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Analytic mappings of a Riemann surface of finite type into a torus

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Introduction

The study on the existence of analytic mappings of a Riemann surface into another generally involves much difficulty. Obstruction lies, above all, in the non-planer character of the image surface. The purpose of the present paper is to investigate analytic mappings of what we call a Riemann surface of finite type into a torus. Namely, the domain surface R_N is a closed Riemann surface with a finite number $(=N)$ of points (punctures) removed. The image surface, on the other hand, is a closed surface T of genus one. In such a case, we can make use of the finite complex plane which is the universal covering surface of T.

After preparing some fundamental facts, we shall first prove a theorem (Theorem 1) which gives a necessary and sufficient condition for the existence of an analytic mapping f of R_N into T with two prescribed properties; one is purely topological and the other is purely analytic. The topological condition imposed on f is the assignment of the homomorphism between the first homology groups which is to be induced by f, and the analytic condition is the pre-assignment of the behavior of f near the punctures (which are the isolated singular points of f). Theorem 1 is proved by means of real normalization of periods of Abelian differentials, while we shall later make use of complex normalization to prove a corresponding theorem (Theorem 5). These two results, Theorems 1 and 5, are thus the same in essence. Each of them has, however, an advantage over the other in applications. Compare Theorem 6 with Theorem 7.

Historically, such a problem was first considerd for closed surfaces $(N=0)$. The existence and the determination of explicit form of the mapping f were mainly studied. See Krazer [8], the last chapter. We shall also recall some relevant known facts: For any homomorphism between the first homology groups of a closed surface of positive genus and a torus, there always exists a *continuous* mapping which induces the homomorphism (H. Hopf [7]). An *analytic* mapping, however, does not necessarily exist (Gerstenhaber [4]). On the contrary, if the domain surface has N punctures, $N\geqq 1$, then every homomorphism between the homology groups is induced by an analytic mapping of R_N into T. In other words, for noncompact surfaces, homomorphisms arising from analytic mappings are subject to no restrictions. This can be easily seen if we use the well-known theorem due to Behnke and Stein (see Kusunoki-Sainouchi [11]; cf. also [10]). We are thus particularly interested in the behavior of an analytic mapping f which induces the given homomorphism between the homology groups. See Theorem 9. By a theorem of Ohtsuka ([13]) f assumes every value in T whithout exception (in any neighborhood of punctures to which f cannot be extended holomorphically).

We shall also prove that the analytic singularity of f is (uniquely) determined in a canonical manner by the induced homormorphism of the homology groups. See section 11. We shall prove furthermore that, for a fixed homomorphism, the singularity can be chosen so as to be a meromorphic function on the Bers fiber space over the Teichmüller space (Theorem 10).

Beside these, we shall discuss some other related topics such as the uniqueness (Theorem 4), relations of our results to the classical theory (Theorems 2, 11, 11′ etc.), the existence of analytic mappings with a simpler topological property (Theorems 6 and 7) and so forth. It can be easily seen that there is a close connection between Theorems 6, 7 and the classical Abel's theorem (cf. [9], [15]). When the domain surface is also of genus one, Theorem 2 reduces to a theorem which is stated in Helfenstein [6]. See Theorem 2′.

Through the paper $\boldsymbol{C}$ (resp. $\boldsymbol{R}$) denotes the complex (resp. real) number system. We shall also use the letter $\boldsymbol{Z}$ (resp. $\boldsymbol{Q}$) to denote the set of all integers (resp. rational numbers).

I. Preliminaries

1. Let R_0 be a closed Riemann surface of genus $g\geqq 1$. We fix a canonical homology basis $\{A_j, B_j\}_{j=1}^{g}$ ([15]). Namely,

$$\left.\begin{aligned} &A_i\times B_j=\delta_{ij}{}^{1)} \\ &A_i\times A_j=B_i\times B_j=0 \end{aligned}\right. \qquad i, j=1, 2, \cdots, g\,.$$

Furthermore we may assume that A_j, B_j are analytic Jordan curves and $A_i\cap A_j=B_i\cap B_j=A_i\cap B_j=\emptyset$, $i\neq j$; $A_j\cap B_j$ consists of a single point. Let $p_1, p_2, \cdots, p_N$ be N distinct points of R_0. We shall allow N to be zero. Then the surface $R_N=R_0-\{p_1, p_2, \cdots, p_N\}$ is a Riemann surface of finite type (g, N). Without loss of generality, we may assume that p_k does not lie on A_j and B_j. Take parametric disks U_k about p_k so small that $U_k\cap U_l=\emptyset$ $(k\neq l)$ and $U_k\cap A_j=U_k\cap B_j=\emptyset$ $(k=1, 2, \cdots N;\ j=1, 2, \cdots, g)$. If we set $D_k=-\partial U_k$, it is easily seen that $\{A_j, B_j; D_k\}_{\substack{j=1,2,\cdots,g\\ k=1,2,\cdots,N-1}}$ forms a canonical homology basis of R_N. If R_N is

1) The intersection number $\gamma\times\delta$ of two 1-cycles γ, δ is defined to be $+1$ if the cycle δ crosses γ from its right to left. This definition agrees to those in [10], [15] etc.. Note that the definition in [1] has the opposite sign and that my former paper [14] adopts the definition of [1].

viewed as an abstract open surface, the ideal boundary of R_N is denoted by ∂R_N. Clearly, ∂R_N consists of N components $\Gamma_1, \cdots, \Gamma_N$ corresponding to the points $p_1, \cdots, p_N$.

Denote by $H_1(R_N)$ the one-dimensional homology group of R_N with integral coefficients, $N \geqq 0$. We denote by $H_1(R_N, \partial R_N)$ the one-dimensional homology group of R_N modulo dividing cycles (cf. [1]). The group $H_1(R_N, \partial R_N)$ is isomorphic to $H_1(R_0)$ and is generated by the equivalence classes of A_j, B_j, $j=1, 2, \cdots, g$. On the other hand, $H_1(R_N)$ is the free abelian group generated by the homology classes of A_j, B_j and D_k, $j=1, 2, \cdots, g$; $k=1, 2, \cdots, N-1$.

Let z_k be a fixed local variable which maps U_k onto the unit disk $|z_k|<1$ and $z_k(p_k)=0$, $k=1, 2, \cdots, N$. We shall call a differential φ on R_N an *Abelian differential* if (i) φ is meromorphic on R_N, (ii) the polar singularities of φ are finite in number. Since R_N is of finite type, φ can be expanded in a Laurent series of powers of z_k about p_k:

$$\varphi(z_k)=\left[\sum_{n=1}^{\infty} \frac{a_n^{(k)}}{z_k^n}+\sum_{n=0}^{\infty} b_n^{(k)} z_k^n\right] dz_k ,$$

$k=1, 2, \cdots, N$. Every Abelian differential on R_0 can be naturally identified with an Abelian differential on R_N. However, not every Abelian differential on R_N necessarily rises from an Abelian differential on R_0. Although every Abelian differential on R_N is analytic on R_0 except for a finite number of isolated singularities, it may have *essential* singularities at some points of $\{p_1, p_2, \cdots, p_N\}$. If φ is an Abelian differential on R_N, the *residue* of φ at Γ_k is defined by

$$\operatorname*{Res}_{\Gamma_k} \varphi=-\frac{1}{2\pi i} \int_{D_k} \varphi=a_1^{(k)} .$$

It is known that the residue of φ at Γ_k is determined uniquely regardless of the choice of a local variable about p_k.

An Abelian differential φ on R_N is called *of the first kind* if φ can be identified with an Abelian differential on R_0 which is of the first kind (in the classical sense). [On the contrary, the differentials of the second and third kinds may be allowed to have non-polar (essential) singularities at the points of $\{p_1, p_2, \cdots, p_N\}$. Namely, φ is called *of the second kind* if φ has no non-zero residues on R_N (including the residue at Γ_k, $k=1, 2, \cdots, N$); otherwise φ is called *of the third kind.*] Note that we have defined the class of the differentials of the first kind independently of the existence of the set $\{p_1, p_2, \cdots, p_N\}$.

Proposition 1. *Let φ, ψ be Abelian differentials on R_N.*

(i) *If φ is of the first kind, we have (Riemann's inequality)*

$$-2 \cdot \operatorname{Im} \sum_{j=1}^{g} \int_{A_j} \varphi \int_{B_j} \bar{\varphi}=i \sum_{j=1}^{g}\left(\int_{A_j} \varphi \int_{B_j} \bar{\varphi}-\int_{B_j} \varphi \int_{A_j} \bar{\varphi}\right)>0 ,$$

provided that $\varphi \not\equiv 0$.

(ii) *If φ is of the first or second kind, then*

$$2\pi i\cdot\sum \mathrm{Res}\,\Phi\psi^{2)}=\sum_{j=1}^{g}\left(\int_{A_j}\varphi\int_{B_j}\psi-\int_{B_j}\varphi\int_{A_j}\psi\right),$$

Φ being a single-valued integral of φ on the planar surface $R_N'=R_N-\bigcup_{j=1}^{g}(A_j\cup B_j)$.

Corollary. *Suppose that an Abelian differential φ of the first kind satisfies*

$$\int_{A_j}\varphi=s_j\zeta_j,\quad \int_{B_j}\varphi=t_j\zeta_j,\quad \zeta_j\in\boldsymbol{C},\ s_j,\ t_j\in\boldsymbol{R},\ j=1,2,\cdots,g.$$

Then $\varphi\equiv 0$.

2. Let $\varphi_1^*,\cdots,\varphi_g^*$ be the normal differentials of the first kind with respect to the homology basis $\{A_j, B_j\}_{j=1}^{g}$. That is,

$$\int_{A_j}\varphi_i^*=\delta_{ij},\quad i,\ j=1,2,\cdots,g.$$

We set

$$\tau_{ij}=\int_{B_j}\varphi_i^*,\quad i,\ j=1,2,\cdots,g.$$

Then, as is well known, $\mathfrak{T}=(\tau_{ij})_{i,j=1,2,\cdots,g}$ is a $g\times g$ symmetric matrix with complex entries and furthermore $\mathrm{Im}\,\mathfrak{T}=(\mathrm{Im}\,\tau_{ij})_{i,j=1,2,\cdots,g}$ is positive definite. See e.g., [15], Vol. II, p. 114.

Set $C_g=\overbrace{\boldsymbol{C}^*\times\cdots\times\boldsymbol{C}^*}^{g\text{-times}}$, $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$, and $\boldsymbol{C}^g=\overbrace{\boldsymbol{C}\times\cdots\times\boldsymbol{C}}^{g\text{-times}}$.

We shall first prove the following

Proposition 2. *Let a g-row vector $\mathfrak{z}=(\zeta_1,\cdots,\zeta_g)\in C_g$ be fixed. Then for any α_k, $\beta_k\in\boldsymbol{C}^*$ which are not real multiples of ζ_k, $k=1,2,\cdots,g$, there are Abelian differentials of the first kind $\phi_{\mathfrak{z}}(A_k,\alpha_k)$ and $\phi_{\mathfrak{z}}(B_k,\beta_k)$ such that*

$$\int_{A_j}\phi_{\mathfrak{z}}(A_k,\alpha_k)/\zeta_j,\quad \left(\int_{B_j}\phi_{\mathfrak{z}}(A_k,\alpha_k)+\alpha_k\delta_{jk}\right)/\zeta_j,$$

$$\left(\int_{A_j}\phi_{\mathfrak{z}}(B_k,\beta_k)-\beta_k\delta_{jk}\right)/\zeta_j,\quad \int_{B_j}\phi_{\mathfrak{z}}(B_k,\beta_k)/\zeta_j$$

are all real numbers, $j=1,2,\cdots,g$. Each differential $\phi_{\mathfrak{z}}(A_k,\alpha_k)$ is determined uniquely by $\mathfrak{z}$ and α_k. Similarly $\phi_{\mathfrak{z}}(B_k,\beta_k)$ is uniquely determined by $\mathfrak{z}$ and β_k.

Proof. For each $k=1,2,\cdots,g$, consider the system of linear equations

$$(1)\qquad \left\{\begin{array}{l}\displaystyle\sum_{i=1}^{g}x_i\zeta_i\tau_{ij}-x_{g+j}\zeta_j=-\alpha_k\delta_{jk}\\ \\ \displaystyle\sum_{i=1}^{g}x_i\bar{\zeta}_i\bar{\tau}_{ij}-x_{g+j}\bar{\zeta}_j=-\bar{\alpha}_k\delta_{jk}\end{array}\right.\qquad j=1,2,\cdots,g.$$

2) The sum of residues here includes the residue of $\Phi\psi$ at Γ_k, $k=1,2,\cdots,N$. (We can obviously define $\mathrm{Res}_{\Gamma_k}\Phi\psi$ etc. despite the fact that $\Phi\psi$ is *not* an Abelian differential on R_N. Cf. [14]).

Since the determinant of the matrix

$$\begin{pmatrix} \zeta_i\tau_{ij}, & -\zeta_i\delta_{ij} \\ \bar{\zeta}_i\bar{\tau}_{ij}, & -\bar{\zeta}_i\delta_{ij} \end{pmatrix}_{i,j=1,2,\cdots,g}$$

is $2i(-1)^g|\zeta_1\cdots\zeta_g|^2\det(\operatorname{Im}\mathfrak{T})\neq 0$, system (1) has a unique solution $x_j=\alpha_{kj}$, $x_{g+j}=\alpha'_{kj}$, $j=1, 2, \cdots, g$. Clearly $(\bar{\alpha}_{k1}, \cdots, \bar{\alpha}_{kg}; \bar{\alpha}'_{k1}, \cdots, \bar{\alpha}'_{kg})$ is also a solution of (1). Hence by uniqueness α_{kj} and α'_{kj} are all real numbers. We can easily verify that the holomorphic differential

$$\phi_\mathfrak{z}(A_k, \alpha_k)=\alpha_{k1}\zeta_1\varphi_1^*+\cdots+\alpha_{kg}\zeta_g\varphi_g^*$$

satisfies the following period conditions:

$$(2)\qquad \begin{cases} \displaystyle\int_{A_j}\phi_\mathfrak{z}(A_k, \alpha_k)=\alpha_{kj}\zeta_j \\ \displaystyle\int_{B_j}\phi_\mathfrak{z}(A_k, \alpha_k)=\alpha'_{kj}\zeta_j-\alpha_k\delta_{jk} \end{cases} \qquad j=1, 2, \cdots, g.$$

A similar reasoning yields that the system of equations

$$(1')\qquad \begin{cases} \displaystyle\sum_{i=1}^{g} x_i\zeta_i\tau_{ij}-x_{g+j}\zeta_j=-\beta_k\tau_{kj} \\ \displaystyle\sum_{i=1}^{g} x_i\bar{\zeta}_i\bar{\tau}_{ij}-x_{g+j}\bar{\zeta}_j=-\bar{\beta}_k\bar{\tau}_{kj} \end{cases} \qquad j=1, 2, \cdots, g$$

has a unique real solution $x_j=\beta_{kj}$, $x_{g+j}=\beta'_{kj}$, $j=1, 2, \cdots, g$ (for each fixed $k=1, 2, \cdots, g$). Setting

$$\phi_\mathfrak{z}(B_k, \beta_k)=\sum_{i=1}^{g}(\beta_{ki}\zeta_i+\beta_k\delta_{ik})\varphi_i^*,$$

we obtain a holomorphic differential $\phi_\mathfrak{z}(B_k, \beta_k)$ which satisfies

$$(2')\qquad \begin{cases} \displaystyle\int_{A_j}\phi_\mathfrak{z}(B_k, \beta_k)=\beta_{kj}\zeta_j+\beta_k\delta_{jk} \\ \displaystyle\int_{B_j}\phi_\mathfrak{z}(B_k, \beta_k)=\beta'_{kj}\zeta_j \end{cases} \qquad j=1, 2, \cdots, g.$$

Thus we have proved the proposition.

As a simple corollary of Proposition 2 we have

Proposition 3. *Let $\mathfrak{z}=(\zeta_1, \cdots, \zeta_g)\in C_g$ be a fixed g-row vector and $\mathfrak{a}=(\alpha_1, \cdots, \alpha_g)$, $\mathfrak{b}=(\beta_1, \cdots, \beta_g)$ any two g-row vectors. Then there are 2g real numbers $x_1, \cdots, x_g; y_1, \cdots, y_g$ such that*

$$\int_{A_j}\varphi=\alpha_j+x_j\zeta_j\,,\quad \int_{B_j}\varphi=\beta_j+y_j\zeta_j\,,\quad j=1,\,2,\,\cdots,\,g$$

hold for some Ablian differential φ of the first kind. (*Of course, φ may be identically zero.*)

We can also prove

Proposition 4. *Let $\mathfrak{z}$, $\mathfrak{a}$, $\mathfrak{b}$ be as in Proposition 2. Then*
(i) *$\phi_{\mathfrak{z}}(A_k,\,\alpha_k)$, $\phi_{\mathfrak{z}}(B_k,\,\beta_k)$, $k=1,\,2,\,\cdots,\,g$ are linearly independent over reals, and*
(ii) *every Abelian differential of the first kind (on R_N) can be written as a linear combination of $\phi_{\mathfrak{z}}(A_k,\,\alpha_k)$ and $\phi_{\mathfrak{z}}(B_k,\,\beta_k)$ with real coefficients.*

Proof. To show the linear independence, let $c_1,\,\cdots,\,c_g\,;\,c_1',\,\cdots,\,c_g'$ be real numbers such that

$$\sum_{i=1}^{g}c_i\phi_{\mathfrak{z}}(A_i,\,\alpha_i)+\sum_{i=1}^{g}c_i'\phi_{\mathfrak{z}}(B_i,\,\beta_i)=0\,.$$

Then, computing the A_j-period of the differential on the left hand side, we have

$$\sum_{i=1}^{g}c_i\alpha_{ij}\zeta_j+\sum_{i=1}^{g}c_i'(\beta_{ij}\zeta_j+\beta_i\delta_{ij})=0$$

for certain real numbers α_{ij} and β_{ij} (see equation (2′)), or

$$\zeta_j\sum_{i=1}^{g}(c_i\alpha_{ij}+c_i'\beta_{ij})+\beta_jc_j'=0\,.$$

Because β_j/ζ_j is a non-real number, it follows that $c_j'=0$. By a similar argument we know that $c_j=0$. Thus we have proved (i).

In order to prove (ii), let φ be any Abelian differential of the first kind. Since α_j/ζ_j and β_j/ζ_j are non-real numbers, there are $4g$ real numbers $x_j,\,y_j,\,x_j',\,y_j'$ $(j=1,\,2,\,\cdots,\,g)$ such that

$$\int_{A_j}\varphi=x_j'\zeta_j+x_j\beta_j\,,\quad \int_{B_j}\varphi=y_j'\zeta_j+y_j\alpha_j\,.$$

It is easy to verify that the A_j- and B_j-periods of the Abelian differential

$$\varphi+\sum_{i=1}^{g}[y_i\phi_{\mathfrak{z}}(A_i,\,\alpha_i)-x_i\phi_{\mathfrak{z}}(B_i,\,\beta_i)]$$

are both real multiples of ζ_j, $j=1,\,2,\,\cdots,\,g$. Hence by Corollary to Proposition 1, we conclude that

$$\varphi=\sum_{i=1}^{g}[x_i\phi_{\mathfrak{z}}(B_i,\,\beta_i)-y_i\phi_{\mathfrak{z}}(A_i,\,\alpha_i)]\,.\qquad\text{q.e.d.}$$

It is clear that for fixed $\mathfrak{z}=(\zeta_1,\,\cdots,\,\zeta_g)$ we can take $\mathfrak{a}=(\alpha_1,\,\cdots,\,\alpha_g)$, $\mathfrak{b}=(\beta_1,\,\cdots,\,\beta_g)$ as $\alpha_j=\beta_j=2\pi i/\bar{\zeta}_j$, $j=1,\,2,\,\cdots,\,g$. In this case we shall write simply $\phi_{\mathfrak{z}}(A_k)$, $\phi_{\mathfrak{z}}(B_k)$ instead of $\phi_{\mathfrak{z}}(A_k,\,\alpha_k)$, $\phi_{\mathfrak{z}}(B_k,\,\beta_k)$ and call them the $\mathfrak{z}$-*basis* for the class of Abelian differentials of the first kind. When $\mathfrak{z}$ is chosen so that $\zeta_j=1$, $j=1, 2, \cdots, g$, we shall omit the index $\mathfrak{z}$, too. Thus $\phi(A_k)$ stands for $\phi_{\mathfrak{z}}(A_k, 2\pi i)$ with $\mathfrak{z}=(1, 1, \cdots, 1)$. We note that $\tilde{\phi}(A_k)=\frac{1}{2\pi i}\phi(A_k)$, $\tilde{\phi}(B_k)=\frac{1}{2\pi i}\phi(B_k)$ are holomorphic differentials on R_0 such that

$$\mathrm{Re}\int_{\gamma'}\tilde{\phi}(\gamma)=\gamma'\times\gamma\,,$$

where γ, γ' represent any two of the cycles $A_1, \cdots, A_g$; $B_1, \cdots, B_g$. Namely, $\tilde{\phi}(A_1), \cdots, \tilde{\phi}(A_g)$; $\tilde{\phi}(B_1), \cdots, \tilde{\phi}(B_g)$ are the elementary differentials of the first kind in the classical sense (cf. [9]).

Let $\Phi_{\mathfrak{z}}(A_k)$, $\Phi_{\mathfrak{z}}(B_k)$ be the integrals of $\phi_{\mathfrak{z}}(A_k)$, $\phi_{\mathfrak{z}}(B_k)$, $k=1, 2, \cdots, g$:

$$\Phi_{\mathfrak{z}}(A_k)(p)=\int^{p}\phi_{\mathfrak{z}}(A_k)\,,\quad \Phi_{\mathfrak{z}}(B_k)(p)=\int^{p}\phi_{\mathfrak{z}}(B_k)\,,$$

So as to make these integrals single-valued, we consider them on the planar surface $R_0'=R_0-\bigcup_{j=1}^{g}(A_j\cup B_j)$ only.

Now recall that the local parameter z_k about the point p_k on R_0 is fixed for each $k=1, 2, \cdots, N$. We set

$$(3)\qquad \begin{cases} a_{j\nu}(\mathfrak{z}, p_k)=\dfrac{1}{\nu!}\left[\dfrac{d^{\nu}\Phi_{\mathfrak{z}}(A_j)}{dz_k^{\nu}}\right]_{z_k=0} & j=1, 2, \cdots, g;\ k=1, 2, \cdots N; \\ b_{j\nu}(\mathfrak{z}, p_k)=\dfrac{1}{\nu!}\left[\dfrac{d^{\nu}\Phi_{\mathfrak{z}}(B_j)}{dz_k^{\nu}}\right]_{z_k=0} & \nu=0, 1, 2, \cdots. \end{cases}$$

Then the integrals $\Phi_{\mathfrak{z}}(A_j)$ and $\Phi_{\mathfrak{z}}(B_j)$ are expanded in Taylor series:

$$(4)\qquad \begin{cases} \Phi_{\mathfrak{z}}(A_j)(z_k)=\sum_{\nu=0}^{\infty}a_{j\nu}(\mathfrak{z}, p_k)z_k^{\nu} \\ \qquad\qquad\qquad\qquad j=1, 2, \cdots, g,\ k=1, 2, \cdots, N. \\ \Phi_{\mathfrak{z}}(B_j)(z_k)=\sum_{\nu=0}^{\infty}b_{j\nu}(\mathfrak{z}, p_k)z_k^{\nu} \end{cases}$$

Notice that the numbers a_{j0}, b_{j0} are not determined uniquely.

In accordance with the aforementioned convention, we write $a_{j\nu}(p_k)$, $b_{j\nu}(p_k)$ for $a_{j\nu}(\mathfrak{z}, p_k)$, $b_{j\nu}(\mathfrak{z}, p_k)$ when $\mathfrak{z}=(1, 1, \cdots, 1)$.

II. Existence theorems

3. Let T be a closed Riemann surface of genus one (torus). We fix a canonical homology basis $\{C_0, C_1\}$ of T. Let dE_0 be the normal differential of the first kind with respect to the homology basis $\{C_0, C_1\}$:

$$\int_{C_0}dE_0=1\,,\quad \int_{C_1}dE_0=\tau\,,$$

where τ is a complex number with positive imaginary part. Without loss of generality, we may assume that $w=E_0(q)$, $q\in T'=T-C_0\cup C_1$ maps the cut surface T' onto a rectilinear parallerogram on the w-plane. We may also assume that the image of T' under the mapping $w=E_0(q)$ is exactly the parallerogram with vertices at 0, 1, $1+\tau$, τ in this order (see Siegel [15], Vol. I, pp. 48-55). Denote by

$$\Pi=\{z\in \boldsymbol{C} \mid z=m+n\tau,\ m,\ n\in \boldsymbol{Z}\}$$

the period module. As is well known, T is identified with $\boldsymbol{C}/\Pi$. The inverse mapping of $w=E_0(q)$ defines the natural projection mapping which we shall denote by ρ. In order to make clear the dependence of T upon τ, we sometimes write $T=T(1, \tau)$. It should be noted that when we write $T=T(1, \tau)$ we have fixed a canonical homology basis on T.

We also note that the projection mapping ρ depends on the choice of a canonical homology basis $\{C_0, C_1\}$ of T. As is well known, for a single torus T there are infinitely many distinct ρ's which are the projection mapping $\boldsymbol{C}\to T$. In many cases, however, we may fix a canonical homology basis $\{C_0, C_1\}$ of T once and for all. Thus ρ in the sequel means, if not mentioned further, the projection mapping which is associated with this fixed canonical homology basis.

Let $T=T(1, \tau)$ be a torus with the canonical homology basis $\{C_0, C_1\}$ and dE_0 the normal holomorphic differential. Any group homomorphism $\eta: H_1(R_N)\to H_1(T)$, $N\geqq 0$, can be explicitly written as

$$(5)\qquad \begin{cases} \eta([A_j])=m_{j0}[C_0]+m_{j1}[C_1] & \\ & j=1, 2, \cdots, g \\ \eta([B_j])=n_{j0}[C_0]+n_{j1}[C_1] & \\ \eta([D_k])=l_{k0}[C_0]+l_{k1}[C_1] & k=1, 2, \cdots, N-1, \end{cases}$$

for some integers m_{ji}, n_{ji}, l_{ki}, $j=1, 2, \cdots, g$; $k=1, 2, \cdots, N-1$; $i=0, 1$. Here $[X]$ denotes the homology class determined by the cycle X (on R_N or T).

With every $\eta: H_1(R_N)\to H_1(T)$ we can associate a unique linear mapping

$$L_\eta: H_1(R_N) \longrightarrow \Pi$$

defined by $L_\eta[X]=\int_{\eta([X])}dE_0$, $[X]\in H_1(R_N)$.[3] If η is given by (5), we have

$$(6)\qquad \begin{cases} L_\eta[A_j]=m_{j0}+m_{j1}\tau & \\ & j=1, 2, \cdots, g \\ L_\eta[B_j]=n_{j0}+n_{j1}\tau & \\ & k=1, 2, \cdots, N-1. \\ L_\eta[D_k]=l_{k0}+l_{k1}\tau & \end{cases}$$

We shall call L_η the *linear mapping associated with* η.

If f is a continuous mapping of R_N into T, then it is well known that f induces a homomorphism of $H_1(R_N)$ into $H_1(T)$. The induced homomorphism

3) More precisely, let Y be a cycle on T such that $[Y]=\eta([X])$. Then we define $L_\eta[X]=\int_Y dE_0$, which is determined uniquely regardless of the choice of a cycle Y.

will be denoted by f_*. Conversely, every homomorphism $\eta: H_1(R_0)\to H_1(T)$ is always induced by a continuous mapping of R_0 into T (Hopf [7]). In particular, for each homomorphism $\eta: H_1(R_N, \partial R_N)\to H_1(T)$ there is a continuous mapping $f: R_N\to T$ such that $f_*=\eta$. On the other hand, we know that not every homomorphism between $H_1(R_0)$ and $H_1(T)$ is induced by an *analytic* mapping of R_0 into T (Gerstenhaber [4]; see also Proposition 5 below).

We shall investigate conditions for the existence of an analytic mapping $f: R_N\to T$, $N\geqq 0$, which induces a given homomorphism between the homology groups and has a prescribed behavior near ∂R_N. In order to describe the boundary behavior, it is sufficient to consider the following type of analytic singularities (cf. Ahlfors-Sario [1], p. 299, for example). By an *analytic singularity* S which is given at ∂R_N we mean a collection of N functions $S_1, \cdots, S_N$ such that

(i) each S_k is a multi-valued analytic function on a punctured neighborhood of p_k,

(ii) $dS_k/dz_k=\sum_{\nu=0}^{\infty} s_\nu(p_k)/z_k^{\nu+1}$, uniformly convergent on $r_k\leqq|z_k|\leqq 2r_k$ for every small $r_k>0$, and

(iii) $\sum_{k=1}^{N} s_0(p_k)=0$.

We shall write as

$$S=\{S_k\}_{k=1}^{N}$$

and denote by $\mathfrak{S}(R_N)$ the totality of all analytic singularities at ∂R_N. We simply say that an analytic mapping $f: R_N\to T$ has the singularity $S=\{S_k\}_{k=1}^{N}\in\mathfrak{S}(R_N)$ if

$$d(\rho^{-1}\circ f)-dS_k$$

can be extended holomorphically to the point p_k for every $k=1, 2, \cdots, N$[4]. If this is the case, we shall write as $\sigma(f)=S$.

For every g-row vector $\mathfrak{z}=(\zeta_1, \cdots, \zeta_g)\in C_g$, we know that

$$\mathcal{P}_j^{\mathfrak{z}}(S)=-\mathrm{Re}\sum_{k=1}^{N}\sum_{\nu=0}^{\infty} a_{j\nu}(\mathfrak{z}, p_k)s_\nu(p_k) \tag{7-1}$$

is convergent for each $S\in\mathfrak{S}(R_N)$ and $j=1, 2, \cdots, g$, since

$$\sum_{k=1}^{N}\sum_{\nu=0}^{\infty} a_{j\nu}(\mathfrak{z}, p_k)s_\nu(p_k)=\sum_{k=1}^{N}\frac{1}{2\pi i}\int_{|z_k|=r_k}\Phi_{\mathfrak{z}}(A_j)(z_k)\frac{dS_k}{dz_k}dz_k$$

with small $r_k>0$.

Similarly

$$Q_j^{\mathfrak{z}}(S)=-\mathrm{Re}\sum_{k=1}^{N}\sum_{\nu=0}^{\infty} b_{j\nu}(\mathfrak{z}, p_k)s_\nu(p_k) \tag{7-2}$$

is convergent for every $S\in\mathfrak{S}(R_N)$ and $j=1, 2, \cdots, g$.

We also set

$$\mathcal{R}_k(S)=-2\pi i s_0(p_k), \quad k=1, 2, \cdots, N. \tag{8}$$

4) $d(\rho^{-1}\circ f)=f^*(dE_0)$, the pull-back of dE_0 by the mapping f.

Then $\mathcal{P}_j^{\mathfrak{z}}$, $Q_j^{\mathfrak{z}}$ and $\mathcal{R}_k$ give linear mappings of $\mathfrak{S}(R_N)$ into $\boldsymbol{R}$ and $\boldsymbol{C}$ respectively. If $\mathfrak{z}=(1, 1, \cdots, 1)$ we shall write, as was noticed earlier, $\mathcal{P}_j(S)$ and $Q_j(S)$ instead of $\mathcal{P}_j^{\mathfrak{z}}(S)$ and $Q_j^{\mathfrak{z}}(S)$.

4. In this section we shall prove the following

Theorem 1. *Let $R_N=R_0-\{p_k\}_{k=1}^N$ be a Riemann surface of finite type, $N\geqq 0$,*[5] *and $T=T(1, \tau)$ be a torus. Then for any group homomorphism $\eta: H_1(R_N)\to H_1(T)$ and any analytic singularity $S\in\mathfrak{S}(R_N)$, the two assertions below are equivalent to each other.*

(I) *There exists an analytic mapping $f: R_N\to T$ such that*
 (i) *$f_*=\eta$, and*
 (ii) *$\sigma(f)=S$.*

(II) (i) *$\mathcal{R}_k(S)=L_\eta[D_k]$, $k=1, 2, \cdots, N$, and*
 (ii) *there exists a holomorphic differential φ on R_0 such that*

$$\left.\begin{aligned}\int_{A_j}\varphi&=L_\eta[A_j]+\mathcal{P}_j(S)\\ \int_{B_j}\varphi&=L_\eta[B_j]+Q_j(S)\end{aligned}\right\}\quad j=1, 2, \cdots, g.$$

Proof. (I)$\Rightarrow$(II): Suppose that there is an analytic mapping $f: R_N\to T$ which induces η. Then

$$\phi=d(\rho^{-1}\circ f)$$

is an Abelian differential on R_N (not on R_0, in general, for it may have essential singularities at some of $p_1, \cdots, p_N$!). If η is given by equation (5), we have

$$\left.\begin{aligned}\int_{A_j}\phi&=\int_{f(A_j)}d\rho^{-1}=\int_{\eta([A_j])}dE_0=m_{j0}+m_{j1}\tau=L_\eta[A_j]\\ \int_{B_j}\phi&=\int_{f(B_j)}d\rho^{-1}=\int_{\eta([B_j])}dE_0=n_{j0}+n_{j1}\tau=L_\eta[B_j]\end{aligned}\right\}\quad j=1, \cdots, g,$$

and

$$\int_{D_k}\phi=\int_{f(D_k)}d\rho^{-1}=\int_{\eta([D_k])}dE_0=l_{k0}+l_{k1}\tau=L_\eta[D_k], \quad k=1, \cdots, N.$$

Because f has the singularity S, we have first

$$\mathcal{R}_k(S)=\int_{D_k}\phi=L_\eta[D_k], \quad k=1, 2, \cdots, N.$$

Next, due to Proposition 3, we can find an Abelian differential φ of the first kind which satisfies

$$\int_{A_j}\varphi=\xi_j+m_{j1}\tau, \quad \int_{B_j}\varphi=\eta_j+n_{j1}\tau$$

5) If $N=0$, conditions (I), (ii) and (II), (i) become vacuous (and (II), (ii) is simplified). Cf. Theorem 2.

for some real ξ_j, η_j, $j=1, 2, \cdots, g$.

Applying Proposition 1, we have

$$2\pi i \sum_{k=1}^{N} \operatorname*{Res}_{\Gamma_k} \Phi(A_j)(\psi-\varphi)$$

$$= \sum_{\nu=1}^{g} \left[\int_{A_\nu}\phi(A_j)\int_{B_\nu}(\psi-\varphi)-\int_{B_\nu}\phi(A_j)\int_{A_\nu}(\psi-\varphi)\right]$$

$$= \sum_{\nu=1}^{g} [\alpha_{j\nu}(n_{\nu 0}-\eta_\nu)-(\alpha'_{j\nu}-2\pi i\delta_{j\nu})(m_{\nu 0}-\xi_\nu)]$$

$$= \sum_{\nu=1}^{g} [(\alpha_{j\nu}(n_{\nu 0}-\eta_\nu)-\alpha'_{j\nu}(m_{\nu 0}-\xi_\nu)]+2\pi i(m_{j0}-\xi_j),$$

or

$$\operatorname{Re} \sum_{k=1}^{N} \operatorname*{Res}_{\Gamma_k} \Phi(A_j)(\psi-\varphi)=m_{j0}-\xi_j=\int_{A_j}(\psi-\varphi)$$

$$=L_\eta[A_j]-\int_{A_j}\varphi.$$

On the other hand, we have for a small positive number r_k

$$\sum_{k=1}^{N} \operatorname*{Res}_{\Gamma_k} \Phi(A_j)(\psi-\varphi)= \sum_{k=1}^{N} \frac{1}{2\pi i}\int_{|z_k|=r_k} \Phi(A_j)(z_k)\frac{dS_k}{dz_k}dz_k$$

$$= \sum_{k=1}^{N} \sum_{\nu=0}^{\infty} a_{j\nu}(p_k)s_\nu(p_k),$$

since the mapping f has the singularity S. It follows that

$$\int_{A_j}\varphi=L_\eta[A_j]+\mathcal{P}_j(S), \quad j=1, 2, \cdots, g.$$

Similar reasoning yields

$$\int_{B_j}\varphi=L_\eta[B_j]+Q_j(S), \quad j=1, 2, \cdots, g.$$

Thus we have proved (I)$\Rightarrow$(II).

We shall now prove the converse. Assume that an analytic singularity $S=\{S_k\}_{k=1}^{N}\in\mathfrak{S}(R_N)$ satisfies (II). In the first place we construct a holomorphic differential ψ' on R_N such that $\psi'-dS$ has a holomorphic extension to each point of $p_1, p_2, \cdots, p_N$. This is achieved by using the classical Dirichlet principle. (For the detail, see, e.g., [1], [10], [15] etc..) If we normalize the periods of ψ' so that $\int_{A_j}\psi'$ and $\int_{B_j}\psi'$ are all real, then ψ' is uniquely determined (cf. Proposition 4 and Corollary to Proposition 1). An application of Proposition 1 implies

$$2\pi \operatorname{Re} \sum_{k=1}^{N} \operatorname*{Res}_{\Gamma_k} \Phi(A_j)dS=2\pi \operatorname{Re} \sum_{k=1}^{N} \operatorname*{Res}_{\Gamma_k} \Phi(A_j)\psi'$$

$$=\operatorname{Im} \sum_{\nu=1}^{g} \left(\int_{A_\nu}\phi(A_j)\int_{B_\nu}\psi'-\int_{B_\nu}\phi(A_j)\int_{A_\nu}\psi'\right)$$

$$=2\pi\int_{A_j}\psi',$$

and hance

$$\mathcal{P}_j(S)+\int_{A_j}\psi'=0, \quad j=1, 2, \cdots, g.$$

Similarly we have

$$Q_j(S)+\int_{B_j}\psi'=0, \quad j=1, 2, \cdots, g.$$

By our assumption, there exists an Abelian differential φ of the first kind whose periods satisfy

$$\int_{A_j}\varphi=L_\eta[A_j]+\mathcal{P}_j(S), \quad \int_{B_j}\varphi=L_\eta[B_j]+Q_j(S), \quad j=1, 2, \cdots, g.$$

If we set

$$\psi=\psi'+\varphi,$$

we have

$$\int_{A_j}\psi=L_\eta[A_j], \quad \int_{B_j}\psi=L_\eta[B_j], \quad j=1, 2, \cdots, g.$$

Next, for each k we have

$$\int_{D_k}\psi=\int_{D_k}\psi'=\int_{D_k}dS_k=\mathcal{R}_k(S),$$

and hence

$$\int_{D_k}\psi=L_\eta[D_k], \quad k=1, 2, \cdots, N.$$

Now we set $f=\rho\circ\left(\int^p\psi\right)$. Then it is almost clear that f defines a single-valued analytic mapping of R_N into T such that $f_*=\eta$ and $\sigma(f)=S$. This completes the proof of Theorem 1.

Remark. Since η is a homomorphism of $H_1(R_N)$ into $H_1(T)$, the sum

$$\eta([D_1])+\cdots+\eta([D_N])$$

is zero in $H_1(T)$, and hence

$$\sum_{k=1}^{N} L_\eta[D_k]=0.$$

On the other hand, we have

$$\sum_{k=1}^{N}\mathcal{R}_k(S)=0,$$

for S is an analytic singularity given at ∂R_N. Thus the condition $\mathcal{R}_k(S)=L_\eta[D_k]$ for $k=1, 2, \cdots, N-1$ implies $\mathcal{R}_N(S)=L_\eta[D_N]$. Therefore condition (II) (i) in the above theorem can be replaced by an apparently weaker condition (i′) $\mathfrak{R}_k(S)=L_\eta[D_k], \quad k=1, 2, \cdots, N-1.$

5. The foregoing theorem has an important corollary which is derived from Weierstrass (see Krazer's monograph [8], in particular, pp. 469-471). Namely, setting $N=0$ in Theorem 1, we have

Theorem 2. *For a closed Riemann surface R_0 of positive genus g, a torus T and a homomorphism $\eta: H_1(R_0)\to H_1(T)$, the following two statements are equivalent:*

(I) *There exists an analytic mapping $f: R_0\to T$ such that $f_*=\eta$.*

(II) *There is an Abelian differential φ of the first kind on R_0 such that*

$$\left.\begin{aligned}\int_{A_j}\varphi&=L_\eta[A_j]\\ \int_{B_j}\varphi&=L_\eta[B_j]\end{aligned}\right\} \quad j=1, 2, \cdots, g.$$

If this is the case, $d(\rho^{-1}\circ f)=\varphi$.

If condition (I) in Theorem 2 is satisfied (with an $\eta\neq 0$), then R_0 and hence R_N, $N\geqq 1$, is realized as a finitely many sheeted covering surface over T. The mapping f in such a case is given by a so-called rational transformation if we represent R_0 and T as irreducible algebraic curves. For this reason we shall say that R_N is *rationally realizable* over T if there exists a non-constant analytic mapping $f: R_N\to T$ which is a restriction of an analytic mapping $f_0: R_0\to T$ onto R_N.

In terms of rational realizability we can give a version of a classical result which was first proved by Poincaré (cf. [8]) and later improved by Haupt and Wirtinger (Haupt [5]).

Proposition 5. *A Riemann surface R_N of type (g, N), $g>1$, $N\geqq 0$, is rationally realizable over a torus if and only if the period matrix of the Abelian differentials of the first kind with respect to some (appropriately chosen) canonical homology basis of R_0 is*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \tau' & \pm 1/\nu & 0\cdots 0 \\ 0 & 1 & \cdots & 0 & \pm 1/\nu & & \\ \vdots & & 1 \ddots & \vdots & 0 & & \\ \vdots & & \ddots\, 1 & 0 & \vdots & & * \\ 0 & \cdots & 0 & 1 & 0 & & \end{pmatrix}.$$

where $\operatorname{Im}\tau'>0$ *and ν is a positive integer* $(\neq 1)$.

For the proof of this proposition, see Krazer [8], esp. p. 474, and Haupt [5]; cf. also Gerstenhaber [4].

Remark. If the torus over which R_0 is realized is $T(1, \tau)$, τ' is of the form $(m+n\tau)/\mu$, where m, n and μ are integers such that μ is a multiple of ν.

6. In this section we shall mention some other immediate consequences of Theorem 1.

Theorem 3. *Let R_N be a Riemann surface of type (g, N) and $T=T(1, \tau)$ a torus, $g\geqq 1$, $N\geqq 0$. Let there be given a group homomorphism $\eta: H_1(R_N)\to H_1(T)$. Let the linear mapping L_η associated with η is described in*

$$(6)\quad \begin{cases} L_\eta[A_j]=m_{j0}+m_{j1}\tau \\ L_\eta[B_j]=n_{j0}+n_{j1}\tau \\ L_\eta[D_k]=l_{k0}+l_{k1}\tau \end{cases} \quad \begin{array}{l} j=1, 2, \cdots, g, \\ k=1, 2, \cdots, N-1. \end{array}$$

If $S\in\mathfrak{S}(R_N)$ satisfies

$$\sum_{j=1}^{g}(m_{j0}n_{j1}-m_{j1}n_{j0})<\sum_{j=1}^{g}(Q_j(S)m_{j1}-\mathcal{P}_j(S)n_{j1}),$$

then there is no analytic mapping $f: R_N\to T$ such that $f_*=\eta$ and $\sigma(f)=S$.

Proof. If, contrary to the assertion, there exists such an analytic mapping f as in the theorem, then the numbers

$$\begin{array}{l} L_\eta[A_j]+\mathcal{P}_j(S)=(m_{j0}+\mathcal{P}_j(S))+m_{j1}\tau \\ L_\eta[B_j]+Q_j(S)=(n_{j0}+Q_j(S))+n_{j1}\tau \end{array} \quad j=1, 2, \cdots, g$$

are, by Theorem 1, the moduli of periodicity of some holomorphic differential on R_0 along A_j, B_j respectively. By what is known as the Riemann's inequality (Proposition 1), we have

$$\mathrm{Im}\sum_{j=1}^{g}(m_{j0}+\mathcal{P}_j(S)+m_{j1}\tau)\overline{(n_{j0}+Q_j(S)+n_{j1}\tau)}\leqq 0.$$

Hence

$$\mathrm{Im}\,\tau\cdot\sum_{j=1}^{g}[(m_{j0}n_{j1}-m_{j1}n_{j0})+(\mathcal{P}_j(S)n_{j1}-Q_j(S)m_{j1})]\geqq 0,$$

which leads to a contradiction, since $\mathrm{Im}\,\tau>0$. q.e.d.

Corollary. *Let a homomorhism $\eta: H_1(R_N)\to H_1(T)$ and an $S\in\mathfrak{S}(R_N)$ satisfy any one of the following conditions:*

(I) $$\begin{array}{l} C_1\times\eta([A_j])=C_1\times\eta([B_j])=0 \\ \{C_0\times\eta([B_j])\}\cdot\mathcal{P}_j(S)<0<\{C_0\times\eta([A_j])\}\cdot Q_j(S) \end{array} \quad j=1, 2, \cdots, g.$$

(II) $C_0\times\eta([A_j])$ *and* $C_0\times\eta([B_j])$ *are all positive, and*

$$\begin{array}{l} \mathcal{P}_j(S)<C_1\times\eta([A_j]) \\ Q_j(S)>C_1\times\eta([B_j]) \end{array} \quad j=1, 2, \cdots, g.$$

Then there is no analytic mapping of R_N into T which induces η and has the singularity S.

Remark. In Theorem 3 and its Corollary, we have apparently used only the property of the induced homomorphism of $H_1(R_N, \partial R_N)$ into $H_1(T)$. But this is not the case; indeed, the condition concerning dividing cycles are included in $\mathcal{P}_j(S)$ and $Q_j(S)$.

7. Let f_1, f_2 be two analytic mappings of $R_N(N \geqq 0)$ into T such that $(f_1)_* = (f_2)_* = \eta$ and both of f_1, f_2 have the same singularity $S \in \mathfrak{S}(R_N)$[6]. Then

$$\phi_1 = d(\rho^{-1} \circ f_1), \quad \phi_2 = d(\rho^{-1} \circ f_2)$$

are two Abelian differentials on R_N and they have the same singularity at ∂R_N. Therefore

$$\phi_0 = \phi_1 - \phi_2$$

is an Abelian differential of the first kind. By our hypothesis that $(f_1)_* = (f_2)_* = \eta$, we have

$$\left.\begin{aligned} \int_{A_j} \phi_0 &= \int_{A_j} (\phi_1 - \phi_2) = L_\eta[A_j] - L_\eta[A_j] = 0 \\ \int_{B_j} \phi_0 &= \int_{B_j} (\phi_1 - \phi_2) = L_\eta[B_j] - L_\eta[B_j] = 0 . \end{aligned}\right\} \quad j = 1. 2, \cdots, g,$$

and hence

$$\phi_0 \equiv 0$$

This means that $d(\rho^{-1} \circ f_1) = d(\rho^{-1} \circ f_2)$ on R_N.

Accordingly, we may conclude that

$$f_1 \equiv f_2 \quad \text{on } R_N,$$

provided $f_1(p_0) = f_2(p_0)$ for some point $p_0 \in R_N$. We have obtained

Theorem 4. *An analytic mapping f of a Riemann surface R_N of finite type (g, N), $g \geqq 1$, $N \geqq 0$, into a torus T is, if it exists, uniquely determined by the following three kinds of data:*

(i) *the induced group homomorphism f_*,*

(ii) *the singularity $\sigma(f)$ of f, and*

(iii) *the image point $f(p_0)$ under f of a fixed reference point p_0 on R_0.*

For any homomorphism η: $H_1(R_N) \to H_1(T)$ we denote by η^A the restriction of η onto the subgroup of $H_1(R_N)$ generated by $[A_1], \cdots, [A_g]$. We set

$$\mathfrak{F}_A(\eta, S) = \{f : R_N \to T \mid f \text{ is analytic}, (f_*)^A = \eta^A, \text{ and } \sigma(f) = S\}$$

and

$$\mathfrak{F}(\eta, S) = \{f \in \mathfrak{F}_A(\eta, S) \mid f_* = \eta\} .$$

Clearly

$$\mathfrak{F}(\eta, S) \subseteqq \mathfrak{F}_A(\eta, S) .$$

6) When $N=0$, this condition becomes vacuous.

The proof of Theorem 4 actually shows a stronger result. Namely we have

Proposition 6. *Let* $\eta: H_1(R_N)\to H_1(T)$, $S\in\mathfrak{S}(R_N)$, *and* $(p_0, q_0)\in R_N\times T$ *be given. Then there is at most a single element* f *in* $\mathfrak{F}_A(\eta, S)$ *such that* $f(p_0)=q_0$.

We note that $\mathfrak{F}(\eta, S)=\emptyset$ for a large number of pairs (η, S). This is exactly the content of Theorem 1. However, as we shall see later (Theorems 8 and 9; cf. also Theorem 1 in [14]), for any η we can find an $S\in\mathfrak{S}(R_N)$ for which $\mathfrak{F}(\eta, S)=0$. It is also easy to construct an example which shows that $\mathfrak{F}(\eta, S)$ is, in general, a proper subclass of $\mathfrak{F}_A(\eta, S)$.

For the completeness we shall also include the following

Proposition 7. *Let there be given a group homomorphism* $\eta: H_1(R_N)\to H_1(T)$ *and an analytic singularity* S *at* ∂R_N, $N\geqq 0$. *Then, either* $\mathfrak{F}_A(\eta, S)$ *is empty or it contains uncountably many distinct elements. If* $\mathfrak{F}_A(\eta, S)$ *is not empty, every* f *in* $\mathfrak{F}_A(\eta, S)$ *is decomposed into a form*

$$f=\chi\circ f_0,$$

where χ *denotes a fixed-point free conformal automorphism of* T *and* f_0 *is a fixed element of* $\mathfrak{F}_A(\eta, S)$.

8. The surface R_N as an open Riemann surface has a very small ideal boundary. Therefore the complex normalization of Abelian differentials is also available for the present problem. (In this connection, see Kusunoki [9], [10]; cf. also [14].)

Recall that $\varphi_1^*, \cdots, \varphi_g^*$ are the normalized differentials of the first kind with respect to the homology basis $\{A_j, B_j\}_{j=1}^g$:

$$\int_{A_j}\varphi_i^*=\delta_{ij}, \quad i, j=1, 2, \cdots, g.$$

The matrix $\mathfrak{T}=(\tau_{ij})_{i,j=1,\dots,g}$ with $\tau_{ij}=\int_{B_j}\varphi_i^*$ is uniquely determined. Let Φ_j^* be integrals of φ_j^* on $R_N'=R_N-\bigcup_{i=1}^{g}(A_i\cup B_i)$ and suppose that

$$\Phi_j^*(z_k)=\sum_{\nu=0}^{\infty} a_{j\nu}^*(p_k)z_k^\nu \quad \text{about } p_k,\ k=1, 2, \cdots, N. \tag{9}$$

Then, as in the real case (cf. equations (7-1) and (7-2)),

$$\mathcal{P}_j^*(S)=-2\pi i\sum_{k=1}^{N}\sum_{\nu=0}^{\infty} a_{j\nu}^*(p_k)s_\nu(p_k) \tag{10}$$

is well defined for every $S=\{S_k\}_{k=1}^N\in\mathfrak{S}(R_N)$, $dS_k/dz_k=\sum_{\nu=0}^{\infty}s_\nu(p_k)/z_k^{\nu+1}$.

We are now ready to prove

Theorem 5. *Let* R_N *be a Riemann surface of type* (g, N), $g\geqq 1$, $N\geqq 0$, *and* $T=T(1, \tau)$ *a torus. Let* $S\in\mathfrak{S}(R_N)$ *and* η *be a homomorphism of* $H_1(R_N)$ *into*

$H_1(T)$. *Then the statements below are equivalent to one another.*

(I*) *There exists an analytic mapping* $f: R_N \to T$ *such that*

(i) $f_* = \eta$, *and*

(ii) $\sigma(f) = S$.

(II*) (i) $\mathfrak{R}_k(S) = L_\eta[D_k]$, $k=1, 2, \cdots, N$, *and*

$$\text{(ii)}\quad \mathfrak{P}_j^*(S) = \sum_{i=1}^{g} L_\eta[A_i]\tau_{ij} - L_\eta[B_j], \quad j=1, 2, \cdots, g. \tag{11}$$

First we note that condition (II*) is equivalent to

(II**) *There exists an Abelian differential* φ *of the first kind such that*

$$\int_{A_j} \varphi = L_\eta[A_j], \quad \int_{B_j} \varphi = L_\eta[B_j] + \mathfrak{P}_j^*(S), \quad j=1, \cdots, g. \tag{12}$$

Since the equivalence of (I*) and (II**) can be shown by word-for-word modification of the proof of Theorem 1, we accomplish the proof of Theorem 5.

Another way to prove Theorem 5 is to show the following

Proposition 8. *Among the functionals* $\mathfrak{P}_j$, Q_j *and* $\mathfrak{P}_j^*$ *the following identities hold*:

$$\mathfrak{P}_j^* = Q_j - \sum_{i=1}^{g} \tau_{ij} \mathfrak{P}_i, \quad j=1, 2, \cdots, g. \tag{13}$$

Proof. We recall equations (1) and (1′). Since, in the present context, $\zeta_1 = \cdots = \zeta_g = 1$ and $\alpha_1 = \cdots = \alpha_g = \beta_1 = \cdots = \beta_g = 2\pi i$, we have

$$\begin{cases} \sum_{i=1}^{g} \alpha_{ki} \operatorname{Im} \tau_{ij} = -2\pi \delta_{kj} \\ \\ \sum_{i=1}^{g} \beta_{ki} \operatorname{Im} \tau_{ij} = -2\pi \operatorname{Re} \tau_{kj} \end{cases} \qquad j, k=1, 2, \cdots, g. \tag{14}$$

We also have

$$\begin{cases} \phi(A_j) = \sum_{i=1}^{g} \alpha_{ji} \varphi_i^* \\ \\ \phi(B_j) = \sum_{i=1}^{g} (\beta_{ji} + 2\pi i \delta_{ji}) \varphi_i^* \end{cases} \qquad j=1, 2, \cdots, g. \tag{15}$$

For the sake of simplicity we use the matrix notation.

Set

$$\mathfrak{A} = (\alpha_{ij}), \quad \mathfrak{B} = (\beta_{ij}), \quad i, j=1, 2, \cdots, g.$$

Divide $\mathfrak{T}$ into its real and imaginary parts:

$$\mathfrak{T} = \mathfrak{T}' + i\mathfrak{T}'',$$

where $\mathfrak{T}'=(\mathrm{Re}\,\tau_{ij})$ and $\mathfrak{T}''=(\mathrm{Im}\,\tau_{ij})$, $i, j=1, 2, \cdots g$. Then equations (14) take the form

(16) $$\mathfrak{A}\mathfrak{T}''=-2\pi\mathfrak{J}_g\,,\quad \mathfrak{B}\mathfrak{T}''=-2\pi\mathfrak{T}'\,,$$

$\mathfrak{J}_g$ being the $g\times g$ identity matrix. In particular, $\mathfrak{A}$ is non-singular.

We also set

$$\mathfrak{P}=\begin{pmatrix}\mathcal{P}_1\\ \vdots\\ \mathcal{P}_g\end{pmatrix},\quad \mathfrak{Q}=\begin{pmatrix}Q_1\\ \vdots\\ Q_g\end{pmatrix}$$

and

$$\mathfrak{P}^*=\begin{pmatrix}\mathcal{P}_i^*\\ \cdots\\ \mathcal{P}_g^*\end{pmatrix}=\mathfrak{P}^{*\prime}+i\mathfrak{P}^{*\prime\prime}\,,\quad \mathfrak{P}^{*\prime}=\begin{pmatrix}\mathrm{Re}\,\mathcal{P}_1^*\\ \vdots\\ \mathrm{Re}\,\mathcal{P}_g^*\end{pmatrix},\quad \mathfrak{P}^{*\prime\prime}=\begin{pmatrix}\mathrm{Im}\,\mathcal{P}_1^*\\ \vdots\\ \mathrm{Im}\,\mathcal{P}_g^*\end{pmatrix}.$$

Then, by noting (15), we can easily

(17) $$\mathfrak{P}=\frac{1}{2\pi}\mathfrak{A}\mathfrak{P}^{*\prime\prime}\,,\quad \mathfrak{Q}=\frac{1}{2\pi}\mathfrak{B}\mathfrak{P}^{*\prime\prime}+\mathfrak{P}^{*\prime}\,.$$

Combination of the first equations of (16) and (17) yields

$$\mathfrak{P}^{*\prime\prime}=2\pi\mathfrak{A}^{-1}\mathfrak{P}=-\mathfrak{T}''\mathfrak{P}\,.$$

Substituting this into the second equation of (17), we obtain

$$\mathfrak{Q}=\frac{1}{2\pi}\mathfrak{B}(-\mathfrak{T}''\mathfrak{P})+\mathfrak{P}^{*\prime}=\mathfrak{T}'\mathfrak{P}+\mathfrak{P}^{*\prime}\,.$$

It follows that

$$\mathfrak{P}^*=\mathfrak{P}^{*\prime}+i\mathfrak{P}^{*\prime\prime}=(\mathfrak{Q}-\mathfrak{T}'\mathfrak{P})-i\mathfrak{T}''\mathfrak{P}=\mathfrak{Q}-\mathfrak{T}\mathfrak{P}\,.$$

q. e. d.

If the A_j-periods of an Abelian differential φ of the first kind are known to be $L_\eta[A_j]+\mathcal{P}_j(S)$, then we have

$$\varphi=\sum_{i=1}^{g}\{L_\eta[A_i]+\mathcal{P}_i(S)\}\,\varphi_i^*\,.$$

φ satisfies (II), (ii) in Theorem 1 if and anly if

$$L_\eta[B_j]+Q_j(S)=\sum_{i=1}^{g}\{L_\eta[A_i]+\mathcal{P}_i(S)\}\,\tau_{ij}\,,$$

or

(18) $$Q_j(S)-\sum_{i=1}^{g}\mathcal{P}_i(S)\tau_{ij}=\sum_{i=1}^{g}L_\eta[A_i]\tau_{ij}-L_\eta[B_j]\,.$$

Equation (18), together with (13), now proves Theorem 5.

9. We shall say that a continuous mapping $f: R_N\to T$ is of *null type* (relative to $(\{A_j, B_j\}_{j=1}^g, \{C_0, C_1\})$), if

(19) $$\prod_{\nu=0,1}[(f(A_j)\times C_\nu)^2+(f(B_j)\times C_\nu)^2]=0\,,\quad j=1, 2, \cdots, g\,.$$

This condition means that for each $j=1, 2, \cdots, g$ we need only one of C_0 or C_1 to express the image classes $[f(A_j)]$ and $[f(B_j)]$ (cf. [14]).

The following proposition is an easy consequence of Theorem 2.

Proposition 9. *A Riemann surface of finite type cannot be rationally realizable over a torus by means of any analytic mapping of null type.*

Now we shall prove

Theorem 6. *Let R_N be a Riemann surface of type (g, N) and $T=T(1, \tau)$ be a torus, $N\geqq 1$, $g\geqq 1$. Let a non-degenerate analytic singularity S at ∂R_N be given (i.e., $S\in\mathfrak{S}(R_N)$, $dS\not\equiv 0$). Then the following two conditions are equivalent:*

(I) *There exists an analytic mapping $f: R_N\to T$ such that*

(i) *f is of null type, and*

(ii) *$\sigma(f)=S$.*

(II) *There is a g-row vector $\mathfrak{z}=(\zeta_1, \cdots, \zeta_g)\in C_g$, $\zeta_j=1$ or τ, such that*

$$\mathcal{P}_j^{\bar{\mathfrak{z}}}(S)\equiv\mathcal{Q}_j^{\bar{\mathfrak{z}}}(S)\equiv 0 \mod \boldsymbol{Z}, \quad j=1, 2, \cdots, g, \tag{20}$$

and

$$\mathcal{R}_k(S)\equiv 0 \mod \Pi, \quad k=1, 2, \cdots, N-1. \tag{21}$$

Proof. We shall only give a proof of (I)$\Rightarrow$(II), for the converse is similarly proved. Let $f: R_N\to T$ be an analytic mapping of null type which has the singularity S. Then there exists a g-row vector $\mathfrak{z}=(\zeta_1, \cdots, \zeta_g)$, $\zeta_j=1$ or τ, such that

$$L_{f*}[A_j]=m_j\zeta_j, \quad L_{f*}[B_j]=n_j\zeta_j$$

for some integers m_j, n_j; $j=1, 2, \cdots, g$.

If we take $\bar{\mathfrak{z}}$-basis $\{\phi_{\bar{\mathfrak{z}}}(A_j), \phi_{\bar{\mathfrak{z}}}(B_j)\}_{j=1}^{g}$ for the class of Abelian differentials of the first kind and form the corresponding linear mappings $\mathcal{P}_j^{\bar{\mathfrak{z}}}$, $\mathcal{Q}_j^{\bar{\mathfrak{z}}}$ (cf. equations (7-1) and (7-2)), then the same argument as in the proof of Theorem 1 implies (II). q.e.d.

Remarks. (1) A more general theorem which replaces R_N by an arbitrary open Riemann surface can be found in [14].

(2) It is not difficult to give a similar theorem for an analytic mapping f which satisfies, in addition to (19),

$$\prod_{\nu=0,1}(f(D_k)\times C_\nu)=0, \quad k=1, \cdots, N.$$

We may as well consider another speciality of topological properties (of analytic mappings). A continuous mapping $f: R_N\to T$ will be called, for lack of an appropriate name, *A-null type,* if

$$[f(A_j)]=0$$

for each $j=1, 2, \cdots, g$. As a counterpart of Theorem 6, we have by Theorem 5

Theorem 7. *Let R_N, T and S be as above. Then there is an analytic mapping $f: R_N\to T$ of A-null type which has the singularity S, if and only if*

$$(20^*) \qquad \mathfrak{P}_j^*(S) \equiv 0 \mod \Pi\,, \quad j=1, 2, \cdots, g\,,$$

$$(21) \qquad \mathfrak{R}_k(S) \equiv 0 \mod \Pi\,, \quad k=1, 2, \cdots, N-1\,.$$

It should be noted that Theorems 6 and 7 have very close form to the classical theorem of Abel (cf. [9], [14] etc.).

III. Analytic mappings which induce the prescribed homomorphism

10. Contrary to the compact case ($N=0$)—see Theorem 2 and Proposition 5—we can prove the following

Theorem 8[7]. *Assume $N \geqq 1$. Let R_N be a Riemann surface of type (g, N) and T a torus. Then for any given homomorphism $\eta: H_1(R_N) \to H_1(T)$ there always exists an analytic mapping f of R_N into T such that $f_* = \eta$.*

Proof. Let T be represented as $T=T(1, \tau)$ and η be described in equations (5). It suffices to show the existence of N sequences $\{s_\nu(p_k)\}_{\nu=0}^\infty$, $k=1, 2, \cdots, N$, of complex numbers such that

$$\left.\begin{aligned} &\limsup_{\nu\to\infty} \sqrt[\nu]{|s_\nu(p_k)|}=0 \\ &2\pi i s_0(p_k) = -L_\eta[D_k] \end{aligned}\right\} \quad k=1, 2, \cdots, N-1\,,$$

$$(22) \qquad 2\pi i \sum_{k=1}^N \sum_{\nu=0}^\infty a_{j\nu}^*(p_k) s_\nu(p_k) = L_\eta[B_j] - \sum_{i=1}^g L_\eta[A_i]\tau_{ij}\,, \quad j=1, 2, \cdots, g\,.$$

Once such sequences can be found, it is then clear by Theorem 5 that there is an analytic mapping $f: R_N \to T$ which induces η and has the singularity S defined by

$$S = \{S_k\}_{k=1}^N\,,$$

where $dS_k/dz_k = \sum_{\nu=0}^\infty s_\nu(p_k)/z_k^{\nu+1}$, $k=1, 2, \cdots, N$.

We are now to show that system of equations (22) always has a solution. First of all, we note that the first term of each sequence $\{s_\nu(p_k)\}_{\nu=0}^\infty$ is uniquely determined:

$$s_0(p_k) = -L_\eta[D_k]/2\pi i\,, \quad k=1, 2, \cdots, N\,.$$

We set

$$s_\nu(p_k)=0\,, \quad \nu=1, 2, \cdots,\ ; \ k=2, 3, \cdots, N\,.$$

and furthermore, for simplicity, we set

$$c_j = \frac{1}{2\pi i}\left\{L_\eta[B_j] - \sum_{i=1}^g L_\eta[A_i]\tau_{ij} + \sum_{k=1}^N a_{j0}^*(p_k) L_\eta[D_k]\right\} \qquad j=1, 2, \cdots, g\,,$$

$$\left.\begin{aligned} &a_{j\nu}^* = a_{j\nu}^*(p_1) \\ &s_\nu = s_\nu(p_1) \end{aligned}\right\} \qquad \nu=1, 2, \cdots\,.$$

---

7) Similar statement for a more general open Riemann surface is valid as well. In fact, the pure-existence part of these theorems is a simple consequence of Behnke-Stein theorem (cf. [14]).

Now we have to solve the following system of linear equations with an infinite number of unknowns s_ν.

(22′) $$\sum_{\nu=1}^{\infty} a_{j\nu}^{*} s_\nu = c_j, \quad j=1, 2, \cdots, g,$$

with additional condition

(22″) $$\limsup_{\nu\to\infty} \sqrt[\nu]{|s_\nu|} = 0.$$

Since $a_{j\nu}^{*}$ is the ν-th Taylor coefficient of the holomorphic function $\Phi_j^{*}(z_1)$ on the unit disk $U_1=\{|z_1|<1\}$, we know

$$\sum_{\nu=1}^{\infty} |a_{j\nu}^{*}|^2 < \infty.$$

In fact, we may assume that $\Phi_j^{*}(z_1)$ belongs to the class $H^2(U_1)$, the Hardy class [of index two]. Hence, for each $j=1, 2, \cdots, g$, the sequence $(a_{j1}^{*}, a_{j2}^{*}, \cdots)$ is considered as an element of the complex Hilbert space $l^2=\{(t_\nu)_{\nu=1}^{\infty} \mid t_\nu \in \boldsymbol{C}, \sum_{\nu=1}^{\infty} |t_\nu|^2 < \infty\}$. What is more, it is obvious that these g vectors are linearly independent in l^2. Therefore by using the method due to E. Schmidt, we know the existence of an l^2-solution $(s_1, s_2, \cdots)$. Namely, there always exists a sequence $\{s_\nu\}_{\nu=1}^{\infty}$ which satisfies

(23) $$\sum_{\nu=1}^{\infty} a_{j\nu}^{*} s_\nu = c_j, \quad j=1, 2, \cdots, g$$

and

(23′) $$\sum_{\nu=1}^{\infty} |s_\nu|^2 < \infty.$$

The l^2-condition (23′) is, however, much weaker than our demand. We have to show the existence of such a sequence as satisfies even (22″). To this end, we note that the number of equations in (22′) is only finite. It follows that we can actually find a solution $(s_1, s_2, \cdots)$ of (22′) such that $s_\nu=0$ for all $\nu \geqq \nu_0$, ν_0 being a sufficiently large positive integer. This completes the proof of Theorem 8.

Incidentally, we have shown that there are infinitely many distinct mappings f's which induce the given $\eta: H_1(R_N) \to H_1(T)$, for there are infinitely many distinct ways of choosing s_ν for large ν's.

11. We could follow the Schmidt's procedure to obtain an estimate for the number ν_0. But we prefer to make use of a function-theoretic property of the coefficients $a_{j\nu}^{*}$ in system (22′). To do this we shall first prove the following

Proposition 10. *Let $a_{j\nu}^{*}$ be the ν-th Taylor coefficient of Φ_j^{*} at the point p_1 (with respect to the local parameter z_1), $\nu=0, 1, 2, \cdots$; $j=1, 2, \cdots, g$. Then, for every $n \geqq 2g-1$* $\operatorname{rank}(a_{j\nu}^{*})_{\substack{j=1,2,\cdots,g \\ \nu=1,2,\cdots,n}} = g$.

Proof. Suppose that, contrary to the conclusion, there is an integer $n \geqq 2g-1$

for which $\operatorname{rank}(a^*_{j\nu})_{\substack{j=1,2,\cdots,g\\ \nu=1,2,\cdots,n}}$ is strictly less than g. Then we can find g complex numbers $t_1, t_2, \cdots, t_g$ such that

(i) $|t_1|+|t_2|+\cdots+|t_g|>0$, and

(ii) $t_1a^*_{1\nu}+t_2a^*_{2\nu}+\cdots+t_ga^*_{g\nu}=0$, $\nu=1, 2, \cdots, n$.

If we set

$$d\Phi^*=t_1d\Phi^*_1+t_2d\Phi^*_2+\cdots+t_gd\Phi^*_g,$$

$d\Phi^*$ is an Abelian differential of the first kind. Since $\sum_{i=1}^g|t_i|>0$, we know $d\Phi^*\not\equiv0$. Therefore, the degree of the divisor $(d\Phi^*)$ is exactly $2g-2$. On the other hand, the first n derivatives of the integral Φ^* (with respect to z_1) vanish at the point p_1, for by property (ii) we have

$$\left[\frac{d^\nu\Phi^*}{dz_1^\nu}\right]_{z_1=0}=\nu!\cdot(t_1a^*_{1\nu}+\cdots+t_ga^*_{g\nu})=0\,,\quad \nu=1, 2, \cdots, n\,.$$

Hence $\deg(d\Phi^*)\geqq n\geqq 2g-1$. This contradiction proves the proposition.

Now we are ready to prove

Theorem 9. *Let $R_N=R_0-\{p_1, p_2, \cdots, p_N\}$ be a Riemann surface of type (g, N) and T a torus, $N\geqq1$, $g\geqq1$. Then, for any homomorphism $\eta: H_1(R_N)\to H_1(T)$, there is an analytic mapping $f: R_N\to T$ such that*

(i) $f_*=\eta$,

(ii) *at an arbitrarily chosen one of N points $p_1, p_2, \cdots, p_N$, $d(\rho^{-1}\circ f)$ has a pole of order not exceeding $2g$, and*

(ii′) *at each of the other $N-1$ points, $d(\rho^{-1}\circ f)$ has at worst a simple pole.*

In particular, if the point chosen in (ii) *is a non-Weirestrass point, the number $2g$ in* (ii) *can be replaced by $g+1$.*

Proof. Because $\operatorname{rank}(a^*_{j\nu})_{\substack{j=1,2,\cdots,g\\ \nu=1,2,\cdots,2g-1}}=g$ by the preceding proposition, there is at least one minor of order g which is different from zero. Let $\det(a^*_{j\nu_i})\neq0$, where $i, j=1, 2, \cdots, g$ and $1\leqq\nu_1<\nu_2<\cdots<\nu_g\leqq 2g-1$.

Choose $s_\nu=0$ if $\nu\neq0$, ν_j (in particular, $s_\nu=0$ for all $\nu\geqq 2g$). Then $s_{\nu_1}, s_{\nu_2}, \cdots, s_{\nu_g}$ are uniquely determined.

We have gotten a solution of (22′) such that at most the first $2g-1$ terms are different from zero. This means that f can be taken so as to satisfy (i), (ii) and (ii′) in the theorem.

For the last part of the theorem, it is sufficient to note that if p_1 is not a Weierstrass point, then

$$\det(a^*_{j\nu})_{\substack{j=1,2,\cdots,g\\ \nu=1,2,\cdots,g}}\neq0 \quad \text{(see,e.g., [1], p. 330; [10], p. 148).}$$

q. e. d.

Remark. The mapping f in Theorem 9 assumes, in each neighborhood of the point p_1, every value on T infinitely often, since $\rho^{-1}\circ f$ has a pole at p_1. In this connection see Theorem 1 in Ohtsuka [13].

Since the total number of Weierstrass points on a closed Riemann surface of genus $g\geqq 1$ is at most $(g-1)g(g+1)$, we have the following

Corollary. *If* $N>(g-1)g(g+1)$, *then there always exists an analytic mapping* $f: R_N\rightarrow T$ *such that* (i) f *induces the prescribed homomorphism between their homology groups and* (ii) *the singularity of* $d(\rho^{-1}\circ f)$ *at* p_i *is at most a simple pole except for a single* p_k *where* $d(\rho^{-1}\circ f)$ *has a pole of order not exceeding* $g+1$.

We have shown the following: Setting $s_\nu(p_2)=s_\nu(p_3)=\cdots=s_\nu(p_N)=0$, $\nu\geqq 1$, we can associate precisely $g+N$ complex numbers with every homomorphism $\eta: H_1(R_N)\rightarrow H_1(T)$, whatever the point p_1 may be (i.e., regardless whether p_1 is a Weierstrass point or not). These $g+N$ complex numbers are *canonically* determined by η. Namely, if the Weierstrass gap sequence at p_1 is given by

$$(1=)\ \nu_1<\nu_2<\cdots<\nu_g\ (\leqq 2g-1),$$

we can set $s_\nu=s_\nu(p_1)=0$ for $\nu\neq 0, \nu_j$. Since $\nu_1, \nu_2, \cdots, \nu_g$ are gap values at p_1, there are g holomorphic differentials $\psi_1, \psi_2, \cdots, \psi_g$ on R_0 such that ψ_j has a zero at p_1 whose order is exactly ν_j-1. Such differentials $\psi_1, \psi_2, \cdots, \psi_g$ span the class of Abelian differentials of the first kind. It follows that $\det(a^*_{j\nu_i})_{i,j=1,2,\cdots,g}\neq 0$. Thus $s_{\nu_1}, s_{\nu_2}, \cdots, s_{\nu_g}$ are uniquely determined. (The remaining N numbers $s_0(p_k)$, $k=1, 2, \cdots, N$, are always unique.) These $g+N$ complex numbers serve as the coefficients of the singularity of an analytic mapping $f: R_N\rightarrow T$ with $f_*=\eta$.

12. Let $N\geqq 1$ and $R_N=R_0-\{p_1, p_2, \cdots, p_N\}$. We set $R^0=R_0-\{p_2, p_3, \cdots, p_N\}$. Denote by U the upper half plane $\{z\in C\,|\,\mathrm{Im}\,z>0\}$ and by L the lower half plane. The surface R^0 can be represented as U/G by a Fuchsian group G without elliptic elements. After Bers (cf. [2], [3]), we define the Teichmüller space of R^0 as follows.

First we set

$$B_2(L, G)=\left\{\phi\ \middle|\ \begin{array}{l}\phi(z) \text{ is holomorphic in } L,\ \sup_{z=x+iy\in L}|y^2\phi(z)|<\infty, \text{ and}\\ \phi(\gamma(z))\gamma'^2(z)=\phi(z) \text{ for every } \gamma\in G.\end{array}\right\}$$

For each $\phi\in B_2(L, G)$, consider the ordinary differential equation of the second order

$$2\eta''(z)+\phi(z)\eta(z)=0\,,\quad z\in L\,.$$

Let $\eta_1(z)$, $\eta_2(z)$ be two linearly independent solutions normalized by the condition $\eta_1(-i)=\eta_2'(-i)=1$, $\eta_1'(-i)=\eta_2(-i)=0$, and set

$$W_\phi(z)=\eta_1(z)/\eta_2(z)\,.$$

The *Teichmüller space* $T(G)$ of G is defined as

$T(G)=\{\phi\in B_2(L, G)\mid W_\phi$ is univalent on L and $W_\phi(L)$ is a Jordan domain$\}$.

We finally set

$$D(\phi)=\hat{C}-\overline{W_\phi(L)}, \quad \phi\in T(G).$$

The space $F(G)=\{(\phi, z)\mid \phi\in T(G), z\in D(\phi)\}$ is called the *Bers fiber space* over the Teichmüller space $T(G)$; it is known to be a bounded domain (and even a domain of holomorphy) in the complex $3g-3+N$ space $B_2(L, G)\oplus C$ ([3]). As usual, we shall write $T(R^0)$ for $T(G)$ and $F(R^0)$ for $F(G)$. We denote by R^ϕ the Riemann surface corresponding to $\phi\in T(R^0)$. Namely, R^ϕ is a Riemann surface of type $(g, N-1)$ which has the same marking as R^0. Note that the point $\phi=0$ represents the surface $R^0=R_0-\{p_2, \cdots, p_N\}$.

As before, let $\varphi_1^*, \varphi_2^*, \cdots, \varphi_g^*$ be the normalized Abelian differentials of the first kind. On any R^ϕ there are differentials of the same nature which we denote by $\varphi_1^*(\phi), \varphi_2^*(\phi), \cdots, \varphi_g^*(\phi)$. The corresponding B_j-periods are denoted by

$$\tau_{ij}(\phi).$$

Due to Ahlfors, Bers, and Rauch, we know that $\tau_{ij}(\phi)$ are holomorphic functions on $T(R^0)$. What is more, Bers showed (see [3]) that $D(\phi)$ is the (holomorphic) universal covering surface of the surface R^ϕ; $D(\phi)/G^\phi$ is conformally equivalent to R^ϕ, where $G^\phi=\{\hat{\gamma}\in PSL(2, C)\mid$ there is a $\gamma\in G$ such that $W_\phi\circ\gamma=\hat{\gamma}\circ W_\phi\}$. We denote by π the projection $D(\phi)\to R^\phi$, $\phi\in T(R^0)$. For every $z\in D(\phi)$, we denote by $R^{\phi,z}$ the surface $R^\phi-\{\pi(z)\}$. Then $R^{\phi,z}$ is a surface of type (g, N).

Now, using the variable $z\in D(\phi)$ as the uniformizing parameter of R^ϕ, we set $\varphi_j^*(\phi)=f_j(\phi, z)dz$, $j=1, 2, \cdots, g$. The functions $f_j(\phi, z)$ are holomorphic on $F(R^0)$ (Bers [2]). In particular, the Wronskian

$$W(\phi, z)=\begin{vmatrix} f_1(\phi, z), f_1'(\phi, z), \cdots\cdots, f_1^{(g-1)}(\phi, z) \\ f_2(\phi, z), f_2'(\phi, z), \cdots\cdots, f_2^{(g-1)}(\phi, z) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_g(\phi, z), f_g'(\phi, z), \cdots\cdots, f_g^{(g-1)}(\phi, z) \end{vmatrix}$$

is a holomorphic function on $F(R^0)$. Here $f_j^{(\nu)}(\phi, z)=\dfrac{\partial^\nu}{\partial z^\nu}f_j(\phi, z)$, $\nu=0, 1, \cdots, g-1$; $f_j'(\phi, z)=f_j^{(1)}(\phi, z)$.

Let R' (resp. T') denote a generic Riemann surface of type $(g, N-1)$ (resp. $(1, 0)$) with the same marking as R^0 (resp. T).[8] Then any group homomorphism $\eta: H_1(R_N)\to H_1(T)$ obviously defines a homomorphism of $H_1(R')$ into $H_1(T')$, which we still denote by η.

Now we shall consider the case $N=1$ and then we have[8]

Theorem 10. *Let R^0 be a fixed closed Riemann surface of positive genus g and T be a fixed torus. Let $R_1=R^0-\{p_1\}$ (with p_1 kept fixed) and $\eta: H_1(R_1)\to H_1(T)$ be a given homomorphism.*

8) Here it is assumed tacitly that $\{A_j, B_j\}_{j=1,2,\cdots,g}$ has been modified so as to determine a marking of R_0.

Then we can find g functions

$$s_1, s_2, \cdots, s_g$$

defined on $F(R^0)\times U$ *which satisfy the following conditions*:

(I) $s_1, s_2, \cdots, s_g$ *are meromorphic functions of* $\phi\in T(R^0)$, $z\in D(\phi)$ *and* $\tau\in U$; *furthermore, they are holomorphic except for those* (ϕ, z, τ) *such that* $\pi(z)$ *is a Weierstrass point on the surface* R^ϕ, τ *being any point in* U.

(II) *If* $s_1(\phi, z, \tau), s_2(\phi, z, \tau), \cdots, s_g(\phi, z, \tau)$ *assume finite values*[9], *then for sufficiently small* $r>0$

$$S_1(\zeta; \phi, z, \tau)=-\sum_{\nu=1}^{g}\frac{s_\nu(\phi, z, \tau)}{\nu(\zeta-z)^\nu}, \quad |\zeta-z|<r$$

defines an element $S(\zeta; \phi, z, \tau)=\{S_1(\zeta; \phi, z, \tau)\}$ *of* $\mathfrak{S}(R^{\phi,z})$.

(III) *For any* $\phi\in T(R^0)$, $z\in D(\phi)$ *and* $\tau\in U$ *such that* $s_j(\phi, z, \tau)$, $j=1, 2, \cdots, g$, *are finite there exists an analytic mapping*

$$f(\phi, z, \tau): R^{\phi,z}\longrightarrow T(1, \tau)$$

such that $(f(\phi, z, \tau))_*=\eta$ *and* $\sigma(f(\phi, z, \tau))=S(\zeta; \phi, z, \tau)$.

In other words, the singularity of analytic mapping of R_1 into T which induces the prescribed homomorphism between their homology groups can be chosen so as to depend meromorphically on the moduli of R_1 and T. (Note that $F(R^0)$ is biholomorphically isomorphic to $T(R_1)$.)

We omit the proof. We only note that the functionals $\mathcal{P}_j^*(S)$, $S\in\mathfrak{S}(R^{\phi,z})$ can be computed by means of the global uniformizer $z(\in D(\phi))$ of R^0 (cf. equation (10)). The functions $s_1, s_2, \cdots, s_g$ can be explicitly given by

$$s_j(\phi, z, \tau)=W_j(\phi, z, \tau)/W(\phi, z), \quad j=1, 2, \cdots, g,$$

where

$$W_j(\phi, z, \tau)=\det\left(c_{\mu\nu}^{(j)}(\phi, z, \tau)\right)_{\mu,\nu=1,2,\cdots,g},$$

$$c_{\mu\nu}^{(j)}(\phi, z, \tau)=\begin{cases} f_\mu^{(\nu-1)}(\phi, z), & \nu\neq j \\ j!c_\mu=\dfrac{j!}{2\pi i}\left(L_\eta[B_\mu]-\sum_{i=1}^{g}L_\eta[A_i]\tau_{i\mu}\right), & \nu=j. \end{cases}$$

IV. Some additional remarks

13. We shall now point out some relations of our results to the classical theory. In this connection, see also Theorem 2 and Proposition 5.

First we recall that in the definitions of the linear mappings $\mathcal{P}_j$, Q_j and $\mathcal{P}_j^*$ $(j=1, 2, \cdots, g)$ a canonical homology basis of R_0 is fixed. The notation $\sigma(f)=S$ is, on the other hand, makes sense only when we have fixed a canonical homology basis of T (hence a projection mapping $\rho: \boldsymbol{C}\to T$). We shall henceforth use the notation $\sigma_{\{C_0, C_1\}}(f)=S$ if the reference to the basis $\{C_0, C_1\}$ is necessary.

9) This is the case if the point $\pi(z)$ is not a Weierstrass point of the surface R^ϕ (cf. (I); see also Bers [2]).

In order to see the effect produced by the exchange of canonical homology basis of R_0, we rewrite equations (7-1), (7-2) and (10) as follows:

$$\mathcal{P}_j(S)=\mathrm{Re}\ \frac{1}{2\pi i}\int_d \Phi(A_j)dS=-\mathrm{Re}\ \sum_{k=1}^{N}\operatorname*{Res}_{\Gamma_k}\Phi(A_j)dS\,, \tag{7-1$'$}$$

$$Q_j(S)=\mathrm{Re}\ \frac{1}{2\pi i}\int_d \Phi(B_j)dS=-\mathrm{Re}\ \sum_{k=1}^{N}\operatorname*{Res}_{\Gamma_k}\Phi(B_j)dS\,, \tag{7-2$'$}$$

$$\mathcal{P}_j^*(S)=\int_d \Phi_j^* dS=-2\pi i\sum_{k=1}^{N}\operatorname*{Res}_{\Gamma_k}\Phi_j^* dS\,, \tag{10$'$}$$

where $d=D_1+D_2+\cdots+D_N$. (Of course, $\operatorname*{Res}_{\Gamma_k}\Phi(A_j)dS$, $\operatorname*{Res}_{\Gamma_k}\Phi(B_j)dS$ etc. should be understood in the sense of footnote 2), p. 594.)

We shall now prove the following

Proposition 11. *Let two canonical homology bases* $\{A_j, B_j\}_{j=1}^{g}$ *and* $\{\tilde{A}_j, \tilde{B}_j\}_{j=1}^{g}$ *of* R_0 *be related by*

$$\left.\begin{aligned}\tilde{A}_j&=\sum_{i=1}^{g}(\mu_{ij}A_i+\mu'_{ij}B_i) &&(24\text{-}1)\\ \tilde{B}_j&=\sum_{i=1}^{g}(\nu_{ij}A_i+\nu'_{ij}B_i) &&(24\text{-}2)\end{aligned}\right\}\quad j=1, 2, \cdots, g\,.$$

Let $\mathcal{R}_j$, Q_j, $\mathcal{P}_j^*$ *(resp.*$\tilde{\mathcal{P}}_j$, $\tilde{Q}_j$, $\tilde{\mathcal{P}}_j^*$*) be the linear mappings which are associated with the basis* $\{A_j, B_j\}_{j=1}^{g}$ *(resp.* $\{\tilde{A}_j, B_j\}_{j=1}^{g}$*). Then we have the following identities:*

$$\tilde{\mathcal{P}}_j=\sum_{i=1}^{g}(\mu_{ij}\mathcal{P}_i+\mu'_{ij}Q_i) \tag{25-1}$$

$$\tilde{Q}_j=\sum_{i=1}^{g}(\nu_{ij}\mathcal{P}_i+\nu'_{ij}Q_i)\quad j=1, 2, \cdots, g\,. \tag{25-2}$$

$$\tilde{\mathcal{P}}_j^*=\sum_{i=1}^{g}\kappa_{ij}\mathcal{P}_i^* \tag{25*}$$

Here $(\kappa_{ij})_{i,j=1,2,\dots,g}$ *denotes a* $g\times g$ *complex matrix which is uniquely determined by* (24-1) *and* (24-2).

Proof. Let $\mathfrak{z}_0=\underbrace{(1, 1, \cdots, 1)}_{g}$ and let $\phi(A_j)=d\Phi(A_j)$, $\phi(B_j)=d\Phi(B_j)$ (resp. $\phi(\tilde{A}_j)=d\Phi(\tilde{A}_j)$, $\phi(\tilde{B}_j)=d\Phi(\tilde{B}_j)$) be the $\mathfrak{z}_0$-basis for the class of Abelian differentials of the first kind corresponding to the basis $\{A_j, B_j\}_{j=1}^{g}$ (resp. $\{\tilde{A}_j, \tilde{B}_j\}_{j=1}^{g}$). (See section 2.) Then it is easy to verify that

$$\left.\begin{aligned}\phi(\tilde{A}_j)&=\sum_{i=1}^{g}[\mu_{ij}\phi(A_i)+\mu'_{ij}\phi(B_i)] &&(26\text{-}1)\\ \phi(\tilde{B}_j)&=\sum_{i=1}^{g}[\nu_{ij}\phi(A_i)+\nu'_{ij}\phi(B_i)] &&(26\text{-}2)\end{aligned}\right\}\quad j=1, 2, \cdots, g\,.$$

Indeed, we have first

$$\sum_{i=1}^{g}(\mu_{ij}\nu_{ik}'-\mu_{ij}'\nu_{ik})=\delta_{jk} \qquad j,\ k=1,\ 2,\ \cdots,\ g,$$

$$\sum_{i=1}^{g}(\mu_{ij}\mu_{ik}'-\mu_{ij}'\mu_{ik})=\sum_{i=1}^{g}(\nu_{ij}\nu_{ik}'-\nu_{ij}'\nu_{ik})=0$$

since $\{A_j,\ B_j\}_{j=1}^{g}$ and $\{\tilde{A}_j,\ \tilde{B}_j\}_{j=1}^{g}$ are both canonical. Now we see that

$$\int_{\tilde{A}_k}\Big(\sum_{i=1}^{g}[\mu_{ij}\phi(A_i)+\mu_{ij}'\phi(B_i)]\Big)\equiv\int_{\tilde{B}_k}\Big(\sum_{i=1}^{g}[\nu_{ij}\phi(A_i)+\nu_{ij}'\phi(B_i)]\Big)\equiv 0$$

and that

$$\int_{\tilde{B}_k}\Big(\sum_{i=1}^{g}[\mu_{ij}\phi(A_i)+\mu_{ij}'\phi(B_i)]\Big)\equiv-\int_{\tilde{A}_k}\Big(\sum_{i=1}^{g}[\nu_{ij}\phi(A_i)+\nu_{ij}'\phi(B_i)]\Big)\equiv-2\pi i\delta_{jk}$$

modulo real numbers ($j,\ k=1,\ 2,\ \cdots,\ g$). Hence by Proposition 2 we have (26-1) and (26-2).

Take an $S\in\mathfrak{S}(R_N)$. Then for every $j=1,\ 2,\ \cdots,\ g$ we have

$$\begin{aligned}\tilde{\mathcal{P}}_j(S)&=\mathrm{Re}\Big(\frac{1}{2\pi i}\int_d\Phi(\tilde{A}_i)dS\Big)\\&=\sum_{i=1}^{g}\Big[\mu_{ij}\,\mathrm{Re}\Big(\frac{1}{2\pi i}\int_d\Phi(A_i)dS\Big)+\mu_{ij}'\,\mathrm{Re}\Big(\frac{1}{2\pi i}\int_d\Phi(B_i)dS\Big)\Big]\\&=\sum_{i=1}^{g}[\mu_{ij}\mathcal{P}_i(S)+\mu_{ij}'Q_i(S)]\,.\end{aligned}$$

Similarly we have

$$\tilde{Q}_j(S)=\sum_{i=1}^{g}[\nu_{ij}\mathcal{P}_i(S)+\nu_{ij}'Q_i(S)]\,,\quad S\in\mathfrak{S}(R_N),\ j=1,\ 2,\ \cdots,\ g\,.$$

Let $\Phi_1^*,\ \Phi_2^*,\ \cdots,\ \Phi_g^*$ be the normal integrals of the first kind on R_0 with respect to the basis $\{A_j,\ B_j\}_{j=1}^{g}$ and $\mathfrak{T}=(\tau_{ij})_{i,j=1,2,\cdots,g}$, $\tau_{ij}=\int_{B_j}d\Phi_i^*$, as before. If $\tilde{\Phi}_1^*,\ \tilde{\Phi}_2^*,\ \cdots,\ \tilde{\Phi}_g^*$ denote the normal integrals of the first kind with respect to the basis $\{\tilde{A}_j,\ \tilde{B}_j\}_{j=1}^{g}$, then there are uniquely determined g^2 complex numbers κ_{ij} ($i,\ j=1,\ 2,\ \cdots,\ g$) for which

$$\tilde{\Phi}_j^*=\sum_{i=1}^{g}\kappa_{ij}\Phi_i^*\,,\quad j=1,\ 2,\ \cdots,\ g\,.$$

It follows immediately that for every $S\in\mathfrak{S}(R_N)$

$$\tilde{\mathcal{P}}_j^*(S)=\sum_{i=1}^{g}\kappa_{ij}\mathcal{P}_i^*(S)\,,\quad j=1,\ 2,\ \cdots,\ g\,.$$

q.e.d.

Remark. Computing the A_i- and B_i-periods of $d\tilde{\Phi}_j^*$, we see that the matrix $\mathfrak{K}=(\kappa_{ij})_{i,j=1,2,\cdots,g}$ satisfies

$${}^t\mathfrak{K}(\mathfrak{J}_g,\ \mathfrak{T})\mathfrak{M}=(\mathfrak{J}_g,\ \tilde{\mathfrak{T}})\,,$$

where

$$\mathfrak{J}_g=(\delta_{ij})_{i,j=1,2,\cdots,g},$$

$$\mathfrak{M}=\begin{pmatrix}\mu_{ij} & \nu_{ij}\\ \mu'_{ij} & \nu'_{ij}\end{pmatrix}_{i,j=1,2,\cdots,g},$$

$$\tilde{\mathfrak{T}}=(\tilde{\tau}_{ij})_{i,j=1,2,\cdots,g},\quad \tilde{\tau}_{ij}=\int_{\widetilde{B_j}}d\tilde{\Phi}_i^*,$$

and ${}^t\mathfrak{K}$ stands for the transpose of $\mathfrak{K}$. It is also well-known that the matrices $\mathfrak{K}$ and $\mathfrak{M}$ are both non-singular (cf. [15], Vol. II).

Proposition 12. ⟨1⟩ *Let* $S\in\mathfrak{S}(R_N)$ *and* $\mathcal{P}_j$, Q_j, $\mathcal{P}_j^*$ *be the linear mappings which correspond to the same canonical homology basis of* R_0. *Then*

$$\mathcal{P}_j(S)=Q_j(S)=0,\quad j=1,2,\cdots,g \tag{27}$$

if and only if

$$\mathcal{P}_j^*(S)=0,\quad j=1,2,\cdots,g. \tag{27*}$$

⟨2⟩ *The property that an* $S\in\mathfrak{S}(R_N)$ *satisfies condition* (27) *does not depend on the choice of a canonical homology basis of* R_0.

⟨2*⟩ *The same is true of condition* (27*).

Proof. Assertions ⟨2⟩, ⟨2*⟩ are simple consequences of the preceding proposition. For the proof of ⟨1⟩ we only need to recall Propositien 8, equation (13). Using the matrix notation, we see that (13) is equivalent to

$$\begin{pmatrix}\mathcal{P}_1^*(S)\\ \vdots\\ \mathcal{P}_g^*(S)\\ \overline{\mathcal{P}_1^*(S)}\\ \vdots\\ \overline{\mathcal{P}_g^*(S)}\end{pmatrix}=\begin{pmatrix}\mathfrak{J}_g & -\mathfrak{T}\\ \mathfrak{J}_g & -\bar{\mathfrak{T}}\end{pmatrix}\begin{pmatrix}Q_1(S)\\ \vdots\\ Q_g(S)\\ \mathcal{P}_1(S)\\ \vdots\\ \mathcal{P}_g(S)\end{pmatrix}. \tag{13'}$$

(Here, of course, $\mathfrak{T}$ denotes the period matrix with respect to the basis which is now considered and the bar stands for the complex conjugation.)
Since the $2g\times 2g$ matrix

$$\begin{pmatrix}\mathfrak{J}_g & -\mathfrak{T}\\ \mathfrak{J}_g & -\bar{\mathfrak{T}}\end{pmatrix}$$

is obviously non-singular, we know ⟨1⟩ is valid. q.e.d.

Now we have

Definition. An analytic singularity S is called *trivial* if it is of the first or second kind (i.e., $\mathcal{R}_k(S)=0$, $k=1,2,\cdots,N$) and (27) is satisfied for some (hence for every) canonical homology basis of R_0. We shall denote by $\mathfrak{S}_0(R_N)$ the class of trivial analytic singularities at ∂R_N.

Due to Proposition 12, we may as well define $\mathfrak{S}_0(R_N)$ to be the class

$$\left\{S\in\mathfrak{S}(R_N)\,\middle|\,\begin{array}{ll}\mathcal{P}_j^*(S)=0, & j=1, 2, \cdots, g\\ \mathfrak{R}_k(S)=0, & k=1, 2, \cdots, N-1\end{array}\right\}$$

with an arbitrarily fixed canonical homology basis of R_0.

14. In this section we assume $N\geqq 1$. We have already known (Theorem 6) that if $S\in\mathfrak{S}_0(R_N)$ then there always exists an analytic mapping of R_N into T such that (i) $\sigma(f)=S$ and (ii) f *is* of null type. The following theroem gives a counterpart to this fact.

Theorem 11. *Let R_0 be a closed Riemann surface of positive genus g and T a torus. Let $\{A_j, B_j\}_{j=1}^g$ (resp. $\{C_0, C_1\}$) be an arbitrarily fixed canonical homology basis of R_0 (resp. T). Let R_N be a Riemann surface obtained by deleting N distinct points from R_0, $N\geqq 1$. Then for every $S\in\mathfrak{S}_0(R_N)$ the following two statements are equivalent*:

(I) *R_0 admits an analytic mapping onto T.*

(II) *There exists an analytic mapping $f: R_N\to T$ such that*

(i) *$\sigma_{\{C_0, C_1\}}(f)=S$, and*

(ii) *f is "not" of null type relative to $(\{A_j, B_j\}_{j=1}^g, \{C_0, C_1\})$.*

Proof. Assume (I). Let τ be the C_1-period of the holomorphic differential dE_0 such that $\int_{C_0} dE_0=1$. Then by Theorem 2 there exists an Abelian differential $\varphi\not\equiv 0$ of the first kind (on R_0) such that

$$\left.\begin{array}{l}\int_{A_j}\varphi=m_{j0}+m_{j1}\tau\\ \int_{B_j}\varphi=n_{j0}+n_{j1}\tau\end{array}\right. \qquad j=1, 2, \cdots, g$$

for appropriate integers m_{jk}, n_{jk} ($j=1, 2, \cdots, g$; $k=0, 1$). Since S is a trivial analytic singularity, $\mathcal{P}_j(S)=Q_j(S)=0$, $j=1, 2, \cdots, g$, and $\mathfrak{R}_k(S)=0$, $k=1, 2, \cdots, N-1$, where $\mathcal{P}_j$, Q_j are the linear mappings corresponding to the basis $\{A_j, B_j\}_{j=1}^g$. Thus we have

$$\left.\begin{array}{l}\int_{A_j}\varphi=m_{j0}+m_{j1}\tau+\mathcal{P}_j(S)\\ \int_{B_j}\varphi=n_{j0}+n_{j1}\tau+Q_j(S)\end{array}\right. \qquad j=1, 2, \cdots, g.$$

Using Theorem 1, we may conclude that there is an analytic mapping $f: R_N\to T$ such that $\sigma(f)=S$. Moreover, the induced homomorphism $f_*: H_1(R_N)\to H_1(T)$ is given by

$$f_*([A_j])=m_{j0}[C_0]+m_{j1}[C_1] \qquad j=1, 2, \cdots, g,$$
$$f_*([B_j])=n_{j0}[C_0]+n_{j1}[C_1]$$

and hence f is not of null type relative to $(\{A_j, B_j\}_{j=1}^g, \{C_0, C_1\})$. Indeed, $\sum_{i=1}^{g}(m_{j0}^2+n_{j0}^2)(m_{j1}^2+n_{j1}^2)>0$, since $\varphi\not\equiv 0$. Thus we have proved the implication (I)$\Rightarrow$(II).

Suppose, conversely, that there is an analytic mapping $f: R_N\to T$ which has the following properties:

(i) $\sigma(f)=S$,

(ii) f is not of null type relative to $(\{A_j, B_j\}_{j=1}^g, \{C_0, C_1\})$.

Then by Theorem 1 we can find an Abelian differential φ of the first kind (on R_N) such that

$$(28)\qquad \begin{cases} \int_{A_j}\varphi=L_{f_*}[A_j]+\mathcal{P}_j(S)=L_{f_*}[A_j] \\ \int_{B_j}\varphi=L_{f_*}[B_j]+Q_j(S)=L_{f_*}[B_j] \end{cases} \qquad j=1, 2, \cdots, g,$$

where L_{f_*} is the linear mapping [of $H_1(R_N)$ into the module $\Pi=\{z=m+n\tau \mid m, n\in \boldsymbol{Z}\}$] associated with the induced homomorphism $f_*: H_1(R_N)\to H_1(T)$. Because f is not of null type, we can find $4g$ integers m_{jk}, n_{jk} ($j=1, 2, \cdots, g$; $k=0, 1$) such that

$$(28\text{-}1)\qquad \begin{aligned} L_{f_*}[A_j]&=m_{j0}+m_{j1}\tau \\ L_{f_*}[B_j]&=n_{j0}+n_{j1}\tau \end{aligned} \qquad j=1, 2, \cdots, g,$$

$$(28\text{-}2)\qquad \sum_{j=1}^{g}(m_{j0}^2+n_{j0}^2)(m_{j1}^2+n_{j1}^2)>0,$$

Condition (28-2) implies that $\varphi\not\equiv 0$. Taking account of this fact, we conclude from equations (28), (28-1) that there is an analytic mapping of R_0 onto T (cf. Theorem 2). q.e.d.

Remarks. (1) If S degenerates (i.e., if $dS\equiv 0$), then Theorem 11 still remains true (cf. Proposition 9 and footnote 10)).

(2) The implication (II)$\Rightarrow$ (I) can be proved under weaker conditions

$$(29\text{-}1)\qquad \mathcal{P}_j(S)\equiv Q_j(S)\equiv 0 \mod \boldsymbol{Z}, \quad j=1, 2, \cdots, g,$$

$$(29\text{-}2)\qquad \mathcal{R}_k(S)=0, \quad k=1, 2, \cdots, N-1.$$

It follows from Proposition 11 that (29-1) does not depend on the special choice of a canonical homology basis of R_0.

(3) The preceding theorem shows that there is a Riemann surface R_N which admits two analytic mappings f_1, f_2 with the following properties: (i) f_1, f_2 are mappings of R_N into the same torus; (ii) they have the same singularity; and (iii) f_1 is of null type, while f_2 is not.

Now let a non-degenerate[10] $S\in\mathfrak{S}(R_N)$ be given. Let $\{A_j, B_j\}_{j=1}^{g}$ (resp. $\{C_0, C_1\}$) be a canonical homology basis of R_0 (resp. T). Then we shall say, in relation to "rational realizability", that the Riemann surface R_N is *S-transcendentally realizable* over T relative to $(\{A_j, B_j\}_{j=1}^{g}, \{C_0, C_1\})$, if we can find an analytic mapping $f: R_N\to T$ such that

(i) $\sigma_{\{C_0, C_1\}}(f)=S$, and

(ii) f is not of null type relative to $(\{A_j, B_j\}_{j=1}^{g}, \{C_0, C_1\})$.

If this is the case, f is an analytic mapping of R_N into T which can never be extended (holomorphically) to the whole of R_0. We say, for short, that $(R_N, \{A_j, B_j\}_{j=1}^{g})$ is S-transcendentally realizable over $(T, \{C_0, C_1\})$.

Suppose now that S is trivial. Then Theorem 11 asserts that $(R_N, \{A_j, B_j\}_{j=1}^{g})$ is S-transcendentally realizable over $(T, \{C_0, C_1\})$ if and only if R_N is rationally realizable over T. We have hence

Proposition 13. *Let R_N, T be as before and S a non-degenerate trivial analytic singularity at $\partial R_N: S\in\mathfrak{S}_0(R_N)$, $dS\not\equiv 0$. Let $\{C_0, C_1\}$ be a (fixed) canonical homology basis of T. Then for any two canonical homology bases $\{A_j, B_j\}_{j=1}^{g}$, $\{\tilde{A}_j, \tilde{B}_j\}_{j=1}^{g}$ of R_0 statements* (I), (II) *below are equivalent*:

(I) *$(R_N, \{A_j, B_j\}_{j=1}^{g})$ is S-transcendentally realizable over $(T, \{C_0, C_1\})$.*

(II) *$(R_N, \{\tilde{A}_j, \tilde{B}_j\}_{j=1}^{g})$ is S-transcendentally realizable over $(T, \{C_0, C_1\})$.*

This proposition yields the following conclusion: So long as we restrict ourselves to the trivial singularities, it is of less importance to refer to the canonical homology basis of R_0. On the other hand, we cannot dispense with the canonical homology basis of T, for we do need a normal integral on T to describe the (analytic) behavior of the mappings near ∂R_N. It is therefore convenient to use, as in the Teichmüller theory, the notion of *marked* tori. Then Theorem 11 becomes

Theorem 11′. *Let R_N be a Riemann surface of finite type, T a marked torus, and $S\in\mathfrak{S}_0(R_N)$, $dS\not\equiv 0$. Then R_N is S-transcendententally realizable over T if and only if it is rationally realizable over T.*

Let R_N be rationally realizable over a (marked) torus T. Then by Theorem 8 (see also the end of section 10) there exist infinitely many distinct non-degenerate $S\in\mathfrak{S}_0(R_N)$ such that R_N is S-transcendentally realizable over T. The same is true, even if we restrict the realization mappings to those which induce the same homomorphism as the rational realization mapping does.

15. We shall now mention some consequences of the preceding theorems. Combining Theorem 11′ with a result of Martens [12], we have

10) Many of the results of this section will be true of the degenerate singularity S_0 (which obviously belongs to the class $\mathfrak{S}_0(R_N)$). See, for instance, the preceding Remark (1). By Theorem 11′ which we shall prove later it will be very reasonable to agree to the following convention: To say that R_N is S_0-transcendentally realizable over a (marked) torus T is nothing other than saying that R_N is rationally realizable over T. (The same is true of the case $N=0$, for $\mathfrak{S}_0(R_0)=\{S_0\}$.)

Proposition 14. *If there is a non-degenerate* $S\in\mathfrak{S}_0(R_N)$ *such that* R_N *is* S-*transcendentally realizable over a marked torus, then any normalized period matrix of* R_0 *is singular (in the sense of Scorza).*

Actually, using theorems of Poincaré and Haupt-Wirtinger (see Proposition 5; cf. also [8]), we can determine the precise form of the period matrix.

Proposition 15. *Let* $S\in\mathfrak{S}_0(R_N)$, $N\geqq 1$. *Then* R_N *is* S-*transcendentally realizable over a marked torus if and only if the period matrix of* R_0 *with respect to "some" canonical homology basis is*

$$\begin{pmatrix} 1\ 0\ 0 \cdot\cdot 0 & \tau' & 1/m\ 0 \cdot\cdot\cdot 0 \\ 0\ 1\ 0 \cdot\cdot 0 & 1/m & \\ 0\ 0\ 1 \cdot\cdot 0 & 0 & \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot & * \\ 0 \cdot\cdot\cdot 1\ 0 & \cdot & \\ 0 \cdot\cdot\cdot 0\ 1 & 0 & \end{pmatrix},$$

where $m\in \boldsymbol{Z}-\{0, \pm 1\}$ *and* $\tau'\in\boldsymbol{C}$, $\operatorname{Im}\tau'>0$.

16. Let R_N, T be as before, $N\geqq 1$. Let $\eta: H_1(R_N, \partial R_N)\rightarrow H_1(T)$ be a homomorphism. Clearly there is a natural injection

$$i: H_1(R_N, \partial R_N)\longrightarrow H_1(R_N).$$

Also we can find a homomorphism $\eta': H_1(R_N)\rightarrow H_1(T)$ such that $\eta'\circ i=\eta$, $\eta'([D_k])=0$, $k=1, 2, \cdots, N-1$. Hence the results so far obtained can be translated to the case of $H_1(R_N, \partial R_N)$ in an obvious manner. For instance, Theorem 9 becomes

Theorem 9′. *Let* R_N, T *be as in Theorem* 9. *Then for any homomorphism* $\eta: H_1(R_N, \partial R_N)\rightarrow H_1(T)$ *we can find an analytic mapping* $f: R_N\rightarrow T$ *such that*

(i) $f_*=\eta$, *and*

(ii) f *can be extended holomorphically to the whole* R_0 *except for a single point* p_k *where* $d(\rho^{-1}\circ f)$ *has a pole of order not exceeding* $2g$.

If p_k *is a non-Weierstrass point,* $2g$ *can be replaced by* $g+1$. *The point* p_k *may be arbitrarily chosen.*

We omit the details. Note that Corollary to Theorem 9 as well as other theorems can be similarly rephrased.

17. Consider now a particular case where R_0 is also of genus one: $R_0=T'=T(1, \tau')$, $\operatorname{Im}\tau'>0$.

Let $\boldsymbol{Q}^*$ be the set of non-zero rational numbers and set

$$GL^+(2, \boldsymbol{Q})=\{\mathfrak{G}\in GL(2, \boldsymbol{Q})\,|\,\det\mathfrak{G}>0\},$$

$$\mathfrak{Q}=GL^+(2, \boldsymbol{Q})/\{\lambda\mathfrak{J}_2\,|\,\lambda\in\boldsymbol{Q}^*\},$$

$\mathfrak{J}_2$ being the 2×2 identity matrix. An element $\mathfrak{G}$ of $\mathcal{G}$ operates on the upper half plane U as

$$\mathfrak{G}\tau=\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \quad \mathfrak{G}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \quad \tau\in U.$$

Then Theorem 2 reduces to

Theorem 2′ (Helfenstein [6]). *There is an analytic mapping of a torus* $T'=T(1, \tau')$ *into another torus* $T=T(1, \tau)$ *if and only if* $\tau'=\mathfrak{G}\tau$ *for some* $\mathfrak{G}\in\mathcal{G}$.

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