J. Math. Kyoto Univ. (JMKYAZ)
20-4 (1980) 709-721

On certain boundary points on the Wiener's compactifications of open Riemann surfaces

By

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(Received Aug. 4, 1979)

Introduction

On a bordered Riemann surface, where the border is attached a priori to the surface, one can characterize a point on the border by the existence of a halfdisclike neighbourhood. Extending this idea to any compactification of a Riemann surface, we may regard an ideal boundary point having a halfdisclike neighbourhood as a borderlike boundary point. (For the precise definition, see § 2.) In this paper we treat such borderlike boundary points on the Wiener's comapctifications of arbitrary Riemann surfaces. Known facts about the Wiener's compactification necessary in our investigation are summarized in § 1.

Now three specific classes of Riemann surfaces can be considered in connection with the set of borderlike boundary points. Among them, them, the class SO_W (resp. SO'_W) is defined as the class of Riemann suafaces such that the set of borderlike boundary points are coincident with (resp. dense in) the whole harmonic boundary. In § 2 we see that the class SO_W can be considered as the class of nearly finite bordered Riemann surfaces whenever the genus is finite (Proposition 4), and that the class SO'_W is precisely the class of Riemann surfaces such that the limit set of each corresponding fuchsian group has vanishing linear measure (Theorem 2).

In § 3 we consider the double of Riemann surfaces of above classes and show a system of strict inclusion relations between these classes and those of Riemann surfaces whose doubles belong to well-known classes. For the explicit statement, see Theorem 4.

Finally in § 4 we consider, as the third class, the one of Riemann surfaces which have no borderlike boundary points. This class coincides with the class of Riemann surfaces such that each corresponding fuchsian group is of the first kind, except for a few trivial surfaces. (See Proposition 6.) And we give a characterization of Riemann surfaces of genus zero belonging to this class (Theorem 6).

The author wishes to thank Professors Y. Kusunoki and T. Fujiie for very helpful conversations.

§ 1. Notations and known facts

In this section we shall state several known facts about the Wiener's compactification of an open Riemann surface which are used in the sequel. For the details, see for example [2] and [6]. For an open Riemann surface R, let R_w^* be the Wiener's comapctification of R, and $\Gamma_w=\Gamma_w(R)$ $(\subset R_w^*-R)$ be the harmonic boundary of R_w^*. This Wiener's compactification R_w^* is finer than other compactifications such as Martin's and Kerékjártó-Stoilow's. More precisely, the following fact is known.

Projection Theorem. *Let R_q^* be the Martin's or the Kerékjártó-Stoilow's compactification of R, and J the identical automorphism of R. Then J can be extended to a continuous mapping from R_w^* onto R_q^*.*

In particular, in case of the unit disk $U=\{|z|<1\}$, it is known that the Martin's compactification of U can be considered as the usual closure $\bar{U}$ in the complex plane. Hence we have the following

Corollary A. *The identical automorphism J_U can be extended to a continuous mapping from U_w^* onto $\bar{U}$.*

Next excluding several trivial cases, U can be considered as the universal covering surface of R. Let G $(=G(R))$ be a fuchsian group on U associated with R and π_G be the projection from U onto R with respect to G, that is, $\pi_G\circ g=\pi_G$ for every $g\in G$. Then we have the following

Covering Theorem. *Let R be a hyperbolic Riemann surface (i.e. $R\notin O_G$) and U and π_G be as above. Then π_G is a Fatou mapping. In particular, π_G can be extended to a continuous mapping from U_w^* onto R_w^*.*

Remark. If R is parabolic (i.e. $R\in O_G$), the above π_G is not a Fatou mapping. In fact, let $R\in O_G$ and suppose that π_G is a Fatou mapping. Then by [2] Satz 10.2, there would exist a non-polar closed set F in R such that $1_{\pi^{-1}(F)}$ is a potential. While, $1_{\pi^{-1}(F)}$ is a non-constant positive superharmonic function on U which is invariant under every g in G, so we have a non-constant positive superharmonic function on R. This is a contradiction, for $R\in O_G$.

Now as relations between R and its subregions, the following facts are well-known.

Localization Theorem. *Let D be a subregion on R, $F=R-D$ and η_D be the identical mapping from D into R. Then η_D can be extended to a continuous mapping from D_w^* into R_w^*, which we denote also by η_D. Moreover, letting $D_1=R_w^*-\bar{F}^w$, and $D_2=\eta_D^{-1}(D_1)$, η_D gives a homeomorphism from D_2 onto D_1 such that $\eta_D(\Gamma_w(D)\cap D_2)=\Gamma_w(R)\cap D_1$ and a measurable set A in $\Gamma_w(D)\cap D_2$ has positive* *harmonic measure if and only if so does* $\eta_D(A)$. *Here (and in the sequel)* $\bar{X}^w$ *means the closure of* X *in the Wiener's compactification.*

Characterization Theorem. *Every component of an open set* $D\ (\notin O_G)$ *on* R *is a region of type* SO_{HB} *if and only if* $\Gamma_w \cap (\bar{D}^w - \overline{(R-D)}^w)\ (=\Gamma_w \cap (\bar{D}^w - \overline{\partial D}^w)) = 0$, *where* ∂D *is the relative boundary of* D *in* R.

Moreover from Localization Theorem we can show the following

Proposition A. *Let* R *and* R' *be arbitrary Riemann surfaces,* D *be a subregion on* R *such that* ∂D *consists of countably many simple curves not accumulating to any point of* R, *and* f *be an analytic mapping from* R *into* R' *which is univalent on* D *and satisfies the condition that* $f(\partial D) = \partial D'$, *where* $D' = f(D)$. *Then the continuous extension of* f *gives a homeomorphism from* $(\bar{D}^w - \overline{\partial D}^w) \cap \Gamma_w(R)$ *onto* $(\bar{D}'^w - \overline{\partial D'}^w) \cap \Gamma_w(R')$.

Proof. First note that $f|_D$ can be extended to a homeomorphism from D^*_w onto D'^*_w, and it is known that $f(\Gamma_w(D)) = \Gamma_w(D')$ (cf. [6] IV 11.A). Now by Localization Theorem we have the continuous mapping $\tilde{\eta} = \eta_{D'} \circ f \circ \eta_D^{-1}$ from $D_1 = (\bar{D}^w - \overline{\partial D}^w) \cap \Gamma_w(R)$ into $\bar{D}'^w \cap \Gamma_w(R')$. For every $p \in D_1$ there is a positive bounded harmonic function u on D such that $u=0$ on ∂D and $u(\eta_D^{-1}(p)) = 1$. Set $u'=0$ on $R'-D'$ and $u' = u \circ f^{-1}$ on D', then because $f^{-1}(\partial D') = \partial D$, we see that u' is a positive bounded continuous subharmonic fuhction on R', hence can be extended to a continuous function on R'^*_w. And it is obvious that $u'=0$ on $\overline{(R'-D')}^w$ and $u'(\tilde{\eta}(p)) = 1$. Hence $\tilde{\eta}(p) \in D'_1 = (\bar{D}'^w - \overline{\partial D'}^w) \cap \Gamma_w(R')$. So $\tilde{\eta}(D_1) \subset D'_1$. Similarly we can show that the continuous mapping $\tilde{\eta}^{-1}$ maps D'_1 into D_1, hence the assertion follows. q.e.d.

In the sequel we shall use, as above, the same notation as the original one for the extended function or mapping.

§ 2. Borderlike boundary points

In this section we shall define certain classes of open Riemann surfaces which concern with the existence of such a part of the ideal boundary as the usual border. First for an arbitrary Riemann surface R, we call a point p in $\Gamma_w(R)$ *a borderlike boundary point,* or simply *a b-point,* of R if there exists a neighbourhood V of p in R^*_w with the properties: (i) $V = \overline{(V \cap R)}^w - \overline{\partial (V \cap R)}^w$, (ii) $V \cap R$ is simply connected, (iii) $\partial(V \cap R)$ is a open simple curve. We call such a neighbourhood of a b-point p *a distinguished neighbourhood of* p. Now set

$$d_w R = \{p \in \Gamma_w(R) : p \text{ is a } b\text{-point of } R\}.$$

Roughly speaking, $d_w R$ is the borderlike part of $\Gamma_w(R)$, and we can see easily that every point of $d_w R$ has vanishing harmonic measure.

Now set

$$SO_W = \{R \notin O_G : d_w R = \Gamma_w(R)\}, \quad \text{and}$$

$$SO'_W = \{R \notin O_G : d_w R \text{ is dense in } \Gamma_w(R)\}.$$

Obviously we have that $SO_W \subset SO'_W$. Also the following Proposition is easily seen.

Proposition 1. *A surface R belongs to SO'_W if and only if $\Gamma_w(R) - d_w R$ has vanishing harmonic measure.*

Proof. Let μ_w be the harmonic measure on Γ_w, then because $d_w R$ is clearly open, it holds that $\mu_w(d_w R) = \mu_w(\overline{d_w R}^w) = \mu_w(\Gamma_w)$, that is $\mu_w(\Gamma_w - d_w R) = 0$. q. e. d.

Proposition 2. *Let D be a subregion of R such ∂D consists of a countable number of disjoint simple curves which are not accumulating to any point of R. Further, if D is of type SO_{HB}, then D belongs to SO'_W as a Riemann surface.*

Proof. First it is easily seen that $\eta_D^{-1}(\partial D) \cap \Gamma_w(D)$ is contained in $d_w D$. Hence we can show the assertion by the definition of the SO_{HB} and Lemma 1 below. q. e. d.

Remark. If we permit components of ∂D to accumulate in R, then it is clear that the assertion of Proposition 2 does not hold.

Lemma 1. *Let I be an open arc on ∂U, U being the unit disc, and u a bounded harmonic function U. Then $u=0$ on $J_U^{-1}(I) \cap \Gamma_w(U)$ if and only if u has the vanishing bonndary value everywhere on I.*

Proof. It is well-known that u has the non-tangential boundary value u^* almost everywhere on ∂U and u is uniquely determined as the Poisson integral of u^*. Let $u_1^* = u^*$ on I and $=0$ on $\partial U - I$, $u_2^* = u^* - u_1^*$, and u_i the Poisson integral of u_i^* ($i=1, 2$). Then u_1 is harmonic on $\bar{C} - \bar{I}$ and u_2 is harmonic on $\bar{C} - (\partial U - I)$.

Now if $u^*=0$ on I (, i. e. $u_1 \equiv 0$), then it is ovbious (cf. Corollary A) that $u=0$ on $J_U^{-1}(I) \cap \Gamma_w$. On the other hand, suppose that $u=0$ on $J_U^{-1}(I) \cap \Gamma_w$. First note that $\mu_w(J_U^{-1}(\bar{I} - I)) = 0$ and $u_1 = 0$ on $J_U^{-1}(U - \bar{I})$. Hence by the assumption and the fact that $u_2 = u - u_1 = 0$ on I, we have that $u_1 = 0$ almost everywhere on $\Gamma_w(U)$ with respect to μ_w. Thus we conclude that $u_1 \equiv 0$, that is, $u \equiv u_2$ has the vanishing boundary value everywhere on I. q. e. d.

Next we note the following

Proposition 3. *SO_W is a proper subset of SO'_W.*

Before proving Proposition 3, we consider here a simply connected subregion D in the unit dise U such that $\gamma = \partial D$ is a simply (open) curve and D is contained in $\{z \in U : \operatorname{Re} z > 0\}$. First note that each end of γ clusters onto a closed subarc, say I_1 and I_2, on ∂U, and $(\bar{D} - \bar{\gamma}) \cap \partial U = \bar{D} \cap \partial U - (I_1 \cup I_2)$ is an open arc, say I, on

∂U (which may be empty). We shall denote by E the set $(\bar{D}^w-\bar{\gamma}^w)\cap\Gamma_w(U)$ and by I_i° the interior of I_i with respect to ∂U.

Lemma 2. *$E\cap J_{\bar{U}}^{-1}(I_i^\circ)=\emptyset$ for each i.*

Proof. If I_i is a single point, then there is nothing to prove. Hence suppose that $I_i^\circ\neq\emptyset$. Then for every $\zeta\in I_i^\circ$, we can find a sequence $\{\gamma_n\}_{n=1}^\infty$ of subarcs of γ such that for a sufficiently small $\varepsilon>0$ every component of $\{z\in U: |\zeta-z|<\varepsilon\}-\bigcup_n\gamma_n$ is relatively compact in U. In particular, every component of $D_\varepsilon=D\cap\{z\in U: |\zeta-z|<\varepsilon\}$ is also relatively compact in U, hence D_ε is of type SO_{HB}. Thus by Characterization Theorem we can conclude that $J_{\bar{U}}^{-1}(\zeta)\cap E=\emptyset$. Because $\zeta\in I_i^\circ$ is arbitrary, we have the assertion. q.e.d.

Lemma 3. *$J_U(E)=\bar{I}$.*

Proof. By Projection Theorem $J_U(\bar{D}^w)\subset\bar{D}$, and by Lemma 2 we have that $J_U(E)\subset\bar{D}\cap\partial U-(I_1^\circ\cup I_2^\circ)$. Also as in the proof of Lemma 2, we can see that $J_U(E)\cap(I_i-(I_i^\circ\cup\bar{I}))=\emptyset$ for each i, hence $J_U(E)$ is contained in $\bar{I}$.

On the other hand it is obvious that $I\subset J_U(E)$. And $s=1_{U-D}$ is a continuous superharmonic function on U such that $s=0$ on $J_{\bar{U}}^{-1}(I)$ and $s=1$ on $\bar{\gamma}^w$. Because s in continuous on U_W^*, we see that $\overline{J_{\bar{U}}^{-1}(I)}^w\cap\Gamma_w\subset E$, hence $J_U(E)\supset\bar{I}$, for $J_U(\overline{J_{\bar{U}}^{-1}(I)})^w$ is compact. q.e.d.

Corollary 1. *D is of type SO_{HB} if and only if $I=\emptyset$.*

Lemma 4. *$\overline{J_{\bar{U}}^{-1}(I)}^w\cap\Gamma_w=E$.*

Proof. In the proof Lemma 3, we have seen that $\overline{J_{\bar{U}}^{-1}(I)}^w\cap\Gamma_w\subset E$. If $F=E-\overline{J_{\bar{U}}^{-1}(I)}^w\cap\Gamma_w\neq\emptyset$, then because F is open in Γ_w, we could construct a non-constant bounded harmonic function u on U such that $u=0$ on Γ_w-F. But by Lemma 1 and 3 $u^*\equiv 0$ on $\partial U-(\bar{I}-I)$, hence $u\equiv 0$ on U, which is a contradiction. Thus we have the assertion. q.e.d.

Now Proposition 3 follows from the following

Example 1. Let $R=U-E$, where $E=\left\{\exp\left[-\frac{1}{n}+\sqrt{-1}\frac{1}{k}\right]: n\in\boldsymbol{Z}^+,\ k\in\boldsymbol{Z}\right\}$, then $R\in SO'_W-SO_W$. To show this, first consider the mapping $\tilde{\eta}=J_U\circ\eta_R$ from R_w^* onto $\bar{U}$, where η_R and J_U are as in Localization Theorem and Corollary A, respectively. Because E is polar, $\bar{E}^w\cap\Gamma_w(U)=\emptyset$ (cf. [2] Satz 9.7), we see from Localization Theorem that η_R gives a homeomorphism from $\Gamma_w(R)$ onto $\Gamma_w(U)$. In particular, $\tilde{\eta}^{-1}(\{1\})$ is not empty. Suppose that there is a point p in d_wR such that $\tilde{\eta}(p)=1$, and take a distinguished neighbourhood V_p of p. Then by Characterization Theorem $V_p\cap R$, hence $V_p\cap U$ is not of type SO_{HB}, so we conclude from Corollary 1 and Lemma 3 that $I=(\overline{V_p}-\partial V_p)\cap\partial U$ is a non-empty open subarc on ∂U such that $\bar{I}\ni 1$. Hence $I\ni\exp\left(\sqrt{-1}\frac{1}{k}\right)$ or $\exp\left(-\sqrt{-1}\frac{1}{k}\right)$ with a sufficiently large k, and E clusters to an interior point of I, which contradicts to the assumption that V_p is simply connected. Thus $\tilde{\eta}^{-1}(\{1\})\cap d_w R=0$, that is, $R\notin SO_W$.

On the other hand, it is obvious that $d_w(R)\supset\tilde{\eta}^{-1}\left(\partial U-\bigcup_k\left\{\exp\left(\sqrt{-1}\right)\frac{1}{k}\right\}\cup\{1\}\right)$ $\cap\Gamma_w(R)$ and every bounded harmonic function u on R (which can be considered as defined on U) such that $u^*=0$ on $\partial U-\bigcup_k\left\{\exp\left(\sqrt{-1}\frac{1}{k}\right)\right\}\cup\{1\}$ vanishes identically on U. Hence $d_w(R)$ is dense in $\Gamma_w(R)$ (cf. Lemma 1), that is $R\in SO'_W$.

For the class SO_W, we can show the following

Theorem 1. *Let $R\in SO_W$, R_s^* be the Kerékjártó-Stoilow's compactification of R and J be as in Projection Theorem. Then $J(\Gamma_w(R))$ is a finite set of points in R_s^*-R.*

Proof. Let $E=J(\Gamma_w)$ $(\subset R_s^*-R)$, and suppose that E contains infinite number of points. Because R_s^* is compact and metrizable and E is compact, we can find a sequence $\{P_n\}_{n=1}^{\infty}$ of points in E converging to a point P in E such that $P_n\neq P_m$ if $n\neq m$ and $P\neq P_n$ for every n. Let $\Gamma=J^{-1}(P)\cap\Gamma_w(R)$ and $\Gamma_n=J^{-1}(P_n)\cap\Gamma_w(R)$ for every n, then we see that every Γ_n is open and closed in the compact Hausdorff subspace $(\bigcup_n\Gamma_n)\cup\Gamma$ of R_w^*, and it is easily seen that $\overline{\bigcup_n\Gamma_n}^w-\bigcup_n\Gamma_n\neq 0$. Let $p\in\overline{\bigcup_n\Gamma_n}^w-\bigcup_n\Gamma_n$ $(\subset\Gamma)$ and suppose that $p\in d_w R$. Then for any distinguished neighbourhood V_p of p, $V_p\cap\Gamma_n\neq 0$ for infinitely many n. Hence we can see either that $V_p\cap R$ is not simply connected, or that $\partial(V_p\cap R)$ is not connected, hence not a simple arc (cf. Example 1). This is a contradiction, hence $(\overline{\bigcup_n\Gamma_n}^w-\bigcup_n\Gamma_n)\cap d_w R=\emptyset$. So $\Gamma_w(R)\neq d_w R$, which contradicts to the assumption that $R\in SO_W$. Thus we conclude that E consists of a finite number of points.

q. e. d.

Now the following is in a sense well-known.

Lemma 5. *Let E be a closed polar subset in U such that $\bar{E}\cap\partial U$ consists of a single point, say 1. Then $R=U-E\in SO_W$.*

Proof. Because E is polar, η_R gives a homeomorphism from $\Gamma_w(R)$ onto $\Gamma_w(U)$ (cf. Example 1). Hence we only need to find a distinguished neighbourhood V_p of P for every point p of $\Gamma_w(U)$ such that $V_p\cap\bar{E}=\emptyset$. Let F be the convex hull of E (minimal convex set containing E), then we can see that $\bar{F}\cap\partial U=\{1\}$ and $U-F$ is connected and simply connected. Hence $E'=\{z\in U: 1_F(z)\geqq 1/2\}$ is a closed set containing E such that $\partial E'$ is a simle analytic arc and $\bar{E}'\cap\partial U=\{1\}$. Also it is seen that $1_{E'}$ is a potential, hence by [2] Satz 9.7, $\bar{E}'^w\cap\Gamma_w(U)=\emptyset$.

Thus $U_w^*-\bar{E}'^w$ is a distinguished neighbourhood of every point of $\Gamma_w(U)$, hence of $\Gamma_w(R)$. q. e. d.

Proposition 4. *Let R be of finite genus g. Then $R\in SO_W$ if and only if R can be considered as a subregion on a compact Riemann surface of genus g such that ∂R consists of a finite set B of analytic simple closed curves and a relatively closed polar set E on the surface $\bar{R}-B$ such that $\bar{E}\cap B$ is a finite set of points.*

Proof. Because the "if" part can be shown similarly as Lemma 5, we shall prove only the converse. Suppose that $R\in SO_W$, then by Theorem 1, the set $J(\Gamma_w)$ is a finite set, say $\{P_n\}_{n=1}^m$, of points in R_s^*-R, where R_s^* and J is as in Theorem 1. Denote $J^{-1}(P_n)\cap\Gamma_w$ by Γ_n for every n, then $\Gamma_w=\bigcup_{n=1}^m\Gamma_n$, hence every Γ_n is open and closed in Γ_w. In particular, $\mu_w(\Gamma_n)>0$ for every n, where μ_w is the harmonic measure on Γ_w.

First from the assumption R can be embedded in a compact Riemann surface, say R_0, of the same finite genus as R and $R\notin O_G$. And it is seen that the component, say E_n, of R_0-R corresponding to P_n can not consist of a single point for every n, for if so, E_n is a polar set in R_w^*, hence $\mu_w(\Gamma_n)=0$, which is a contradiction. Thus we may assume that every ∂E_n (in R_0) is an analytic simple closed curve. Moreover it is easily seen that any other component of R_0-R than $\{E_n\}_{n=1}^m$ consists of a single point.

Next because Γ_w is compact and there is a distinguished neighbourhood of p for every $p\in\Gamma_w$ $(=d_wR$ by assumption), we can find an open neighbourhood V of Γ_w such that $J(V)\cap(R_s^*-R)=\{P_n\}_{n=1}^m$ (cf. the proof of Lemma 5). So $\overline{J^{-1}(R_s^*-R-\cup\{P_n\})}^w$ is polar in R_w^*, hence $E=S-R$ is also polar in S, where $S=R_0-\bigcup_{n=1}^m E_n$.

Finally if $\bar{E}\cap\partial S$ contains infinitely many points, hence has an accumulating point on ∂S, then by a slight modification of the argument in Example 1, we can show that $d_wR\neq\Gamma_w$. Hence $\bar{E}\cap\partial S$ is a finite set of points. q. e. d.

Now let R be a Riemann surface with the hyperbolic universal covering surface in the sequel, and $G=G(R)$ be a fuchsian group on U associated with R. We call R *of type I or type II*, respectively, according as G is of the first kind or the second kind (cf. [3], [5]). Also, if the limit set $L(G)$ $(\subset\partial U)$ of G has vanishing linear measure, we call R *of type II_0*. Recall that if R is of type II, then $R\notin O_G$ (cf. [3]) hence $\Gamma_w(R)\neq\emptyset$. And we can show the following

Theorem 2. *A Riemann surface R belongs to SO'_W if and only if R is of type II_0.*

Remark. It is known that the limit set of every non-elementary fuchsian group has positive capacity ([4]).

To prove Theorem 2, we shall state several lemmas. Let $R\notin O_G$, $G=G(R)$

and $B=\partial U-L(G)$. Then B is empty or consists of countably many open arcs in ∂U, which we denote by $\{I_n\}_{n=1}^{\infty}$.

Now let $p\in d_w R$ and V a distinguished neighbourhood of p, then because $V\cap R$ is simply connected, any component, say D, of $\pi_G^{-1}(V\cap R)$ is mapped conformally onto $V\cap R$ by $f_D=\pi_G|_D$, where π_G is as in Covering Theorem. Also it can be seen from Proposition A that f_D gives a homeomorphism from $(\bar{D}^w-\overline{\partial D}^w)\cap\Gamma_w(U)$ onto $V\cap\Gamma_w(R)$, for $V\cap\Gamma_w(R)=(\overline{(V\cap R)}^w-\overline{\partial(V\cap R)}^w)\cap\Gamma_w(R)$. And as before we have the following

Lemma 6. *Let $I=(\bar{D}-\overline{\partial D})\cap\partial U$, then I is an open arc contained in B, hence in some I_n.*

Proof. Because $D\cap g(D)=\emptyset$, hence $I\cap g(I)=\emptyset$ for every $g\in G-\{g_0\}$, we can see that I contains none of fixed points of elements of $G-\{g_0\}$, where g_0 is the identity of the group G. But the set of fixed points of elements of $G-\{g_0\}$ is dense in $L(G)$ and I is open. Hence we conclude that $I\cap L(G)=\emptyset$. q.e.d.

Lemma 7. *Let $E_w=\pi_G^{-1}(d_w R)\cap\Gamma_w(U)$, $F=J_U(E_w)$ and $B'=\bigcup_n \bar{I}_n$, then*

$$B\subset F\subset B'.$$

Proof. Take $q\in E_w$ arbitrary, and let $\pi_G(q)=p$, V a distinguished neighbourhood of p and D the component of $\pi_G^{-1}(V\cap R)$ corresponding to q. Then by Lemma 6, $I=(\bar{D}-\overline{\partial D})\cap\partial U\subset I_n$ with some n. Hence by Lemma 3, $J_U(q)\in\bar{I}\subset\bar{I}_n\subset B'$. Because q is arbitrary, we have that $F\subset B'$.

On the other hand, for any point $\zeta\in B$ we can take a sufficiently small closed disk U_ζ with the center ζ such that ∂U_ζ is orthogonal to ∂U, $U_\zeta\cap\partial U\subset B$, and $U_\zeta\cap g(U_\zeta)=\emptyset$ for every $g\in G-\{g_0\}$. Then π_G gives a conformal mapping from $U_\zeta\cap U$ onto $V_\zeta=\pi_G(U_\zeta\cap U)$ and $\bar{V}_\zeta{}^w-\overline{\partial V_\zeta}{}^w$ is a distinguished neighbourhood of every point of $\pi_G\circ J_U^{-1}(\zeta)\cap\Gamma_w(R)$ $(\neq\emptyset)$. Thus we conclude that $B\subset F$. q.e.d.

Lemma 8. *Let $B_w=J_U^{-1}(B)\cap\Gamma_w(U)$ and $B'_w=\bigcup_n\overline{J_U^{-1}(I_n)\cap\Gamma_w(U)}^w$, then $B_w\subset E_w\subset B'_w$.*

Proof. By Lemma 7 we see that B_w is an open subset in E_w. Let $q\in E_w$, and p and I be as above, then by Lemma 4 and 6 we have that $q\in\overline{J_U^{-1}(I)}^w\cap\Gamma_w(U)\subset B'_w$. Thus the assertion follows, q.e.d.

Proof of Theorem 2. First suppose that $R\in SO'_W$ and u be the Poisson integral of the function $u^*=0$ on B and $=1$ on $\partial U-B$ $(=L(G))$. Recall that R is of type II_0 if and only if $u\equiv 0$, and that $u\circ g\equiv u$ for every $g\in G$, hence u can be considered also as a function on R. Now by Lemma 1 $u=0$ on B_w, hence on B'_w. So by Lemma 8 we can see that $u=0$ on $d_w R$. Thus we conclude from the assumption that $u\equiv 0$, that is, R is of type II_0.

Conversely, suppose that R is of type II_0, and let u be a bounded harmonic function on R vanishing on d_wR. Then the bounded harmonic function $u\circ\pi_G$ on U vanishes on E_w, and by Lemma 1 and 7 $u^*=0$ on B, hence almost everywhere on ∂U by assumption. Thus $u\equiv 0$, and we conclude that $R\in SO'_W$. q. e. d.

Corollary 2. *Let D be a parabolic end of a Riemann surface such that ∂D is a finite set of simple closed curves. Then D is, as a Riemann surface, of type II_0.*

Proof. D is of type SO_{HB} by definition and we see from Proposition 2 that $D\in SO'_W$ as a Riemann surface. Thus the assertion follows from Theorem 2.
q. e. d.

Corollary 3. *Let D be as in Proposition 2. If D is of type SO_{HB}, then D is of type II_0 as a Riemann surface.*

Finally we show by an example that the class SO'_W is not quasiconformally invariant.

Example 2. Let f be a quasiconformal automorphism of U such that there is a compact set F on ∂U of linear measure zero whose image $f(F)$ has positive linear measure (cf. [1]). Let F' be the union of F and a discrete set on $\partial U-F$ such that every point of F is clustered by points of F' from both sides. And let E be a countable set of points on U such that $\bar{E}\cap\partial U=F'$, then $R=U-E$ and $R'=U-f(E)$ are mutually quasiconformally equivalent. And it is obvious that R is of type SO_{HB} as a subregion of $\bar{C}-\bar{E}$, hence by Proposition 2 we see that $R\in SO'_W$.

On the other hand, we can see as in Example 1 that $(J_U\circ\eta_{R'})^{-1}(f(F'))$, which obviously has positive harmonic measure, is disjoint from d_wR', hence from Proposition 1 we conclude that $R'\notin SO'_W$.

§ 3. On the double of a Riemann surface.

For a Riemann surface of type II, we can consider the double, which is denoted by $\hat{R}$, of R. In this section we are concerned with the double of a surface in SO_W or SO'_W. And set

$$DO_X=\{R \text{ is of type II and } \hat{R}\in O_X\}.$$

Then first we show the following

Theorem 3. *The following strict inclusion relations hold;*

$$DO_{HB}\longrightarrow SO'_W\longrightarrow DO_{AB},$$

where $A\rightarrow B$ means that A is a proper subset of B.

Proof. First suppose that $R\in DO_{HB}$, that is, $\Gamma_w(\hat{R})$ is empty or consists of a single point. Suppose $(\bar{R}^w-\overline{\partial R}^w)\cap\Gamma_w(\hat{R})\neq\emptyset$ or $(\hat{R}^*_w-\overline{(R\cup\partial R)}^w)\cap\Gamma_w(\hat{R})\neq\emptyset$, where the closure is taken in $\hat{R}_w^*$. Then $\Gamma_w(\hat{R})$ must contains at least two points which is a contradiction. Hence $(\bar{R}^w-\overline{\partial R}^w)\cap\Gamma_w(\hat{R})=\emptyset$, that is, R is of type SO_{HB} (as a subregion of $\hat{R}$), as is seen from Characterization Theorem. Because R satisfies the condition in Proposition 2, we conclude that $R\in SO'_W$.

Next suppose that $R\in SO'_W$. Then by Theorem 2, R is of type II_0. Let f be a bounded analytic function on $\hat{R}$ and $G=G(R)$. Then $\tilde{f}$ can be lifted to a bounded analytic function, say $\tilde{f}$, on $\boldsymbol{C}-L(G)$. Because $L(G)$ has vanishing linear measure on ∂U, $\tilde{f}$ can be extended to a bounded analytic function on $\boldsymbol{C}$, hence a constant. Thus f is also the constant, and we conclude that $R\in DO_{AB}$.

Finally the strictness follows from following examples. q. e. d.

Example 3. Let F and E be as in Example 2. Then we have seen that $R=U-E$ belongs to SO'_W. Here we can see that F has positive capacity. Then $\hat{R}=\boldsymbol{C}-\bar{E}\cup\{z: 1/\bar{z}\in E\}\notin O_{HB}$.

Remark. This Example 3 also show that the double of a subregion of type SO_{HB} (on some Riemann surface) does not necessarily belong to the class O_{HB}. By modifying [3] Example 2, we can also show that the double of a subregion of type SO_{HB} does not necessarily belong even to the class O_{AD}.

Example 4. Let R_0 be the famous example of a susface in $O_{HP}-O_G$ (cf. [6] V 7C), and D be a compact disk on R_0. Then $R=R_0-D$ belongs to U_{HB}, hence $\hat{R}\in U_{HB}\subset O_{AB}$. But because $\Gamma_w(R)$ contains a point having positive harmonic measure, we can see that $R\notin SO'_W$, for such a point can not belong to d_wR.

Moreover we can show the following

Proposition 5. *Let $R\in SO'_W$ and $\hat{R}$ be the double of R. Considering R as a subergion of $\hat{R}$, the relative boundary ∂R is a countable set of analytic simple arcs not accumulating any point of $\hat{R}$, which we denote by $\{\gamma_n\}_{n=1}^{\infty}$. Then*

$$\Gamma_w(\hat{R})\subset\overline{\partial R}^w-\bigcup_n\bar{\gamma}_n^w .$$

Proof. First note that $\hat{R}=(\boldsymbol{C}-L(G))/G$, where $G=G(R)$, and ∂R corresponds to $\partial U-L(G)$ under this covering. From the assumption and Theorem 2 we see that $L(G)$ has vanishing linear measure. Hence we can show that R is of type SO_{HB} as a subregion of $\hat{R}$. So we have from Charabterization Theorem that $(\bar{R}^w-\overline{\partial R}^w)\cap\Gamma_w(\hat{R})=\emptyset$. And similarly we have that $(\bar{R}'^w-\overline{\partial R'}^w)\cap\Gamma_w(\hat{R})=\emptyset$, where $R'=\hat{R}-\bar{R}$ (, hence in particular, $\partial R'=\partial R$). Thus we conclude that $\Gamma_w(\hat{R})\subset\overline{\partial R}^w$.

Next fix a γ_n and a component, say I_n, of $\partial U-L(G)$ corresponding to γ_n arbitrarily. Here note that if γ_n is compact then $\bar{\gamma}_n^w=\gamma_n$, hence it is obvious that $\bar{\gamma}_n^w\cap\Gamma_w(\hat{R})=\emptyset$. So suppose that γ_n is not compact. Then it is seen that I_n is projected univalently onto γ_n. Now let U_n be the open disk on $\boldsymbol{C}$ such that $U_n\cap\partial U=I_n$ and ∂U_n is orthogonal to ∂U. Because $g(U_n)\cap U_n=\emptyset$ for every g in $G-\{g_0\}$, we see that U_n is projected univalently onto a region, say D_n, on $\hat{R}$.

Then by Proposition A it holds that $(\bar{D}_n^w-\overline{\partial D}_n^w)\cap\Gamma_w(\hat{R})$ is homeomorphic to $(\bar{U}_n^w-\overline{\partial U}_n^w)\cap\Gamma_w(C-L(G))$, and it is easily seen that the latter set is empty and $\bar{\gamma}_n^w\subset(\bar{D}_n^w-\overline{\partial D}_n^w)$, hence we conclude that $\bar{\gamma}_n^w\cap\Gamma_w(\hat{R})=\emptyset$. Thus we have that $\Gamma_w(R)$ is contains in $\overline{\partial R}^w-\bigcup_n\bar{\gamma}_n^w$, q. e. d.

Next we show the follwing refinement of Theorem 1.

Lemma 9. *If $R\in SO_W$, then the number of components of ∂R (on $\hat{R}$) is finite.*

Proof. Let $\partial R=\overset{\infty}{\underset{n}{\bigcup}}\gamma_n$ and I_n, U_n, and D_n be as in the proof of Proposition 5. Then by Proposition A and Lemma 8 we can see that $\{\overline{(D_n\cap R)}^w-\overline{\partial(D_n\cap R)}^w)\}_{n=1}^\infty$ is an open covering of $d_wR=\Gamma_w(R)$. Because these sets are mutually disjoint and $\Gamma_w(R)$ is compact, the number of γ_n must be finite. q. e. d.

Theorem 4. *The following system of strict inclusion relation holds;*

$$SO_W \longrightarrow DO_G \longrightarrow DO_{HB} \longrightarrow SO'_W \longrightarrow DO_{AB}\,.$$

Proof. If $R\in SO_W$, then from Proposition 5 and Lemma 9, we see that $\Gamma_w(\hat{R})$ is contained in $\overline{\partial R}^w-\bigcup_n\bar{\gamma}_n^w=\overline{\partial R}^w-\overline{\bigcup_n\gamma_n}^w=\emptyset$, that is $\hat{R}\in O_G$. And let R be the surface in Example 1, then it is obvious that $\hat{R}\in O_G$, hence we conclude that $SO_W\to DO_G$. Next because the famous example of a surface in $O_{HP}-O_G$ have an anticonformal involusion which leaves analytic simple arcs fixed, we can see that $DO_G\to DO_{HB}$. Thus the assertion follows from Theorem 3. q. e. d.

Theorem 5. *The class SO_W is quasiconformally invariant.*

Proof. Let $R\in SO_W$ and R' be quasiconformally equivalent to R. Then $\hat{R}'$ is quasiconformally equivalent to $\hat{R}$. And by Theorem 4, we see that $\hat{R}$, hence $\hat{R}'$ belongs to O_G. Because R' is of type SO_{HB} on $\hat{R}'$, the subset $\eta_{\bar{R}'}^{-1}(\partial R')\cap\Gamma_w(R')$ of d_wR' is dense in $\Gamma_w(R')$.

On the other hand, we see from Lemma 9 that the number of components of $\partial R'$ (in R') is finite, and let $\partial R'=\{\gamma_i\}_{i=1}^n$. Then as in the proof of Proposition 5, we can conclude that $\bigcup_{i=1}^n\overline{\eta_{\bar{R}'}^{-1}(\gamma_i)}^w\cap\Gamma_w(R')\subset d_wR'$. Thus we have that $\Gamma_w(R')=\overline{\eta_{\bar{R}'}^{-1}(\partial R')}^w\cap\Gamma_w(R')=\bigcup_{i=1}^n\overline{\eta_{\bar{R}'}^{-1}(\gamma_j)}^w\cap\Gamma_w(R')=d_wR'$, that is, $R'\in SO_W$. q. e. d.

§4. On surfaces of type I

In this section we consider the other extremal class, that is, the class of Riemann surfaces R such that $d_wR=\emptyset$. First note the following

Proposition 6. *R is of type I if and only if R has the hyperbolic universal covering surface and $d_wR=\emptyset$.*

Proof. Assume that $R \notin O_G$, for if $R \in O_G$ then it is well-known that R is of type I. And then the assertion follows from Lemma 7. q.e.d.

Corollary 4 (*cf.* [3]). *If $R \in O_{HB}$ and has the hyperbolic universal covering surface, then R is of type I.*

Proof. Noting that $R \in O_{HB}$ if and only if $\Gamma_w(R)$ consists at most a single point, we have the assertion. q.e.d.

Corollary 5. *The class $\{R : d_w R = 0\}$ is quasiconformally invariant.*

Now it seems difficult to determine the class of general Riemann surfaces of type I (, or such that $d_w R = 0$). But we can determine the class of planar such surfaces. For this purpose, in the sequel, we always suppose that R is planar and a (not necessarily extremal) vertical slit region (, that is, $\boldsymbol{C} - R$ consists of vertical slits and points). Here recall that it is well-known that any planar region is conformally equivalent to such a region. We call a point $p \in \boldsymbol{C} - R$ *a faint point* if the component of $\boldsymbol{C} - R$ containing p is either $\{P\}$, or a slit, say E, such that each component of $V - E$ contains a point in $\boldsymbol{C} - \{R \cup E\}$ for every neighbourhood V of p in $\boldsymbol{C}$. Then we have the following

Theorem 6. *A planar Riemann surface R is of type I if and only if $\boldsymbol{C} - R$ contains at least three points and every point of $\boldsymbol{C} - R$ is a faint point.*

Proof. Suppose that R is of type II, then $G = G(R)$ is of the second kind, hence there is a disk V on $\boldsymbol{C}$ such that $\bar{V} \cap L(G) = 0$, ∂V is orthogonal to ∂U, and π_G gives a conformal mapping from $V_1 = V \cap U$ into R, which in turn can be extended to a continuous mapping, say f, from $\bar{V}_1^w$ into R_s^*. Now it is easily seen that the image of $E_1 = \bar{V}_1^w - (\bar{V}_1^w \cap U)$ by f is a single point, say P_V, of $R_s^* - R$. Let E be the component of $\boldsymbol{C} - R$ corresponding to P_V. If E consists of a single point, say c, then $f \equiv c$ on E_1, hence by Lemma 1, f has non-tangential boundary value c on $\bar{V} \cap \partial U$ $(\neq 0)$. Then $f \equiv c$ on V_1, which is a contradiction. Hence E is a slit.

Thus by mapping $\boldsymbol{C} - E$ conformally onto U, we may assume that $|f| \equiv 1$ on E_1. Then taking a smaller disk if necessary, we may assume that $\pi_G|_{V_1}$ is a continuous mapping from $\bar{V}_1$ into $\bar{U}$. And it is easily seen that there is a non-faint point on E (cf. [3] Example 2). Because the converse is obvious, we have the assertion. q.e.d.

Corollary 6. *If E is a totally disconnected compact set on $\boldsymbol{C}$ and contains at least three points, then $\boldsymbol{C} - E$ is of type I.*

Remark. There is a Riemann surface R having only one ideal boundary component such that $R \notin O_{AD}$ and R is of type I. In fact, let E be a countable subset of U such that E does not cluster in U and $\bar{E} \supset \partial U$, and consider the twosheeted covering surface R of U branching at every point of E. Then we can show that R is of type I, and other assertions are clear.

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