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Periodic families in homotopy groups

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§ 1. Introduction

Let p be a prime number. We consider a homomorphism

$$\phi: \pi_*(X: p) \longrightarrow {}_p\pi_*^S$$

of degree $-l$ from the p-primary components of the homotopy groups of a space X to the p-primary components of the stable homotopy groups of spheres such that one or both of the following conditions are satisfied for compositions or secondary compositions:

$$\phi(\alpha \circ E^k \beta) = \phi(\alpha) \circ E^\infty \beta,$$

$$\phi\{\alpha, E^k\beta, E^k\gamma\}_k \subset \pm\{\phi(\alpha), E^\infty\beta, E^\infty\gamma\},$$

where the notation is due to H. Toda [14]. (We notice that the homomorphisms used in [12, 13] have these common properties.)

In this paper we define families of elements by compositions and secondary compositions and see how to detect them by d- and e-invariants of Adams-Toda [1, 15] after applying the homomorphism ϕ with the above properties. This is a generalization of the method used in [12, 13].

This paper is organized as follows. In § 2 we define a family of elements in $\pi_*(X)$. The construction is a generalization of the ones in [1, 9, 12, 13]. Then in § 3 we see how the family of elements is detected by making use of the homomorphism ϕ and the invariants of Adams-Toda [1, 15]. In § 4 we obtain more results in the case $p=2$ in connection with the results in § 12 of [1]. Some examples of the candidate for ϕ will be given in § 5. We study in the last section, § 6, the unstable homotopy groups of Stiefel manifolds of 2-frames $V_{n,2}=SO(n)/SO(n-2)$, $W_{n,2}=SU(n)/SU(n-2)$ and $X_{n,2}=Sp(n)/Sp(n-2)$. The results in § 6 are summarized in the following theorems.

Theorem 1. *Let* $n \geqq 5$. *Then*

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$\pi_{n-2+r}(V_{n,2}: 2)\neq 0$ *for all* $r\geqq 9$ *with* $r\equiv 0, 1, 2, 3, 4 \mod 8$.

Theorem 2. *Let* $n\geqq 5$. *Then*

$$\pi_{2n-3+r}(W_{n,2}: 2)\neq 0 \quad \text{for all} \quad r\geqq 9 \quad \text{with} \quad r\equiv 1, 3, 5, 7 \mod 8.$$

Theorem 3. *Let* $n\geqq 3$. *Then*

$$\pi_{4n-5+r}(X_{n,2}: 2)\neq 0 \qquad \text{for all} \quad r\geqq 5.$$

§2. A family of elements

We use the notation of the generators of H. Toda [14], for example, $\pi_n(S^n)=\{\iota_n\}\cong Z$ $(n\geqq 1)$ and $\pi_{n+1}(S^n)=\{\eta_n\}\cong Z_2$ $(n\geqq 3)$.

Let m be an integer. Assume that an element γ of $\pi_{j+h}(S^j)$ $(h\geqq 1)$ satisfies the following condition

(2.1) $m\gamma=0$ *in case* $m\not\equiv 2 \mod 4$

$m\gamma=0$ *and* $E\gamma\circ\eta_{j+h+1}=0$ *in case* $m\equiv 2 \mod 4$.

Under this condition we have

(2.2) $\{m\iota_{j+t}, E^t\gamma, m\iota_{j+h+t}\}_r\subset m\iota_{j+t}\circ\pi_{j+h+t+1}(S^{j+t})+\pi_{j+h+t+1}(S^{j+t})\circ m\iota_{j+h+t+1}$ *for* $t\geqq 1$ *and* $t\geqq r\geqq 0$.

Proof. By (1.15) of [14] we obtain

$$\{m\iota_{j+t}, E^t\gamma, m\iota_{j+h+t}\}_r\subset\{m\iota_{j+t}, E^t\gamma, m\iota_{j+h+t}\}\supset\{m\iota_{j+t}, E^t\gamma, m\iota_{j+h+t}\}_1.$$

The last Toda bracket contains the zero element by Corollary 3.7 of [14] and our assumption (2.1). It follows then that the middle Toda bracket is equal to

$$m\iota_{j+t}\circ\pi_{j+h+t+1}(S^{j+t})+\pi_{j+h+t+1}(S^{j+t})\circ m\iota_{j+h+t+1}.$$

This completes the proof. Q. E. D.

We define periodic families by the construction given by the Toda brackets of the following type.

Proposition 2.3. *Let* m *and* k *be positive integers and* γ *an element of* $\pi_{j+h}(S^j)$ $(h\geqq 1$ *and* $j\geqq 2)$ *satisfying the condition* (2.1). *Assume that* α *is an element of* $\pi_i(X)$ $(i\geqq j+k+1)$ *such that* $m\alpha=0$. *Then there exists a family of elements* $\alpha^{(n)}$ *of* $\pi_{i+n(h+1)}(X)$ *for all* $n\geqq 0$ *such that*

$$\alpha^{(0)}=\alpha,$$

$$\alpha^{(n)}\in\{\alpha^{(n-1)}, m\iota_{i+(n-1)(h+1)}, E^{i-j+(n-1)(h+1)}\gamma\}_k\subset\pi_{i+n(h+1)}(X) \text{ for } n\geqq 1.$$

Moreover, $m\alpha^{(n)}=0$ *for all* $n\geqq 0$.

Proof. Suppose, for the inductive hypothesis, that $m\alpha^{(n)}=0$. Then we see that the Toda bracket

$$\{\alpha^{(n)}, m\iota_{i+n(h+1)}, E^{i-j+n(h+1)}\gamma\}_k \subset \pi_{i+(n+1)(h+1)}(X)$$

is well defined. We choose an element $\alpha^{(n+1)}$ from this Toda bracket. It follows then that

$$\begin{aligned}
m\alpha^{(n+1)} &= \alpha^{(n+1)}\circ m\iota_{i+(n+1)(h+1)} \\
&\in \{\alpha^{(n)}, m\iota_{i+n(h+1)}, E^{i-j+n(h+1)}\gamma\}_k \circ m\iota_{i+(n+1)(h+1)} \\
&= \pm\alpha^{(n)}\circ E^k\{m\iota_{i+n(h+1)-k}, E^{i-j+n(h+1)-k}\gamma, m\iota_{i+(n+1)(h+1)-k-1}\} \\
&\subset \alpha^{(n)}\circ E^k\{m\iota_{i+n(h+1)-k}\circ\pi_{i+(n+1)(h+1)-k}(S^{i+n(h+1)-k}) \\
&\qquad + \pi_{i+(n+1)(h+1)-k}(S^{i+n(h+1)-k})\circ m\iota_{i+(n+1)(h+1)-k}\} \\
&= \alpha^{(n)}\circ m\iota_{i+n(h+1)}\circ E^k\pi_{i+(n+1)(h+1)-k}(S^{i+n(h+1)-k}) \\
&= m\alpha^{(n)}\circ E^k\pi_{i+(n+1)(h+1)-k}(S^{i+n(h+1)-k}) = 0
\end{aligned}$$

by Proposition 1.4 of [14] and (2.2). Thus we have a family of elements $\alpha^{(n)}$ of $\pi_{i+n(h+1)}(X)$ such that $m\alpha^{(n)}=0$ for all $n \geqq 0$. Q.E.D.

§3. The homomorphism ϕ and the Adams-Toda invariant

Let us now consider a homomorphism

$$\phi\colon \pi_*(X: p) \longrightarrow {}_p\pi_*^S$$

of degree $-l$, namely, a collection of homomorphisms $\phi=\{\phi_i\}$ such that

$$\phi_i\colon \pi_i(X: p) \longrightarrow {}_p\pi_r^S \qquad (r=i-l)$$

for all $i \geqq N$, where N is a fixed integer. We write simply ϕ instead of the above ϕ_i, since there would arise no confusion. Then the homomorphism we consider is the following:

(3.1) $$\phi\colon \pi_i(X: p) \longrightarrow {}_p\pi_r^S \qquad (r=i-l,\ i \geqq N).$$

Let $\alpha \in \pi_i(X: p)$, $\beta \in \pi_a(S^{i-k})$ and $\gamma \in \pi_b(S^a)$. We consider further the following conditions for compositions and secondary compositions:

(3.2) $$\phi(\alpha\circ E^k\beta) = \phi(\alpha)\circ E^\infty\beta \qquad \text{for some } k,$$

(3.3) $$\phi\{\alpha, E^k\beta, E^k\gamma\}_k \subset \pm\{\phi(\alpha), E^\infty\beta, E^\infty\gamma\} \qquad \text{for some } k,$$

where $k \geqq 0$ is a fixed integer.

We remark that the condition (3.2) implies:

$$\text{If } \alpha\circ E^k\beta=0, \text{ then } \phi(\alpha)\circ E^\infty\beta = \phi(\alpha\circ E^k\beta)=0.$$

Thus if the Toda bracket $\{\alpha, E^k\beta, E^k\gamma\}_k$ is well defined, then the Toda bracket $\{\phi(\alpha), E^\infty\beta, E^\infty\gamma\}$ is also well defined. But, for $\beta=m\iota_{i-k}$, $\phi(\alpha)\circ E^\infty\beta=0$ always holds, because $\phi(\alpha)\circ m\iota = m\phi(\alpha) = \phi(m\alpha)=0$.

Making use of the homomorphism ϕ mentioned above, we shall now state the main theorem that enables us to detect some periodic families in the homotopy group $\pi_*(X: p)$.

We consider the following condition:

$$\text{(3.4)}\qquad \begin{array}{lll} \Lambda=C & & \textit{then}\ \ h, r\equiv 1 \bmod 2, \\ \Lambda=R & \textit{and}\ \ p\neq 2 & \textit{then}\ \ h, r\equiv 3 \bmod 4, \\ \Lambda=R & \textit{and}\ \ p=2 & \textit{then}\ \ r\equiv 3 \bmod 4\ \ \textit{and}\ \ h\equiv 7 \bmod 8. \end{array}$$

Let e_Λ be the Adams-Toda invariant e_C, e_R, e'_R or e''_R [1, 15]. Then we have the following theorem.

Theorem 3.5. *Let $m=p^f$ and $d=p^g$ for a prime number p and integers $f\geqq g\geqq 1$. Let Λ, h and r be as in* (3.4). *Let the elements γ, α and $\alpha^{(n)}$ be those in Proposition* 2.3. *Assume further that*

$$e_\Lambda(E^\infty\gamma)\equiv q/m \bmod p/m \qquad (1\leqq q\leqq p-1).$$

Let the homomorphism ϕ of (3.1) *satisfy the condition* (3.3). *Then*

$$e_\Lambda\phi(\alpha)\equiv q_0/d \bmod p/d \qquad (1\leqq q_0\leqq p-1)$$

implies

$$e_\Lambda\phi(\alpha^{(n)})\equiv q_n/d \bmod p/d \qquad (1\leqq q_n\leqq p-1) \qquad \textit{for all } n\geqq 0.$$

Moreover, if $m=d$, then the element $\alpha^{(n)}$ is of order d for all $n\geqq 0$.

Proof. We prove the theorem by induction on n. In case $n=0$, we have $\alpha^{(0)}=\alpha$ and hence $e_\Lambda\phi(\alpha^{(0)})\equiv q_0/d \bmod p/d$ $(1\leqq q_0\leqq p-1)$. Now let us suppose that

$$e_\Lambda\phi(\alpha^{(n)})\equiv q_n/d \bmod p/d \qquad (1\leqq q_n\leqq p-1).$$

Then we obtain

$$\begin{aligned} e_\Lambda\phi(\alpha^{(n+1)}) &\in e_\Lambda\phi\{\alpha^{(n)}, m\iota_{i+n(h+1)}, E^{i-j+n(h+1)}\gamma\}_k \\ &\subset \pm e_\Lambda\{\phi(\alpha^{(n)}), m\iota, E^\infty\gamma\} \qquad \text{by (3.3)} \\ &= \pm m e_\Lambda(E^\infty\gamma)e_\Lambda\phi(\alpha^{(n)}) \qquad \text{by Theorem 11.1 of [1]} \\ &\equiv k_{n+1}/d \bmod p/d \qquad (1\leqq k_{n+1}\leqq p-1) \end{aligned}$$

by our inductive hypothesis and Proposition 7.14 of [1].

Moreover we see $m\alpha^{(n)}=0$ for all $n\geqq 0$ by Proposition 2.3. It follows then that $\alpha^{(n)}$ is an element of order d if $m=d$. Q.E.D.

Remark. In case $p\neq 2$ and $\Lambda=R$, we may take $e_\Lambda=e'_R$ or e''_R with different values under the condition (3.4). But the choice does not affect the situation, since we have

$$e'_R(E^\infty\gamma)\equiv q/m \bmod p/m \qquad (1\leqq q\leqq p-1)$$

if and only if

$$e''_R(E^\infty\gamma)\equiv q'/m \mod p/m \qquad (1\leqq q'\leqq p-1)$$

by virtue of Proposition 7.14 of [1]. In case $p=2$, we assume the condition that $h\equiv 7 \mod 8$ of (3.4) for the same reason.

Remark. In the proof of Theorem 3.5, the term $me_A(E^\infty\gamma)e_A\phi(\alpha^{(n)})$ should be understood to be evaluated on the suitable spheres (as in Theorem 11.1 of [1]) choosing the representatives of the homotopy classes.

§4. More about the case $p=2$

In this section we consider the 2-primary components $\pi_*(X:2)$. Then the homomorphism we consider is

(4.1) $$\phi:\pi_i(X:2)\longrightarrow {}_2\pi_r^S \qquad (r=i-l \text{ and } i\geqq N).$$

We put $m=2^f$ and $d=2^g$ for integers $f\geqq g\geqq 1$. We take an element $\gamma\in\pi_{j+h}(S^j)$ ($j\geqq 2$ and $h\equiv 7 \mod 8$) satisfying (2.1). Moreover we assume that

(4.2) $$e'_R(E^\infty\gamma)\equiv 1/m \mod 2/m.$$

Notation. For an element $\alpha\in\pi_i(X:2)$ ($i\geqq j+k+1$) satisfying $m\alpha=0$, $\alpha^{(n)}$ is the family of elements given by Proposition 2.3 with γ satisfying (4.2).

Using the notation mentioned above, we have the following

Proposition 4.3. *Let ϕ satisfy* (3.3). *Let $r=i-l\equiv 1$ or 2 mod 8 and let $m\alpha=0$. Then*

$$d_R\phi(\alpha)\neq 0 \text{ implies } d_R\phi(\alpha^{(n)})\neq 0 \qquad \text{for all } n\geqq 0.$$

Proposition 4.4. *Let $r=i-l\equiv 1$ mod 8. Let δ be an element of $\pi_{a+b}(S^a)$ ($b\equiv 1$ mod 8 and $a\leqq i-k$) such that $d_R(E^\infty\delta)\neq 0$.*

(1) *Let ϕ satisfy* (3.2). *Then*

$$d_R\phi(\alpha)\neq 0 \text{ implies } d_R\phi(\alpha\circ E^{i-a}\delta)\neq 0.$$

(2) *Let ϕ satisfy* (3.2) *and* (3.3). *Assume that $m\alpha=0$. Then*

$$d_R\phi(\alpha)\neq 0 \text{ implies } d_R\phi(\alpha^{(n)}\circ E^{i+n(h+1)-a}\delta)\neq 0 \qquad \text{for all } n\geqq 0.$$

Proposition 4.5. *Let $r=i-l\equiv 2$ mod 8. Let δ be an element of $\pi_{a+b}(S^a)$ ($b\equiv 1$ mod 8 and $a\leqq i-k$) such that $d_R(E^\infty\delta)\neq 0$.*

(1) *Let ϕ satisfy* (3.2). *Then*

$$d_R\phi(\alpha)\neq 0 \text{ implies } e'_R\phi(\alpha\circ E^{i-a}\delta)\equiv 1/2 \mod 1.$$

(2) *Let ϕ satisfy* (3.2) *and* (3.3). *Assume that $m\alpha=0$. Then*

$$d_R\phi(\alpha)\neq 0 \text{ implies } e'_R\phi(\alpha^{(n)}\circ E^{i+n(h+1)-a}\delta)\equiv 1/2 \mod 1 \qquad \text{for all } n\geqq 0.$$

Let $\nu_p(n)$ be the exponent to which the prime p occurs in the decomposition of n into powers, so that $n=2^{\nu_2(2)}3^{\nu_3(n)}5^{\nu_5(n)}\cdots$. Define a function $m(t)$ as follows:

For p odd

$$\nu_p(m(t))=\begin{cases} 0 & \text{if } t\not\equiv 0 \bmod (p-1) \\ 1+\nu_p(t) & \text{if } t\equiv 0 \bmod (p-1). \end{cases}$$

For $p=2$

$$\nu_2(m(t))=\begin{cases} 1 & \text{if } t\not\equiv 0 \bmod 2 \\ 2+\nu_2(t) & \text{if } t\equiv 0 \bmod 2. \end{cases}$$

Proposition 4.6. *Let $r=i-l\equiv 1$ or $2 \bmod 8$. Let θ be an element of $\pi_{a+b}(S^a)$ ($b=8t-1$ and $a\leqq i-k$) such that $m(4t)e_R(E^\infty\theta)$ is odd.*
(1) *Let ϕ satisfy* (3.2). *Then*

$$d_R\phi(\alpha)\neq 0 \text{ implies } e_R\phi(\alpha\circ E^{i-a}\theta)\neq 0.$$

(2) *Let ϕ satisfy* (3.2) *and* (3.3). *Let $m\alpha=0$. Then*

$$d_R\phi(\alpha)\neq 0 \text{ implies } e_R\phi(\alpha^{(n)}\circ E^{i+n(h+1)-a}\theta)\neq 0 \text{ for all } n\geqq 0.$$

These propositions correspond to the results in § 12 of J. F. Adams [1]. The proofs are quite similar to those in [12] and [13] and are omitted.

§ 5. Hopf invariant, suspension map and Toda bracket

Consider the homomorphism

$$p_*: \pi_i(X) \longrightarrow \pi_i(S^t)$$

induced by a map $p: X\to S^t$. Then we have by (iv) of Proposition 1.2 in [14]

$$(5.1)\qquad p_*(\alpha\circ E^k\beta)=p_*(\alpha)\circ E^k\beta \qquad (k\geqq 0),$$

$$p_*\{\alpha, E^k\beta, E^k\gamma\}_k\subset\{p_*(\alpha), E^k\beta, E^k\gamma\}_k \qquad (k\geqq 0),$$

for elements $\alpha\in\pi_i(X)$, $\beta\in\pi_a(S^{i-k})$ and $\gamma\in\pi_b(S^a)$.

Consider the homotopy exact sequence associated with the fibering $F\xrightarrow{i}E\xrightarrow{p}B$:

$$\cdots\longrightarrow\pi_i(F)\xrightarrow{i_*}\pi_i(E)\xrightarrow{p_*}\pi_i(B)\xrightarrow{\Delta'}\pi_{i-1}(F)\longrightarrow\cdots.$$

Then we have

$$(5.2)\qquad \Delta'(\alpha\circ E^k\beta)=\Delta'(\alpha)\circ E^{k-1}\beta \qquad (k\geqq 1),$$

$$\Delta'\{\alpha, E^k\beta, E^k\gamma\}_k\subset\{\Delta'(\alpha), E^{k-1}\beta, E^{k-1}\gamma\}_{k-1} \qquad (k\geqq 1)$$

for $\alpha\in\pi_i(B)$, $\beta\in\pi_a(S^{i-k})$ and $\gamma\in\pi_b(S^a)$.

The formula (5.2) for $k=1$ is Theorem 5.2 of [8] and (5.2) for $k\geqq 1$ is proved similarly.

In the $EH\Delta$-sequence (2.11) of [14]:

$$\cdots \xrightarrow{\Delta} \pi_i(S^m) \xrightarrow{E} \pi_{i+1}(S^{m+1}) \xrightarrow{H} \pi_{i+1}(S^{2m+1}) \xrightarrow{\Delta} \pi_{i-1}(S^m) \xrightarrow{E} \cdots,$$

the behavior of compositions and secondary compositions is described by the following formulas.

For the suspension map E, we have by (1.6) of [14] and Proposition 1.3 of [14]

$$(5.3) \qquad E(\alpha\circ E^k\beta)=E\alpha\circ E^{k+1}\beta \qquad (k\geqq 0)$$

$$E\{\alpha, E^k\beta, E^k\gamma\}_k\subset -\{E\alpha, E^{k+1}\beta, E^{k+1}\gamma\}_{k+1} \qquad (k\geqq 0)$$

for elements $\alpha\in\pi_i(S^m)$, $\beta\in\pi_a(S^{i-k})$ *and* $\gamma\in\pi_b(S^a)$.

For the Hopf invariant H, we have by Propositions 2.2 and 2.3 of [14]

$$(5.4) \qquad H(\alpha\circ E^k\beta)=H(\alpha)\circ E^k\beta \qquad (k\geqq 1)$$

$$H\{\alpha, E^k\beta, E^k\gamma\}_k\subset\{H(\alpha), E^k\beta, E^k\gamma\}_k \qquad (k\geqq 1)$$

for elements $\alpha\in\pi_{i+1}(S^{m+1})$, $\beta\in\pi_a(S^{i-k+1})$ *and* $\gamma\in\pi_b(S^a)$.

When we consider the elements in the 2-primary components, we have the following results.

$$(5.5) \qquad \Delta(\alpha\circ E^k\beta)=\Delta(\alpha)\circ E^{k-2}\beta \qquad (k\geqq 2)$$

$$\Delta\{\alpha, E^k\beta, E^k\gamma\}_k\subset -\{\Delta(\alpha), E^{k-2}\beta, E^{k-2}\gamma\}_{k-2} \qquad (k\geqq 2)$$

for $\alpha\in\pi_i(S^{2m+1}: 2)$, $\beta\in\pi_a(S^{i-k}: 2)$ *and* $\gamma\in\pi_b(S^a: 2)$.

Proof. The first formula is due to Proposition 2.5 of [14]. We prove the second one. Let us use the notations in §§ 1 and 2 of [14]. We see $\Delta=\partial\circ h_{m*}^{-1}\circ\Omega_1$ by (2.9) of [14] and $\Omega_1=i_*^{-1}\circ\Omega_0$ by (2.3) of [14]. Then we have

$$\begin{aligned}
&\Delta\{\alpha, E^k\beta, E^k\gamma\}_k \\
&=\partial\circ h_{m*}^{-1}\circ i_*^{-1}\circ\Omega_0\{\alpha, E^k\beta, E^k\gamma\}_k \\
&=-\partial\circ h_{m*}^{-1}\circ i_*^{-1}\{\Omega_0(\alpha), E^{k-1}\beta, E^{k-1}\gamma\}_{k-1} && \text{by Proposition 1.3 of [14]} \\
&=-\partial\circ h_{m*}^{-1}\{i_*^{-1}\circ\Omega_0(\alpha), E^{k-1}\beta, E^{k-1}\gamma\}_{k-1} && \text{by Lemma 2.1 of [14]} \\
&=-\partial\{h_{m*}^{-1}\circ i_*^{-1}\circ\Omega_0(\alpha), E^{k-1}\beta, E^{k-1}\gamma\}_{k-1} && \text{by Theorem 2.4 of [14]} \\
&\subset -\{\partial\circ h_{m*}^{-1}\circ i_*^{-1}\circ\Omega_0(\alpha), E^{k-2}\beta, E^{k-2}\gamma\}_{k-2} && \text{by (5.2)} \\
&=-\{\Delta(\alpha), E^{k-2}\beta, E^{k-2}\gamma\}_{k-2}.
\end{aligned}$$

Q.E.D.

Now let us consider the iterated suspension

$$E^\infty: \pi_{t+r}(S^t)\longrightarrow\pi_r^S$$

from the $(t+r)$-th homotopy group of S^t to the r-stem of the stable homotopy group of spheres. Then we have

(5.6) $$E^\infty(\alpha\circ E^k\beta)=E^\infty\alpha\circ E^\infty\beta,$$

$$E^\infty\{\alpha, E^k\beta, E^k\gamma\}_k\subset\pm\{E^\infty\alpha, E^\infty\beta, E^\infty\gamma\}$$

by (1.6) of [14] and Proposition 1.3 of [14]. Thus we may define homomorphism

$$\phi\colon \pi_i(X)\longrightarrow\pi_r^S$$

by putting $\phi=E^\infty$, $E^\infty\circ H$, $E^\infty\circ p_*$, $E^\infty\circ H\circ p_*$, $E^\infty\circ H\circ\Delta$, $E^\infty\circ H\circ\Delta\circ p_*$, $E^\infty\circ H\circ\Delta'$, $E^\infty\circ\Delta'\circ p_*$ etc. according to the choice of the space X under consideration. So the homomorphism ϕ defined above has the following properties:

(5.7) $$\phi(\alpha\circ E^k\beta)=\phi(\alpha)\circ E^\infty\beta \qquad (k\geqq N),$$

$$\phi\{\alpha, E^k\beta, E^k\gamma\}_k\subset\pm\{\phi(\alpha), E^\infty\beta, E^\infty\gamma\} \qquad (k\geqq N).$$

J. F. Adams [1] obtained the periodic families in the stable homotopy groups of spheres. M. G. Barratt [2] determined the spheres of origin of these periodic families. In connection with these families, there are some periodic families in other spaces as well as the unstable periodic families in the homotopy groups of spheres. In many cases, they can be constructed by compositions and secondary compositions and detected by d- and e-invariants of Adams-Toda [1], [15] after applying some homomorphisms ϕ as in [12, 13]. Before we mention some examples we list some of the results of J. F. Adams [1] and M. G. Barratt [2].

Proposition 5.8. $\pi_*^n=\pi_*(S^n\colon 2)$ *has the following direct summands*:

(1) $8s$ stem: $\pi_{n+8s}^n\supset\{\mu_{s-1,n}\circ\sigma_{n+8s-7}\}\cong Z_2$ $(n\geqq 3,\ s\geqq 2)$

(2) $8s+1$ stem: $\pi_{n+8s+1}^n\supset\{\eta_n\circ\mu_{s-1,n+1}\circ\sigma_{n+8s-6}\}\cong Z_2$ $(n\geqq 2,\ s\geqq 2)$

$\pi_{n+8s+1}^n\supset\{\mu_{s,n}\}\cong Z_2$ $(n\geqq 3,\ s\geqq 0)$

(3) $8s+2$ stem: $\pi_{n+8s+2}^n\supset\{\eta_n\circ\mu_{s,n+1}\}\cong Z_2$ $(n\geqq 2,\ s\geqq 0)$

(4) $8s+3$ stem: $\pi_{n+8s+3}^n\supset\{\zeta_{s,n}\}\cong Z_8$ $(n\geqq 5,\ s\geqq 0)$

(5) $8s+7$ stem: $\pi_{8s+12}^5\supset\{\alpha_s'\}\cong Z_2$,

$\pi_{8s+13}^6\supset\{\alpha_s''\}\cong Z_4$,

$\pi_{8s+14}^7\supset\{\alpha_s'''\}\cong Z_8$,

$\pi_{8s+16}^9\supset\{\alpha_s^{IV}\}\cong Z_{16}$ $(s\geqq 0)$,

where $\pi_{n+1}^n=\{\eta_n\}\cong Z_2$ $(n\geqq 3)$ *and* $\pi_{n+7}^n=\{\sigma_n\}\cong Z_{16}$ $(n\geqq 9)$.

The other elements are defined in the following way. There exist elements $\mu_3\in\pi_{12}^3$, $\nu_5\in\pi_8^5$ and $\zeta_5\in\pi_{16}^5$ of orders $m=2$, 8 and 8 respectively. Then applying Proposition 2.3 with $\gamma=(16/m)\sigma_j$ and $k=1$, we have elements $\mu_3^{(s)}\in\pi_{8s+12}^3$ and

$\zeta_5^{(s)} \in \pi_{8s+16}^5$ for all $s \geqq 0$. We write $\mu_{0,3} = \eta_3$, $\mu_{s,3} = \mu_3^{(s-1)}$, $\zeta_{0,5} = \nu_5 \in \pi_8^5$ and $\zeta_{s,5} = \zeta_5^{(s-1)}$ for $s \geqq 1$. Moreover, we write $\mu_{s,n} = E^{n-3}\mu_{s,3}$ for $n \geqq 4$, $\zeta_{s,n} = E^{n-5}\zeta_{s,5}$ for $n \geqq 6$, $\mu_s = E^\infty \mu_{s,3}$ and $\zeta_s = E^\infty \zeta_{s,5}$ by definition. Applying the same construction to the cases $\alpha'_0 = \sigma'''$, $\alpha''_0 = \sigma''$, $\alpha'''_0 = \sigma'$ and $\alpha^{IV}_0 = \sigma_9$, we have the elements α'_s, α''_s, α'''_s and α^{IV}_s.

We note that

(5.9)
$$e'_R(\sigma) \equiv 1/16 \bmod 1/8, \qquad e_C(\mu_s) \equiv 1/2 \bmod 1,$$
$$e'_R(\zeta_s) \equiv 1/8 \bmod 1/4,$$
$$e_C(E^\infty \alpha'_s) \equiv 1/2 \bmod 1, \qquad e_C(E^\infty \alpha''_s) \equiv 1/4 \bmod 1/2,$$
$$e_C(E^\infty \alpha'''_s) \equiv 1/8 \bmod 1/4, \qquad e_C(E^\infty \alpha^{IV}_s) \equiv 1/16 \bmod 1/8.$$

The following examples are obtained by making use of Theorem 3.5 and Propositions 4.3∼4.6. (See [12, 13] for the notations of the generators and the detail of the proofs.)

Example 5.10. $\phi = E^\infty \circ H$;

(1) $e'_R\phi(\alpha^{(n)}) \equiv 1/8 \bmod 1/4$ $(n \geqq 0)$ *for* $\alpha = F_s \in \pi_{16s+14}^{8s+2}$ $(s \geqq 1)$, $G_s \in \pi_{16s+22}^{8s+6}$ $(s \geqq 0)$ *and* $M_s \in \pi_{16s+18}^{8s+8}$ $(s \geqq 0)$.

(2) $e_C\phi(\alpha^{(n)}) \equiv 1/2 \bmod 1$ $(n \geqq 0)$ *for* $\alpha = A_s \in \pi_{16s+8}^{8s+4}$ $(s \geqq 1)$ *and* $B_s \in \pi_{16s+16}^{8s+8}$ $(s \geqq 1)$.

(3) $d_R\phi(\alpha^{(n)}) \neq 0$ $(n \geqq 0)$ *for* $\alpha = C_s \in \pi_{16s+7}^{8s+3}$ $(s \geqq 1)$ *and* $D_s \in \pi_{16s+15}^{8s+7}$ $(s \geqq 1)$.

(4) $d_R\phi(\alpha) \neq 0$ *for* $\alpha = A_s^{(n)} \circ \mu_{s',16s+8n+8}$ *and* $B_s^{(n)} \circ \mu_{s',16s+8n+16}$ $(n \geqq 0,\ s \geqq 1$ *and* $s' \geqq 0)$.

(5) $e_R\phi(\alpha) \neq 0$ *for* $\alpha = A_s^{(n)} \circ \eta_{16s+8n+8} \circ \mu_{s',16s+8n+9}$, $B_s^{(n)} \circ \eta_{16s+8n+16} \circ \mu_{s',16s+8n+17}$, $C_s^{(n)} \circ \mu_{s',16s+8n+7}$ *and* $D_s^{(n)} \circ \mu_{s',16s+8n+15}$ $(n \geqq 0,\ s \geqq 1$ *and* $s' \geqq 0)$.

(6) $e_R\phi(\alpha) \neq 0$ *for* $\alpha = A_s^{(n)} \circ \sigma_{16s+8n+8}$, $B_s^{(n)} \circ \sigma_{16s+8n+16}$, $C_s^{(n)} \circ \sigma_{16s+8n+7}$, $D_s^{(n)} \circ \sigma_{16s+8n+15}$, $A_s^{(n)} \circ \mu_{s',16s+8n+8} \circ \sigma_{16s+8(n+s')+9}$ *and* $B_s^{(n)} \circ \mu_{s',16s+8n+16} \circ \sigma_{16s+8(n+s')+17}$.

Example 5.11. $\phi = E^\infty \circ p_*$;

(1) $e_C\phi([\sigma''']^{(n)}) \equiv 1/2 \bmod 1$ $(n \geqq 0)$ *for* $[\sigma'''] \in \pi_{12}(SU(3)\colon 2)$.

(2) $e'_R\phi([\nu_7]^{(n)}) \equiv 1/8 \bmod 1/4$ $(n \geqq 0)$ *for* $[\nu_7] \in \pi_{10}(Sp(2)\colon 2)$.

$e'_R\phi([2\sigma']^{(n)}) \equiv 1/4 \bmod 1/2$ $(n \geqq 0)$ *for* $[2\sigma'] \in \pi_{14}(Sp(2)\colon 2)$.

(3) $d_R\phi(\langle \eta_6 \circ \mu_7 \rangle^{(n)}) \neq 0$ $(n \geqq 0)$ *for* $\langle \eta_6 \circ \mu_7 \rangle \in \pi_{16}(G_2\colon 2)$,

$e_R\phi(\langle \eta_6 \circ \mu_7 \rangle^{(n)} \circ \sigma_{8n+16}) \neq 0$ $(n \geqq 0)$.

Example 5.12. $\phi=E^{\infty}\circ p_{2*}$;

(1) $e_C\phi([\eta_5\circ\varepsilon_6\oplus\sigma']^{(n)})\equiv 1/8 \mod 1/4$ $(n\geqq 0)$ *for* $[\eta_5\circ\varepsilon_6\oplus\sigma']\in\pi_{14}(SU(4)\colon 2)$,

$e_C\phi([\zeta_5\oplus\mu_7]^{(n)})\equiv 1/2 \mod 1$ $(n\geqq 0)$ *for* $[\zeta_5\oplus\mu_7]\in\pi_{16}(SU(4)\colon 2)$.

(2) $e_R\phi([\zeta_5\oplus\mu_7]^{(n)}\circ\sigma_{8n+16})\neq 0$ $(n\geqq 0)$,

$e_R\phi([\zeta_5\oplus\mu_7]^{(n)}\circ\eta_{8n+16}\circ\sigma_{8n+17})\neq 0$ $(n\geqq 0)$.

Example 5.13. $\phi=E^{\infty}\circ H\circ p_*$;

$e'_R\phi(\langle\zeta'+\mu_6\circ\sigma_{15}\rangle^{(n)})\equiv 1/8 \mod 1/4$ $(n\geqq 0)$ *for* $\langle\zeta'+\mu_6\circ\sigma_{15}\rangle\in\pi_{22}(G_2\colon 2)$.

§6. Periodic families in the homotopy groups of Stiefel manifolds

In this section, we show that some sequences of elements are non-zero in the homotopy groups of Stiefel manifolds $V_{n,2}$, $W_{n,2}$ and $X_{n,2}$. We deal with the unstable range of these homotopy groups. These periodic families come from the periodic families of J. F. Adams [1] and M. G. Barratt [2].

(A) *Real Stiefel manifold* $V_{n,2}=SO(n)/SO(n-2)$

We consider the exact sequence

(6.1) $$\cdots\longrightarrow\pi_{q+1}(S^{n-1})\xrightarrow{\Delta}\pi_q(S^{n-2})$$
$$\xrightarrow{i_*}\pi_q(V_{n,2})\xrightarrow{p_*}\pi_q(S^{n-1})\xrightarrow{\Delta}\pi_{q-1}(S^{n-2})\longrightarrow\cdots$$

associated with the fibering

(6.2) $$S^{n-2}\xrightarrow{i}V_{n,2}\xrightarrow{p}S^{n-1}.$$

As is well known, we have a cross section $s\colon S^{n-1}\to V_{n,2}$ of (6.2) when n is even. Hence we see

(6.3) *For n even,*

$$\pi_{n-2+r}(V_{n,2})=i_*\pi_{n-2+r}(S^{n-2})\oplus s_*\pi_{n-2+r}(S^{n-1}),$$

where i_* and s_* are monomorphisms.

When n is odd, we have to use the following result [4, 5, 6, 10]:

Lemma 6.4. *For n odd,*

(1) $$\Delta(\iota_{n-1})=2\iota_{n-2},$$

(2) $$E^2\Delta(\alpha)=2E\alpha \quad \textit{for} \quad \alpha\in\pi_q(S^{n-1}).$$

We first study the homomorphism $i_*\colon \pi_q(S^{n-2})\to\pi_q(V_{n,2})$ of (6.1).

Proposition 6.5. *The following elements are non-trivial in $\pi_{n-2+r}(V_{n,2})$:*

$r=8s$: $\quad i_*\mu_{s-1,n-2}\circ\sigma_{8s+n-9}$ $\quad (s\geqq 2,\ n\geqq 5)$,

$$r=8s+1: \quad i_*\mu_{s,n-2} \qquad (s\geqq 0,\ n\geqq 5),$$
$$i_*\eta_{n-2}\circ\mu_{s-1,n-1}\circ\sigma_{8s+n-8} \qquad (s\geqq 2,\ n\geqq 4),$$
$$r=8s+2: \quad i_*\eta_{n-2}\circ\mu_{s,n-1} \qquad (s\geqq 0,\ n\geqq 4),$$
$$r=8s+3: \quad i_*\zeta_{s,n-2} \qquad (s\geqq 0,\ n\geqq 7).$$

Proof. When n is even, the results are the immediate consequence of the direct decomposition (6.3). When n is odd, the results are obtained by the following Lemma 6.6. Q.E.D.

Lemma 6.6. *Let n be odd. Let $\alpha\in\pi_{n-2+r}(S^{n-2})$. If $E^\infty\alpha$ is not divisible by 2 in π_r^S, then $i_*\alpha\neq 0$ in $\pi_{n-2+r}(V_{n,2})$.*

Proof. Consider the exact sequence

$$\pi_{n-1+r}(S^{n-1}) \xrightarrow{\Delta} \pi_{n-2+r}(S^{n-2}) \xrightarrow{i_*} \pi_{n-2+r}(V_{n,2}).$$

We show that the element α of $\pi_{n-2+r}(S^{n-2})$ with the required property is not contained in the image of $\Delta: \pi_{n-1+r}(S^{n-1})\to\pi_{n-2+r}(S^{n-2})$ and hence $i_*\alpha\neq 0$ in $\pi_{n-2+r}(V_{n,2})$ by the exactness of the above sequence. Let us suppose that $\alpha=\Delta(\beta)$ for some element $\beta\in\pi_{n-1+r}(S^{n-1})$. We see $E^2\Delta(\beta)=2E\beta$ by (2) of Lemma 6.4. So $E^\infty\alpha=E^\infty\Delta(\beta)=E^\infty(E^2\Delta(\beta))=E^\infty(2E\beta)=2E^\infty\beta$ in π_r^S. This contradicts the assumption that $E^\infty\alpha$ is not divisible by 2 in π_r^S. Hence α is not contained in the image of $\Delta: \pi_{n-1+r}(S^{n-1})\to\pi_{n-2+r}(S^{n-2})$. Q.E.D.

We denote by $[\alpha]\in\pi_{n-2+r}(V_{n,2})$ such an element that $p_*[\alpha]=\alpha\in\pi_{n-2+r}(S^{n-1})$.

Lemma 6.7. *For $s\geqq 0$ and $n\geqq 4$, there exists an element $[\mu_{s,n-1}]$ of $\pi_{8s+n}(V_{n,2})$ such that $p_*[\mu_{s,n-1}]=\mu_{s,n-1}$.*
Moreover, $2[\mu_{s,n-1}]=0$ if n is even;

$$2[\mu_{s,n-1}]\equiv i_*\mu_{s,n-2}\circ\eta_{8s+n-1} \mod 2i_*\pi_{8s+n}(S^{n-2}) \textit{ if } n \textit{ is odd.}$$

Proof. If n is even, the result is immediate from (6.3). For n odd, we consider the following exact sequence

$$\pi_{8s+n}(S^{n-2}) \xrightarrow{i_*} \pi_{8s+n}(V_{n,2}) \xrightarrow{p_*} \pi_{8s+n}(S^{n-1}) \xrightarrow{\Delta} \pi_{8s+n-1}(S^{n-2}).$$

The element $\mu_{s,n-1}$ is in $\pi_{8s+n}(S^{n-1})$. We see

$$\Delta(\iota_{n-1})\circ\mu_{s,n-2}=2\iota_{n-2}\circ\mu_{s,n-2}=2\mu_{s,n-2}=0$$

by (1) of Lemma 6.4. We apply Theorem 2.1 of [7] to the case $\alpha=\iota_{n-1}$, $\beta=\mu_{s,n-2}$ and $\gamma=2\iota_{8s+n-1}$. Then for an element δ of $\{\Delta(\iota_{n-1}), \mu_{s,n-2}, 2\iota_{8s+n-1}\}$, there exists an element $[\mu_{s,n-1}]$ of $\pi_{8s+n}(V_{n,2})$ such that $p_*[\mu_{s,n-1}]=\mu_{s,n-1}$ and $i_*\delta=[\mu_{s,n-1}]\circ 2\iota_{8s+n}=2[\mu_{s,n-1}]$. Now we see

$$\{\Delta(\iota_{n-1}), \mu_{s,n-2}, 2\iota_{8s+n-1}\}=\{2\iota_{n-2}, \mu_{s,n-2}, 2\iota_{8s+n-1}\}\ni\mu_{s,n-2}\circ\eta_{8s+n-1}$$

by Corollary 3.7 of [14] (for $n=5$, consider the suspension map, which is monic). Then we have

$$\delta \equiv \mu_{s,n-2}\circ\eta_{8s+n-1} \mod 2\iota_{n-2}\circ\pi_{8s+n}(S^{n-2})+\pi_{8s+n}(S^{n-2})\circ 2\iota_{8s+n}. \qquad \text{Q. E. D.}$$

Making use of (6.3) and Lemma 6.7, we have the following

Proposition 6.8. *There exist the following non-trivial elements in* $\pi_{n-2+r}(V_{n,2})$:

$$\begin{aligned}
r=8s+1: &\quad [\mu_{s-1,n-1}]\circ\sigma_{8s+n-8} && (s\geqq 2,\ n\geqq 4),\\
r=8s+2: &\quad [\mu_{s,n-1}] && (s\geqq 0,\ n\geqq 4),\\
&\quad [\mu_{s-1,n-1}]\circ\eta_{8s+n-8}\circ\sigma_{8s+n-7} && (s\geqq 2,\ n\geqq 4),\\
r=8s+3: &\quad [\mu_{s,n-1}]\circ\eta_{8s+n} && (s\geqq 0,\ n\geqq 4),\\
r=8s+4: &\quad [\mu_{s,n-1}]\circ\eta^2_{8s+n} && (s\geqq 0,\ n\geqq 4).
\end{aligned}$$

Remark. Computing directly (cf. [11]), we see

$$\pi_{2n+3}(V_{2n+3,2})=\{[\eta_{2n+2}]\}\cong Z_4$$

for all $n\geqq 1$. Then for $n\geqq 4$, applying Proposition 2.3, we may construct elements

$$[\eta_{2n+2}]^{(s)}\in\pi_{8s+2n+3}(V_{2n+3,2}) \qquad \text{for} \quad s\geqq 0.$$

We may use this element instead of $[\mu_{s,2n+2}]$.

(B) *Complex Stiefel manifold* $W_{n,2}=SU(n)/SU(n-2)$

Consider the homotopy exact sequence

$$(6.9) \qquad \cdots \longrightarrow \pi_{q+1}(S^{2n-1}) \xrightarrow{\Delta} \pi_q(S^{2n-3}) \xrightarrow{i_*} \pi_q(W_{n,2}) \xrightarrow{p_*} \pi_q(S^{2n-1}) \xrightarrow{\Delta} \pi_{q-1}(S^{2n-3}) \longrightarrow \cdots$$

associated with the fibering

$$(6.10) \qquad S^{2n-3} \xrightarrow{i} W_{n,2} \xrightarrow{p} S^{2n-1}.$$

There exists a cross section $s\colon S^{2n-1}\to W_{n,2}$ of (6.10) when n is even ([5]). Then we have a direct sum decomposition:

(6.11) *For n even,*

$$\pi_{2n-3+r}(W_{n,2})=i_*\pi_{2n-3+r}(S^{2n-3})\oplus s_*\pi_{2n-3+r}(S^{2n-1}),$$

where i_* *and* s_* *are monomorphisms.*

When n is odd, we use the following result of I. M. James and J. H. C. Whitehead [4, 5, 6].

Lemma 6.12. *Let n be odd. Then*

(1) $\qquad \Delta(\iota_{2n-1})=\eta_{2n-3}$,

(2) $\qquad E^3\Delta(\alpha)=\eta_{2n}\circ E^2\alpha \qquad$ *for* $\alpha\in\pi_q(S^{2n-1})$.

Making use of Lemma 6.12, we first show

Proposition 6.13. *The following elements are non-trivial in* $\pi_{2n-3+r}(W_{n,2})$:

(1) $r=8s+1$: $i_*\mu_{s,2n-3}$ $(s\geqq 1,\ n\geqq 3)$,

$r=8s+3$: $i_*\zeta_{s,2n-3}$ $(s\geqq 0,\ n\geqq 4)$,

$r=8s+7$: $i_*E^{2n-8}\alpha'_s$ $(s\geqq 0,\ n\geqq 4)$.

(2) $r=8s$: $i_*\mu_{s-1,2n-3}\circ\sigma_{2n+8s-10}$ $(s\geqq 2,\ n=4;\ s\geqq 1,$ *even* $n\geqq 6)$,

$r=8s+1$: $i_*\eta_{2n-3}\circ\mu_{s-1,2n-2}\circ\sigma_{2n+8s-9}$ $(s\geqq 2,\ n=4;\ s\geqq 1,$ *even* $n\geqq 6)$,

$r=8s+2$: $i_*\eta_{2n-3}\circ\mu_{s,2n-2}$ $(s\geqq 0,$ *even* $n\geqq 4)$.

Proof. When n is even, the result is immediate from (6.11). Let n be odd. We consider the following exact sequence:

$$\pi_{2n-2+r}(S^{2n-1})\xrightarrow{\Delta}\pi_{2n-3+r}(S^{2n-3})\xrightarrow{i_*}\pi_{2n-3+r}(W_{n,2}).$$

Let $r=8s+1$ $(s\geqq 1)$. Then the element $\mu_{s,2n-3}$ is in $\pi_{2n+8s-2}(S^{2n-3})$. Suppose $\mu_{s,2n-3}=\Delta(\beta)$ for some element $\beta\in\pi_{2n+8s-1}(S^{2n-1})$. Then $\mu_s=E^\infty\mu_{s,2n-3}=E^\infty\Delta(\beta)$ $=E^\infty(E^3\Delta(\beta))=E^\infty(\eta_{2n}\circ E^2\beta)=\eta\circ E^\infty\beta$ by (2) of Lemma 6.12. We see that $e_C(\eta\circ E^\infty\beta)$ $=e_C(\eta)d_C(E^\infty\beta)=0$ by Propositions 3.2 and 7.1 of [1]. On the other hand, we have $e_C(\mu_s)\equiv 1/2 \bmod 1$ by (5.9). This is a contradiction. Hence $\mu_{s,2n-3}$ is not contained in the image of $\Delta\colon\pi_{2n+8s-1}(S^{2n-1})\to\pi_{2n+8s-2}(S^{2n-3})$. Thus we have $i_*\mu_{s,2n-3}\neq 0$.

Let $r=8s+3$ $(s\geqq 0)$. The element $\zeta_{s,2n-3}$ is in $\pi_{2n+8s}(S^{2n-3})$. Suppose $\zeta_{s,2n-3}=\Delta(\beta)$ for some element $\beta\in\pi_{2n+8s+1}(S^{2n-1})$. Then $\zeta_s=E^\infty\zeta_{s,2n-3}=$ $E^\infty(E^3\Delta(\beta))=E^\infty(\eta_{2n}\circ E^2\beta)=\eta\circ E^\infty\beta$ by (2) of Lemma 6.12. We see $2\eta=0$ and hence $2(\eta\circ E^\infty\beta)=0$. This contradicts the fact that ζ_s is an element of order 8. Thus $\zeta_{s,2n-3}$ is not contained in the image of $\Delta\colon\pi_{2n+8s+1}(S^{2n-1})\to\pi_{2n+8s}(S^{2n-3})$. Then we have $i_*\zeta_{s,2n-3}\neq 0$.

Let $r=8s+7$ $(s\geqq 0)$. The element $E^{2n-8}\alpha'_s$ is in $\pi_{2n+8s+4}(S^{2n-3})$. Suppose $E^{2n-8}\alpha'_s=\Delta(\beta)$ for some element $\beta\in\pi_{2n+8s+5}(S^{2n-1})$. Then $E^\infty\alpha'_s=E^\infty\Delta(\beta)=\eta\circ E^\infty\beta$ by (2) of Lemma 6.12. We see that $e_C(E^\infty\alpha'_s)\equiv 1/2 \bmod 1$ and $e_C(\eta\circ E^\infty\beta)=0$ by Propositions 3.2 and 7.1 of [1]. This is a contradiction. It follows then that $E^{2n-8}\alpha'_s$ is not contained in the image of $\Delta\colon\pi_{2n+8s+5}(S^{2n-1})\to\pi_{2n+8s+4}(S^{2n-3})$ and hence $i_*E^{2n-8}\alpha'_s\neq 0$. Q.E.D.

Lemma 6.14. *Let n be odd. Then*

$$\pi_{2n-1}(W_{n,2})=\{[2\iota_{2n-1}]\}\cong Z \qquad (n\geqq 3),$$

$$\pi_{2n+2}(W_{n,2})=\{[\omega_{2n-1}]\}\cong Z_{24} \qquad (n\geqq 5),$$

$$\pi_{2n+2}(W_{n,2}:2)=\{[\nu_{2n-1}]\}\cong Z_8 \qquad (n\geqq 5).$$

The proof is easy and left to the reader (Cf. [3]).

Making use of the result in Lemma 6.14, we may construct a family of elements. We apply Proposition 2.3 to the case $\alpha=[\nu_{2n-1}]$, $m=8$, $\gamma=2\sigma_9$ and $k=1$. Then we obtain a family of elements

$$[\nu_{2n-1}]^{(s)}\in\pi_{2n+8s+2}(W_{n,2}:2) \qquad \text{for}\quad s\geqq 0.$$

By Theorem 3.5 and (6.11), we have the following

Proposition 6.15. *The following elements are non-trivial in* $\pi_{2n-3+r}(W_{n,2})$:

(1) *Let n be odd. Then*

$r=8s+1$: $[2\iota_{2n-1}]\circ E^{2n-10}\alpha^{IV}_{s-1}$ $(s\geqq 1,\ n\geqq 5)$,

$r=8s+5$: $[\nu_{2n-1}]^{(s)}$ $(s\geqq 0,\ n\geqq 5)$.

(2) *Let n be even. Then*

$r=8s+1$: $s_*E^{2n-10}\alpha^{IV}_{s-1}$ $(s\geqq 1,\ n\geqq 6)$,

$r=8s+2$: $s_*\mu_{s-1,2n-1}\circ\sigma_{2n+8s-8}$ $(s=1,\ n\geqq 4;\ s\geqq 2,\ n\geqq 2)$,

$r=8s+3$: $s_*\eta_{2n-1}\circ\mu_{s-1,2n}\circ\sigma_{2n+8s-7}$ $(s=1,\ n\geqq 4;\ s\geqq 2,\ n\geqq 2)$,

$s_*\mu_{s,2n-1}$ $(s\geqq 0,\ n\geqq 2)$,

$r=8s+4$: $s_*\eta_{2n-1}\circ\mu_{s,2n}$ $(s\geqq 0,\ n\geqq 2)$,

$r=8s+5$: $s_*\zeta_{s,2n-1}$ $(s\geqq 0,\ n\geqq 4)$.

(C) *Quaternionic Stiefel manifold* $X_{n,2}=Sp(n)/Sp(n-2)$

Consider the exact sequence

(6.16)
$$\cdots\longrightarrow\pi_{q+1}(S^{4n-1})\xrightarrow{\Delta}\pi_q(S^{4n-5})\xrightarrow{i_*}\pi_q(X_{n,2})\xrightarrow{p_*}\pi_q(S^{4n-1})\xrightarrow{\Delta}\pi_{q-1}(S^{4n-5})\longrightarrow\cdots$$

associated with the fibering

(6.17)
$$S^{4n-5}\xrightarrow{i}X_{n,2}\xrightarrow{p}S^{4n-1}.$$

By the result of I. M. James and J. H. C. Whitehead [4, 5, 6], we have

Proposition 6.18.

(1) $\Delta(\iota_7)=\omega'$ $(n=2)$,

$\Delta(\iota_{4n-1})=n\omega_{4n-5}$ $(n\geqq 3)$,

(2) $E^5\Delta(\alpha)=n\omega_{4n}\circ E^4\alpha$ *for* $\alpha\in\pi_q(S^{4n-1})$ $(n\geqq 2)$,

where $\pi_6(S^3)=\{\omega'\}\cong Z_{12}$ *and* $\pi_{n+3}(S^n)=\{\omega_n\}\cong Z_{24}$ *for* $n\geqq 5$.

Proposition 6.19. *The following elements are non-trivial in* $\pi_{4n-5+r}(X_{n,2})$:

$r=8s$: $i_*\mu_{s-1,4n-5}\circ\sigma_{4n+8s-12}$ $(s=1,\ n\geqq 3;\ s\geqq 2,\ n\geqq 2)$,

$r=8s+1$: $i_*\eta_{4n-5}\circ\mu_{s-1,4n-4}\circ\sigma_{4n+8s-11}$ $(s=1,\ n\geqq 3;\ s\geqq 2,\ n\geqq 2)$,

$i_*\mu_{s,4n-5}$ $(s\geqq 0,\ n\geqq 2)$,

$r=8s+2$: $i_*\eta_{4n-5}\circ\mu_{s,4n-4}$ $(s\geqq 0,\ n\geqq 2)$,

$r=8s+3$: $i_*\zeta_{s,4n-5}$ $(s\geqq 1,\ n\geqq 3)$,

$r=8s+7$: $i_*E^{4n-12}\alpha_s'''$ $(s\geqq 0,\ n\geqq 3)$.

Proof. Consider the following exact sequence

$$\pi_{4n-4+r}(S^{4n-1}) \xrightarrow{\Delta} \pi_{4n-5+r}(S^{4n-5}) \xrightarrow{i_*} \pi_{4n-5+r}(X_{n,2}).$$

For $r=8s$ and $\alpha=\mu_{s-1,4n-5}\circ\sigma_{4n+8s-12}$, we see $e_R(E^\infty(\alpha))\neq 0$ by Example 12.15 of [1]. Suppose $\alpha=\Delta(\beta)$ for some element $\beta\in\pi_{4n-4+r}(S^{4n-1})$. Then by (1) of Proposition 6.18, we have $E^\infty\alpha=E^\infty\Delta(\beta)=n\omega\circ E^\infty\beta$. Then $e_R(E^\infty\alpha)=e_R(n\omega\circ E^\infty\beta)$ $=e_R(n\omega)d_R(E^\infty\beta)=0$ by Propositions 3.2 and 7.1 of [1]. This is a contradiction. Thus we conclude that α is not contained in the image of Δ: $\pi_{4n-4+r}(S^{4n-1})\rightarrow$ $\pi_{4n-5+r}(S^{4n-5})$. Hence we have $i_*\alpha=i_*\mu_{s-1,4n-5}\circ\sigma_{4n+8s-12}\neq 0$. The others are obtained similarly. Q.E.D.

Lemma 6.20. *The following results hold in the 2-components of the homotopy groups of spheres*:

(1) *Let n be even. Then*

$$\{n\nu_k,\ \nu_{k+3},\ 8\iota_{k+6}\}=\{8\sigma_k\} \qquad \textit{for} \quad k\geqq 9.$$

(2) *Let n be an integer. Then*

$$\{n\nu_k,\ 2\nu_{k+3},\ 8\iota_{k+6}\}=\{8\sigma_k\} \qquad \textit{for} \quad k\geqq 9.$$

(3) *Let n be an integer. Then*

$$\{n\nu_k,\ 8\sigma_{k+3},\ 2\iota_{k+10}\}\ni n\zeta_k \bmod 2\zeta_k \qquad \textit{for} \quad k\geqq 10.$$

Proof. (1) Since E^∞: $\pi_{16}^9\rightarrow{}_2\pi_7^S$ is monic, we may prove the statement in the stable range. Let $n=2m$. Then $\langle 2m\nu,\ \nu,\ 8\iota\rangle$ and $\langle\nu,\ 8\iota,\ m\nu\rangle$ have a common element by (3.10) of [14], where $\langle\nu,\ 8\iota,\ m\nu\rangle\supset\langle\nu,\ 8\iota,\ \nu\rangle\circ m\iota$ and $\langle\nu,\ 8\iota,\ \nu\rangle$ contains $E^\infty\sigma'''=8\sigma$ by Lemmas 5.13 and 5.14 of [14], so we have the result.

(2) Similarly we prove the statement in the stable range. We see $\langle n\nu,\ 2\nu,\ 8\iota\rangle$ $\supset\langle 2n\nu,\ \nu,\ 8\iota\rangle\ni 8\sigma$, from which follows the result.

(3) The result is immediate from the definition of ζ_n in p. 59 of [14], namely, $\zeta_5\in\{\nu_5,\ 8\iota_8,\ E\sigma'\}_1$. Q.E.D.

Lemma 6.21. (1) *There exists an element* $[\eta_{4n-1}]$ *of* $\pi_{4n}(X_{n,2})$ *such that* $p_*[\eta_{4n-1}]=\eta_{4n-1}$ *and* $2[\eta_{4n-1}]=0$ *for all* $n\geqq 3$.

(2) *For even* $n\geqq 2$, *there exists an element* $[\nu_{4n-1}]$ *of* $\pi_{4n+2}(X_{n,2})$ *such that* $p_*[\nu_{4n-1}]=\nu_{4n-1}$ *and* $16[\nu_{4n-1}]=0$.

(3) *There exists an element* $[2\nu_{4n-1}]$ *of* $\pi_{4n+2}(X_{n,2})$ *such that* $p_*[2\nu_{4n-1}]$ $=2\nu_{4n-1}$ *and* $8[2\nu_{4n-1}]=0$ *for all* $n\geqq 2$.

(4) *There exists an element* $[E^{4n-6}\sigma''']$ *of* $\pi_{4n+6}(X_{n,2})$ *such that* $p_*[E^{4n-6}\sigma''']$ $=E^{4n-6}\sigma'''$ *and* $16[E^{4n-6}\sigma''']=0$ *for all* $n\geqq 3$.

Proof. (1) Let us consider the exact sequence

$$\pi_{4n}(S^{4n-5}) \xrightarrow{i_*} \pi_{4n}(X_{n,2}) \xrightarrow{p_*} \pi_{4n}(S^{4n-1}) \xrightarrow{\Delta} \pi_{4n-1}(S^{4n-5}).$$

We see that $\pi_{4n}(S^{4n-5})=0$, $\pi_{4n-1}(S^{4n-5})=0$ and $\pi_{4n}(S^{4n-1})=\{\eta_{4n-1}\}\cong Z_2$ for $n\geqq 3$. Then we have the result.

(2) Consider the exact sequence

$$\pi_{4n+2}(S^{4n-5}) \xrightarrow{i_*} \pi_{4n+2}(X_{n,2}) \xrightarrow{p_*} \pi_{4n+2}(S^{4n-1}) \xrightarrow{\Delta} \pi_{4n+1}(S^{4n-5}),$$

where $\pi_{4n+2}(S^{4n-1}: 2)=\{\nu_{4n-1}\}\cong Z_8$ for $n\geqq 2$, and $\pi_{4n+1}(S^{4n-5}: 2)=\{\nu^2_{4n-5}\}\cong Z_2$ for $n\geqq 3$. When $n=2$, we remark that $\pi_9(S^3: 2)=0$ and $\pi_{10}(S^3: 2)=0$. Then we have an element $[\nu_7]$ such that $p_*[\nu_7]=\nu_7$ and $8[\nu_7]=0$.

Let $n\neq 2$, then $\Delta(\nu_{4n-1})=\Delta(\iota_{4n-1})\circ\nu_{4n-2}=n\omega_{4n-5}\circ\nu_{4n-2}=0$, since $2\nu^2_{4n-5}=0$ for $n\geqq 6$ by Proposition 5.11 of [14]. Moreover, by Theorem 2.1 of [7] and Lemma 6.20, we have $8[\nu_{4n-1}]\in i_*\{\Delta(\iota_{4n-1}), \nu_{4n-2}, 8\iota_{4n+1}\}=i_*\{8\sigma_{4n-5}\}$. Then we have $16[\nu_{4n-1}]=0$.

(3) is proved similarly.

(4) We see $\Delta(E^{4n-6}\sigma''')=\Delta(\iota_{4n-1})\circ E^{4n-7}\sigma'''=n\omega_{4n-5}\circ E^{4n-7}\sigma'''=n\omega_{4n-5}\circ E^{4n-11}E^4\sigma'''=n\omega_{4n-5}\circ E^{4n-11}8\sigma_9=8n\omega_{4n-5}\circ\sigma_{4n-2}=0$. So by Theorem 2.1 of [7] we have $2[E^{4n-6}\sigma''']\in i_*\{\Delta(\iota_{4n-1}), E^{4n-7}\sigma''', 2\iota_{4n+5}\}=i_*\{n\omega_{4n-5}, 8\sigma_{4n-2}, 2\iota_{4n+5}\}\ni i_*n\zeta_{4n-5}$ mod $2i_*\zeta_{4n-5}$. It follows that $16[E^{4n-6}\sigma''']=0$. Q.E.D.

Now, applying the construction given by Proposition 2.3 to the elements in Lemma 6.21 with $\gamma=(16/m)\sigma_j$, we have the following elements:

$$[\eta_{4n-1}]^{(s)}\in\pi_{4n+8s}(X_{n,2}) \quad \text{for } s\geqq 0,$$
$$[\nu_{4n-1}]^{(s)}, [2\nu_{4n-1}]^{(s)}\in\pi_{4n+8s+2}(X_{n,2}) \quad \text{for } s\geqq 0,$$
$$[E^{4n-6}\sigma''']^{(s)}\in\pi_{4n+8s+6}(X_{n,2}) \quad \text{for } s\geqq 0.$$

Proposition 6.22. *The following elements are non-trivial in* $\pi_{4n-5+r}(X_{n,2})$:

$r=8s+4$:	$[\eta_{4n-1}]^{(s-1)}\circ\sigma_{4n+8s-8}$	$(s\geqq 1, n\geqq 3)$,
$r=8s+5$:	$[\eta_{4n-1}]^{(s)}$	$(s\geqq 0, n\geqq 3)$,
	$[\eta_{4n-1}]^{(s-1)}\circ\eta_{4n+8s-8}\circ\sigma_{4n+8s-7}$	$(s\geqq 1, n\geqq 3)$,
$r=8s+6$:	$[\eta_{4n-1}]^{(s)}\circ\eta_{4n+8s}$	$(s\geqq 0, n\geqq 3)$,
$r=8s+7$:	$[\nu_{4n-1}]^{(s)}$	$(s\geqq 0, n$: *even* $\geqq 2)$,
	$[2\nu_{4n-1}]^{(s)}$	$(s\geqq 0, n$: *odd* $\geqq 3)$,
$r=8s+11$:	$[E^{4n-6}\sigma''']^{(s)}$	$(s\geqq 0, n\geqq 3)$.

Proof. Let $\phi=E^\infty\circ p_*$. We remark that

$$d_R\phi[\eta_{4n-1}]=d_R(\eta)\neq 0,$$
$$e'_R\phi[\nu_{4n-1}]=e'_R(\nu)\equiv 1/8 \mod 1/4,$$
$$e'_R\phi[2\nu_{4n-1}]=e'_R(2\nu)\equiv 1/4 \mod 1/2,$$
$$e_C\phi[E^{4n-6}\sigma''']=e_C(8\sigma)\equiv 1/2 \mod 1.$$

Then we obtain the required results by Propositions 4.3～4.6. Q.E.D.

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