J. Math. Kyoto Univ. (JMKYAZ)
20-4 (1980) 691-708

On Rees algebras over Buchsbaum rings

By

Shiro GOTO and Yasuhiro SHIMODA

(Communicated by Prof. M. Nagata, Aug. 2, 1979)

1. Introduction

The purpose of this paper is to prove the following

Theorem (1.1). *For a Noetherian local ring A with maximal ideal m the following two conditions are equivalent.*

(1) *A is a Buchsbaum ring and $H_{\mathfrak{m}}^{i}(A)=(0)$ for $i\neq 1$,* $\dim A$.

(2) *The Rees algebra $R(\mathfrak{q})=\bigoplus_{n\geq 0}\mathfrak{q}^{n}$ is a Cohen-Macaulay ring for every parameter ideal $\mathfrak{q}$ of A.*

In this case $R(\mathfrak{q}^{n})$ is also a Cohen-Macaulay ring for every parameter ideal $\mathfrak{q}$ and for every integer $n>0$.

Here $H_{\mathfrak{m}}^{i}(A)$ denotes the i-th local cohomology module. Now recall the definition of Buchsbaum rings. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then A is called a *Buchsbaum* ring if the difference

$$l_{A}(A/\mathfrak{q})-e_{A}(\mathfrak{q})$$

is an invariant $I(A)$ of A not depending on the particular choice of a parameter ideal $\mathfrak{q}$ of A, where $e_{A}(\mathfrak{q})$ denotes the multiplicity of A relative to $\mathfrak{q}$. This is equivalent to the condition that the equality

$$(a_{1}, a_{2}, \cdots, a_{i}):a_{i+1}=(a_{1}, a_{2}, \cdots, a_{i}):\mathfrak{m}$$

holds for every $0\leqq i<d$ and for every system $a_{1}, a_{2}, \cdots, a_{d}$ of parameters for A, where $d=\dim A$ (c.f. [14], Satz 10). The theory of Buchsbaum rings has started from an answer of W. Vogel [18] to a problem of D. A. Buchsbaum [2] (c.f. p. 228). The basic properties of Buchsbaum rings were discovered by J. Stückrad and W. Vogel ([14] and [15]), and our theorem (1.1) guarantees that certain Buchsbaum rings are characterized by the behaviour of Rees algebras relative to parameter ideals. This is a new point of view in the study of Buchsbaum singularities (c.f. [4] and [16]).

Recently G. Valla [17] proved that, if a Noetherian local ring A is Cohen-Macaulay, then so is the Rees algebra $R(\mathfrak{q}^{n})$ for every parameter ideal $\mathfrak{q}$ of A and for every integer $n>0$ (c.f. [3] for a shorter proof). Our research was motivated by a partial answer of Y. Shimoda [12] to the question whether the converse of Valla's result is true. He solved this problem in case A is an integral local domain of dimension 2. A complete answer comes from our theorm (1.1) and is stated as follows.

Corollary (1.2). *Let A be a Noetherian local ring and assume that depth $A\neq 1$. Then A is a Cohen-Macaulay ring if and only if so is the Rees algebra $R(\mathfrak{q}^n)$ for every parameter ideal $\mathfrak{q}$ of A and for every integer $n>0$.*

Of course this is not true in case depth $A=1$ (c.f. (5.6)).

Our theorem (1.1) will be proved in Section 4. In Section 2 we will give some results on Buchsbaum modules which we need in Section 4 in order to compute the depth of Rees algebras relative to parameter ideals. In Section 3 we will show that every Noetherian local ring is at least Buchsbaum if all the Rees algebras relative to parameter ideals are Cohen-Macaulay. In Section 5 we assume that A is a Buchsbaum local ring with canonical module K_A. The aim of this section is to prove that K_A is a Cohen-Macaulay module if (and only if) $H_{\mathfrak{m}}^i(A)=(0)$ for every $1<i<\dim A$. Of course this is the same condition as (1) of Theorem (1.1) in case depth $A>0$.

In the following we denote by A a Noetherian local ring of dimension d and with maximal ideal $\mathfrak{m}$. $H_{\mathfrak{m}}^i(\cdot)$ will always stand for the i-th local cohomology functor.

2. $U(aM)$ as a Buchsbaum module

First we recall the definition of Buchsbaum rings, or more generally that of Buchsbaum modules. Let M be a finitely generated A-module of dimension r.

Definition (2.1) M is called a *Buchsbaum* module if the difference

$$l_A(M/\mathfrak{q}M)-e_M(\mathfrak{q})$$

is an invariant $I(M)$ of M not depending on the choice of a parameter ideal $\mathfrak{q}$ of M, where $e_M(\mathfrak{q})$ denotes the multiplicity of M relative to $\mathfrak{q}$.

This is equivalent to the condition that every system $a_1, a_2, \cdots, a_r$ of parameters for M is a weak sequence, i.e., the equality

$$(a_1, a_2, \cdots, a_i)M : a_{i+1}=(a_1, a_2, \cdots, a_i)M : \mathfrak{m}$$

holds for every $0\leqq i<r$ (c.f. [14], Satz 10). A Noetherian local ring is said to be a Buchsbaum ring if it is a Buchsbaum module over itself.

Examples (2.2). (1) A finitely generated module M is Cohen-Macaulay if and only if M is Buchsbaum and $I(M)=0$.

(2) Suppose that A is a Buchsbaum ring with $\dim A=d>0$. Then the maximal ideal $\mathfrak{m}$ of A is a Buchsbaum module and $I(\mathfrak{m})=I(A)+d-1$ (c.f. [5], (2.4)).

In particular, if A is a Cohen-Macaulay ring of dimension 2, then $\mathfrak{m}$ is a Buchsbaum module with $I(\mathfrak{m})=1$. This seems to be a simplest example of Buchsbaum modules which are not Cohen-Macaulay.

(3) Suppose that $d=\dim A>0$ and let V be a t-dimensional vector space over $A/\mathfrak{m}$. Let $B=A\times V$ be the idealization of V by A. Then B is a Buchabaum ring if and only if so is A. In this case $\dim B=d$ and $I(B)=I(A)+t$ (c.f. [5], (2.8)). In particular, if A is a Cohen-Macaulay ring, then B is a Buchsbaum ring with $I(B)=t$. Thus for arbitrary integers $d>0$ and $t\geqq 0$ there is a Buchsbaum local ring B such that

$$\dim B=d \quad \text{and} \quad I(B)=t.$$

(4) Suppose that A is a Buchsbaum ring which is not Cohen-Macaulay. Then any formal power series ring over A is not a Buchsbaum ring (c.f. [11], (4.6)).

(5) Let k be a field and $R=k[|s, t|]$ a formal power series ring. We put $A=k[|s^4, s^3t, st^3, t^4|]$ in R. Then it is well-known that A is not a Cohen-Macaulay ring. However A is Buchsbaum and $I(A)=1$.

(6) Let k be a field and $R=k[|x_1, x_2, \cdots, x_d, y_1, y_2, \cdots, y_d|]$ a formal power series ring. We put $A=R/\mathfrak{a}$ where

$$\mathfrak{a}=(x_1, x_2, \cdots, x_d)\cap(y_1, y_2, \cdots, y_d).$$

Then A is a d-dimensional Buchsbaum ring and $I(A)=d-1$. Moreover

$$H_{\mathfrak{m}}^i(A)=\begin{cases} A/\mathfrak{m} & (i=1) \\ (0) & (i\neq 1,\ d) \end{cases}$$

(c.f. [10], p. 469).

(7) Let $d>0$ and $h_0, h_1, \cdots, h_{d-1}\geqq 0$ be integers. Then there exists a Buchsbaum local ring A such that

$$\dim A=d \quad \text{and} \quad \dim_{A/\mathfrak{m}} H_{\mathfrak{m}}^i(A)=h_i \qquad \text{for all } 0\leqq i<d.$$

(Here $\dim_{A/\mathfrak{m}} H_{\mathfrak{m}}^i(A)$ denotes the dimension of $H_{\mathfrak{m}}^i(A)$ as a vector space over $A/\mathfrak{m}$. See (2.6), (3).) Moreover it is known that, if $h_0=0$ (resp. $d\geqq 2$ and $h_0=h_1=0$), then the ring A may also be taken to be an integral domain (resp. a normal domain). See [5].

Let M be a finitely generated A-module.

Definition (2.3). $\mathrm{Assh}_A M=\{\mathfrak{p}\in\mathrm{Supp}_A M;\ \dim A/\mathfrak{p}=\dim_A M\}$. Notice that, for an element a of $\mathfrak{m}$, $\dim_A M/aM=\dim_A M-1$ if and only if $a\notin \bigcup_{\mathfrak{p}\in\mathrm{Assh}_A M}\mathfrak{p}$. Let N be an A-submodule of M and

$$N=\bigcap_{\mathfrak{p}\in\mathrm{Ass}_A M/N} N(\mathfrak{p})$$

a primary decomposition of N in M.

Definition (2.4). $U_M(N)=\bigcap_{\mathfrak{p}\in\mathrm{Assh}_A M/N} N(\mathfrak{p})$.

As every $\mathfrak{p}\in \mathrm{Assh}_A M/N$ is a minimal element of $\mathrm{Supp}_A M/N$, this definition does not depend on the choice of a primary decomposition of N. Usually we denote $U_M(N)$ simply by $U(N)$. Notice that $\mathrm{Ass}_A M/U(N)=\mathrm{Assh}_A M/N$.

Now we are prepared to state the main result of this section.

Theorem (2.5). *Suppose that M is a Buchsbaum A-module of dimension $r>0$. Let a be an element of $\mathfrak{m}$ and assume that $\dim_A M/aM=r-1$. Then*

(1) *$U(aM)$ is also a Buchsbaum module and $\dim_A U(aM)=r$.*

(2) $$I(U(aM))=\begin{cases} I(M)-(r-1)\cdot\dim_{A/\mathfrak{m}} H_{\mathfrak{m}}^1(M) & (r\geqq 2)\\ 0 & (r=1).\end{cases}$$

(3) $$\mathrm{depth}_A U(aM)=\begin{cases} \min\{2\leqq i\leqq r\,;\, H_{\mathfrak{m}}^i(M)\neq(0)\} & (r\geqq 2 \text{ and } \mathrm{depth}_A M>0)\\ 0 & (r\geqq 2 \text{ and } \mathrm{depth}_A M=0)\\ 1 & (r=1).\end{cases}$$

In order to prove this assertion we need some results on Buchsbaum modules.

Lemma (2.6). *Suppose that M is a Buchsbaum A-module of dimension $r>0$. Let $U=U((0))$ in M. Then*

(1) $\mathrm{Assh}_A M=\mathrm{Ass}_A M\setminus\{\mathfrak{m}\}$.

(2) *M/U is again a Buchsbaum module with $\dim_A M/U=r$ and $\mathrm{depth}_A M/U>0$.*

(3) *$\mathfrak{m}\cdot H_{\mathfrak{m}}^i M=(0)$ for all $0\leqq i<r$. In particular $H_{\mathfrak{m}}^0(M)=[0:\mathfrak{m}]_M=U$.*

(4) *Let a be an element of $\mathfrak{m}$ and assume that $\dim_A M/aM=r-1$. Then M/aM is again a Buchsbaum module. Moreover*

$$H_{\mathfrak{m}}^i(M/aM)=H_{\mathfrak{m}}^i(M)\oplus H_{\mathfrak{m}}^{i+1}(M)$$

for all $0\leqq i<r-1$, and there is an exact sequence

$$0\longrightarrow H_{\mathfrak{m}}^{r-1}(M)\longrightarrow H_{\mathfrak{m}}^{r-1}(M/aM)\longrightarrow H_{\mathfrak{m}}^r(M)\stackrel{a}{\longrightarrow} H_{\mathfrak{m}}^r(M)\longrightarrow 0$$

of A-modules.

(5) $I(M)=\sum_{i=0}^{r-1}\binom{r-1}{i}\cdot\dim_{A/\mathfrak{m}} H_{\mathfrak{m}}^i(M)$. *(Here $\dim_{A/\mathfrak{m}} H_{\mathfrak{m}}^i(M)$ denotes the dimension of $H_{\mathfrak{m}}^i(M)$ as a vector space over $A/\mathfrak{m}$.)*

Proof. (1) This is trivial since $\mathrm{Ass}_A M/U=\mathrm{Assh}_A M$ and since $\mathfrak{m}\cdot U=(0)$ (c.f. [14], Satz 5).

(2) See [14], Korollar 13.

(3) See [10], Hilfsatz 3 and its proof.

(4) See [14], Korollar 6 for the first assertion. Consider the second one. Fist notice that $U=[0:a]_M$. Then we have two exact sequences

$$0\longrightarrow U\longrightarrow M\stackrel{g}{\longrightarrow} aM\longrightarrow 0 \quad\text{and}\quad 0\longrightarrow aM\stackrel{f}{\longrightarrow} M\longrightarrow M/aM\longrightarrow 0$$

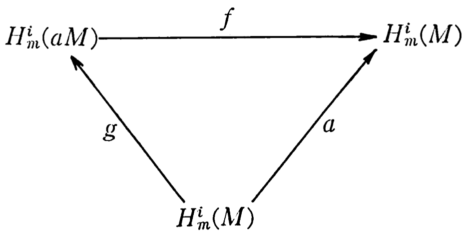
where $f\cdot g(x)=ax$ for all $x\in M$. Apply the functor $H_{\mathfrak{m}}^i(\cdot)$ to the second sequence and we get a long exact sequence

$$(*)\quad \cdots \longrightarrow H^i_{\mathfrak{m}}(aM) \xrightarrow{f} H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(M/aM) \longrightarrow H^{i+1}_{\mathfrak{m}}(aM) \xrightarrow{f} H^{i+1}_{\mathfrak{m}}(M) \longrightarrow \cdots .$$

On the other hand, as $\mathfrak{m}\cdot U=(0)$,

$$H^i_{\mathfrak{m}}(M) \xrightarrow{g} H^i_{\mathfrak{m}}(aM)$$

is an epimorphism (res. an isomorphism) for $i=0$ (resp. $i>0$). Thus, considering the following commutative triangle



we conclude that the map $H^i_{\mathfrak{m}}(aM) \xrightarrow{f} H^i_{\mathfrak{m}}(M)$ is 0 for every $0 \leqq i < r$ because $aH^i_{\mathfrak{m}}(M)=(0)$ for $0 \leqq i < r$ by (3). Hence from the long exact sequence (*) we obtain exact sequences

$$(**)\quad 0 \longrightarrow H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(M/aM) \longrightarrow H^{i+1}_{\mathfrak{m}}(M) \longrightarrow 0 \quad (0 \leqq i < r-1)$$

and

$$0 \longrightarrow H^{r-1}_{\mathfrak{m}}(M) \longrightarrow H^{r-1}_{\mathfrak{m}}(M/aM) \longrightarrow H^r_{\mathfrak{m}}(M) \xrightarrow{a} H^r_{\mathfrak{m}}(M) \longrightarrow 0 .$$

Of course the sequence (**) splits as $H^i_{\mathfrak{m}}(M/aM)$ is a vector space over $A/\mathfrak{m}$.

(5) See [10], Satz 2.

The following striking result is due to J. Stückrad and W. Vogel [16] and J. Stückrad [13].

Lemma (2.7). *Let M be a finitely generated A-module. If the canonical homomorphisms*

$$h^i_M : \mathrm{Ext}^i_A(A/\mathfrak{m}, M) \longrightarrow H^i_{\mathfrak{m}}(M) = \varinjlim_n \mathrm{Ext}^i_A(A/\mathfrak{m}^n, M)$$

are surjective for all $i \neq \dim_A M$, then M is a Buchsbaum module. In case A is a regular local ring, the converse is also true.

Proof of Theorem (2.5).

If $r=1$, then the assertions are trivial because $aM=M/H^0_{\mathfrak{m}}(M)$ and $U(aM)=aM$ in this case. Now consider the case $r \geqq 2$. First we will show that

(a) $H^0_{\mathfrak{m}}(U(aM))=H^0_{\mathfrak{m}}(M)$,

(b) $H^1_{\mathfrak{m}}(U(aM))=(0)$,

and (c) $H^i_{\mathfrak{m}}(U(aM))=H^i_{\mathfrak{m}}(M)$ $(i \geqq 2)$.

Apply the functor $H^i_{\mathfrak{m}}(\cdot)$ to the following two exact sequences

$$(*)\quad 0 \longrightarrow aM \longrightarrow U(aM) \longrightarrow U(aM)/aM \longrightarrow 0,$$

and

$$0 \longrightarrow U((0)) \longrightarrow M \longrightarrow aM \longrightarrow 0.$$

Then we see

$$H^i_{\mathfrak{m}}(aM)=H^i_{\mathfrak{m}}(U(aM)) \quad \text{for} \quad i\geqq 2$$

and

$$(**)\quad H^i_{\mathfrak{m}}(M)=H^i_{\mathfrak{m}}(aM) \qquad \text{for} \quad i>0,$$

because $U((0))$ and $U(aM)/aM$ $(=U_{M/aM}((0)))$ are vector spaces over $A/\mathfrak{m}$ (c. f. [14], Satz 5). Summarizing them we have the assertion (c). Moreover, applying the functor $H^i_{\mathfrak{m}}(\cdot)$ to the exact sequence

$$0 \longrightarrow U(aM) \longrightarrow M \longrightarrow M/U(aM) \longrightarrow 0,$$

we have the assertion (a) because $\operatorname{depth}_A M/U(aM)>0$ (c. f. (2.6), (2)).

Now let us prove the assertion (b). Apply the functor $H^i_{\mathfrak{m}}(\cdot)$ to the sequence (*) and we have an exact sequence

$$(***)\quad 0 \longrightarrow H^0_{\mathfrak{m}}(U(aM)) \longrightarrow U(aM)/aM \longrightarrow H^1_{\mathfrak{m}}(aM) \longrightarrow H^1_{\mathfrak{m}}(U(aM)) \longrightarrow 0.$$

On the other hand we see

$$U(aM)/aM=H^0_{\mathfrak{m}}(M)\oplus H^1_{\mathfrak{m}}(M)$$

by (2.6), (4) because $U(aM)/aM=U_{M/aM}((0))$ and $U_{M/aM}((0))=H^0_{\mathfrak{m}}(M/aM)$. Thus, recalling $H^1_{\mathfrak{m}}(aM)=H^1_{\mathfrak{m}}(M)$ by (**) and $H^0_{\mathfrak{m}}(U(aM))=H^0_{\mathfrak{m}}(M)$ by (a), we conclude that

$$H^1_{\mathfrak{m}}(U(aM))=(0)$$

by the exact sequence (***) of vector spaces over $A/\mathfrak{m}$.

Now let us prove Theorem (2.5). It follows from (a), (b) and (c) that

$$\sum_{i=0}^{r-1}\binom{r-1}{i}\cdot\dim_{A/\mathfrak{m}}H^i_{\mathfrak{m}}(U(aM))=I(M)-(r-1)\cdot\dim_{A/\mathfrak{m}}H^1_{\mathfrak{m}}(M)$$

(c. f. (2.6), (5)). Moreover we have by (a), (b) and (c) that

$$\dim_A U(aM)=r$$

and

$$\operatorname{depth}_A U(aM)=\begin{cases}\min\{2\leqq i\leqq r\,;\,H^i_{\mathfrak{m}}(M)\neq(0)\} & (\operatorname{depth}_A M>0)\\ 0 & (\operatorname{depth}_A M=0).\end{cases}$$

Thus it suffices to show that $U(aM)$ is a Buchsbaum module. For this purpose, after passing through the completion of A, we may assume without loss of generality that A is a regular local ring.

Now apply the functor $\operatorname{Ext}^i_A(A/\mathfrak{m}, \cdot)$ to the sequence (*) and we obtain a commutative diagram

$$\begin{array}{ccccc}\operatorname{Ext}^i_A(A/\mathfrak{m}, aM) & \longrightarrow & \operatorname{Ext}^i_A(A/\mathfrak{m}, U(aM)) & \longrightarrow & \operatorname{Ext}^i_A(A/\mathfrak{m}, U(aM)/aM)\\ \downarrow h^i_{aM} & & \downarrow h^i_{U(aM)} & & \downarrow\\ H^i_{\mathfrak{m}}(aM) & \longrightarrow & H^i_{\mathfrak{m}}(U(aM)) & \longrightarrow & H^i_{\mathfrak{m}}(U(aM)/aM)=(0)\end{array}$$

with exact rows for every $0<i<r$, where the vertical maps are canonical homomorphisms. On the other hand, as $aM=M/U((0))$ is a Buchsbaum module of dimension r (c. f. (2.6), (2)), we see by (2.7) that h_{aM}^{i} is a surjection for every $i\neq r$. Hence so is $h_{U(aM)}^{i}$ by the above diagram and we conclude again by (2.7) that $U(aM)$ is also a Buchsbaum module. This completes the proof of our assertion.

Corollary (2.8). *Under the same situation as* (2.5), $U(aM)$ *is a Cohen-Macaulay module if and only if* $r=1$ *or*

$$H_{\mathfrak{m}}^{i}(M)=(0) \qquad \textit{for} \quad i\neq 1, r.$$

Remark (2.9). Let M be a finitely generated A-module of dimension 2 and suppose that $\mathfrak{m}\cdot H_{\mathfrak{m}}^{0}(M)=\mathfrak{m}\cdot H_{\mathfrak{m}}^{1}(M)=(0)$. Then $U(aM)$ is a Buchsbaum module with

$$I(U(aM))=\dim_{A/\mathfrak{m}}H_{\mathfrak{m}}^{0}(M)$$

for every element a of $\mathfrak{m}$ such that $\dim_{A}M/aM=1$. But such M is not necessarily a Buchsbaum module. For example, let

$$A=k[|x, y, z, w|]/(x, y)\cap(z, w)\cap(x^{2}, y, z^{2}, w)$$

where $k[|x, y, z, w|]$ is a formal power series ring over a field k. Then $\dim A=2$ and $H_{\mathfrak{m}}^{0}(A)=H_{\mathfrak{m}}^{1}(A)=k$. As W. Vogel mentioned in [19], A is not a Buchsbaum ring.

3. In this section we will prove the following

Theorem (3.1). *Suppose that the Rees algebra* $R(\mathfrak{q})=\bigoplus_{n\geq 0}\mathfrak{q}^{n}$ *is a Cohen-Macaulay ring for every parameter ideal* $\mathfrak{q}$ *of* A. *Then* A *is a Buchsbaum ring.*

For this purpose we need a few lemmas. Of course we may assume $d=\dim A>0$. For a moment let $a_{1}, a_{2}, \cdots, a_{d}$ be a system of parameters for A. We put $\mathfrak{q}=(a_{1}, a_{2}, \cdots, a_{d})$ and $R=R(\mathfrak{q})$. Notice that the ring R can be canonically identified with the graded A-subalgebra

$$A[a_{1}X, a_{2}X, \cdots, a_{d}X]$$

of $A[X]$, where X is an indeterminate over A. By $\mathfrak{M}$ we denote the unique graded maximal ideal of R, i. e.,

$$\mathfrak{M}=(\mathfrak{m}, a_{1}X, a_{2}X, \cdots, a_{d}X).$$

Recall that

$$\dim R=\dim R_{\mathfrak{M}}=d+1$$

(c. f. [9] and [17]). We put

$$\mathfrak{Q}=(a_{1}, a_{2}+a_{1}X, \cdots, a_{d}+a_{d-1}X, a_{d}X).$$

Lemma (3.2). $\mathfrak{M}=\sqrt{\mathfrak{Q}}$. *In particular,*

$$a_{1}, a_{2}+a_{1}X, \cdots, a_{d}+a_{d-1}X, a_{d}X$$

is a system of parameters for $R_{\mathfrak{M}}$.

Proof. Suppose $a_iX\in\sqrt{\mathfrak{Q}}$ for some i. Then $a_{i-1}X\in\sqrt{\mathfrak{Q}}$, as

$$(a_{i-1}X)^2=(a_i+a_{i-1}X)\cdot a_{i-1}X-a_{i-1}\cdot a_iX\,.$$

Hence it follows by induction on i that $a_iX\in\sqrt{\mathfrak{Q}}$ for all $1\leqq i\leqq d$, which yields also $\mathfrak{q}\subset\sqrt{\mathfrak{Q}}$ as $a_i+a_{i-1}X\in\mathfrak{Q}$ by definition. Thus $\mathfrak{M}\subset\sqrt{\mathfrak{Q}}$, which implies $\mathfrak{M}=\sqrt{\mathfrak{Q}}$.

Corollary (3.3). *R is a Cohen-Macaulay ring if and only if*

$$a_1,\ a_2+a_1X,\ \cdots a_d+a_{d-1}X,\ a_dX$$

is an $R_{\mathfrak{M}}$*-sequence.*

Proof. If $a_1,\ a_2+a_1X,\ \cdots a_d+a_{d-1}X,\ a_dX$ forms an $R_{\mathfrak{M}}$-sequence, then $R_{\mathfrak{M}}$ is a Cohen-Macaulay local ring by (3.2). Thus R is globally a Cohen-Macaulay ring by virtue of [9], Theorem. The converse is trivial.

Lemma (3.4). *Suppose that R is a Cohen-Macaulay ring. Then*

$$(a_1,\ a_2,\ \cdots,\ a_{d-1}):a_d=(a_1,\ a_2,\ \cdots,\ a_{d-1}):a_d^n$$

for every integer $n>0$.

Proof. It suffices to show $\mathfrak{a}:a_d^2\subset\mathfrak{a}:a_d$ where $\mathfrak{a}=(a_1,\ a_2,\ \cdots,\ a_{d-1})$. If $d=1$, this is trivial as a_1 is A-regular. Consider the case $d=2$. Let r be an element of A and assume that $ra_2^2=sa_1$ for some $s\in A$. Then we have $s\in a_2R$ since $a_2(r\cdot a_2X)=s\cdot a_1X$ and since $a_2,\ a_1X$ is an R-sequence by (3.3). Let $s=ta_2$ for some $t\in A$, and we have $ra_2=ta_1$ as $ra_2^2=as=a_2(ta_1)$.

For the case $d\geqq3$ we need the following

Claim. Let c be an element of $\mathfrak{a}$ and assume that $c\in\mathfrak{q}^2$. Then $c\cdot a_d^{d-3}\in\mathfrak{a}\mathfrak{q}^{d-2}$.

Proof of the claim.

Let us express $c=\sum_{i=1}^{d-1}a_ib_i$ and put

$$I=(\{a_i-a_{i+1}X\}_{1\leq i\leq d-2},\ a_1X)\,.$$

Then

$$a_d\cdot ca_d^{d-3}X^{d-1}=\sum_{i=1}^{d-1}a_iX\cdot b_ia_d^{d-2}X^{d-2}$$

and $a_j\equiv a_{j+1}X \bmod I$ for every $1\leqq j\leqq d-2$. Observe the equations

$$a_iX\cdot b_ia_d^{d-2}X^{d-2}\equiv a_{i-1}X\cdot b_ia_d^{d-2}X^{d-3}\equiv\cdots\equiv a_1X\cdot b_ia_d^{d-2}X^{d-i-1}\equiv0 \bmod I$$

$(1\leqq i\leqq d-1)$, and we have

$$a_d\cdot ca_d^{d-3}X^{d-1}\in I\,.$$

On the other hand we see by (3.3) that $a_d,\ a_{d-1}-a_dX,\ \cdots,\ a_1-a_2X,\ a_1X$ is an $R_{\mathfrak{M}}$-sequence. Thus $ca_d^{d-3}X^{d-1}\in IR_{\mathfrak{M}}$, i.e.,

$$f\cdot ca_d^{d-3}X^{d-1}\in I$$

for some $f\in R\backslash\mathfrak{M}$. Now let us express

$$(*)\quad f\cdot ca_d^{d-3}X^{d-1}=\sum_{i=1}^{d-2}(a_i-a_{i+1}X)g^{(i)}+a_1X\cdot g$$

with $g^{(i)}$, $g\in R$. Let $g_j^{(i)}$ (resp. g_j) denote the coefficient of the term X^j in $g^{(i)}$ (resp. g). Then, comparing the term X^{d-1} in the equation (*), we see

$$f_0\cdot ca_d^{d-3}=\sum_{i=1}^{d-2}a_ig_{d-1}^{(i)}-\sum_{i=1}^{d-2}a_{i+1}g_{d-2}^{(i)}+a_1g_{d-2}\,.$$

As f_0 is a unit of A, this equation implies that

$$ca_d^{d-3}\in\mathfrak{a}\mathfrak{q}^{d-2}$$

as desired.

Proof of Lemma (3.4) (*Continued*).

Let r be an element of A and assume that $ra_d^2\in\mathfrak{a}$. We put

$$I=(a_1,\ \{a_i-a_{i-1}X\}_{2\leq i\leq d-1},\ a_dX)\,.$$

First notice that

$$\begin{aligned}ra_d\cdot(a_d-a_{d-1}X)^{d-2}&=ra_d\cdot\sum_{i=0}^{d-2}(-1)^i\binom{d-2}{i}a_d^{d-2-i}\cdot(a_{d-1}X)^i\\&=ra_d^{d-1}+\sum_{i=1}^{d-2}(-1)^i\binom{d-2}{i}a_dX\cdot ra_{d-1}^ia_d^{d-2-i}X^{i-1}\\&\equiv ra_d^{d-1}\ \mathrm{mod}\ I\,.\end{aligned}$$

On the other hand, as $ra_d^2\in\mathfrak{a}\cap\mathfrak{q}^2$, we have $ra_d^{d-1}\in\mathfrak{a}\mathfrak{q}^{d-2}$ by the above claim. Now let us express $ra_d^{d-1}=\sum_{i=1}^{d-1}a_ib_i$ with $b_i\in\mathfrak{q}^{d-2}$. Then, since $a_j\equiv a_{j-1}X\ \mathrm{mod}\ I$, we observe that $a_ib_i\equiv a_{i-1}\cdot b_iX\equiv\cdots\equiv a_1\cdot b_iX^{i-1}\equiv 0\ \mathrm{mod}\ I$ $(1\leqq i\leqq d-1)$, which implies

$$ra_d^{d-1}\equiv 0\ \mathrm{mod}\ I\,.$$

Thus

$$ra_d\cdot(a_d-a_{d-1}X)^{d-2}\in I\,,$$

and so we have $ra_d\in IR_{\mathfrak{M}}$ because $a_d-a_{d-1}X$ is $R_{\mathfrak{M}}/IR_{\mathfrak{M}}$-regular by (3.3). Hence

$$f\cdot ra_d\in I$$

for some $f\in R\backslash\mathfrak{M}$. Comparing the constant term similarly as in the proof of the above claim, we see that

$$ra_d\in\mathfrak{a}$$

as required. This completes the proof of our assertion.

Proof of Theorem (3.1).

Let $a_1, a_2, \cdots, a_{d-1}, a$ and $a_1, a_2, \cdots, a_{d-1}, b$ be two systems of parameters for A. In order to prove A is a Buchsbaum ring, it suffices to show

$$\mathfrak{a} : a = \mathfrak{a} : b$$

where $\mathfrak{a} = (a_1, a_2, \cdots, a_{d-1})$ (c. f. [14], Satz 5). Of course, by the symmetry between a and b, we have only to prove

$$\mathfrak{a} : a \subset \mathfrak{a} : b .$$

Let $n > 0$ be an integer such that $b^n \in \mathfrak{a} + aA$ and express $b^n = \sum_{i=1}^{d-1} a_i x_i + ax$ with x_i, $x \in A$.

Now let r be an element of A and assume that $ra \in \mathfrak{a}$. Then we have $rb^n \in \mathfrak{a}$ as $rb^n = \sum_{i=1}^{d-1} a_i \cdot rx_i + ra \cdot x$ and as $ra \in \mathfrak{a}$ by the assumption. Hence $r \in \mathfrak{a} : b^n$ and so $r \in \mathfrak{a} : b$ by (3.4). Thus we have $\mathfrak{a} : a \subset \mathfrak{a} : b$ as desired, and this completes the proof of Theorem (3.1).

Remark (3.5). *A* Noetherian local ring A is not necessarily a Buchsbaum ring even if $R(\mathfrak{q})$ is a Cohen-Macaulay ring for some parameter ideal $\mathfrak{q}$ of A. For example, let $k[|s, t|]$ be a formal power series ring over a field k and put

$$A = k[|s^2, st, t, s^5|]$$

in $k[|s, t|]$. Then $R((s^4, t))$ is a Cohen-Macaulay ring but A is not a Buchsbaum ring.

4. The depth of $R(\mathfrak{q})$

In this section suppose that A is a Buchsbaum ring and let $a = a_1, a_2, \cdots, a_d$ be a system of parameters for A. We put

$$\mathfrak{q} = (a_1, a_2, \cdots, a_d) \quad \text{and} \quad R = R(\mathfrak{q}) .$$

For a finitely generated R-module E we denote $\dim_{R_{\mathfrak{M}}} E_{\mathfrak{M}}$ (resp. $\operatorname{depth}_{R_{\mathfrak{M}}} E_{\mathfrak{M}}$) simply by

$$\dim E \quad (\text{resp. } \operatorname{depth} E)$$

where $\mathfrak{M} = (\mathfrak{m}, a_1X, a_2X, \cdots, a_dX)$, the unique graded maximal ideal of R. The main purpose of this section is to prove the following

Theorem (4.1).

$$\operatorname{depth} R = \begin{cases} \operatorname{depth}_A U(aA) + 1 & (\operatorname{depth} A > 0) \\ 0 & (\operatorname{depth} A = 0) . \end{cases}$$

We put $\mathfrak{q}_i = (a_1, a_2, \cdots, a_i)$ $(0 \leqq i \leqq d)$ and begin with

Lemma (4.2). *$U(\mathfrak{q}_i) \cap \mathfrak{q}^n = \mathfrak{q}_i \mathfrak{q}^{n-1}$ for every integer $n > 0$ and for every $0 \leqq i \leqq d$.*

Proof. This is trivial in case $i = d$.
Suppose $i < d$ and that the assertion holds for $i+1$. First notice that

$$U(\mathfrak{q}_i) \cap \mathfrak{q}^n \subset U(\mathfrak{q}_{i+1}) \cap \mathfrak{q}^n .$$

In fact, if $i=d-1$, then $U(\mathfrak{q}_{i+1})=\mathfrak{q}$. Hence $U(\mathfrak{q}_i)\cap\mathfrak{q}^n\subset\mathfrak{q}^n=U(\mathfrak{q}_{i+1})\cap\mathfrak{q}^n$ clearly. In case $i<d-1$, we have $U(\mathfrak{q}_i)=\mathfrak{q}_i:\mathfrak{m}$ and $U(\mathfrak{q}_{i+1})=\mathfrak{q}_{i+1}:\mathfrak{m}$ by (2.6), (3). So $U(\mathfrak{q}_i)\subset U(\mathfrak{q}_{i+1})$ as $\mathfrak{q}_i\subset\mathfrak{q}_{i+1}$, and the claim follows.

Let x be an element of $U(\mathfrak{q}_i)\cap\mathfrak{q}^n$. Then $x\in U(\mathfrak{q}_{i+1})\cap\mathfrak{q}^n$ as we have remarked above. On the other hand we know

$$U(\mathfrak{q}_{i+1})\cap\mathfrak{q}^n=\mathfrak{q}_i\mathfrak{q}^{n-1}+a_{i+1}\mathfrak{q}^{n-1}$$

by the assumption on i. Thus x may be expressed as

$$x=y+a_{i+1}f$$

where $y\in\mathfrak{q}_i\mathfrak{q}^{n-1}$ and $f\in\mathfrak{q}^{n-1}$. Recalling $a_{i+1}f=x-y\in U(\mathfrak{q}_i)$, we get $f\in U(\mathfrak{q}_i)$ because a_{i+1} is $A/U(\mathfrak{q}_i)$-regular.

If $n=1$, then $a_{i+1}f\in\mathfrak{q}_i$ since $U(\mathfrak{q}_i)=\mathfrak{q}_i:\mathfrak{m}$. Therefore $x=y+a_{i+1}f\in\mathfrak{q}_i$, and so we have

$$U(\mathfrak{q}_i)\cap\mathfrak{q}=\mathfrak{q}_i$$

in this case. Now suppose $n\geqq 2$ and assume that

$$U(\mathfrak{q}_i)\cap\mathfrak{q}^{n-1}=\mathfrak{q}_i\mathfrak{q}^{n-2}.$$

Then, as $f\in U(\mathfrak{q}_i)\cap\mathfrak{q}^{n-1}$, we see $f\in\mathfrak{q}_i\mathfrak{q}^{n-2}$ and hence $a_{i+1}f\in\mathfrak{q}_i\mathfrak{q}^{n-1}$. Thus $x=y+a_{i+1}f\in\mathfrak{q}_i\mathfrak{q}^{n-1}$ as required. This completes the proof of our assertion.

Corollary (4.3). *$U(aA)\cap\mathfrak{q}^n=a\mathfrak{q}^{n-1}$ for every integer $n>0$.*

Let $h:R\to A$ be the canonical projection. We denote $U(aA)$ by ${}_hU(aA)$ when we consider it via h an R-module. Moreover we regard ${}_hU(aA)$ as a graded module trivially, i.e.,

$$[{}_hU(aA)]_0=U(aA) \quad\text{and}\quad [{}_hU(aA)]_n=(0) \quad\text{for}\quad n\neq 0.$$

Proposition (4.4). *There is an exact sequence*

$$0\longrightarrow {}_hU(aA)\longrightarrow R/(aX)\longrightarrow R((q+U(aA))/U(aA))\longrightarrow 0$$

of graded R-modules.

Proof. Let $f:R\to R((q+U(aA))/U(aA))$ be the canonical epimorphism and put $I=\operatorname{Ker} f$. Then $I\ni aX$, and I is a graded ideal of R. Let z be an element of I_n $(n>0)$ and express $z=bX^n$ $(b\in\mathfrak{q}^n)$. Then $b\in U(aA)$ and so, by (4.3), we have $b=ca$ for some $c\in\mathfrak{q}^{n-1}$. Hence

$$z=aX\cdot cX^{n-1}$$

and this implies that $\sum_{n>0}I_n=(aX)$. Of course $I_0=U(aA)$ and it is a routine work to check

$${}_hU(aA)\cong I/(aX)$$

as graded R-modules.

Cerollary (4.5) ([1]). *R is a Cohen-Macaulay ring if so is A.*

This is proved by induction on dim A. But we omit the detail as this fact has been already known by J. Barshay [1].

We note

Lemma (4.6). *Suppose that* depth $A>0$. *Then aX is a non-zerodivisor of R.*

Lemma (4.7). dim ${}_hU(aA)=\dim A$ *and* depth ${}_hU(aA)=\mathrm{depth}_A U(aA)$.

Proof. These follow from the isomorphisms

$$H^i_{\mathfrak{M}}({}_hU(aA))\cong {}_hH^i_m(U(aA)),$$

where ${}_hH^i_{\mathfrak{M}}(U(aA))$ denotes $H^i_m(U(aA))$ considered an R-module via $h:R\to A$. For the first assertion recall that $\dim_A U(aA)=\dim A$ by (2.5), (1).

Proposition (4.8). *Suppose that* dim $A=2$. *Then, if* depth $A>0$, *R is a Cohen-Macaulay ring.*

Proof. We put $\bar{A}=A/U(aA)$ and $\bar{q}=\mathfrak{q}\bar{A}$. Then $R(\bar{\mathfrak{q}})$ is a Cohen-Macaulay ring of dimension 2 by (4.5) because $\bar{A}$ is a Cohen-Macaulay local ring of dimension 1. Consider this fact together with the exact sequence

$$0\longrightarrow {}_hU(aA)\longrightarrow R/(aX)\longrightarrow R(\bar{\mathfrak{q}})\longrightarrow 0$$

given by (4.4). Then we see depth $R/(aX)=2$ as depth ${}_hU(aA)=2$ by (2.5) and (4.7). Therefore depth $R=3$ since aX is a regular element of R (c.f. (4.6)). Thus $R_{\mathfrak{M}}$ is a Cohen-Macaulay local ring. Hence the assertion follows from [9], Theorem.

Remark (4.9). Let A be the example given by (2.2), (5). Then M. Hochster and J. Roberts [9] showed that $R(\mathfrak{q})$ is a Cohen-Macaulay ring for the parameter ideal $\mathfrak{q}=(s^4, t^4)$, and mentioned by this example that a ring retract of a Cohen-Macaulay ring is not necessarily Cohen-Macaulay. Our result (4.8) guarantees that the Rees algebra $R(\mathfrak{q})$ is a Cohen-Macaulay ring for *every* parameter ideal $\mathfrak{q}$ of A. See also Y. Shimoda [12].

Proof of Theorem (4.1).

(1) (depth $A>0$) We have to show

$$\mathrm{depth}\, R=\mathrm{depth}_A U(aA)+1\,.$$

Assume the contray and choose $d=\dim A$ as small as possible among such counterexamples. We put

$$\bar{A}=A/U(aA)\,,\quad \bar{\mathfrak{q}}=(\mathfrak{q}+U(aA))/U(aA)\quad \text{and}\quad \bar{R}=R(\bar{\mathfrak{q}})\,.$$

Then $d\geqq 3$ by (4.8) and, by the minimality of d, we see

$$\mathrm{depth}\,\bar{R}=\mathrm{depth}_{\bar{A}} U(b\bar{A})+1$$

where $b=a_2$ mod $U(aA)$. We put $s=\mathrm{depth}_{\bar{A}} U(b\bar{A})$. Notice $s\geqq 2$ by (2.5).

If $d=s+1$, then depth $\bar{R}=d$ and so, by the exact sequence

$$0 \longrightarrow {}_hU(aA) \longrightarrow R/(aX) \longrightarrow \bar{R} \longrightarrow 0$$

given by (4.4), we have

$$\operatorname{depth} R/(aX) = \operatorname{depth} {}_hU(aA).$$

Hence $\operatorname{depth} R = \operatorname{depth}_A U(aA)+1$, but this contradicts the choice of d. Thus we conclude $s<d-1$.

Claim. $H^i_{\mathfrak{m}}(A)=(0)$ for $2 \leqq i \leqq s$ and $H^{s+1}_{\mathfrak{m}}(A) \neq (0)$.

Proof of claim.

Apply the functor $H^i_{\mathfrak{m}}(\cdot)$ to the following two exact sequences

$$0 \longrightarrow b\bar{A} \longrightarrow U(b\bar{A}) \longrightarrow U(b\bar{A})/b\bar{A} \longrightarrow 0$$

and

$$0 \longrightarrow U(aA)/aA \longrightarrow A/aA \longrightarrow \bar{A} \longrightarrow 0.$$

Then we get

$$H^i_{\mathfrak{m}}(U(b\bar{A})) = H^i_{\mathfrak{m}}(\bar{A}) \quad (\text{resp. } H^i_{\mathfrak{m}}(\bar{A}) = H^i_{\mathfrak{m}}(A/aA))$$

for every $i \geqq 2$ by the first (resp. second) sequence. Thus we see by (2.6), (4) that

$$H^i_{\mathfrak{m}}(U(b\bar{A})) = H^i_{\mathfrak{m}}(A) \oplus H^{i+1}_{\mathfrak{m}}(A)$$

for every $2 \leqq i < d-1$.

Suppose $s=2$. If $H^2_{\mathfrak{m}}(A) \neq (0)$, then $\operatorname{depth}_A U(aA)=2$ by (2.5). So we have $\operatorname{depth} R/(aX)=2$ by (4.4), because $\operatorname{depth} \bar{R}=3$. This asserts $\operatorname{depth} R=3=\operatorname{depth}_A U(aA)+1$, which is impossible. Thus we conclude $H^2_{\mathfrak{m}}(A)=(0)$ in this case. Of course

$$H^3_{\mathfrak{m}}(A) = H^2_{\mathfrak{m}}(U(b\bar{A})) \neq (0).$$

Now consider the case $s \geqq 3$. Then

$$H^i_{\mathfrak{m}}(U(b\bar{A})) = H^i_{\mathfrak{m}}(A) \oplus H^{i+1}_{\mathfrak{m}}(A) = (0)$$

for $2 \leqq i \leqq s-1$ and

$$H^s_{\mathfrak{m}}(U(b\bar{A})) = H^s_{\mathfrak{m}}(A) \oplus H^{s+1}_{\mathfrak{m}}(A) \neq (0).$$

Hence $H^i_{\mathfrak{m}}(A)=(0)$ $(2 \leqq i \leqq s)$ and $H^{s+1}_{\mathfrak{m}}(A) \neq (0)$ as required.

Now back to the proof of Theorem (4.1). It follows from the above claim and (2.5) that

$$\operatorname{depth}_A U(aA) = s+1.$$

On the other hand

$$\operatorname{depth} \bar{R} = s+1.$$

Hence $\operatorname{depth} \bar{R} = \operatorname{depth}_A U(aA)$, which implies by (4.6) that

$$\operatorname{depth} R = \operatorname{depth}_A U(aA)+1$$

—— this is a contradiction.

(2) ($\operatorname{depth} A=0$) Let x be a non-zero element of A such that $x\mathfrak{m}=(0)$. Then $x\mathfrak{M}=(0)$ and so we have $\operatorname{depth} R=0$ in this case.

Example (4.10). Let d and t be integers with $d>t\geqq 2$. Then there is a Buchsbaum ring A such that

$$\dim A=d \quad \text{and} \quad \operatorname{depth} A=t$$

(c.f. (2.2), (7) and [16], Theorem 3). In this case

$$\dim R(\mathfrak{q})=d+1 \quad \text{and} \quad \operatorname{depth} R(\mathfrak{q})=t+1$$

for every parameter ideal $\mathfrak{q}$ of A. Of course $R(\mathfrak{q})$ is not a Cohen-Macaulay ring.

Corollary (4.11). *Suppose that* depth $A>0$. *Then the following condition are equivalent.*

(1) $H_{\mathfrak{m}}^{i}(A)=(0)$ *for* $i\neq 1, d$.

(2) *The Ress algebra* $R(\mathfrak{q})$ *is a Cohen-Macaulay ring for every parameter ideal* $\mathfrak{q}$ *of* A.

(3) *There is a parameter ideal* $\mathfrak{q}$ *of* A, *for which the Rees algebra* $R(\mathfrak{q})$ *is a Cohen-Macaulay ring.*

(4) *The* A*-module* $U(aA)$ *is a Cohen-Macaulay module for every element* a *of* $\mathfrak{m}$ *such that* $\dim A/aA=d-1$.

(5) *There is an element* a *of* $\mathfrak{m}$ *such that* $\dim A/aA=d-1$, *for which the* A*-module* $U(aA)$ *is Cohen-Macaulay.*

If A *has the canonical module* K_A, *one may add further*

(6) K_A *is a Cohen-Macaulay module.*

Proof. The equivalence of the conditions from (1) to (5) follows from (2.5), (2.8) and (4.1). The proof of the equivalence of the conditions (1) and (6) will be found in the next section.

Proof of Theorem (1.1).

The equivalence of the conditions (1) and (2) is now clear by (3.1) and (4.11). Now consider the last assertion. Let $\mathfrak{q}$ be a parameter ideal of A and let $n>0$ an integer. Then $R(\mathfrak{q}^n)=\bigoplus_{i\geqq 0}\mathfrak{q}^{in}$ is a direct summand of $R(\mathfrak{q})$ as an $R(\mathfrak{q}^n)$-module. Moreover $R(\mathfrak{q})$ is a module-finite extension of $R(\mathfrak{q}^n)$. Thus the result follows from [7], Proposition 12. This completes the proof of Theorem (1.1).

5. The canonical modules of Buchsbaum rings

The purpose of this section is to prove the equivalence of the conditions (1) and (6) in Corollary (4.11). Now suppose that A is a Buchsbaum ring.

First we recall the definition of canonical modules. Let $\hat{A}$ (resp. E) denote the completion of A (resp. the injective envelope $E_{\hat{A}}(\hat{A}/\hat{\mathfrak{m}})$ of the residue field $\hat{A}/\hat{\mathfrak{m}}$).

Definition (5.1) ([6]). An -module K_A is called the canonical module if

$$\hat{A}\otimes_A K_A\cong \operatorname{Hom}_{\hat{A}}(H_{\hat{\mathfrak{m}}}^{d}(\hat{A}), E)$$

as $\hat{A}$-modules.

The canonical module is uniquely determined up to isomorphisms if it exists. In case A is a homomorphic image of a Gorenstein local ring B, then A has the canonical module and it is given by

$$K_A = \mathrm{Ext}_B^s(A, B)$$

where $s = \dim B - \dim A$ (c.f. [6], Satz 5.12).

In what follows we assume that A has the canonical module K_A. Recall that $\dim K_A = \dim A$ (c.f. [6]).

Lemma (5.2). *Suppose A is complete and $d = \dim A > 0$. Let a be an element of $\mathfrak{m}$ such that $\dim A/aA = d-1$. Then*

(1) *a is K_A-regular. In particular,* depth $K_A > 0$.

(2) *There is an exact sequence*

$$0 \longrightarrow K_A/aK_A \longrightarrow K_{A/aA} \longrightarrow H_{\mathfrak{m}}^{d-1}(A) \longrightarrow 0$$

of A-modules.

Proof. Apply the functor $\mathrm{Hom}_A(\cdot, E)$ to the sequence given by (2.6), (4). Then we obtain an exact sequence

$$0 \longrightarrow K_A \xrightarrow{\ a\ } K_A \longrightarrow K_{A/aA} \longrightarrow H_{\mathfrak{m}}^{d-1}(A) \longrightarrow 0\,,$$

because $H_{\mathfrak{m}}^{d-1}(A) \cong \mathrm{Hom}_A(H_{\mathfrak{m}}^{d-1}(A), E)$. This yields all the results we claimed.

Corollary (5.3). depth $K_A \geqq 2$ *if* $\dim A \geqq 2$. *In particular K_A is a Cohen-Macaulay module if* $\dim A = 2$.

Proof. We may assume that A is complete. Let a be an element of $\mathfrak{m}$ such that $\dim A/aA = \dim A - 1$. Then depth $K_{A/aA} > 0$, and K_A/aK_A is contained in $K_{A/aA}$ (c.f. (5.2)). Hence depth $K_A \geqq 2$ as a is K_A-regular. The second assertion is obvious.

The equivalence of the conditions (1) and (6) in Corollary (4.11) comes from the next

Theorem (5.4). *K_A is a Cohen-Macaulay module if and only if*

$$H_{\mathfrak{m}}^i(A) = (0) \qquad \text{for} \quad 1 < i < \dim A\,.$$

Proof. We may assume A is complete. By (5.3) we may assume further $d = \dim A \geqq 3$. Let a be an element of $\mathfrak{m}$ such that $\dim A/aA = d-1$.

First notice that K_A is a Cohen-Macaulay module if and only if $K_{A/aA}$ is a Cohen-Macaulay module and $H_{\mathfrak{m}}^{d-1}(A) = (0)$. For, suuppose that K_A is Cohen-Macaulay. Then depth $K_A/aK_A = d-1 \geqq 2$. On the other hand depth $K_{A/aA} \geqq 2$ by (5.3). Hence we see by the exact sequence given in (5.2), (2) that

$$H_{\mathfrak{m}}^{d-1}(A) = (0)\,,$$

because the length of $H_{\mathfrak{m}}^{d-1}(A)$ is finite. Thus $K_{A/aA}=K_A/aK_A$, and hence $K_{A/aA}$ is Cohen-Maulay. The converse is trivial.

In case $d=3$, we have by (5.3) and the above fact that K_A is a Cohen-Macaulay module if and only if $H_{\mathfrak{m}}^2(A)=(0)$. Now suppose $d\geqq 4$ and assume that our assertion holds for $d-1$. Then

K_A is a Cohen-Macaulay module
$\Leftrightarrow K_{A/aA}$ is Cohen-Macaualy, and $H_{\mathfrak{m}}^{d-1}(A)=(0)$
$\Leftrightarrow H_{\mathfrak{m}}^i(A/aA)=(0)$ for $1<i<d-1$, and $H_{\mathfrak{m}}^{d-1}(A)=(0)$ (by the assumption on d)
$\Leftrightarrow H_{\mathfrak{m}}^i(A)\oplus H_{\mathfrak{m}}^{i+1}(A)=(0)$ for $1<i<d-1$ (by (2.6), (4))
$\Leftrightarrow H_{\mathfrak{m}}^i(A)=(0)$ for $1<i<d$.

This completes the proof of Theorem (5.4).

Question (5.5). Is K_A a Buchsbaum module? If $\dim A=3$, this is true and $I(K_A)=\dim_{A/\mathfrak{m}}H_{\mathfrak{m}}^2(A)$.

Proof. As usual we may assume that A is complete. Let a be an element of $\mathfrak{m}^2$ such that $\dim A/aA=2$. Then a is K_A-regular and there is an exact sequence

$$0 \longrightarrow K_A/aK_A \longrightarrow K_{A/aA} \longrightarrow H_{\mathfrak{m}}^2(A) \longrightarrow 0$$

of A-modules (c. f. (5.2)). Apply the functor $H_{\mathfrak{m}}^i(\cdot)$ to this sequence and we have that

$$(*) \quad H_{\mathfrak{m}}^1(K_A/aK_A)=H_{\mathfrak{m}}^2(A),$$

as $K_{A/aA}$ is a Cohen-Macaulay module of dimension 2 by (5.3) and as $\mathfrak{m}\cdot H_{\mathfrak{m}}^2(A)=(0)$. This yields by [10], Satz 3 that K_A/aK_A is a Buchsbaum module, and hence so is K_A by the choice of a (c. f. [19], Theorem). For the second assertion notice that

$$H_{\mathfrak{m}}^1(K_A/aK_A)=H_{\mathfrak{m}}^2(K_A)$$

(c. f. (2. 6), (4)). Then we see by the equality (*) that

$$I(K_A)=\dim_{A/\mathfrak{m}}H_{\mathfrak{m}}^2(A)$$

because $I(K_A)=\dim_{A/\mathfrak{m}}H_{\mathfrak{m}}^2(K_A)$ by (2.6), (5). This completes the proof of our assertion.

We will close this paper with the following

Theorem (5.6). *Let $d\geqq 2$ and $h\geqq 1$ be integers. Then there is a Buchsbaum complete local domain A which satisfies the following conditions:* (1) $\dim A=d$. (2) $H_{\mathfrak{m}}^i(A)=(0)$ *for* $i\neq 1, d$. (3) $\dim_{A/\mathfrak{m}}H_{\mathfrak{m}}^1(A)=h$. *Hence depth* $A=1$. (4) *The normalization B of A is a regular local ring and $\mathfrak{m}B\subset A$. In particular Sing $A=\{\mathfrak{m}\}$.* (5) $K_A=B$.

Proof. Let K/k be an extension of fields with $[K:k]=h+1$ and $B=K[|x_1, x_2, \cdots, x_d|]$ a formal power series ring over K. We put

$$A=\{f\in B\,;\, f(0, 0, \cdots, 0)\in k\}$$

and $P=k[|x_1, x_2, \cdots, x_d|]$. Then A is an intermediate ring between P and B. Moreover A is a Noetherian complete local ring with $\dim A=d$, because B is a module-finite extension of P. Let $\mathfrak{m}$ (resp. $\mathfrak{n}$) denote the maximal ideal of A (resp. B). Then $\mathfrak{n}=\mathfrak{m}$, since

$$\mathfrak{n}=\{f\in B\,;\,f(0, 0, \cdots, 0)=0\}\subset A$$

by definition. In particular $\mathfrak{m}B\subset A$ and so B coincides with the normalization of A.

Consider the exact sequence

$$0\longrightarrow A\longrightarrow B\longrightarrow B/A\longrightarrow 0$$

of A-modules. Then, applying the functor $H_{\mathfrak{m}}^i(\cdot)$ to this, we see that

$$H_{\mathfrak{m}}^i(A)=\begin{cases} H_{\mathfrak{n}}^d(B) & (i=d)\\ B/A & (i=1)\\ (0) & (i\neq 1,\ d)\,. \end{cases}$$

Hence it follows from [10], Satz 3 that A is a Buchsbaum local ring. Of course

$$\begin{aligned}\dim_{A/\mathfrak{m}} H_{\mathfrak{m}}^1(A)&=\dim_{A/\mathfrak{m}} B/A\\ &=[K:k]-1\\ &=h\,.\end{aligned}$$

Thus we have proved the assertions from (1) to (4).

Now consider the last one. Let E_A (resp. E_B) denote the injective envelope $E_A(A/\mathfrak{m})$ (resp. $E_B(B/\mathfrak{n})$). Then

$$K_A=\mathrm{Hom}_A(H_{\mathfrak{m}}^d(A),\ E_A)$$

by definition. On the other hand

$$\begin{aligned}\mathrm{Hom}_A(H_{\mathfrak{m}}^d(A),\ E_A)&=\mathrm{Hom}_A(H_{\mathfrak{n}}^d(B),\ E_A)\\ &\cong\mathrm{Hom}_B(H_{\mathfrak{n}}^d(B),\ E_B)\\ &\cong B\,,\end{aligned}$$

and so we have $K_A=B$ as required. This completes the proof of Theorem (5.6).

Remark (5.7). Together with the example given by (2.2), (6) the example in the proof of Theorem (5.6) is obtained by "glueing". In general, certain glueings are always Buchsbaum and satisfy the condition (1) of Theorem (1.1). We will prove this in a subsequent paper.

DEPARTMENT OF MATHEMATICS
NIHON UNIVERSITY

DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY

References

[1] J. Barshay, Graded algebras of powers of ideals generated by A-sequences, J. Algebra, **25** (1973), 90-99.

[2] D.A. Buchsbaum, Complexes in local ring theory, In: Some aspects of ring theory, C.I.M.E., Rome 1965.

[3] S. Goto, On the Rees algebras of the powers of an ideal generated by a regular sequence, Proceedings of the Institute of Natural Sciences, Nihon University, **13** (1978), 9-11.

[4] ———, On the Cohen-Macaulayfication of certain Buchsbaum rings, in preprint.

[5] ———, On Buchsbaum rings, in preprint.

[6] J. Herzog and E. Kunz, Der kanonische Modul eines Cohen-Macaulay-Rings, Lecture Notes in Mathematics **238**, Springer Verlag 1971.

[7] M. Hochster and J.A. Eagon, Cohen-Macaulay rings, invariant theory, and generic perfection of determinantal loci, Amer. J. Math., **93** (1971), 1020-1058.

[8] M. Hochster and J. L. Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay, Advances in Math., **13** (1974), 115-175.

[9] J. Matijevic and P. Roberts, A conjecture of Nagata on graded Cohen-Macaulay rings, J. Math. Kyoto Univ., **14** (1974), 125-128.

[10] B. Renschuch, J. Stückrad, and W. Vogel, Weitere Bemerkungen zu einem Problem der Schnittheorie und über ein Maβ von A. Seidenberg für die Imperfektheit, J. Algebra, **37** (1975), 447-471.

[11] P. Schenzel, N.V. Trung, and N.T. Guong, Verallgemeinerte Cohen-Macaulay-Moduln, Math. Nachr., **85** (1978), 57-73.

[12] Y. Shimoda, A note on Rees algebras of two dimensional local domains, to appear in J. Math., Kyoto Univ.

[13] J. Stückrad, Üper die kohomologische Charakterisierung von Buchsbaum-Moduln, to appear in Math. Nachr.

[14] J. Stückrad and W. Vogel, Eine Verallgemeinerung der Cohen-Macaulay Ringe und Anwendungen auf ein Problem der Multiplizitätstheorie, J. Math. Kyoto Univ., **13** (1973), 513-528.

[15] ———, Über das Amsterdamer Programm von W. Gröbner und Buchsbaum Varietäten, Monatshefte fur Mathematik, **78** (1974), 433-445.

[16] ———, Toward a theory of Buchsbaum singularities, Amer. J. Math., **100** (1978), 727-746.

[17] G. Valla, Certain graded algebras are always Cohen-Macaulay, J. Algebra, **42** (1976), 537-548.

[18] W. Vogel, Über eine Vermutung von D.A. Buchsbaum, J. Algebra, **25** (1973), 106-112.

[19] ———, A non-zero-divisor characterization of Buchsbaum modules, in preprint.