)\mathscr{H}_2 = \{0\}$,

(b) the naturally defined representation τ of G on the Banach space $\mathscr{H}_1/\mathscr{H}_2$ is topologically irreducible, and SF-equivalent to $\{\mathfrak{H}, T_x\}$.

Therefore we have proved the following main theorem.

Theorem 5. *Let G be a locally compact unimodular group with a continuous decomposition $G = SK$, where S is a closed subgroup and K a compact subgroup of G such that $S \cap K = \{1\}$. Let $\{\mathfrak{H}, T_x\}$ be a topologically iirreducble representation of G on a Banach space $\mathfrak{H}$ which contains $\delta \in \hat{K}$ finitely many times. Then,*

(I) *there exists a topologically irreducible representation $\Lambda(s)$ of S on a Banach space with the following property; for the induced representation $\{\mathfrak{H}^{\Lambda}, T_x^{\Lambda}\}$ of G, there exist closed G-invariant subspaces $\mathscr{H}_1$, $\mathscr{H}_2$ of $\mathfrak{H}^{\Lambda}$ such that*

(a) $\mathscr{H}_1 \supset \mathscr{H}_2$, $E^{\Lambda}(\delta)\mathscr{H}_2 = \{0\}$,

(b) *the naturally defined representation τ of G on the Banach space $\mathscr{H}_1/\mathscr{H}_2$ is topologically irreducible, and SF-equivalent to $\{\mathfrak{H}, T_x\}$.*

(II) *One of topologically irreducible representations $\Lambda(s)$ of S which satisfy (I) is algebraically equivalent to the naturally defined representation of S on $\mathfrak{H}_{\rho}/\mathscr{K}$, where $\rho(x) = \|T_x\|$ and $\mathscr{K}$ is a non-trivial maximal $L_{\rho}(S)$-invariant subspace of $\mathfrak{H}_{\rho}$.*

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