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The method of linear operators for square integrable differentials on open Riemann surfaces and its applications

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Introduction

In the study of open Riemann surfaces, the method of orthogonal decompositions and that of linear operators are fundamental. As for the latter, the normal operators introduced by L. Sario play an important role. H. Yamaguchi [24] modified the normal operators and defined regular operators by means of a formal Green's formula. M. Yoshida [25] discussed some relations between those two methods by using the concept of regular operators; he, furthermore, studied meromorphic functions whose boundary behavior is given by a regular operator. M. Shiba [18] introduced another type of boundary behavior which is not necessarily given by regular operators, and he gave an extension of the Riemann-Roch theorem of Kusunoki's type.

In this paper we shall introduce some new linear operators. Our operators generalize the notion of regular operators and act on square integrable differentials (not on functions!). The concept of principal differentials with respect to this operator is similarly defined as that of principal functions. We shall show several properties of principal differentials. As an application of principal differentials we shall give a formulation of the Riemann-Roch theorem on an arbitrary open Riemann surface. In a similar situation Abel's theorem will be proved.

Up to now we have two types of formulations for these theorems. L. Ahlfors [2], [3] and H.L. Royden [16] formulated the theorems in complex form (cf. B. Rodin [13], Y. Sainouchi [17], O. Watanabe [21] etc.). But these theorems are, as was pointed out by R. Accola [1], meaningful only for Riemann surfaces with small boundaries, say, those of the class O_{KD}. Y. Kusunoki [6], on the other hand, used real normalization and the results are valid for general surfaces (cf. H.L. Royden [16], M. Yoshida [25], M. Shiba [18], [19] and O. Watanabe [22]). Our present formulation is rather close to the former in the sense that it is described in complex form, but it seems to be meaningful also for Riemann surfaces with large boundaries. Furthermore, some infinite divisors are allowed in our theory.

In § 1 the definition and the fundamental properties of linear operators are given. Our operator maps a differential into another which has minimal Dirichlet integral among a certain class of differentials. It can be seen that the image differential has a similar property as Kuramochi functions (cf. [4]). In §2 we shall give the definition of principal differentials for our operators. We first investigate the relationships between principal differentials and reproducing kernels for subspaces of the Hilbert space consisting of square integrable harmonic differentials (cf. [14], [15], [24], [25]). Next we shall give the extremal property of principal differentials (cf. [6]) and the conditions for principal differentials to be analytic. We know that some principal differentials are closely related to differentials with Shiba's boundary behavior. In §3 we investigate semiexact principal differentials and observe the vanishing property of certain integrals along the ideal boundary. This vanishing property has close connections with the bilinear relation, the characterization of semiexact canonical differentials and so forth (cf. [8], [13], [20]). In §4 we shall show on a general open Riemann surface the existence of behavior spaces which correspond to Shiba's behavior spaces. Using these behavior spaces, we shall formulate in §5 the theorem of Riemann-Roch and Abel's theorem.

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§1. Linear operators on spaces of differentials

1.1 The definition of operators

Let R be an arbitrary Riemann surface. As usual, two Lebesgue measurable complex differentials on R are identified if they are equal almost everywhere. With this convention, the set of square integrable complex differentials on R forms a Hilbert space $\Gamma=\Gamma(R)$ over the complex number field $\boldsymbol{C}$ with respect to Dirichlet's inner product:

$$(\omega_1, \omega_2)=(\omega_1, \omega_2)_R=\int_R \omega_1\bar{\omega}_2^*,$$

where $\omega_1\bar{\omega}_2^*$ is the exterior product for ω_1 and $\bar{\omega}_2^*$. (We denote by $\bar{\omega}$ the complex conjugate of ω, and by ω^* the conjugate differential of ω.) M. Shiba considered the same set as a Hilbert space $\Lambda=\Lambda(R)$ over the real number field $\boldsymbol{R}$ with the inner product $\langle\omega_1, \omega_2\rangle=\mathrm{Re}(\omega_1, \omega_2)$ (cf. M. Shiba [18]). In what follows, almost all the statements and proofs are equally valid for Γ and Λ. So we shall be mainly concerned with Γ and refer to Λ if a statement has different forms for Γ and Λ. As for the notations of subspaces in Γ we follow Ahlfors-Sario [3]; for instance, Γ_c, Γ_h, Γ_{se}, Γ_{hse}, and Γ_{eo} denote the spaces of closed, harmonic, semiexact, harmonic semiexact differentials, and the space of differentials of Dirichlet potentials (cf. [4]) respectively.

Let Γ_x be a closed subspace of Γ_h, and F be a closed set on R. We consider the following classes of differentials:

$$X=\Gamma_x+\Gamma_{eo}\,, \qquad X_F=\{\omega\in X\,;\,\omega=0 \text{ on } F\}\,.$$

For a measurable set $\Omega(\subset R)$, $(\omega_1, \omega_2)_\Omega$ denotes $\int_\Omega \omega_1\bar{\omega}_2^*$, and $\|\omega\|_\Omega^2=(\omega, \omega)_\Omega$.

Lemma 1. *The classes X and X_F are closed linear subspaces of Γ.*

Proof. We only show that they are complete. Let $\{\omega_n\}$ be a Cauchy sequence in X. Each ω_n has a unique orthogonal decomposition: $\omega_n=\omega_{n,1}+\omega_{n,0}$ with $\omega_{n,1}\in\Gamma_x$ and $\omega_{n,0}\in\Gamma_{eo}$. Since $\|\omega_{n,i}-\omega_{m,i}\|\leqq\|\omega_n-\omega_m\|$ for $i=0, 1$, $\{\omega_{n,1}\}$ and $\{\omega_{n,0}\}$ are also Cauchy sequences. Suppose they converge to $\omega_1\in\Gamma_x$ and $\omega_0\in\Gamma_{eo}$ respectively. Then $\{\omega_n\}$ converges to $\omega_1+\omega_0\in X$. In case that $\{\omega_n\}$ is contained in X_F, we have, furthermore, $0=\lim_{n\to\infty}\|\omega_1+\omega_0-\omega_n\|\geqq\lim_{n\to\infty}\|\omega_1+\omega_0-\omega_n\|_F=\|\omega_1+\omega_0\|_F \geqq 0$. Thus the differential $\omega_1+\omega_0$ belongs to X_F.

Let X^F be the orthogonal complement of X_F in X and $\omega^F=\omega_x^F\in X^F$ denotes the projection of $\omega\in X$ to X^F. Now we define an operator on X by the projection from X to X^F. The operator $\omega\to\omega_x^F$ is linear over $\boldsymbol{C}$. From the definition we have

Lemma 2. *If $\omega\in X$, then $\omega_x^F-\omega\in X_F$ and $\omega_x^F\perp X_F$ (i.e., ω_x^F is orthogonal to the space X_F). Conversely, if $\omega, \omega'\in X$ satisfy that $\omega'-\omega\in X_F$ and $\omega'\perp X_F$, then $\omega'=\omega_x^F$.*

We also have the following (cf. [4]).

Proposition 1. *For ω in X, ω_x^F has the following properties:*
(i) *$\omega_x^F=\omega$ on F,*
(ii) *ω_x^F is harmonic in $R-F$,*
(iii) *$\|\omega_x^F\|=\inf\{\|\omega'\|\,;\,\omega'\in X$ and $\omega'=\omega$ on $F\}$.*

Proof. (i) Since $\omega^F-\omega$ belongs to X_F, $\omega^F-\omega=0$ on F. (ii) For a connected component G of $R-F$, an infinitely differentiable function $f\in C_0^\infty(G)$ with compact support in G is regarded as a function on R by setting $f=0$ on $R-G$. Since $\mathrm{d}f\in\Gamma_{eo}(R)$ and $\mathrm{d}f\in X_F$, we have $0=(\omega^F, \mathrm{d}f)=(\omega^F, \mathrm{d}f)_G$ and $0=(\omega^F, \mathrm{d}f^*)=(\omega^F, \mathrm{d}f^*)_G$. By Weyl's lemma ω^F is harmonic in G, hence in $R-F$. (iii) If $\omega'-\omega$ belongs to X_F, then $\omega'-\omega^F=\omega'-\omega-(\omega^F-\omega)\in X_F$, and $\|\omega'\|^2=\|\omega'-\omega^F+\omega^F\|^2=\|\omega'-\omega^F\|^2+\|\omega^F\|^2\geqq\|\omega^F\|^2$, which implies (iii).

Remark. If $\omega_1, \omega_2\in X$ coincide on F, then $(\omega_1)_x^F=(\omega_2)_x^F$. Thus we can define $\omega^F=\omega_x^F$ by $(\omega_1)_x^F$ for any differential ω such that $\omega=\omega_1$ on F for some $\omega_1\in X$. Note that ω need not belong to X.

For later use we show a sufficient condition for $\omega_x^F=\omega$. We shall denote by $\Gamma_x^\perp$ the orthogonal complement of Γ_x in Γ_h.

Lemma 3. *If a differential $\omega \in X$ satisfies that $\omega=\tau$ in $R-F$ with $\tau \in \Gamma_x^{\perp}+\Gamma_{eo}^*$, then ω_x^F is equal to ω.*

Proof. It suffices to note that for any $\sigma \in X_F$,

$$(\omega, \sigma)=(\omega, \sigma)_F+(\tau, \sigma)_{R-F}=(\tau, \sigma)=0.$$

1.2 The dependence of the operator on F and X

The operator: $\omega \rightarrow \omega_x^F$ is clearly continuous. Now this operator changes as F and X vary. So we shall study the dependence of the operator on F and X. First let a space X be fixed. Then we have

Lemma 4. (i) *If $F' \supset F$, then $\omega_x^F=(\omega_x^F)_x^{F'}=(\omega_x^{F'})_x^F$.*
(ii) *Let $\{F_n\}$ be an increasing sequence of closed sets. If $F=\bigcup_n F_n$ is closed, then $\{\omega_x^{F_n}\}$ converges to ω_x^F (in the sense of norm).*

Proof. (i) Since $X_{F'}$ is contained in X_F, ω^F is orthogonal to $X_{F'}$. By Lemma 2 we have $(\omega^F)^{F'}=\omega^F$. Next, we see that $(\omega^{F'})^F-\omega$ belongs to X_F and that $(\omega^{F'})^F$ is orthogonal to X_F. Then we have $(\omega^{F'})^F=\omega^F$.

(ii) Since $(\omega^{F_m}, \omega^{F_n}-\omega^{F_m})=0$ for $m \leqq n$, we have

$$0 \leqq \|\omega^{F_n}-\omega^{F_m}\|^2=\|\omega^{F_n}\|^2-\|\omega^{F_m}\|^2.$$

Similarly we have $0 \leqq \|\omega^F\|^2-\|\omega^{F_n}\|^2$. Now we can easily see that $\{\omega^{F_n}\}$ is a Cauchy sequence. Let $\omega_0 \in X$ be the limit. Then for every m $\|\omega_0-\omega\|_{F_m} =\lim_{n\to\infty} \|\omega^{F_n}-\omega\|_{F_m}=0$ and $\omega_0=\omega$ on F_m. Hence $\omega_0=\omega$ on F. On the other hand, since $X_F \subset X_{F_n}$, $0=\lim_{n\to\infty}(\omega^{F_n}, \omega')=(\omega_0, \omega')$ for every $\omega' \in X_F$. By Lemma 2 we conclude that $\omega_0=\omega^F$.

As for the dependence of X, we have

Lemma 5. *Let $\{\Gamma_{x_n}\}$ be a decreasing sequence of subspaces of Γ_h. Let $X_n=\Gamma_{x_n}+\Gamma_{eo}$ and $X=\bigcap_n X_n$. Then for any ω in X, $\{\omega_{x_n}^F\}$ converges to ω_x^F.*

Proof. Since $(X_m)_F$ contains $(X_n)_F$ for $m \leqq n$, we can deduce that $\{\omega_{x_n}^F\}$ is a Cauchy sequence (cf. the proof of Lemma 4 (ii)). The limit differential ω_0 belongs to X_n for every n. Hence ω_0 belongs to X. The inequality $\|\omega_0\| =\lim_{n\to\infty} \|\omega_{x_n}^F\| \leqq \|\omega_x^F\|$ gives $\omega_0=\omega_x^F$ (cf. Proposition 1 (iii)).

Now let $\{R_n\}$ be a non decreasing sequence of regions of R such that $\bigcup_n R_n=R$. Let F_n be a relatively closed set of R_n such that $F_n \subset F_{n+1}$ $(n=1, 2, \cdots)$, and $\bigcup_n F_n=F$ is closed in R. Let $\Gamma_{x_n}(R_n)$ and $\Gamma_x(R)$ be subspaces of $\Gamma_h(R_n)$ and $\Gamma_h(R)$ respectively. By the way every $\omega \in \Gamma(R_n)$ can be extended to an $\tilde{\omega} \in \Gamma(R)$ by setting $\tilde{\omega}=\omega$ on R_n, $=0$ on $R-R_n$. Hence $\Gamma(R_n)$ can be regarded as a subspace of $\Gamma(R)$. We shall make use of this convention.

Theorem 1. (*cf. H. Yamaguchi* [24]) *Suppose that* $X_n=\Gamma_{x_n}(R_n)+\Gamma_{eo}(R_n)$ *and* $X=\Gamma_x(R)+\Gamma_{eo}(R)$ *satisfy the following conditions*:
(i) *If a sequence* $\{\omega_\nu\}$ *in* $\bigcup_n X_n$ *is weakly convergent to an* ω_0 *in* $\Gamma(R)$, *i.e.,* $\lim_{\nu\to\infty}(\omega_\nu, \sigma)=(\omega_0, \sigma)$ *for any* $\sigma\in\Gamma(R)$, *then* ω_0 *belongs to* X.
(ii) *For every* $\omega\in X$, *there exists* $\{\omega_n:\omega_n\in X_n\}_n$ *such that*

$$\omega_n=\omega \quad \text{on} \quad F_n \quad \text{and} \lim_{n\to\infty}\|\omega_n-\omega\|_{R_n}=0\,.$$

Then we have that for every $\omega\in X$

$$\lim_{n\to\infty}\|(\omega_n)_{x_n}^{F_n}-\omega_x^F\|=0\,,$$

where $\{\omega_n\,;\,\omega_n\in X\}$ *is a sequence given in condition* (ii).

Proof. Since ω_n and $(\omega_n)_{x_n}^{F_n}$ are uniformly bounded in norm, we can find a subsequence $\{(\omega_\nu)_{x_\nu}^{F_\nu}\}$ which converges to an ω_0 in $\Gamma(R)$ weakly. We note that $\omega_0\in X$ by condition (i). Let m be a fixed positive integer and f_m be the characteristic function of the F_m. We have

$$\begin{aligned}(\omega_0-\omega, \omega_0-\omega)_{F_m}&=(\omega_0-\omega, f_m\cdot(\omega_0-\omega))\\ &=\lim_{\nu\to\infty}((\omega_\nu)_{x_\nu}^{F_\nu}-\omega, f_m\cdot(\omega_0-\omega))\\ &=\lim_{\nu\to\infty}((\omega_\nu)_{x_\nu}^{F_\nu}-\omega, f_m\cdot(\omega_0-\omega))_{F_m}=0\,.\end{aligned}$$

Since m is arbitrary fixed, we conclude that $\omega_0=\omega$ on F. Next by condition (ii) we can find $\sigma_n\in X_n$ such that $\sigma_n=\omega_x^F$ on F_n and $\lim_{n\to\infty}\|\sigma_n-\omega_x^F\|_{R_n}=0$. Since $\sigma_n=\omega=\omega_n$ on F_n, we have $\|(\omega_n)_{x_n}^{F_n}\|\leqq\|\sigma_n\|$. Observe that

$$\|\omega_x^F\|\leqq\|\omega_0\|\leqq\underline{\lim}_{\nu\to\infty}\|(\omega_\nu)_{x_\nu}^{F_\nu}\leqq\lim_{\nu\to\infty}\|\sigma_\nu\|=\|\omega_x^F\|\,.$$

We find that $\omega_x^F=\omega_0$ and $\{(\omega_\nu)_{x_\nu}^{F_\nu}\}$ converges to ω_0. This completes the proof.

§ 2. Principal differentials

2.1 The existence and uniqueness of principal differentials

Using a normal operator M. Nakai & L. Sario (cf. [15]) defined "principal forms". In the present section we shall use the operator in § 1 to define "principal differentials". We shall also discuss their existence and uniqueness.

Let θ be a differential on a Riemann surface R and F be a closed subset of R.

Definition. We call a differential ω on R to have $(X, F;\theta)$-behavior if $\omega-\theta$ belongs to X and $(\omega-\theta)_x^F=\omega-\theta$. Further, the ω is called a $(X, F;\theta)$-principal differential (or $(X, F;\theta)$-p.d.) if it is harmonic in some open set which contains F.

We note that we can consider $(X, F; \theta)$-p.d. only if θ is supposed to be harmonic in a open subset of $R-F$, because $\omega-\theta=(\omega-\theta)_x^F$ is harmonic in $R-F$.

Hereafter we assume that the closed set F satisfies: F is a disjoint countable union of closed regions F_i each of whose relative boundary ∂F_i consists of a finite number of analytic closed Jordan curves. Let $\mathfrak{F}$ be the class of those F's. For each $F\in\mathfrak{F}$ we set $\mathfrak{P}=\mathfrak{P}(F)=\{P; P$ is a discrete closed set of R which does not meet $F.\}$. Further, for $F\in\mathfrak{F}$ and $P\in\mathfrak{P}(F)$ we consider

$\Theta=\Theta(F, P)=\{\theta; \theta$ is a closed C^1-differential in $R-P$ such that (i) θ is harmonic on $(R-F-P)\cup\partial F$, (ii) $\|\theta\|_F^2<\infty$, (iii) $\int_{\partial F_i}\theta=\int_{\partial F_i}\theta^*=0$ for every component F_i of $F.\}$.

Let θ and θ_1 be differentials in $\Theta(F, P)$ such that $\theta+\theta_1^*=0$ in $R-F-P$. Clearly $\theta+\theta_1^*\in\Gamma$. Let ω be the projection of $\theta+\theta_1^*$ to X and set $\phi(\theta, \theta_1; X)=\theta-\omega$.

Proposition 2. *The differential $\phi(\theta, \theta_1; X)$ is $(X, F; \theta)$-principal.*

Proof. By means of orthogonal decomposition of Γ, we can write as $\theta+\theta_1^*=\omega+\tau$ with $\omega\in\Gamma_x+\Gamma_{eo}$ and $\tau\in\Gamma_x^\perp+\Gamma_{eo}^*$. Since $\theta-\omega$ is closed in $R-P$ and $\tau-\theta_1^*$ is coclosed in $R-P$, $\phi(\theta, \theta_1; X)=\theta-\omega=\tau-\theta_1^*$ is harmonic in $R-P$. We have then $\phi-\theta=-\omega\in X$. Furthermore, since $\phi-\theta=\tau\in\Gamma_x^\perp+\Gamma_{eo}^*$ in $R-F$, we know by Lemma 3 $(\phi-\theta)^F=\phi-\theta$. Hence $\phi(\theta, \theta_1; X)$ is a $(X, F; \theta)$-p. d..

The $(X, F; \theta)$-p.d. $\phi(\theta, \theta_1; X)$ which is constructed as above will be called a $(X, F; \theta, \theta_1)$-principal differential (or $(X, F; \theta, \theta_1)$-p.d.).

Remark. We have the following equalities:

(1) $a\phi(\theta, \theta_1; X)+b\phi(\theta', \theta_1'; X)=\phi(a\theta+b\theta', a\theta_1+b\theta_1'; X)$,

(2) $\overline{\phi(\theta, \theta_1; X)}=\phi(\bar{\theta}, \bar{\theta}_1; \bar{X})$,

(3) $\phi(\theta, \theta_1; X)^*=\phi(\theta_1, -\theta; X^{\perp *})$ and $\phi(\theta_1, -\theta; X^*)=\phi(\theta, \theta_1; X^\perp)^*$

where $\bar{X}=\{\bar{\omega}; \omega\in X\}$, $X^\perp=\Gamma_x^\perp+\Gamma_{eo}$ and $X^*=\Gamma_x^*+\Gamma_{eo}$.

Next we shall use

Lemma 6. (*cf.* [18], [24], [25]). *Let G be a regulary imbedded connected subregion of R whose relative boundary ∂G is compact, and let V be the complement of $\bar{G}$ in R. For any closed C^1-differential σ defined in a neighbourhood of $\bar{V}$, the following two statements are equivalent*:

(i) *$\sigma|_V$, the restriction of σ to V, can be extended as a closed C^1-differential $\tilde{\sigma}$ on R so that the support of $\tilde{\sigma}$ has a compact intersection with $\bar{G}$.*

(ii) $\int_{\partial G}\sigma=0$.

Theorem 2. *Let $F=\bigcup_i^n F_i\in\mathfrak{F}$, $n<\infty$, and θ be a harmonic differential on*

$(R-F-P)\cup\partial F$ such that $\int_{\partial F_i}\theta=\int_{\partial F_i}\theta^*=0$, $1\leqq i\leqq n$. Then there exist a $\tilde{\theta}\in\Theta(F, P)$ and a $(X, F;\tilde{\theta})$-principal differential such that $\tilde{\theta}=\theta$ on $R-F-P$.

Proof. By Lemma 6 we can find $\tilde{\theta}$, $\tilde{\theta}^*\in\Theta(F, P)$ such that $\tilde{\theta}=\theta$, $\tilde{\theta}^*=\theta^*$ on $R-F-P$. Then $\phi(\tilde{\theta}, \tilde{\theta}^*; X)$ is clearly a requested differential (cf. Proposition 2).

There generally exist many $(X, F;\theta)$-p.d.'s for a given θ in $\Theta(F, P)$. In order to discuss the uniqueness of a $(X, F;\theta)$-p.d., we shall first observe

Lemma 7. *Let $F\in\mathfrak{F}$ be compact. If $\sigma\in\Gamma_x$ is exact on F and satisfies $\sigma=\sigma_x^F$, then $\sigma=0$.*

Proof. There is a relatively compact neighbourhood V of F on which σ is exact. Let s be a harmonic function in V such that $ds=\sigma$. Take a function $k\in C_0^1(R)$ such that $k=1$ on F, $=0$ on $R-V$. We define a differential σ' so that $\sigma'=d(ks)$ in V, $=0$ on $R-V$. Then σ' belongs to Γ_{eo} and $\sigma-\sigma'$ belongs to X_F. Hence $(\sigma, \sigma')=0$ and $(\sigma_x^F, \sigma-\sigma')=0$. Since $\sigma=\sigma_x^F$, we have $(\sigma, \sigma)=0$.

We set $\Gamma_x^{FI}=\{\sigma\in\Gamma_x;\ \sigma_x^F\equiv\sigma\}$. The Γ_x^{FI} is clearly a subspace of Γ_x. The $(X, F;\theta)$-p.d. is uniquely determined up to the elements of Γ_x^{FI}. In fact, if ϕ_1 and ϕ_2 are $(X, F;\theta)$-p.d., then

(i) $\phi_1-\phi_2=\phi_1-\theta-(\phi_2-\theta)$ belongs to $X\cap\Gamma_h=\Gamma_x$,

(ii) $(\phi_1-\phi_2)^F=(\phi_1-\theta)^F-(\phi_2-\theta)^F=(\phi_1-\theta)-(\phi_2-\theta)=\phi_1-\phi_2$.

Therefore $\phi_1-\phi_2\in\Gamma_x^{FI}$.

Let γ be a non-dividing oriented closed curve on R. We take a ring domain V such that γ is a boundary component of V and V lies on the left side of γ. There is a function $f_\gamma\in C^1(R-\gamma)$ such that

$$f_\gamma=\begin{cases}1 \text{ in a neighbourhood of } \gamma \text{ in } V\\ 0 \text{ in a neighbourhood of the other component of } \partial V\\ 0 \text{ on } R-\bar{V}.\end{cases}$$

The df_γ belongs to $\Theta(\bar{V}, \emptyset)$. We set $\sigma_{\gamma,x}=-\phi(0, df_\gamma; X)$. If we replace V and f_γ by another ring domain V' and a function f'_γ which satisfy the same conditions, we have that the $(X, \bar{V}'; 0, df'_\gamma)$-p.d. is equal to the $(X, \bar{V}; 0, df_\gamma)$-p.d.. Hence $\sigma_{\gamma,x}$ is independent of the choice of V and f_γ.

Let $\Xi(F)=\{A_j, B_j\}$ be a canonical homology basis of F modulo ∂F such that (i) $A_j\cap B_j$ consists of a point, (ii) $(A_i\cup B_i)\cap(A_j\cup B_j)=\emptyset$ for $i\neq j$ and (iii) $A_i\times A_j=B_i\times B_j=0$, $A_i\times B_j=0$ for $i\neq j$ and $A_j\times B_j=1$, where A_j crosses B_j from right to left. Let $\Xi'(F)=\{A_j, B_j, C_i\}$ be a homology basis of F.

Theorem 3. *Let $F\in\mathfrak{F}$ be compact. Then Γ_x^{FI} is spanned by*

$$\{\sigma_{A_j,x}, \sigma_{B_j,x}, \sigma_{C_i,x};\ A_j, B_j, C_i\in\Xi'(F)\}.$$

Proof. Let $\psi\in\Gamma_x^{FI}$. Suppose that $(\psi, \sigma_{A_j,x})=(\psi, \sigma_{B_j,x})=(\psi, \sigma_{C_i,x})=0$ for every j and i. We cut F along $\bigcup_j (A_j\cup B_j)$ and denote the resulting surface by F'. We can take $\mathrm{d}f_{A_j}^*$ so that its support is contained in F. Then by Green's formula

$$0=(\psi, \sigma_{A_j,x})=(\psi, \mathrm{d}f_{A_j}^*)=\int_{\partial F'}\overline{f_{A_j}}\psi$$

$$=\int_{\partial F}\overline{f_{A_j}}\psi-\sum_i\left\{\int_{A_i}\overline{\mathrm{d}f_{A_j}}\int_{B_i}\psi-\int_{B_i}\overline{\mathrm{d}f_{A_j}}\int_{A_i}\psi\right\}$$

$$=\int_{A_j}\psi.$$

Similarly $\int_{B_j}\psi=0$ and $\int_{C_i}\psi=0$. This shows that ψ is exact on F. By Lemma 7 we have $\psi=0$.

Corollary 3.1 *Let $F\in\mathfrak{F}$ be compact and $\theta\in\Theta(F, P)$. For an arbitrary $(X, F; \theta)$-principal differential ϕ, we can find a closed C^1-differential $\theta_1\in\theta(F, P)$ in $R-P$ such that $\theta_1=\theta^*$ in $R-F-P$ and $\phi=\psi(\theta, \theta_1; X)$. Moreover there exists a differential $\tau\in\Gamma_x^\perp+\Gamma_{eo}^*$ which coincides with $\phi-\theta$ in $R-F$.*

Proof. By Lemma 6 and Theorem 2 we can find θ_1' so that $\phi-\psi(\theta, \theta_1'; X)\in\Gamma_x^{FI}$. Then by Theorem 3 we can write as $\phi-\psi(\theta, \theta_1'; X)=\sum a_j\psi(0, \mathrm{d}f_{A_j}; X)+\sum b_j\psi(0, \mathrm{d}f_{B_j}; X)+\sum c_i\psi(0, \mathrm{d}f_{C_i}; X)$. So $\theta_1=\theta_1'+\sum a_j\mathrm{d}f_{A_j}+\sum b_j\mathrm{d}f_{B_j}+\sum c_i\mathrm{d}f_{C_i}$ satisfies the first statement. Then the last statement is trivial.

2.2 Reproducing differentials

There are some connections between principal differentials and reproducing differentials for any subspace of Γ_h, while B. Rodin, L. Sario, M. Yoshida and H. Yamaguchi discussed for some subspaces of Γ_h. We shall denote by O the space $\{0\}+\Gamma_{eo}$.

Definition. We set $\sigma(\theta, \theta_1; \Gamma_x)=\psi(\theta, \theta_1; X)-\psi(\theta, \theta_1; O)$ and call it a $(\Gamma_x; \theta, \theta_1)$-reproducing differential (or $(\Gamma_x; \theta, \theta_1)$-r. d.).

Proposition 3. (i) $\sigma=\sigma(\theta, \theta_1; \Gamma_x)\in\Gamma_x$.

(ii) $(\omega, \sigma)=-(\omega, \theta+\theta_1^*)$ *for any* $\omega\in\Gamma_x$.

(iii) $\|\omega\|^2+2\,\mathrm{Re}\,(\omega, \theta+\theta_1^*)\geqq\|\sigma\|^2+2\,\mathrm{Re}\,(\sigma, \theta+\theta_1^*)=-\|\sigma\|^2$ *for any* $\omega\in\Gamma_x$.

Proof. (i) We can write as

$$\psi(\theta, \theta_1; X)=\theta-(\omega_x+\omega_0)=\tau-\theta_1^*$$

with $\omega_x\in\Gamma_x$, $\omega_0\in\Gamma_{eo}$ and $\tau\in\Gamma_x^\perp+\Gamma_{eo}^*$.

Then $\psi(\theta, \theta_1; O)=\theta-\omega_0=\omega_x+\tau-\theta_1^*$. Hence $\sigma=\psi(\theta, \theta_1; X)-\psi(\theta, \theta_1; O)=-\omega_x\in\Gamma_x$. (ii) Since $\sigma=-\omega_x=\tau+\omega_0-(\theta+\theta_1^*)$, we have $(\omega, \sigma)=(\omega, \tau+\omega_0)-(\omega, \theta+\theta_1^*)=-(\omega, \theta+\theta_1^*)$. (iii) By (ii) we have $(\sigma, \theta+\theta_1^*)=-(\sigma, \sigma)$ which proves the second equality of (iii). We also have

$$\|\omega\|^2+2\operatorname{Re}(\omega, \theta+\theta_1^*)-[\|\sigma\|^2+2\operatorname{Re}(\sigma, \theta+\theta_1^*)]$$
$$=(\omega, \omega)-[(\omega, \sigma)+(\sigma, \omega)]+(\sigma, \sigma)$$
$$=(\omega-\sigma, \omega-\sigma)\geqq 0.$$

Now we apply Proposition 3 to specific kinds of differentials with singularities. Let $p\in R$ and V_1 be a parametric disc about p with the variable z. We set $V_r=\{p'; |z(p')|<r\}$ $(0<r\leqq 1)$, $F=\bar{V}_1-V_{1/2}$ and $P_n=\begin{cases}\{p, q\} & n=0\\ \{p\} & n\geqq 1\end{cases}$. Take a $q\in V_{1/2}$. Then there exist real valued functions $h_n\in C^2(R-P_n)$ such that

$$h_0=\begin{cases}\dfrac{1}{2\pi}(\log|z|-\log|z-z(q)|) & \text{on } \bar{V}_{1/2}\\ 0 \quad \text{on } R-V_1,\end{cases}$$

$$h_n=\begin{cases}\dfrac{1}{2\pi}\operatorname{Re}\dfrac{1}{z^n} & \text{on } \bar{V}_{1/2}\\ 0 \quad \text{on } R-V_1\end{cases} \qquad (n\geqq 1).$$

Since $\int_{|z|=1/2}d\,h_n^*=\int_{|z|=1}d\,h_n^*=0$ $(n\geqq 0)$, by Lemma 6 there exists a real closed C^1-differential $\widetilde{d\,h_n^*}$ in $R-P_n$ such that $\widetilde{d\,h_n^*}=d\,h_n^*$ on $R-F$. We put $\eta_n=d\,h_n$ and $\xi_n=\widetilde{d\,h_n^*}$. Let $\sigma_n=\sigma_{n,x}=\sigma(\eta_n, \xi_n; \Gamma_x)$ and $\tau_n=\tau_{n,x}=\sigma(\xi_n, -\eta_n; \Gamma_x)$.

Theorem 4. *For $\omega\in\Gamma_x$,*

(i) $$(\omega, \sigma_0)=\int_p^q\omega, \qquad \|\omega\|^2-2\operatorname{Re}\int_p^q\omega\geqq-\operatorname{Re}\int_p^q\sigma_0,$$

(ii) $$(\omega, \tau_0)=\int_p^q\omega^*, \qquad \|\omega^*\|^2-2\operatorname{Re}\int_p^q\omega^*\geqq-\operatorname{Re}\int_p^q\tau_0^*,$$

where the paths of integrations are taken in V_1.

Let $\omega=d\,w$, $\omega^*=d\,w^*$, $\sigma_n=d\,s_n$ and $\tau_n^*=d\,t_n^*$ on V_1. Then for $n\geqq 1$,

(iii) $$(\omega, \sigma_n)=\frac{1}{(n-1)!}\frac{\partial^n w}{\partial x^n}(p), \quad (z=x+iy),$$

$$\|\omega\|^2-\frac{2}{(n-1)!}\frac{\partial^n w}{\partial x^n}(p)\geqq-\frac{1}{(n-1)!}\operatorname{Re}\frac{\partial^n s_n}{\partial x^n}(p),$$

(iv) $$(\omega, \tau_n)=\frac{-1}{(n-1)!}\frac{\partial^n w^*}{\partial x^n}(p),$$

$$\|\omega\|^2+\frac{2}{(n-1)!}\frac{\partial^n w^*}{\partial x^n}(p)\geqq\frac{1}{(n-1)!}\mathrm{Re}\frac{\partial^n t_n^*}{\partial x^n}(p).$$

Proof. (i) For every sufficiently small $\varepsilon>0$, let $V'_\varepsilon=\{p'\in V_1;\ |h_0(p')|>1/\varepsilon\}\cup[p, q]$, where $[p, q]$ is a segment from p to q in V_1. Then we have by Proposition 3

$$(\omega, \sigma_0)=-(\omega, \eta_0+\xi_0^*)=-\lim_{\varepsilon\to 0}(\omega, \eta_0+\xi_0^*)_{V_1-V'_\varepsilon}$$

$$=-[\lim_{\varepsilon\to 0}(\omega, \eta_0)_{V_1-V'_\varepsilon}+\lim_{\varepsilon\to 0}(\omega, \xi_0^*)_{V_1-V'_\varepsilon}]$$

$$=-[\lim_{\varepsilon\to 0}\int_{\partial(V_1-V'_\varepsilon)}\bar{h}_0\omega^*+\lim_{\varepsilon\to 0}-\int_{\partial(V_1-V'_\varepsilon)}w\bar{\xi}_0]$$

$$=w(q)-w(p)=\int_p^q\omega.$$

(ii) In a similar way to (i)

$$(\omega, \tau_0)=-(\omega, \xi_0-\eta_0^*)=-(\omega^*, \eta_0+\xi_0^*)=\int_p^q\omega^*.$$

(iii) Let $\zeta_n=\eta_n+i\xi_n$ and $\widetilde{\mathrm{d}\,h_n^*}=\mathrm{d}(\hat{h}_n^*)$. Then

$$(\omega, \sigma(\overline{\zeta_n}, \overline{-i\zeta_n};\Gamma_x))=-(\omega, \overline{\zeta_n-i\zeta_n^*})=-[(\omega, \overline{\zeta_n})+(i\omega^*, \overline{\zeta_n})]$$

$$=-\lim_{r\to 0}(\zeta_n, \overline{i(\omega+i\omega^*)^*})_{V_1-V_r}$$

$$=-\lim_{r\to 0}-i\int_{\partial(V_1-V_r)}(h_n+i\hat{h}_n^*)(\omega+i\omega^*)$$

$$=-\lim_{r\to 0}\frac{i}{2\pi}\int_{\partial V_r}\frac{1}{z^n}\mathrm{d}(w+iw^*)$$

$$=\frac{1}{(n-1)!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}(w+iw^*)(p),$$

$$(\omega, \sigma(\zeta_n, -i\zeta_n;\Gamma_x))=-(\omega, \zeta_n-i\zeta_n^*)=-\overline{(\overline{\omega}, \overline{\zeta_n-i\zeta_n^*})}$$

$$=\frac{1}{(n-1)!}\overline{\frac{\mathrm{d}^n}{\mathrm{d}z^n}(\overline{w}+i\overline{w}^*)(p)}.$$

Since $\sigma(\eta_n, \xi_n;\Gamma_x)=\sigma((\zeta_n+\bar{\zeta}_n)/2, (-i\zeta_n-\overline{i\zeta_n})/2;\Gamma_x)$, we have

$$(\omega, \sigma_n)=\frac{1}{2(n-1)!}\left[\frac{\partial^n}{\partial x^n}(w+iw^*)(p)+\frac{\partial^n}{\partial x^n}(w-iw^*)(p)\right]$$

$$=\frac{1}{(n-1)!}\frac{\partial^n w}{\partial x^n}(p).$$

(iv) Since $\sigma(\xi_n, -\eta_n;\Gamma_x)=\sigma((\zeta_n-\overline{\zeta_n})/2i, (-i\zeta_n-\overline{i\zeta_n})/2i;\Gamma_x)$, we have

$$(\omega, \tau_n)=\frac{1}{2i(n-1)!}\left[\frac{\partial^n}{\partial x^n}(w-iw^*)(p)-\frac{\partial^n}{\partial x^n}(w+iw^*)(p)\right]$$

$$=\frac{-1}{(n-1)!}\frac{\partial^n w^*}{\partial x^n}(p).$$

By Proposition 3 the second inequalities of (i), (ii), (iii) and (iv) are clearly satisfied.

For a piecewise C^1-chain $\gamma=\sum[p_i, p_{i+1}]$ we consider $\sum\sigma(\eta_0^i, \xi_0^i; \Gamma_x)$, where $\eta_0^i=\eta_0, \xi_0^i=\xi_0$ for $p=p_i$ and $q=p_{i+1}$. Then we see that for $\omega\in\Gamma_x$, $(\omega, \sum\sigma(\eta_0^i, \xi_0^i; \Gamma_x))=\int_\gamma\omega$. Particularly we know that $\sum\sigma(\eta_0^i, \xi_0^i; \Gamma_x)=\sigma_{\gamma,x}$ for a closed curve γ (cf. p. 667).

Let $\psi_{n,x}=\psi(2\pi\eta_n, 2\pi\xi_n; X)$, $\psi'_{n,x}=\psi(2\pi\xi_n, -2\pi\eta_n; X)$, $\tau_{A_j,x}=\psi(\mathrm{d}f_{A_j}, 0; X)$ and $\tau_{B_j,x}=\psi(\mathrm{d}f_{B_j}, 0; X)$, where $\mathrm{d}f_{A_j}$ and $\mathrm{d}f_{B_j}$ are closed differentials given in 2.1. Then $\psi_{0,x}$ (resp. $\psi'_{0,x}$) has the singularities $\mathrm{Re}\frac{\mathrm{d}z}{z}$ at p and $-\mathrm{Re}\frac{\mathrm{d}z}{z-z(q)}$ at q (resp. $\mathrm{Im}\frac{\mathrm{d}z}{z}$ at p and $-\mathrm{Im}\frac{\mathrm{d}z}{z-z(q)}$ at q). The $\psi_{n,x}$ (resp. $\psi'_{n,x}$) $(n\geqq 1)$ has the singularities $\mathrm{Re}\,\mathrm{d}\frac{1}{z^n}$ at p (resp. $\mathrm{Im}\,\mathrm{d}\frac{1}{z^n}$ at p). As for $\tau_{A_j,x}$ and $\tau_{B_j,x}$ we have

$$\int_\gamma(\tau_{A_j,x}+\tau_{A_j,x^\perp})=\int_\gamma \mathrm{d}f_{A_j}=A_j\times\gamma,$$

$$\int_\gamma(\tau_{B_j,x}+\tau_{B_j,x^\perp})=\int_\gamma \mathrm{d}f_{B_j}=B_j\times\gamma.$$

Definition. These $\psi_{n,x}$, $\psi'_{n,x}$, $\tau_{A_j,x}$ and $\tau_{B_j,x}$ are called X-fundamental differentials.

Remark. Let Γ_x have an orthogonal decomposition $\Gamma_x=\sum\Gamma_{x_i}$. Then we have

(1) $$\sigma(\theta, \theta_1; \Gamma_x)=\sum\sigma(\theta, \theta_1; \Gamma_{x_i}).$$

If $\Gamma_x\subset\Gamma_y$,

(2) $$\sigma(\theta, \theta_1; \Gamma_y)-\sigma(\theta, \theta_1; \Gamma_x)=\sigma(\theta, \theta_1; \Gamma_x^\perp\cap\Gamma_y),$$

i.e., $\psi(\theta, \theta_1; Y)-\psi(\theta, \theta_1; X)=\psi(\theta, \theta_1; X^\perp\cap Y)-\psi(\theta, \theta_1; O)$. Further we have

(3) $$\overline{\sigma(\theta, \theta_1; \Gamma_x)}=\overline{\psi(\theta, \theta_1; X)}-\overline{\psi(\theta, \theta_1; O)}=\sigma(\bar\theta, \bar\theta_1; \overline{\Gamma_x}),$$

(4) $$\sigma(\theta, \theta_1; \Gamma_x)^*=\psi(\theta, \theta_1; X)^*-\psi(\theta, \theta_1; O)^*$$

$$=\psi(\theta_1, -\theta; X^{\perp *})-\psi(\theta_1, -\theta; H) \qquad (H=\Gamma_h+\Gamma_{eo})$$

$$=\sigma(\theta_1, -\theta; \Gamma_x^{\perp *})-\sigma(\theta_1, -\theta; \Gamma_h)$$

$$=-\sigma(\theta_1, -\theta; \Gamma_x^*).$$

Particularly,

(5) $$\overline{\sigma(\zeta_n, -i\zeta_n, \Gamma_x)}=\sigma(\overline{\zeta_n}, -\overline{i\zeta_n}; \overline{\Gamma_x}),$$

(6) $$\tau_{n,x}=\sigma(\xi_n, -\eta_n; \Gamma_x)=-\sigma(\eta_n, \xi_n; \Gamma_x^*)^*=-(\sigma_{n,x^*})^*,$$

(7) $$\tau_{A_j,x}=-(\sigma_{A_j,x^{\perp *}})^*,$$

(8) $$\psi'_{n,x}=(\psi_{n,x^{\perp *}})^*.$$

These equalities allow us to construct some reproducing differentials from exact principal differentials (cf. [14], [15], [24], [25]).

2.3 Extremal properties of principal differentials

A principal differential has the minimum Dirichlet integral among a certain class of differentials. So it is expected that principal differentials have some other extremal properties.

Let V_1 be a parametric disc about $p\in R$ and θ be a closed C^1-differential on $R-\{p\}$ such that

$$\theta=\begin{cases} d\left[\sum a_n\dfrac{1}{z^n}+\sum b_m\dfrac{1}{\bar{z}^m}\right] & \text{on} \quad V_{1/2}-\{p\} \\ 0 & \text{on} \quad R-V_1 \end{cases}$$

Let $\lambda_\theta=\lambda_{\theta,x}$ be the uniquely determined $(X, \bar{V}_1-V_{1/2}; \theta)$-p. d. (cf. Lemma 7). To study extremal properties of λ_θ which are similar to Kusunoki [6], we consider the following classes of differentials:

$$Q'_x=\{\lambda; \lambda \text{ is a harmonic differential on } R-\{p\} \text{ and } \lambda-\theta\in X\},$$

$$Q''_x=\{\lambda; \lambda \text{ is a harmonic differential on } R-\{p\} \text{ and } \lambda-\theta=\tau \text{ on } R-V_1$$
$$\text{for some } \tau\in\Gamma_x^{\perp}+\Gamma_{eo}^*\}.$$

A differential λ in $Q'_x\cup Q''_x$ can be written as

$$\lambda=df=d\left[\sum a_n\frac{1}{z^n}+\sum b_m\frac{1}{\bar{z}^m}+\sum c_i z^i+\sum d_j\bar{z}^j\right]$$

on $\bar{V}_1-\{p\}$. Let

$$Q_x^1=\left\{\lambda\in Q'_x; (\lambda, \lambda)_{R-V_1}\leqq-\mathrm{Re}\int_{\partial V_1}f\bar{\lambda}^*\right\},$$

$$Q_x^2=\left\{\lambda\in Q''_x; (\lambda, \lambda)_{R-V_1}\leqq-\mathrm{Re}\int_{\partial V_1}f\bar{\lambda}^*\right\}.$$

The λ_θ belongs to Q_x^1 and Q_x^2. In fact we can easily see

(1) $$(\lambda_\theta, \lambda_\theta)_{R-V_1}=-\int_{\partial V_1}f_\theta\bar{\lambda}_\theta^*,$$

where $\lambda_\theta = \mathrm{d}f_\theta$. Hence Q_x^1 and Q_x^2 are non empty.

Theorem 5. *The* $(X, \bar{V}_1 - V_{1/2}; \theta)$*-principal differential is a unique differential which maximizes (resp. minimizes) the expression* $J(\lambda) = \mathrm{Re}[\sum n a_n \bar{d}_n + \sum m b_m \bar{c}_m]$ *among the class* Q_x^1 *(resp.* Q_x^2*).*

Proof. By Corollary 3.1 there exists a differential $\tau \in \Gamma_x^1 + \Gamma_{eo}^*$ such that $\lambda_\theta = \tau$ on $R - V_1$. Take $\lambda_i \in Q_x^i$ $(i=1, 2)$. Let $\mathrm{d}s = \theta$, $\mathrm{d}f_1 = \lambda_1$ and $\mathrm{d}f_2 = \lambda_2$ on $\bar{V} - \{p\}$. Then we have

(2) $$(\lambda_1, \lambda_\theta)_{R-V_1} = (\lambda_1 - \theta, \tau)_{R-V_1} = -\int_{\partial V_1} (f_1 - s)\bar{\tau}^* = -\int_{\partial V_1} f_1 \bar{\lambda}_\theta^*,$$

(3) $$(\lambda_2, \lambda_\theta)_{R-V_1} = (\lambda_2, \lambda_\theta - \theta)_{R-V_1} = -\int_{\partial V_1} \overline{(f_\theta - s)}\lambda_2^* = -\int_{\partial V_1} \bar{f}_\theta \lambda_2^*.$$

Now let $\lambda, \lambda' \in Q_x^1 \cup Q_x^2$, $\lambda = \mathrm{d}f$, and $0 < r < 1$. Then

(4) $$(\lambda, \lambda')_{V_1 - V_r} = \int_{\partial(V_1 - V_r)} f \bar{\lambda}'^*.$$

Hence we know

(5) $$(\lambda, \lambda)_{R-V_r} \leqq -\mathrm{Re}\int_{\partial V_r} f\bar{\lambda}^*,$$

(6) $$(\lambda_\theta, \lambda_\theta)_{R-V_r} = -\int_{\partial V_r} f_\theta \bar{\lambda}_\theta^*,$$

(7) $$(\lambda_1 - \lambda_\theta, \lambda_\theta)_{R-V_r} = -\int_{\partial V_r} (f_1 - f_\theta)\bar{\lambda}_\theta^* \qquad \text{for } \lambda_1 \in Q_x^1,$$

(8) $$(\lambda_2 - \lambda_\theta, \lambda_\theta)_{R-V_r} = -\int_{\partial V_r} \bar{f}_\theta (\lambda_2 - \lambda_\theta)^* \qquad \text{for } \lambda_2 \in Q_x^2.$$

By direct calculation we have

(9) $$\mathrm{Re}\int_{\partial V_r} f\bar{\lambda}^* = -2\pi\left[\sum n|a_n|^2 \frac{1}{r^{2n}} + \sum m|b_m|^2 \frac{1}{r^{2m}}\right] + o(r)$$

for $\lambda \in Q_x^1 \cup Q_x^2$,

(10) $$\mathrm{Re}\,(\lambda_1 - \lambda_\theta, \lambda_\theta)_{R-V_r} = -\mathrm{Re}\int_{\partial V_r} (f_1 - f_\theta)\bar{\lambda}_\theta^* = 2\pi\,\mathrm{Re}\,[\sum n\bar{a}_n(d_n - d_n^o) + \sum m\bar{b}_m(c_m - c_m^o) + o(r) \qquad \text{for } \lambda_1 \in Q_x^1,$$

(11) $$\mathrm{Re}\,(\lambda_2 - \lambda_\theta, \lambda_\theta)_{R-V_r} = -\mathrm{Re}\int_{\partial V_r} \bar{f}_\theta (\lambda_2 - \lambda_\theta)^* = -2\pi\,\mathrm{Re}\,[\sum n\bar{a}_n(d_n - d_n^o) + \sum m\bar{b}_m(c_m - c_m^o) + o(r) \qquad \text{for } \lambda_2 \in Q_x^2,$$

where $f_\theta=\sum a_n \frac{1}{z^n}+\sum b_m \frac{1}{\bar{z}^m}+\sum c_i^0 z^i+\sum d_j^0 \bar{z}^j$.

By (9) we know that

$$\lim_{r\to 0}\left[-\mathrm{Re}\int_{\partial V_r} f\bar{\lambda}^*+\mathrm{Re}\int_{\partial V_r} f_\theta \bar{\lambda}_\theta^*\right]=0 .$$

By the way we have

$$0\leqq \|\lambda-\lambda_\theta\|^2=\lim_{r\to 0}\|\lambda-\lambda_\theta\|^2_{R-V_r}$$

$$=\lim_{r\to 0}[\|\lambda\|^2_{R-V_r}-\|\lambda_\theta\|^2_{R-V_r}-2\,\mathrm{Re}\,(\lambda-\lambda_\theta,\ \lambda_\theta)_{R-V_r}]$$

$$\leqq \lim_{r\to 0}\left[-\mathrm{Re}\int_{\partial V_r} f\bar{\lambda}^*+\mathrm{Re}\int_{\partial V_r} f_\theta\bar{\lambda}_\theta^*-2\,\mathrm{Re}\,(\lambda-\lambda_\theta,\ \lambda_\theta)_{R-V_r}\right].$$

Hence

$$0\leqq -\lim_{r\to 0} 2\,\mathrm{Re}\,(\lambda-\lambda_\theta,\ \lambda_\theta)_{R-V_r} .$$

From (10) and (11) the assertion follows.

Remark. Suppose $\Gamma_x\subset\Gamma_y$. Then $\lambda_{\theta,x}\in Q_y^1$ and $\lambda_{\theta,y}\in Q_x^2$. By Theorem 5 we know that $\mathrm{J}(\lambda_{\theta,x})=\mathrm{J}(\lambda_{\theta,y})$ implies $\lambda_{\theta,x}=\lambda_{\theta,y}$. Further, by Corollary 3.1 we can find some $\theta_1\in\Theta(\bar{V}_1-V_{1/2},\ \{p\})$ and write as $\lambda_{\theta,x}=\psi(\theta,\theta_1;X)$, $\lambda_{\theta,y}=\psi(\theta,\theta_1;Y)$. Here we see $\theta=\sum 2\pi a_n\zeta_n+\sum 2\pi b_m\bar{\zeta}_m$, $\theta_1=\sum -2\pi i a_n\zeta_n+\sum 2\pi i b_m\bar{\zeta}_m$. Hence we see that for $\omega=d[\sum c_i z^i+\sum d_j\bar{z}^j]\in\Gamma_x^\perp\cap\Gamma_y$

$$0=(\omega,\ \psi(\theta,\theta_1;Y)-\psi(\theta,\theta_1;X))$$

$$=(\omega,\ \sigma(\theta,\theta_1;\Gamma_x^\perp\cap\Gamma_y))$$

$$=4\pi[\sum n\bar{a}_n d_n+\sum m\bar{b}_m c_m]$$

(see the remark of 2.2 and the proof of Theorem 4).

Now suppose that θ and $\lambda_{\theta,x}$ are both real differentials. Let

$\mathcal{M}Q^1=\{\omega;\ \omega$ is a meromorphic differential such that $\omega+\bar{\omega}\in Q_x^1\}$,

$\mathcal{M}Q^2=\{\omega;\ \omega$ is a meromorphic differential such that $\omega+\bar{\omega}\in Q_x^2\}$.

Then $\omega_\theta=(\lambda_\theta+i\lambda_\theta^*)/2\in MQ^1\cap MQ^2$. Let $\omega\in\mathcal{M}Q^1\cup\mathcal{M}Q^2$ and write it $\omega=d\left[\sum a_n\frac{1}{z^n}+\sum c_m(\omega)z^m\right]$ on $\bar{V}_1-\{p\}$. Since $\mathrm{J}(\omega+\bar{\omega})=2\,\mathrm{Re}\sum n a_n c_n(\omega)$, we have

Corollary 5.1 (*cf Y. Kusunoki* [6])

$$\max\{\mathrm{Re}\sum n a_n c_n(\omega);\ \omega\in\mathcal{M}Q^1\}=\mathrm{Re}\sum n a_n c_n(\omega_\theta)$$

$$=\min\{\mathrm{Re}\sum n a_n c_n(\omega);\ \omega\in\mathcal{M}Q^2\} .$$

2.4 Meromorphic principal differentials

We investigate some conditions for a principal differential to be meromorphic. We consider the class

$$A(X^F)=\{\omega-i\omega^* ; \omega\in X^F\} .$$

Proposition 4. *Let $\theta\in\Theta(F, P)$ and ϕ be a $(X, F; \theta)$-principal differential.*
(i) *If ϕ is analytic in $R-P$, then $\theta-i\theta^*$ belongs to $A(X^F)$.*
(ii) *If $\theta-i\theta^*$ belongs to $A(X^F)$, there exists a differential $\sigma\in\Gamma_x^{FI}$ such that $\phi-\sigma$ is a $(X, F; \theta)$-principal differential which is analytic in $R-P$.*

Proof. We use the representation $\phi=\theta-\omega$ with $\omega\in X^F$.
(i) Since ϕ is analytic in $R-P$, we have

$$\theta-i\theta^*=(\phi+\omega)-i(\phi+\omega)^*=\omega-i\omega^* \quad \text{on} \quad R-P .$$

Since P is discrete, $\theta-i\theta^*\in A(X^F)$.
(ii) Let $\theta-i\theta^*=\lambda-i\lambda^*$ with $\lambda\in X^F$ and set $\sigma=\lambda-\omega$. Then $\sigma\in X^F$ and $\phi-\sigma$ is closed in $R-P$. Furthermore,

$$\begin{aligned}\phi-i\phi^*&=\theta-i\theta^*-(\omega-i\omega^*)\\&=\lambda-i\lambda^*-(\omega-i\omega^*)=\sigma-i\sigma^* ,\end{aligned}$$

so that $\phi-\sigma=i(\phi-\sigma)^*$. Hence $\phi-\sigma$ is analytic in $R-P$ and σ is harmonic there. We see $\sigma\in\Gamma_x^{FI}$. It follows that $\phi-\sigma$ is a $(X, F; \theta)$-p. d. which is analytic in $R-P$.

Theorem 6. *Let ϕ be a $(X, F; \theta, \theta_1)$-principal differential for $F\in\mathfrak{F}$ and $\theta, \theta_1\in\Theta(F, P)$ ($\theta_1=\theta^*$ in $R-F-P$). If $\theta+\theta_1^*$ is orthogonal to any differential of $(\Gamma_x-\Gamma_x^{\perp *})\cup(\Gamma_x^{\perp}-\Gamma_x^*)$, then $\phi+i\phi^*$ is a $(X, F; \theta+i\theta_1, \theta_1-i\theta)$-principal differential which is analytic in $R-P$.*

Proof. From the representation

$$\phi=\theta-(\omega_x+\omega_0)=\tau_x+\tau_0^*-\theta_1^*$$

with $\omega_x\in\Gamma_x$, $\tau_x\in\Gamma_x^{\perp}$ and ω_0, $\tau_0\in\Gamma_{eo}$, we have

$$\theta^*-\theta_1=\tau_x^*-\tau_0+\omega_x^*+\omega_0^* .$$

To show $\tau_x^*\in\Gamma_x$, if $\tau_x\notin\Gamma_x^*$, then by the assumption

$$\begin{aligned}0&=(\theta+\theta_1^*, \tau_x)=(\theta^*-\theta_1, \tau_x^*)\\&=(\tau_x^*-\tau_0+\omega_x^*+\omega_0^*, \tau_x^*)=(\tau_x^*, \tau_x^*) .\end{aligned}$$

Hence $\tau_x=0\in\Gamma_x^*$, which is a contradiction. Thus τ_x^* belongs to Γ_x. Similarly we see $\omega_x^*\in\Gamma_x^{\perp}$. We have now the representation

$$i\phi^*=i\theta_1+i\tau_x^*-i\tau_0=-i\omega_x^*-i\omega_0^*+i\theta^*$$

with $i\tau_x^* \in \Gamma_x$, $-i\omega_x^* \in \Gamma_x^{\perp}$. It follows that $i\phi^*$ is a $(X, F; i\theta_1, -i\theta)$-p. d.. This gives the conclusion.

Remark. In case $\Gamma_x^{\perp} \subsetneqq \Gamma_x^*$, the condition of Theorem 6 is equivalent to that $\theta+\theta_1^*$ is orthogonal to Γ_x. Indeed $\Gamma_x \cap \Gamma_x^* \neq \{0\}$, for $\Gamma_x^* = \Gamma_x^{\perp} + \Gamma_x \cap \Gamma_x^*$. So there is a $\sigma \in \Gamma_x \cap \Gamma_x^*$, $\sigma \neq 0$. For every $\omega \in \Gamma_x$, at least one of $\omega+\sigma$ and $\omega-\sigma$ belongs to $\Gamma_x - \Gamma_x^{\perp *}$ because $\sigma \in \Gamma_x - \Gamma_x^{\perp *}$. If $\theta+\theta_1^*$ is orthogonal to $\Gamma_x - \Gamma_x^{\perp *}$, then $(\theta+\theta_1^*, \omega)=0$, for

$$(\theta+\theta_1^*, \omega) = (\theta+\theta_1^*, \omega+\sigma) - (\theta+\theta_1^*, \sigma)$$

$$= (\theta+\theta_1^*, \omega-\sigma) + (\theta+\theta_1^*, \sigma).$$

Corollary 6.1 *If $\Gamma_x = \Gamma_x^{\perp *}$, then for every $(X, F; \theta, \theta_1)$-principal differential ϕ the meromorphic differential $\phi + i\phi^*$ is $(X, F; \theta+i\theta_1, \theta_1 - i\theta)$-principal.*

For example, from the X-fundamental differentials $\tau_{A_j, x}$, $\tau_{B_j, x}$, and $\phi_{n, x}$ we can construct the meromorphic principal differentials $\tau_{A_j, x} + i\tau_{A_j, x}^*$, $\tau_{B_j, x} + i\tau_{B_j, x}^*$, $\phi_{n, x} + i\phi_{n, x}^*$ $(n \geqq 1)$ and $\phi_{0, x} + i\phi_{0, x}^*$. These differentials will play the role of Abelian differentials of the first, the second and the third kind respectively (cf. § 5).

For the real Hilbert space Λ we have analogously the following.

Theorem 6′. *Let ϕ be a $(X, F; \theta, \theta_1)$-principal differential in the real Hilbert space Λ. If $\theta+\theta_1^*$ is orthogonal to $(\Lambda_x - i\Lambda_x^{\perp *}) \cup (\Lambda_x^{\perp} - i\Lambda_x^*)$, then $\phi + i\phi^*$ is a $(X, F; \theta+i\theta_1, \theta_1 - i\theta)$-principal differential which is analytic in $R-P$.*

Corollary 6′.1 *If $\Lambda_x = i\Lambda_x^{\perp *}$, then for every $(X, F; \theta, \theta_1)$-principal differential ϕ the moromorphic differential $\phi + i\phi^*$ is $(X, F; \theta+i\theta_1, \theta_1 - i\theta)$-principal.*

It is noted that $\Lambda_x = i\Lambda_x^{\perp *}$ is one of the conditions for a behavior space by Shiba [18] (cf. Matsui [10]).

§ 3 Semiexact principal differentials

3.1 Differentials with $(X, F; 0)$-behavior

Let $\{G_n\}$ be a canonical exhaustion of R and $\Xi = \Xi(R) = \{A_j, B_j\}$ be an associate canonical homology basis of R modulo the ideal boundary of R. Let

$$\mathfrak{F}' = \Big\{ \bigcup_{i=1}^{\infty} F_i \in \mathfrak{F};\ F_i \text{ is compact and does not meet } \partial G_n.$$
$$\{A_j, B_j \in \Xi(R);\ A_j \cup B_j \subset F_i\} \text{ is a homology basis of } F_i \text{ modulo}$$
$$\partial F_i \text{ and the other } A_j \cup B_j \text{ does not meet } F_i. \Big\}.$$

In this section we assume $F \in \mathfrak{F}'$ and $\Gamma_x \subset \Gamma_{hse}$. For a $\sigma \in \Gamma_{se}$, we often use a representation $\sigma = \mathrm{d}\, s$. We note that s is a function on a region R' obtained by cutting R along $\{A_j \cup B_j\}$ which is determined up to an additive constant.

Lemma 8. *If $\omega \in X \cap \Gamma^1$ is harmonic on ∂F, then $\omega^F|_{R-F}$ has a harmonic extension across ∂F.*

Proof. Let $\omega = \mathrm{d}\,w$ and $\omega^F = \mathrm{d}\,w^F$. Then $w - w^F$ is constant on each F_i. Since $\mathrm{Re}\,w$ and $\mathrm{Im}\,w$ are real analytic along ∂F, so are $\mathrm{Re}\,w^F$ and $\mathrm{Im}\,w^F$. It follows that $w^F|_{R-F}$ and $\omega^F|_{R-F}$ have harmonic extensions across ∂F.

We shall use the following Green's formula.

Lemma 9. (*cf.* [18]) *Let G be a regularly imbedded region. Let σ and ω be closed C^1-differentials on $\bar{G}$ and assume that σ is semiexact. Let $\sigma = \mathrm{d}\,s$ on the planar region $\bar{G}'$ obtained by cutting $\bar{G}$ along a canonical homology basis $\{A'_j, B'_j\}$ of G* (mod ∂G). *Then*

$$(\sigma, \omega^*)_G = -\int_{\partial G} s\bar{\omega} + \sum \left[\int_{A'_j} \sigma \int_{B'_j} \bar{\omega} - \int_{B'_j} \sigma \int_{A'_j} \bar{\omega} \right].$$

Lemma 10. *Let $\omega \in X$ and $F = \bigcup_i F_i \in \mathfrak{F}'$. Then for decreasing sequences F_i^n such that $\bigcap_n F_i^n = F_i$, $\bigcup_i F_i^n \in \mathfrak{F}'$ we have*

$$\lim_{F_i^n \to F_i} \int_{\partial F_i^n} \omega^{F*} = 0$$

for each i.

Proof. There is a function $k_{F_i} \in C_0^2(R)$ such that (i) $k_{F_i} = 1$ in a neighbourhood of F_i, (ii) the support of $\mathrm{d}\,k_{F_i}$ is compact and does not intersect $F - F_i$. Since $\mathrm{d}\,k_{F_i}$ belongs to Γ_{eo} and X_F, by Lemma 2 and Lemma 9,

$$0 = (\omega^F, \mathrm{d}\,k_{F_i}) = (\omega^F, \mathrm{d}\,k_{F_i})_{R-F} = \lim_{n\to\infty} (\omega^F, \mathrm{d}\,k_{F_i})_{R-F\cup F_n^i}$$

$$= \lim_{n\to\infty} ((\omega^F)^{F\cup F_i^n}, \mathrm{d}\,k_{F_i})_{R-F\cup F_i^n} = \lim_{n\to\infty} \int_{\partial F_i^n} \omega^{F*}.$$

We shall write $\lim_{n\to\infty} \int_{\partial F_i^n} \omega^{F*}$ as $\int_{\partial F_i} \omega^{F*}$ hereafter.

Lemma 11. (*cf.* [24]). *For a Dirichlet potential f_0 and a closed differential ω which are continuously differentiable in a neighbourhood of the ideal boundary, we have*

$$\lim_{n\to\infty} \int_{\partial G_n} f_0 \omega = 0.$$

Now we can give a sufficient condition for $\omega^F = \omega$.

Proposition 5. *Let $F \in \mathfrak{F}'$ be compact and $\omega \in X \cap \Gamma^1$ be harmonic on $\overline{R-F}$.*

Further, suppose that ω *satisfies the following*:

(i) $\int_{\partial F_i}\omega^*=0$ *for each* i,

(ii) $(\omega, \sigma)_{R-F}=-\int_{\partial F}\bar{s}\omega^*$ *for any* $\sigma=\mathrm{d}\,s\in\Gamma_x$.

Then we have $\omega^F=\omega$.

Proof. Let $\omega=\mathrm{d}\,w$, $\omega^F=\mathrm{d}\,w^F$ on $\overline{R-F}$. By Lemma 9

$$\|\omega^F-\omega\|^2$$

$$=\lim_{n\to\infty}\Big\{\int_{\partial(G_n-F)}\overline{(w^F-w)}(\omega^F-\omega)^*$$

$$-\sum_{G_n-F}\Big[\int_{A_j}\overline{(\omega^F-\omega)}\int_{B_j}(\omega^F-\omega)^*-\int_{B_j}\overline{(\omega^F-\omega)}\int_{A_j}(\omega^F-\omega)^*\Big]\Big\}$$

$$=\lim_{n\to\infty}\Big\{\int_{\partial(G_n-F)}\overline{(w^F-w)}\omega^{F*}-\sum_{G_n-F}\Big[\int_{A_j}\overline{(\omega^F-\omega)}\int_{B_j}\omega^{F*}-\int_{B_j}\overline{(\omega^F-\omega)}\int_{A_j}\omega^{F*}\Big]\Big\}$$

$$-\lim_{n\to\infty}\Big\{\int_{\partial G_n}\overline{(w^F-w)}\omega^*-\sum_{G_n-F}\Big[\int_{A_j}\overline{(\omega^F-\omega)}\int_{B_j}\omega^*-\int_{B_j}\overline{(\omega^F-\omega)}\int_{A_j}\omega^*\Big]\Big\}$$

$$+\int_{\partial F}\overline{(w^F-w)}\omega^*\,.$$

Now the first term is equal to $(\omega^F, \omega^F-\omega)_{R-F}$ which vanishes by Lemma 2. Next if we write $w^F-w=w_x+w_0$ with $\mathrm{d}\,w_x\in\Gamma_x$ and a Dirichlet potential w_0, we see by (ii)

$$\lim_{n\to\infty}\Big\{\int_{\partial G_n}\overline{w}_x\omega^*-\sum_{G_n-F}\Big[\int_{A_j}\overline{\mathrm{d}\,w_x}\int_{B_j}\omega^*-\int_{B_j}\overline{\mathrm{d}\,w_x}\int_{A_j}\omega^*\Big]$$

$$=(\omega, \mathrm{d}\,w_x)_{R-F}+\int_{\partial F}\overline{w}_x\omega^*=0$$

and by Lemma 11

$$\lim_{n\to\infty}\Big\{\int_{\partial G_n}\overline{w}_0\omega^*-\sum_{G_n-F}\Big[\int_{A_j}\overline{\mathrm{d}\,w_0}\int_{B_j}\omega^*-\int_{B_j}\overline{\mathrm{d}\,w_0}\int_{A_j}\omega^*\Big]=0\,.$$

Hence we can deduce the second term also vanishes. As for the last term also vanishes, we note that w^F-w is constant on each F_i and by (i) $\int_{\partial F_i}\omega^*=0$. Thus we have proved $\omega^F=\omega$.

3.2. The boundary integrals of principal differentials

In the study of bilinear relations on an open Riemann surface, the standard device is to consider classes of differentials whose certain integrals along the ideal boundary vanish. Some principal differentials seem to play a role of now mentioned differentials.

In order to investigate the boundary integrals of principal differentials we first show

Theorem. 7 *Let* $F\in\mathfrak{F}'$ *and* $\sigma, \omega\in X\cap\Gamma^1$ $(\omega=\mathrm{d}\,w)$. *If* σ *is represented by* $\sigma=\tau+\tau_0^*$ *with* $\tau\in\Gamma_x^{\perp}$ *and* $\tau_0\in\Gamma_{eo}$ *on* $\overline{R-F}$, *then*

$$(\sigma^F, \omega^F)_{R-F}=\sum_i\left\{-\int_{\partial F_i}\bar{w}\sigma^{F*}+\sum_{F_i}\left[\int_{A_j}\bar{\omega}\int_{B_j}\tau^*-\int_{B_j}\bar{\omega}\int_{A_j}\tau^*\right]\right\}.$$

Proof. We first note $\sigma=\sigma^F$ by Lemma 3. From Lemma 2 we have

$$(\sigma^F, \omega^F)_{R-F}=(\sigma^F, \omega)_{R-F}=(\tau+\tau_0^*, \omega)_{R-F}=-\sum_i(\tau+\tau_0^*, \omega)_{F_i}.$$

Since ω and $\tau+\tau_0^*$ belong to Γ, we know that $\sum_i(\tau+\tau_0^*, \omega)_{F_i}$ is absolutely convergent. Further, we have

$$-\sum_i(\tau+\tau_0^*, \omega)_{F_i}$$

$$=\sum_i\left\{-\int_{\partial F_i}\bar{w}(\tau+\tau_0^*)^*+\sum_{F_i}\left[\int_{A_j}\bar{\omega}\int_{B_j}(\tau+\tau_0^*)^*-\int_{B_j}\bar{\omega}\int_{A_j}(\tau+\tau_0^*)^*\right]\right.$$

$$=\sum_i\left\{-\int_{\partial F_i}\bar{w}\sigma^{F*}+\sum_{F_i}\left[\int_{A_j}\bar{\omega}\int_{B_j}\tau^*-\int_{B_j}\bar{\omega}\int_{A_j}\tau^*\right]\right\}.$$

Thus the assertion follows.

Remark. The condition for σ is satisfied if σ equals some $(X, F'; \theta)$-p.d. for some $F'\subset F$ and $\theta\in\Theta(F', P)$ such that $P\subset F-F'$ and $\theta=0$ on $\overline{R-F}$.

Corollary 7.1 *Let* $\{a_j, b_j\}$ *be a system of complex numbers such that* $|a_j|+|b_j|\neq 0$. *Suppose that* Γ_x *satisfies the following conditions*: (i) $\Gamma_x^{\perp}\subset\Gamma_x^*$, (ii) $a_j\int_{A_j}\lambda=b_j\int_{B_j}\lambda$ *for any* $\lambda\in\Gamma_x$ *and any* j. *Then under the same assumption as in Theorem 7 we have*

$$(\sigma^F, \omega^F)_{R-F}=-\int_{\partial F}\bar{w}\sigma^{F*}\qquad(\mathrm{d}\,w=\omega),$$

$$\lim_{n\to\infty}\int_{\partial G_n}\overline{w^F}\sigma^{F*}=0\qquad(\mathrm{d}\,w^F=\omega^F).$$

Proof. We note that by the assumption

$$\tau^*\in\Gamma_x\quad\text{and}\quad\int_{A_j}\bar{\omega}\int_{B_j}\tau^*-\int_{B_j}\bar{\omega}\int_{A_j}\tau^*=0.$$

So we have by Theorem 7 the first assertion

$$(\sigma^F, \omega^F)_{R-F}=-\int_{\partial F}\bar{w}\sigma^{F*}.$$

Further we have

$$(\sigma^F,\ \omega^F)_{R-F\cup\bar{G}_n}=((\sigma^F)^{F\cup\bar{G}_n},\ (\omega^F)^{F\cup\bar{G}_n})_{R-F\cup\bar{G}_n}=-\int_{\partial(F\cup\bar{G}_n)}\overline{w^F}\sigma^{*}.$$

$$(\tau+\tau_0^*,\ \omega^F)_{F-G_n}=\int_{\partial(F-G_n)}\overline{w^F}\sigma^{F*}.$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\int_{\partial G_n}\overline{w^F}\sigma^{F*}&=\lim_{n\to\infty}\left\{\int_{\partial(F\cup G_n)}\overline{w^F}\sigma^{F*}-\int_{\partial(F-G_n)}\overline{w^F}\sigma^{F*}\right\}\\&=-\lim_{n\to\infty}(\sigma^F,\ \omega^F)_{R-F\cup\bar{G}_n}-\lim_{n\to\infty}(\tau+\tau_0^*,\ \omega^F)_{F-G_n}\\&=0.\end{aligned}$$

As for a real Hilbert space Λ_x ($\Lambda_x\subset\Lambda_{hse}$) we have

Theorem 7′. *Let $F\in\mathfrak{F}'$ and $\sigma,\ \omega\in(\Lambda_x+\Lambda_{eo})\cap\Lambda^1$ ($\omega=\mathrm{d}\,w$). If σ is represented by $\sigma=\tau+\tau_0^*$ with $\tau\in\Lambda_x^\perp$ and $\tau_0\in\Lambda_{eo}$ on $\overline{R-F}$, then $\sigma^F=\sigma$ and*

$$\langle\sigma^F,\ \omega^F\rangle_{R-F}=-\mathrm{Re}\int_{\partial F}\bar{w}\sigma^*+\sum_F\mathrm{Re}\left[\int_{A_j}\bar{\omega}\int_{B_j}\tau^*-\int_{A_j}\bar{\omega}\int_{B_j}\tau^*\right].$$

Further, suppose that Λ_x satisfies one of the following three conditions:
(i) $\Lambda_x^\perp\subset i\Lambda_x^*$, and there exists a family $L=\{l_j\}$ of straight lines l_j through zero in $\boldsymbol{C}$ such that $\int_{A_j}\lambda,\ \int_{B_j}\lambda\in l_j$ for any $\lambda\in\Lambda_x$ and any j,
(ii) $\Lambda_x^\perp\subset i\Lambda_x^*$ or $\Lambda_x^\perp\subset\Lambda_x^*$, and there exists a system of real numbers $\{a_j,\ b_j\}$ such that $|a_j|+|b_j|\neq 0$ and $a_j\int_{A_j}\lambda=b_j\int_{B_j}\lambda$ for any $\lambda\in\Lambda_x$ and any j,
(iii) $\Lambda_x^\perp\subset i\bar{\Lambda}_x^*$ or $\Lambda_x^\perp\subset\bar{\Lambda}_x^*$, and there exists a system of complex numbers $\{a_j,\ b_j\}$ such that $|a_j|+|b_j|\neq 0$ and $a_j\int_{A_j}\lambda=b_j\int_{B_j}\lambda$ for any $\lambda\in\Lambda_x$ and any j.
Then we have

$$\langle\sigma^F,\ \omega^F\rangle_{R-F}=-\mathrm{Re}\int_{\partial F}\bar{w}\sigma^{F*},$$

$$\lim_{n\to\infty}\mathrm{Re}\int_{\partial G_n}\overline{w^F}\sigma^{F*}=0\qquad(\mathrm{d}\,w^F=\omega^F).$$

Let $\Gamma_x\subset\Gamma_{hse}$, $\Gamma_y\subset\Gamma_h$ and set $X=\Gamma_x+\Gamma_{eo}$, $Y=\Gamma_y+\Gamma_{eo}$. Suppose that (i) $\Gamma_x\perp\bar{\Gamma}_y^*$, (ii) there exists a system of complex numbers $\{a_j,\ b_j\}$ such that $|a_j|+|b_j|\neq 0$ and $a_j\int_{A_j}\lambda=b_j\int_{B_j}\lambda$ for any $\lambda\in\Gamma_x\cup\Gamma_y$ and any j. Now let $\mathrm{d}\,u\in\Theta(F,\ P)$ be a semiexact C^1-differential in $R-P$ and $\nu\in\Theta(F,\ P)$ be a closed C^1-differential in $R-P$ such that $\lim_{n\to\infty}\int_{\partial G_n}u\nu=0$, $\lim_{n\to\infty}\int_{\partial G_n}f\nu=0$ for any $\mathrm{d}\,f\in\Gamma_x$ and $\lim_{n\to\infty}\int_{\partial G_n}u\omega=0$

for any $\omega\in\Gamma_y$. Then for $(X, F; \mathrm{d}\,u)$-p. d. ϕ_o and $(Y, F; \nu)$-p. d. ψ_o

$$\lim_{n\to\infty}\int_{\partial G_n} f_o\psi_o=-(\phi_o-\mathrm{d}\,u,\ \overline{(\psi_o-\nu)^*})=0 \qquad (\mathrm{d}\,f_o=\phi_o)\,.$$

Further, if ϕ_o and ψ_o are meromorphic on R and $f_o\psi_o$ is holomorphic in a neighbourhood of the ideal boundary, we have

$$2\pi i\sum \mathrm{Res}\, f_o\psi_o=-\sum\Big[\int_{A_j}\phi_o\int_{B_j}\psi_o-\int_{B_j}\phi_o\int_{A_j}\psi_o\Big]\,.$$

We shall mention the real case. Let $X=\Lambda_x+\Lambda_{eo}\subset\Lambda_{se}$ and $Y=\Lambda_y+\Lambda_{eo}\subset\Lambda_{co}$ satisfy that (1) $\Lambda_x\perp i\bar{\Lambda}_y^*$, (2) each differential of $\Lambda_x\cup\Lambda_y$ satisfies the period condition in (iii) of Theorem 7′, or each differential of $\Lambda_x\cup\bar{\Lambda}_y$ satisfies the period condition in (i) of Theorem 7′. Here we assume that $\lim_{n\to\infty}\mathrm{Im}\int_{\partial G_n} u\nu=0$, $\lim_{n\to\infty}\mathrm{Im}\int_{\partial G_n} f\nu=0$ for any $\mathrm{d}\,f\in\Lambda_x$ and $\lim_{n\to\infty}\mathrm{Im}\int_{\partial G_n} u\omega=0$ for any $\omega\in\Lambda_y$. Then for $(X, F; \mathrm{d}\,u)$-p. d. ϕ_o and $(Y, F; \nu)$-p. d. ψ_o we have

$$\lim_{n\to\infty}\mathrm{Im}\int_{\partial G_n} f_o\psi_o=0 \qquad (\mathrm{d}\,f_o=\phi_o)\,.$$

If ϕ_o and ψ_o are meromorphic on R and $f_o\psi_o$ is holomorphic in a neighbourhood of the ideal boundary, we have

$$2\pi\ \mathrm{Re}\sum \mathrm{Res}\, f_o\psi_o=-\sum \mathrm{Im}\Big[\int_{A_j}\phi_o\int_{B_j}\psi_o-\int_{B_j}\phi_o\int_{A_j}\psi_o\Big]\,.$$

The subspaces Γ_x with which we have been concerned seem to be interesting complex subspaces which correspond to the behavior spaces treated by M. Shiba [18]. We shall investigate the existence of such a Γ_x in §4.

Finally we refer to the regular operators.

Theorem 8. *Let $F\in\mathfrak{F}'$ be compact and $\sigma\in X$ be a C^1-differential which is harmonic on ∂F.*

(i) *For any ω in $X\cap\Gamma^1$ which is exact $(\omega=\mathrm{d}\,w)$ on F*

$$(\sigma^F,\ \omega^F)_{R-F}=-\int_{\partial F}\bar{w}\,\sigma^{F*}\,.$$

(ii) *In case $\Gamma_x\supset\Gamma_{ho}$, for any ω in $X\cap\Gamma^1$ which is semiexact $(\omega=\mathrm{d}\,w)$ on F*

$$(\sigma^F,\ \omega^F)_{R-F}=-\int_{\partial F}\bar{w}\,\sigma^{F*}\,.$$

Proof. Let k_F be a C^1-function such that $k_F=1$ on F and the support of k_F is compact and does not intersect the homology basis of $R-F$. We define a differential $\omega_1=\mathrm{d}\,(k_F w)$. Then $\omega_1\in\Gamma_{eo}$ in case (i) and $\omega_1\in\Gamma_{ho}+\Gamma_{eo}$ in case (ii). We have

$$(\sigma^F,\ \omega^F)_{R-F}=(\sigma^F,\ \omega_1^F)_{R-F}=(\sigma^F,\ \omega_1)_{R-F}=-\int_{\partial F}\bar{w}\,\sigma^{F*}\,.$$

If $\Gamma_x \subset \Gamma_{he}$, any C^1-differential in X satisfies condition (i) and therefore our operator reduces to a canonical regular operator in the sense of H. Yamaguchi [24].

Corollary 8.1 *Let $\Gamma_x \subset \Gamma_{he}$ or $\Gamma_{ho} \subset \Gamma_x \subset \Gamma_{hse}$. Then for*

$$\omega,\ \sigma \in X, \qquad \omega^F = \mathrm{d}\, w^F, \qquad \lim_{n\to\infty}\int_{\partial G_n} \overline{w^F} \sigma^{F*} = 0\,.$$

§ 4. Existence of some behavior spaces

Shiba showed that the Riemann-Roch theorem of Kusunoki's type can be extended by means of behavior spaces. An example of his behavior spaces is $\Lambda_x = {}_R\Gamma_{hm} + i_R\Gamma_{hse}$, where ${}_R\Gamma$ denotes the space of real differentials. But in our case the existence of a space Γ_x is not always trivial on an arbitrary Riemann surface. To prove the existence, we shall use Zorn's Lemma (cf. [23]).

Theorem 9. *Let $\Xi = \{A_j, B_j\}$ be a canonical homology basis of R modulo dividing cycles. Let $\{a_j, b_j\}$ be a system of complex numbers such that $|a_j| + |b_j| \neq 0$. Then there exists a subspace Γ_x of Γ_{hse} such that* (i) $\Gamma_x = \Gamma_x^{\perp *}$, (ii) $a_j \int_{A_j} \omega = b_j \int_{B_j} \omega$ *for any $\omega \in \Gamma_x$ and any j.*

Proof. We set

$$\Gamma_{x_0} = \left\{\omega \in \Gamma_{hse}\,;\ a_j \int_{A_j} \omega = b_j \int_{B_j} \omega \text{ for any } j\right\}\,.$$

Since Γ_{x_0} contains $\Gamma_{hm} = \Gamma_{hse}^{\perp *}$, subspace $\Gamma_{x_0}^{\perp *}$ is contained in Γ_{hse}. There exist differentials τ_{A_j} and τ_{B_j} in Γ_{ho} such that $\int_\gamma \tau_{A_j} = A_j \times \gamma$ and $\int_\gamma \tau_{B_j} = B_j \times \gamma$ for any cycle γ. Then the differential $\tau_j = a_j \tau_{A_j} - b_j \tau_{B_j}$ belongs to Γ_{x_0}, because τ_j belongs to $\Gamma_{ho} \subset \Gamma_{hse}$ and satisfies that

(i) $\int_{A_i} \tau_j = \int_{B_i} \tau_j = 0 \qquad \text{for} \quad i \neq j\,,$

(ii) $$a_j \int_{A_j} \tau_j = a_j \int_{A_j} (a_j \tau_{A_j} - b_j \tau_{B_j}) = a_j b_j$$
$$= b_j \int_{B_j} (a_j \tau_{A_j} - b_j \tau_{B_j}) = b_j \int_{B_j} \tau_j\,.$$

Let $\{G_n\}$ be a canonical regular exhaustion with which Ξ is associated and write $\tau_j = \mathrm{d}\, t_j$. Then for any $\sigma \in \Gamma_{x_0}^{\perp}$,

$$0 = (\tau_j,\ \sigma) = \lim_{n\to\infty}\left\{\int_{\partial G_n} t_j \bar{\sigma}^* - \sum_{G_n}\left[\int_{A_i} \tau_j \int_{B_i} \bar{\sigma}^* - \int_{B_i} \tau_j \int_{A_i} \bar{\sigma}^*\right]\right\}\,.$$

We can assume that t_j coincides with a Dirichlet potential in a neighbourhood of the ideal boundary. Since $\lim_{n\to\infty}\int_{\partial G_n} t_j \bar{\sigma}^* = 0$,

$$0=\int_{A_j}\tau_j\int_{B_j}\bar{\sigma}^*-\int_{B_j}\tau_j\int_{A_j}\bar{\sigma}^*.$$

It follows that $a_j\int_{A_j}\bar{\sigma}^*=b_j\int_{B_j}\bar{\sigma}^*$ and $\Gamma_{x_0}\supset\bar{\Gamma}_{x_0}^{\perp *}$. Let Υ be the family of subspaces: $\Upsilon=\{\Gamma_y\,;\,\Gamma_y\subset\Gamma_{x_0}$ and $\Gamma_y\supset\bar{\Gamma}_y^{\perp *}\}$. The family Υ is an ordered set with the partial ordering by the set inclusion relation. For any linearly ordered subfamily $\Upsilon_1=\{\Gamma_{y_k}\}_{k\in\kappa}$ of Υ, we put $\Gamma_y=\bigcap_{k\in\kappa}\Gamma_{y_k}$. It is clear that $\Gamma_y\subset\Gamma_{x_0}$. If $\Gamma_{y_k}\supset\Gamma_{y_l}$, then $\bar{\Gamma}_{y_k}^{\perp *}\subset\bar{\Gamma}_{y_l}^{\perp *}\subset\Gamma_{y_l}$. Hence $\bar{\Gamma}_{y_k}^{\perp *}$ is contained in Γ_y. It follows that $\bar{\Gamma}_y^{\perp *}$ is contained in any Γ_{y_k}, i.e., $\bar{\Gamma}_y^{\perp *}\subset\bigcap_{k\in\kappa}\Gamma_{y_k}=\Gamma_y$. The space Γ_y belongs to Υ and satisfies $\Gamma_{y_k}\supset\Gamma_y$ for any $\Gamma_{y_k}\in\Upsilon_1$. By Zorn's Lemma Υ contains at least one minimal element. Let Γ_x be a minimal element in Υ. If a differential ω in Γ_x is orthogonal to $\bar{\Gamma}_x^{\perp *}$, ω belongs to $\bar{\Gamma}_x^*$. We have an orthogonal decomposition:

$$\Gamma_x=\bar{\Gamma}_x^{\perp *}+\Gamma_x\cap\bar{\Gamma}_x^*.$$

Suppose that $\Gamma_x\cap\bar{\Gamma}_x^*\neq\{0\}$ and ω be a non trivial differential in $\Gamma_x\cap\bar{\Gamma}_x^*$. There exists a real number α such that $\omega+e^{i\alpha}\bar{\omega}^*\neq 0$ and $(\omega+e^{i\alpha}\bar{\omega}^*, \overline{(\omega+e^{i\alpha}\bar{\omega}^*)}^*)=0$. Let Γ_x' denote a subspace of Γ_x whose element is orthogonal to $\omega+e^{i\alpha}\bar{\omega}^*$. It is clear that $\Gamma_x'\supset\overline{\Gamma_x'}^{\perp *}$. Since $\overline{(\omega+e^{i\alpha}\bar{\omega}^*)}^*)$ belongs to Γ_x', we have

$$0=(\sigma^*, (\omega+e^{i\alpha}\bar{\omega}^*)^*)=(\sigma, (\omega+e^{i\alpha}\bar{\omega}^*))$$

for any σ in $\overline{\Gamma_x'}^{\perp *}$. Thus $\overline{\Gamma_x'}^{\perp *}\subset\Gamma_x'$ $(\subsetneqq\Gamma_x)$. This is a contradiction.

We can get the following in the same way.

Corollary 9.1 *Let $\{a_j, b_j\}$ be a system of real numbers such that $|a_j|+|b_j|\neq 0$. Then there exists a subspace Γ_x of Γ_{hse} such that* (i) $\Gamma_x=\bar{\Gamma}_x=\Gamma_x^{\perp *}$, (ii) $a_j\int_{A_j}\omega=b_j\int_{B_j}\omega$ *for any $\omega\in\Gamma_x$ and any j.*

§ 5 A formulation of the Riemann-Roch theorem

We shall show that the theorems of Riemann-Roch and of Abel on an arbitrary open Riemann surface can be formulated by means of a behavior space Γ_x in § 4. They are formulated in complex form. Infinite divisors are also allowed. Let $\{V_i\}$ $(V_i=\{|z_i|<1)$ be a family of parametric discs on a Riemann surface R such that $\bar{V}_i\cap\bar{V}_j=\emptyset$ for $i\neq j$ and $\{V_i\}$ has no accumulating point in R. We put $G=\bigcup_i V_i$. Let $\Xi=\{A_j, B_j\}$ be a canonical homology basis of R modulo dividing cycles and $\{G_n\}$ be a canonical regular exhaustion with which Ξ is associated. We assume that A_j, B_j and ∂G_n do not meet G. Now, take a system of real numbers $\{a_j, b_j\}$ $(|a_j|+|b_j|\neq 0)$ and let Γ_x be a subspace of Γ_{hse} such that (i) $\Gamma_x=\bar{\Gamma}_x=\Gamma_x^{\perp *}$, (ii) $a_j\int_{A_j}\omega=b_j\int_{B_j}\omega$ for any $\omega\in\Gamma_x$ and any j (cf. Corollary 9.1).

Definition. A meromorphic differential ϕ is called to have X-behavior if there exist a G_n and an ω in $X=\Gamma_x+\Gamma_{eo}$ such that $\phi=\omega$ on $R-G-G_n$.

Now the existence of meromorphic differentials below is fundamental:

ϕ_{A_j} : a holomorphic differential with X-behavior such that

$$a_i\int_{A_i}\phi_{A_j}=b_i\int_{B_i}\phi_{A_j}-b_i\delta_{ij} \qquad \text{for any } i,$$

ϕ_{B_j} : a holomorphic differential with X-behavior such that

$$a_i\int_{A_i}\phi_{B_j}=b_i\int_{B_i}\phi_{B_j}-a_i\delta_{ij} \qquad \text{for any } i,$$

$\phi_{n,p}$: a meromorphic differential with X-behavior which has the singularity $\mathrm{d}\left(\frac{1}{z^n}\right)$ only at p (z is a fixed local parameter around p and $n\geqq 1$),

$\phi_{p,q}$: a meromorphic differential with X-behavior which has simple poles of residue 1 at p and of residue -1 at q and is regular analytic elsewhere.

The existence of $\phi_{n,p}$ and $\phi_{p,q}$ is evident by Corollary 6.1. So we shall be concerned with ϕ_{A_j} only. We set $\phi_{A_j}=\tau_{A_j,x}+i\tau^*_{A_j,x}$, where $\tau_{A_j,x}$ is the X-fundamental differential. Then ϕ_{A_j} is clearly a holomorphic differential with X-behavior. Next, recall that $\int_\gamma(\tau_{A_j,x}+\tau_{A_j,x^\perp})=A_j\times\gamma$ for any cycle γ and $\tau_{A_j,x}\in\Gamma_x^\perp$, $\tau_{A_j,x^\perp}\in\Gamma_x$. Then we have

$$a_i\int_{A_i}\tau_{A_j,x}=-a_i\int_{A_i}\tau_{A_j,x^\perp}=-b_i\int_{B_i}\tau_{A_j,x^\perp}=b_i\int_{B_i}\tau_{A_j,x}-b_i\delta_{ij}.$$

On the other hand, since $i\tau^*_{A_j,x}$ belongs to $i\Gamma_x^{\perp *}=\Gamma_x$, we have $a_i\int_{A_i}i\tau^*_{A_j,x}=b_i\int_{B_i}i\tau^*_{A_j,x}$ Hence $a_i\int_{A_i}\phi_{A_j}=b_i\int_{B_i}\phi_{A_j}-b_i\delta_{ij}$.

5.1 Riemann-Roch theorem

Let δ be a finite or infinite divisor on R whose support is contained in $G=\bigcup_i V_i$ and has a finitely many points in common with each V_i. Write as $\delta=\delta_p/\delta_q$, where $\delta_p=p_1^{\mu_1}p_2^{\mu_2}\cdots p_m^{\mu_m}\cdots$ and $\delta_q=q_1^{\nu_1}q_2^{\nu_2}\cdots q_n^{\nu_n}\cdots$ are disjoint integral divisors. We consider the following linear spaces over the complex number field:

$S(X;1/\delta)$ $=\{f$; (i) f is a single valued meromorphic function on R, (ii) $\mathrm{d}f$ has X-behavior, (iii) the divisor of f is a multiple of $1/\delta\}$,

$M(X;1/\delta_p)=\{f$; (i) f is a (multi valued) meromorphic function on R such that $a_i\int_{A_i}\mathrm{d}f=b_i\int_{B_i}\mathrm{d}f$ for all i, (ii) $\mathrm{d}f$ has X-behavior, (iii) the divisor of f is a multiple of $1/\delta_p\}$,

$D(X;\delta)$ $=\{\phi$; (i) ϕ is a meromorphic differential which has X-behavior, (ii) the divisor of ψ is a multiple of δ and the number of poles of ψ is finite},

$D(X; 1/\delta_q)=\{\psi$; (i) ψ is a meromorphic differential which has X-behavior, (ii) the divisor of ψ is a multiple of $1/\delta_q$ and the number of poles of ψ is finite}.

Here, in case that $\delta_q\neq 1$ we identify two elements f_1 and f_2 of $M(X; 1/\delta_p)$ if and only if f_1-f_2 is constant.

We consider a bilinear form defined on the product space $M(X; 1/\delta_p)\times D(X; 1/\delta_q)$:

$$h(f, \psi)=2\pi i\lim_{m\to\infty}\sum_{p_j\in G_m}\operatorname{Res}_{p_j} f\psi.$$

We shall see that h is a well defined bilinear form. Since ψ is regular at each p_j, additive constants (including periods) of f have no effect for the residue of $f\psi$ at p_j. So we regard f as a function on $R'=R-\bigcup_i(A_i\cup B_i)$. For any $f\in M(X; 1/\delta_p)$ and any $\psi\in D(X; 1/\delta_q)$, there exist a G_n and differentials ω, $\sigma\in X$ such that $df=\omega$ and $\psi=\sigma$ on $R-G-G_n$. Then by Corollary 7.1 and the period conditions,

$$\begin{aligned} h(f, \psi)&=2\pi i\lim_{m\to\infty}\sum_{p_j\in G_m}\operatorname{Res}_{p_j} f\psi=2\pi i\lim_{m\to\infty}\Big\{\sum_{G_m\cap R'}\operatorname{Res} f\psi-\sum_{G_m\cap R'}\operatorname{Res}_{q_l} f\psi\Big\}\\ &=\lim_{m\to\infty}\Big\{\int_{\partial G_m} f\psi-\sum_{G_m}\Big[\int_{A_i}df\int_{B_i}\psi-\int_{B_i}df\int_{A_i}\psi\Big]\Big\}-2\pi i\sum\operatorname{Res}_{q_l} f\psi\\ &=-\sum_{G_n}\Big[\int_{A_i}df\int_{B_i}\psi-\int_{B_i}df\int_{A_i}\psi\Big]-2\pi i\sum\operatorname{Res}_{q_l} f\psi. \end{aligned}$$

Thus $h(f, \psi)$ is a finite complex value.

Next let f belong to $M(X; 1/\delta_p)$. If df is holomorphic on R,

$$\begin{aligned} (df, df)&=(df, i\,df^*)\\ &=-i\lim_{m\to\infty}\Big\{\int_{\partial G_m} f\overline{df}-\sum_{G_m}\Big[\int_{A_i}df\int_{B_i}\overline{df}-\int_{B_i}df\int_{A_i}\overline{df}\Big]\\ &=-i\lim_{m\to\infty}\int_{\partial G_m} f\overline{df}=0, \end{aligned}$$

and hence df is identically zero.

Thus by a similar argument to Kusunoki [5], [6] (cf. [18]), we have the following

Theorem 10. (*Riemann-Roch theorem*)

$$\dim\frac{M(X; 1/\delta_p)}{S(X; 1/\delta)}=\dim\frac{D(X; 1/\delta_q)}{D(X; \delta)},$$

where the both sides may be infinite.

In particular, if δ_p is a finite divisor,

$$\dim S(X; 1/\delta) = \deg \delta_p + 1 - \min(\deg \delta_q, 1) - \dim \frac{D(X; 1/\delta_q)}{D(X; \delta)}.$$

Corollary 10.1 *Let R be a Riemann surface with a finite genus g and $\delta=\delta_p/\delta_q$ be a finite divisor. Then it holds that*

$$\dim S(X; 1/\delta) = \dim D(X; \delta) + \deg \delta - g + 1.$$

Remark. If Γ_x satisfies that (i) $\Gamma_x = \Gamma_x^{\perp *}$, (ii) there exists a system of non zero complex numbers $\{a_i, b_i\}$ such that $a_i \int_{A_i} \omega = b_i \int_{B_i} \omega$ for any $\omega \in \Gamma_x$ and any i, then we can formulate a Riemann-Roch theorem for $S(X; 1/\delta)$ and $D(\bar{X}; \delta)$, where $\bar{X}$ consists of the complex conjugates of differentials in X.

5.2 Abel's theorem

Let δ_p and δ_q be disjoint (finite or infinite) integral divisors whose supports are contained in $\bigcup_i V_{i,1/2}\left(V_{i,1/2} = \left\{|z_i| < \frac{1}{2}\right\}\right)$. Furthermore, suppose that the restrictions to each V_i of δ_p and δ_q have the same finite degree and write them as $p_{i,1} \cdots p_{i,n(i)}$, $q_{i,1} \cdots q_{i,n(i)}$, where $p_{i,j}$ (resp. $q_{i,j}$) may coincide with $p_{i,k}$ (resp. $q_{i,k}$) for $j \neq k$. We denote by δ the divisor δ_p/δ_q. We assume that there exists a closed C^1-differential θ in $R - \bigcup_i \{p_{i,j}, q_{i,j}\}_{j=1}^{n(i)}$ such that

$$\text{(i)} \quad \theta = \begin{cases} d\left[\sum_{j=1}^{n(i)} \log \dfrac{z_i - z_i(p_{i,j})}{z_i - z_i(q_{i,j})}\right] & \text{on each } V_i \\ 0 & \text{on } R - \bigcup_i V_i, \end{cases}$$

$$\text{(ii)} \quad (\theta, \theta)_{\bigcup_i (V_i - \bar{V}_{i,1/2})} < \infty.$$

Remark. When δ is given, it will be generally difficult to find a θ. However, for a given integral divisor δ_p which is supported in $\bigcup_i V_{i,1/2}$ and has no accumulating point in R, we can find an integral divisor δ_q and a closed C^1-differential θ satisfing conditions (i), (ii). In fact, if we take $q_{i,j}$ close to $p_{i,j}$ so that $d\left[\log \dfrac{z_i - z_i(p_{i,j})}{z_i - z_i(q_{i,j})}\right]$ is close enough to 0 uniformly on $\bar{V}_i - V_{i,1/2}$, then we can find a θ which satisfies (i), (ii).

For the system of real numbers $\{a_j, b_j\}$ which satisfies $a_j \int_{A_j} \omega = b_j \int_{B_j} \omega$ for any $\omega \in \Gamma_x$, we further assume that a_j and b_j are integers. Now we can state a theorem of Abel's type.

Theorem 11. (*Abel's theorem*)
The following two statements are equivalent.
(1) *There exists a single valued meromorphic function such that*

(i) *the divisor of* f *is* δ, (ii) $\mathrm{d}\log f$ *has X-behavior.*
(2) *For any chain* $C=C_1+C_2$ *with* $C_1\subset G_n$, $C_2\subset G\cap(R-G_n)$ *and* $\partial C_1=\sum_{G_n}(p_{i,j}-q_{i,j})$, $\partial C_2=\sum_{R-G_n}(p_{i,j}-q_{i,j})$, *there exist integers* m_k, n_k ((*k runs through a finite subset* $\mathcal{K}$ *of* $\{1,\cdots,g\}$) *such that* $\int_C\phi_{A_j}+\sum_{k\in\mathcal{K}}\left[m_k\int_{A_k}\phi_{A_j}+n_k\int_{B_k}\phi_{A_j}\right]$, $\int_C\phi_{B_j}+\sum_{k\in K}\left[m_k\int_{A_k}\phi_{B_j}+n_k\int_{B_k}\phi_{B_j}\right]$ *are integers for all* j.

Proof. Let $R'=R-\bigcup_i(A_i\cup B_i)$ and $G'_n=G_n\cap R'$. We set $h_{A_j}=\int\phi_{A_j}$ and $h_{B_j}=\int\phi_{B_j}$ in R'. Let f be a meromorphic function which is asserted in (1). Since $\mathrm{d}\log f$ has X-behavior, there exists a finite set of integers $\tilde{\mathcal{K}}$ such that

$$a_i\int_{A_i}\mathrm{d}\log f=b_i\int_{B_i}\mathrm{d}\log f$$

for $i\notin\tilde{\mathcal{K}}$. Then we have for $i\notin\tilde{\mathcal{K}}$

$$\int_{A_i}\phi_{A_j}\int_{B_i}\mathrm{d}\arg f-\int_{B_i}\phi_{A_j}\int_{A_i}\mathrm{d}\arg f=-\delta_{ij}\int_{A_i}\mathrm{d}\arg f.$$

We set $\tilde{m}_k=\frac{1}{2\pi}\int_{B_k}\mathrm{d}\arg f$ and $\tilde{n}_k=\frac{1}{2\pi}\int_{A_k}\mathrm{d}\arg f$ for $k\in\tilde{\mathcal{K}}$. We can take integers m'_i, n'_i $(i\in\mathcal{K}_n)$ so that $\sum_{i\in\mathcal{K}_n}(m'_iA_i+n'_iB_i)$ is homologous to C_1-C' (mod ∂G_n) for some chain C' on G'_n. Set $\mathcal{K}=\mathcal{K}_n\cup\tilde{\mathcal{K}}$, $m_k=m'_k+\tilde{m}_k$ for $k\in\mathcal{K}\cap\tilde{\mathcal{K}}$, $=m'_k$ for $k\in\mathcal{K}_n-\tilde{\mathcal{K}}$, $=\tilde{m}_k$ for $k\in\tilde{\mathcal{K}}-\mathcal{K}_n$ and so with n_k. Then we have

$$\int_C\phi_{A_j}+\sum_{k\in\mathcal{K}}\Big(m_k\int_{A_k}\phi_{A_j}+n_k\int_{B_k}\phi_{A_j}\Big]$$
$$=\int_C\phi_{A_j}+\sum_{i\in\mathcal{K}_n}\Big(m'_i\int_{A_i}\phi_{A_j}+n'_i\int_{B_i}\phi_{A_j}\Big)+\sum_{k\in\tilde{\mathcal{K}}}\Big(m_k\int_{A_k}\phi_{A_j}+n_k\int_{B_k}\phi_{A_j}\Big)$$
$$=\lim_{m\to\infty}\sum_{G_m}\mathrm{Res}\,h_{A_j}\,\mathrm{d}\log f+\sum_{k\in\tilde{\mathcal{K}}}\Big(m_k\int_{A_k}\phi_{A_j}+n_k\int_{B_k}\phi_{A_j}\Big)$$
$$=\frac{1}{2\pi i}\lim_{m\to\infty}\Big\{\int_{\partial G_m}h_{A_j}\,\mathrm{d}\log f-\sum_{G_m}\Big[\int_{A_i}\phi_{A_j}\int_{B_i}\mathrm{d}\log f-\int_{B_i}\phi_{A_j}\int_{A_i}\mathrm{d}\log f\Big]\Big\}$$
$$+\sum_{k\in\tilde{\mathcal{K}}}\frac{1}{2\pi}\Big[\int_{A_k}\phi_{A_j}\int_{B_k}\mathrm{d}\arg f-\int_{B_k}\phi_{A_j}\int_{A_k}\mathrm{d}\arg f\Big]$$
$$=\sum_{i\notin\tilde{\mathcal{K}}}\frac{1}{2\pi}\Big[\int_{A_i}\phi_{A_j}\int_{B_i}\mathrm{d}\arg f-\int_{B_i}\phi_{A_j}\int_{A_i}\mathrm{d}\arg f\Big]$$
$$=\begin{cases}-\frac{1}{2\pi}\int_{A_j}\mathrm{d}\arg f & \text{for } j\notin\tilde{\mathcal{K}}\\ 0 & \text{for } j\in\tilde{\mathcal{K}}.\end{cases}$$

The same can be said for ϕ_{B_j}. Thus we get (2).

Conversely we suppose (2). Let θ_1 (resp. θ_1') be the real (resp. imaginary) part of θ and ϕ_1 be a $(X, F; \theta_1, \theta_1')$-p.d., where $F=\bigcup_i(\bar{V}_i-V_{i,1/2})$. We set $\Phi_1=\phi_1+i\phi_1^*$. Then Φ_1 is a $(X, F; \theta, \theta_1'-i\theta_1)$-p.d. and hence it is a meromorphic function with X-behavior. Furthermore, Φ_1 satisfies $a_i\int_{A_i}\Phi_1=b_i\int_{B_i}\Phi_1$ for any i. Setting $\Phi=\Phi_1+2\pi i\left\{\sum_{i\in\mathcal{K}_n}(m_i'\phi_{A_i}+n_i'\phi_{B_i})+\sum_{k\in\mathcal{K}}(m_k\phi_{A_k}+n_k\phi_{B_k}\right)$, we have

$$\begin{aligned}
&\lim_{m\to\infty}2\pi i\sum_{G_m}\operatorname{Res} h_{A_j}\Phi\\
&=\lim_{m\to\infty}\left\{\int_{\partial G_m}h_{A_j}\Phi-\sum_{G_m}\left[\int_{A_i}\phi_{A_j}\int_{B_i}\Phi-\int_{B_i}\phi_{A_j}\int_{A_i}\Phi\right]\right\}\\
&=-2\pi i\left\{\sum_{i\in\mathcal{K}_n}\left(m_i'\int_{A_i}\phi_{A_j}+n_i'\int_{B_i}\phi_{B_i}\right)+\sum_{k\in\mathcal{K}}\left(m_k\int_{A_k}\phi_{A_j}+n_k\int_{B_k}\phi_{A_j}\right)+\int_{A_j}\Phi\,.\right.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{2\pi i}\int_{A_j}\Phi&=\lim_{m\to\infty}\sum_{G_m}\operatorname{Res} h_{A_j}\Phi+\sum_{i\in\mathcal{K}_n}\left(m_i'\int_{A_i}\phi_{A_j}+n_i'\int_{B_i}\phi_{A_j}\right)\\
&\quad+\sum_{k\in\mathcal{K}}\left(m_k\int_{A_k}\phi_{A_j}+n_k\int_{B_k}\phi_{A_j}\right)\\
&=\int_C\phi_{A_j}+\sum_{k\in\mathcal{K}}\left(m_k\int_{A_k}\phi_{A_j}+n_k\int_{B_k}\phi_{A_j}\right).
\end{aligned}$$

Thus $\frac{1}{2\pi i}\int_{A_j}\Phi$ is an integer. Similarly $\frac{1}{2\pi i}\int_{B_j}\Phi$ is an integer. It follows that $f(p)=\exp\int^p\Phi$ is a single valued meromorphic function and has the required properties in (1).

Corollary 11.1 *The following two statements are equivalent.*
(1′) *There exists a single valued meromorphic function f such that* (i) *the divisor of f is δ,* (ii) $\mathrm{d}\log f$ *has X-behavior, and* (iii) $a_i\int_{A_i}\mathrm{d}\arg f=b_i\int_{B_i}\mathrm{d}\arg f$ *for any i.*
(2′) *For any chain C in* (2) *of Theorem* 11 *which is contained in R', $\int_C\phi_{A_j}$ and $\int_C\phi_{B_j}$ are integers for all j.*

A meromorphic function f in (1′) *is uniquely determined up to a non zero multiplicative constant.*

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