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On ideal-adic completion of noetherian rings

By

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Introduction

In commutative (noetherian) ring theory, complete local rings play many important roles. Thanks to the efforts made by Krull, Zariski, Nagata and Grothendieck, a lot of marvelous properties of complete local rings are known. Moreover, they applied the knowledge to investigate the (local) properties of general noetherian rings, using the maximal ideal-adic completion.

In particular, discovering many beautiful properties of complete local rings, Nagata successfully used them in the investigation, for example, of the finiteness problem of integral closures of noetherian domains. In this work, he found an acceptable class of noetherian rings which possess the (universal) finiteness property for integral closures. He named these rings "pseudo-geometric" (for definition, see [7, (31.A)]). We note here that he found the examples of bad local rings at the same time (cf. [8, Appendix]).

In reconstructing Nagata's work, Grothendieck noticed the importance of the informations included in formal fibres, which connect a local ring with its completion. Developing the concept of formal smoothness, he paid a special attention to the study of noetherian rings whose formal fibres are geometrically regular. He also found a new class of noetherian rings which have algebraic-geometrically reasonable properties. He called them "excellent" (for definition, see [7, (34.A)]).

Since complete local rings are proved to be always excellent, Grothendieck expected that the situation of formal fibres of noetherian rings may become better when one completes the noetherian rings in an ideal-adic topology. He asked if, for a noetherian ring A having good formal fibres and for an (arbitrary) ideal I of A, the completion A^* of A in the I-adic topology has also good formal fibres. More precisely, letting $\boldsymbol{P}$ denote a certain (ring-theoretic) condition, Grothendieck defined a $\boldsymbol{P}$-ring to be a noetherian ring whose formal fibres satisfy the condition. In this terminology, he stated the questions as follows (cf. [3, (7.4.8)]):

Question 1. Let A be a noetherian ring and I an ideal of A. If A is a $\boldsymbol{P}$-ring, is the I-adic completion A^* of A also a $\boldsymbol{P}$-ring?

More generally

Question 2. Let A be a noetherian ring and I an ideal of A. Suppose

a) A is complete and separated in the I-adic topology, and

b) A/I is a $\boldsymbol{P}$-ring.

Does it follow that A is also a $\boldsymbol{P}$-ring?

Marot [6] studied the above questions in case of the class of pseudo-geometric (=universally japanese) rings and obtained a nice answer:

Theorem. (Marot [6]) *Let A be a noetherian ring and I an ideal of A. Suppose*

a) *A is complete and separated in the I-adic topology, and*

b) *A/I is universally japanese.*

Then A is also universally japanese.

We note here that it can be generalized a little more:

Proposition. (cf. Tate [7, (31.C), Theorem 69], [9, Corollary 4]) *Let A be a noetherian domain and x a non-zero element of A. Suppose*

a) *A is complete and separated in the xA-adic topology, and*

b) *$A/\mathfrak{p}$ is a japanese ring for any $\mathfrak{p} \in \mathrm{Ass}(A/xA)$.*

Then A is also a japanese ring.

Recently, Rotthaus [10] succeeded to prove the following:

Theorem. (Rotthaus [10]) *Let A be a semi-local ring and I an ideal of A. Suppose*

a) *A is complete and separated in the I-adic topology, and*

b) *A/I is quasi-excellent.*

Then A is also quasi-excellent.

In this paper, we first study the above questions for semi-local rings in the case where the condition $\boldsymbol{P}$ is being geometrically regular (or geometrically normal, geometrically reduced). We prove:

Theorem. (cf. Marot [6], Rotthaus [10]) *Let A be a semi-local ring and I an ideal of A. Suppose*

a) *A is complete and separated in the I-adic topology, and*

b) *A/I is a G-ring (or a Z-ring, an N-ring).*

Then A is also a G-ring (or a Z-ring, an N-ring, resp.) (for definition, see (0.1)).

The proof of the above theorem, substantially due to Rotthaus [10], is given in section 3.

Section 1 consists of preliminary lemmas. Lemma (1.1) is easy, but we recognize that this lemma is the crucial first step toward the answer. Lemma (1.2), originally obtained by Rotthaus (cf. [10, Lemma 2]), is a direct consequence of Lemma (1.1). Lemma (1.3), which gives an inequality on the depth (=la profondeur) for the chain of prime ideals, is well-known.

Section 2 is devoted to understanding André's Theorem [1]. We could say that this is one of the most brilliant theorems about complete local rings. Some direct corollaries of André's Theorem are noted (e.g. Proposition (2.4), Proposition (2.5)). We also emphasize that these results of this section play a key role in the proof of the theorem.

The efforts to generalize the theorem in the case of general noetherian rings are described in section 4 and some sufficient conditions are proposed. We omit their proofs, because all of our results are derived immediately from the theorem or from our proof of it.

Regretably, the above questions are not affirmative in general. An example, which gives a negative answer to Question 1, is presented in section 5 (cf. Example (5.3)). We note that this gives also a new example of a two-dimensional local domain which has non-noetherian over-rings between the domain and its derived normal ring (cf. Nagata [8, p. 207, Example 4]).

Moreover, in section 6, we show that the same method gives further examples of bad local rings. First we construct an elementary example of a two-dimensional normal local ring which is analytically ramified (cf. Nagata [8, p. 208, Example 6]). Using this, we finally give a new example of a three-dimensional local domain whose derived normal ring is not noetherian (cf. Nagata [8, p. 207, Example 5]).

We conclude this introduction with the following fascinating (still-open) problems (cf. [7, (34.D)]):

Problem 1. Let A be a noetherian ring and I an ideal of A. If A is (quasi-) excellent, is the I-adic completion A^* of A also (quasi-) excellent?

Problem 2. Let A be a noetherian ring and I an ideal of A. Suppose

a) A is complete and separated in the I-adic topology, and

b) A/I is (quasi-) excellent.

Does it follow that A is also (quasi-) excellent?

0. Notation and terminology

In this article, we mean by a ring a commutative ring with identity. By a semi-local ring $(A, \mathfrak{m}_1,\dots, \mathfrak{m}_r)$, we understand that A is a noetherian ring with only a finite number of maximal ideals $\mathfrak{m}_1,\dots, \mathfrak{m}_r$. A local ring is a semi-local ring with only one maximal ideal. For a semi-local ring $(A, \mathfrak{m}_1,\dots, \mathfrak{m}_r)$, $\hat{A}$ (or $A^\wedge$) means the completion of A in the $\mathfrak{m}$-adic topology, where $\mathfrak{m}=\mathfrak{m}_1\cdots\mathfrak{m}_r$. When $\mathfrak{p}$ is a prime ideal of a ring A, the field of quotients of the integral domain $A/\mathfrak{p}$ is denoted by $k(\mathfrak{p})$. The nil-radical of a ring A is denoted by $\mathrm{nil}(A)$.

Let A be a noetherian ring. When the subset $\mathrm{Reg}(A)$ of regular points of the set $\mathrm{Spec}(A)$ (i.e. $\mathrm{Reg}(A)=$ the set of prime ideals $\mathfrak{p}$ of A such that $A_\mathfrak{p}$ is regular) is open in Zariski topology, we denote by $\mathrm{sing}(A)$ the maximal defining ideal of the closed set $\mathrm{Sing}(A)=\mathrm{Spec}(A)-\mathrm{Reg}(A)$, i.e. $\mathrm{sing}(A)=\bigcap_{\mathfrak{p}\in\mathrm{Sing}(A)}\mathfrak{p}$. In the same way, when the set $\mathrm{Nor}(A)$ of normal points of $\mathrm{Spec}(A)$ is open, we denote by non-nor(A) the maximal defining ideal of the closed set non-Nor$(A)=\mathrm{Spec}(A)-\mathrm{Nor}(A)$. Note that nil$(A)$, sing$(A)$ and non-nor(A), when they are well-defined, are all semi-prime ideals.

Let k be a field and A a noetherian k-algebra. We say that A is geometrically regular (or geometrically normal, geometrically reduced) over k if the ring $A\otimes_k k'$ is regular (or normal, reduced, resp.) for every finite extension field k' of k. Let ψ be a ring-homomorphism of a noetherian ring A to a noetherian ring B. We say that ψ is regular (or normal, reduced) if ψ is flat and, for any prime ideal $\mathfrak{p}$ of A, the induced map $\psi\otimes k(\mathfrak{p})$ of $k(\mathfrak{p})$ to $B\otimes_A k(\mathfrak{p})$ makes the $k(\mathfrak{p})$-algebra $B\otimes_A k(\mathfrak{p})$ geometrically regular (or geometrically normal, geometrically reduced, resp.) over $k(\mathfrak{p})$.

Definition. (0.1) A noetherian ring A is called a G-ring (or a Z-ring, an N-ring) if, for any $\mathfrak{p}\in \mathrm{Spec}(A)$, the canonical map $A_{\mathfrak{p}}\to A_{\mathfrak{p}}^{\wedge}$ of a local ring $A_{\mathfrak{p}}$ to its completion is regular (or normal, reduced, resp.).

Remarks. (0.2.1) If A is a G-ring (or a Z-ring, an N-ring), then any its localization and any finite A-algebra are also G-rings (or Z-rings, N-rings, resp.).

(0.2.2) (cf. [3, (7.4.4)]) A noetherian ring A is a G-ring (or a Z-ring, an N-ring) if, for every maximal ideal $\mathfrak{m}$ of A, the canonical map $A_{\mathfrak{m}}\to A_{\mathfrak{m}}^{\wedge}$ of a local ring $A_{\mathfrak{m}}$ to its completion is regular (or normal, reduced, resp.).

(0.2.3) (cf. [7, (30.D), Theorem 68]) A complete semi-local ring is a G-ring.

(0.2.4) (cf. [7, (32.A), (32.C), (33.D)]) If a semi-local ring A is a G-ring (or an N-ring), then Reg(A) (or Nor(A), resp.) is open in Spec(A).

1. Lemmas

Lemma. (1.1) (cf. [8, Proof of (30.1)]) *Let A be a noetherian ring, I an ideal of A and $\{\mathfrak{a}_i\}_{i\in \mathbf{N}}$ a descending sequence of ideals of A. Suppose*

(1.1.1) *A is complete and separated in the I-adic topology,*

(1.1.2) *there is an $r>0$ such that $\mathfrak{a}_m\not\subset I^r$ for any $m>0$, and*

(1.1.3) *for any $n>0$, there exists an integer $t(n)$ such that $\mathfrak{a}_{t(n)}+I^n=\mathfrak{a}_m+I^n$ for any $m>t(n)$.*

Then $\bigcap_n \mathfrak{a}_n\neq(0)$.

Lemma. (1.2) (cf. Rotthaus [10, Lemma 2]) *With A and I as above, let B be an A-algebra, $\mathfrak{b}$ an ideal of B and $\mathfrak{a}=\mathfrak{b}\cap A$. Put $\mathfrak{b}_n=\mathfrak{b}+I^nB$ and $\mathfrak{a}_n=\mathfrak{b}_n\cap A$ ($n\in \mathbf{N}$). Suppose*

(1.2.1) *A is complete and separated in the I-adic topology,*

(1.2.2) *$\mathfrak{b}_n=\mathfrak{a}_nB$ for any $n>0$,*

(1.2.3) *$B/\mathfrak{a}_nB$ is faithfully flat over $A/\mathfrak{a}_n$ for any $n>0$, and*

(1.2.4) $\bigcap_n (\mathfrak{a}B+I^nB)=\mathfrak{a}B$ *and* $\bigcap_n \mathfrak{b}_n=\mathfrak{b}$.

Then $\mathfrak{a}B=\mathfrak{b}$.

Proof. Since $\mathfrak{a}B \cap A=\mathfrak{a}$, we may assume $\mathfrak{a}=(0)$. Suppose $\mathfrak{b}\neq(0)$. Then there is an $r>0$ such that $\mathfrak{b}\not\subset I^rB$ (cf. (1.2.4)). Hence, $\mathfrak{a}_m\not\subset I^r$ for any $m>0$ (cf. (1.2.2)). Moreover, as $(\mathfrak{a}_m+I^n)B=\mathfrak{a}_mB+I^nB=\mathfrak{b}_m+I^nB=\mathfrak{b}_n=\mathfrak{a}_nB$ for any $m>n$, we have $\mathfrak{a}_m+I^n=\mathfrak{a}_n$ (cf. (1.2.3)). Therefore, $\mathfrak{a}=\bigcap_n \mathfrak{a}_n\neq(0)$ by (1.1). Contradiction.

Lemma. (1.3) (cf. [5, Theorem 134, Theorem 127]) *Let* $(A, \mathfrak{m})$ *be a local ring,* $\mathfrak{p}$ *a prime ideal of* A *and* a *an element of* $\mathfrak{m}$. *Let* $J=\mathfrak{p}+aA$. *Suppose* J *is* $\mathfrak{m}$*-primary. Then*

(1.3.1) $\operatorname{prof} A\leq \operatorname{prof} A_{\mathfrak{p}}+1$.

2. André's Theorem

André's Theorem. (2.1) (André [1]) *Let* $(A, \mathfrak{m})$ *and* $(B, \mathfrak{n})$ *be local rings and* ψ *a local homomorphism of* A *to* B. *Suppose*

(2.1.1) ψ *is formally smooth (with respect to the canonical topologies), and*

(2.1.2) A *is a G-ring.*

Then ψ *is regular.*

Remark. (2.2) (cf. [3, $(0_{\text{IV}}$ 19.7.1), $(0_{\text{IV}}$ 22.5.8)]) The condition (2.1.1) is equivalent to the following two-conditions:

(2.2.1) ψ is flat, and

(2.2.2) $\bar{\psi}(=\psi\otimes k(\mathfrak{m}))$ is regular.

Proposition. (2.3) (cf. [7, (33.E), Lemma 3]) *With notation as in* (2.1), *the following are equivalent to each other*:

(2.3.1) ψ *is regular (or normal, reduced).*

(2.3.2) ψ *is flat and, for any prime* $\mathfrak{p}$ *of* A, $\psi\otimes k(\mathfrak{p})$ *makes* $B\otimes_A k(\mathfrak{p})$ *geometrically regular (or geometrically normal, geometrically reduced, resp.) over* $k(\mathfrak{p})$.

(2.3.3) ψ *is flat and, for any finite A-algebra C which is an integral domain with field of quotients L,* $B\otimes_A L$ *is regular (or normal, reduced, resp.).*

Proposition. (2.4) *With notation as in* (2.1) *and* (2.2). *Suppose*

(2.4.1) ψ *is flat,*

(2.4.2) $\bar{\psi}$ *is normal (or reduced), and*

(2.4.3) A *is a Z-ring, (or an N-ring, resp.).*

Then ψ *is normal* (*or reduced, resp.*).

Proof. Let B^* be the $\mathfrak{m}B$-adic completion of B, and $\hat{\psi}$ be the induced map of $\hat{A}$ to B^*. Let ρ (or τ) be the canonical map $A \to \hat{A}$ (or $B \to B^*$, resp.). We have the following commutative diagram:

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\hat{\psi}} & B^* \\ {\scriptstyle\rho}\uparrow & & \uparrow{\scriptstyle\tau} \\ A & \xrightarrow[\psi]{} & B \end{array}$$

Hence, as ρ is normal (or reduced, resp.) and τ is faithfully flat, we may assume

(2.4.4) A is complete (cf. [3, (7.3.4), (7.3.8)]).

By noetherian induction, we may also assume

(2.4.5) A is an integral domain with field of quotients K and, for any non-zero prime $\mathfrak{p}$ of A, $\psi \otimes k(\mathfrak{p})$ makes $B \otimes_A k(\mathfrak{p})$ geometrically normal (or geometrically reduced, resp.) over $k(\mathfrak{p})$.

Let C be an integral domain which is a finite A-algebra. Then C is also a Z-ring (or an N-ring, resp.) (cf. (0.2.1)). Moreover, the induced map $\psi \otimes C$ of C to $B \otimes_A C$ satisfies (2.4.1), (2.4.2) and (2.4.5) (with respect to the pairs of the corresponding maximal ideals). Hence, in order to prove (2.4), it is sufficient to show that $B \otimes_A K$ is normal (or reduced, resp.) (cf. (2.3)). Let $\bar{A}$ be the derived normal ring of A. Then $\bar{A}$ is a finite A-algebra (cf. [8, (32.1)]). Hence the same reasoning as above allows us to assume

(2.4.6) A is normal.

Under these assumptions, we claim

(2.4.7) B is normal (or reduced, resp.), i.e. B satisfies (S_2) and (R_1) (or (S_1) and (R_0), resp.).

Before proving the above claim, we fix some notation. Let P be a prime ideal of B, and $\mathfrak{p} = P \cap A$. Let a be a non-zero element of $\mathfrak{m}$, Q a minimal prime over-ideal of $P + aB$, and $\mathfrak{q} = Q \cap A$. (Note that a may be contained in P. In that case we understand $Q = P$.) Then $\mathfrak{q}$ is a non-zero prime of A. We denote by ψ_Q the induced map of $A_{\mathfrak{q}}$ to B_Q and understand $\bar{\psi}_Q = \psi_Q \otimes k(\mathfrak{q})$.

Proof of (2.4.7). Normal case: Suppose $\operatorname{prof} B_P \leq 1$. Then $\operatorname{prof} B_Q \leq 2$ by (1.3.1). Moreover, $\operatorname{prof} B_Q/\mathfrak{q}B_Q \leq 1$, for $\operatorname{prof} A_{\mathfrak{q}} \geq 1$ (cf. [7, (21.B), Theorem 50]). Hence, $\bar{\psi}_Q$ is regular, i.e. ψ_Q is formally smooth (cf. (2.4.5), (2.2)). Consequently, as $A_{\mathfrak{q}}$ is a G-ring, ψ_Q is also regular by André's Theorem (cf. (2.4.4), (0.2.3)). On the other hand, $A_{\mathfrak{p}}$ is regular, for $\operatorname{prof} A_{\mathfrak{p}} \leq 1$ (cf. (2.4.6)). Therefore, B_P is also regular (cf. [7, (33.B), Lemma 2]).

Reduced case: Suppose $\operatorname{prof} B_P = 0$. Then $\operatorname{prof} B_Q \leq 1$ by (1.3.1). Moreover, $\operatorname{prof} B_Q/\mathfrak{q}B_Q = 0$. Hence, $\bar{\psi}_Q$ is regular. Consequently, ψ_Q is also regular by André's

Theorem. Therefore, as $A_{\mathfrak{p}}=K$, B_P is regular. q.e.d.

Proposition. (2.5) *Let ψ be a ring-homomorphism of a noetherian ring A to another noetherian ring B and, for each prime ideal P of B, let ψ_P be the induced map of $A_{(P\cap A)}$ to B_P. Suppose, for every maximal ideal Q of B with $\mathfrak{q}=Q\cap A$, that*

(2.5.1) *ψ_Q is flat,*

(2.5.2) *$\bar{\psi}_Q(=\psi_Q\otimes k(\mathfrak{q}))$ is regular (or normal, reduced), and*

(2.5.3) *$A_{\mathfrak{q}}$ is a G-ring (or a Z-ring, an N-ring, resp.).*

Then ψ is regular (or normal, reduced, resp.).

3. Proof of the theorem

Since a complete (semi-)local ring is a G-ring (cf. (0.2.3)), by induction on dim A/I, we may assume

c) for any non-maximal prime $\mathfrak{p}$ of A which contains I, the $I_{\mathfrak{p}}$-adic completion $A_{\mathfrak{p}}^*$ of $A_{\mathfrak{p}}$ is a G-ring (or a Z-ring, an N-ring, resp.).

By noetherian induction, we may also assume

d) A is a semi-local domain with field of quotients K and, for any non-zero ideal $\mathfrak{a}$ of A, $A/\mathfrak{a}$ is a G-ring (or a Z-ring, an N-ring, resp.).

Hence it remains to be proved that $\hat{A}\otimes_A K$ is geometrically regular (or geometrically normal, geometrically reduced, resp.) over K. Moreover, by the same argument as in the proof of (2.4) it is sufficient to prove that $\hat{A}\otimes_A K$ is regular (or normal, reduced, resp.).

Let $\mathfrak{b}=\mathrm{sing}(\hat{A})$ (or $\mathfrak{b}=\text{non-nor}(\hat{A})$, $\mathfrak{b}=\mathrm{nil}(\hat{A})$, resp.), $\mathfrak{b}_n=\mathfrak{b}+I^n\hat{A}$ and $\mathfrak{a}_n=\mathfrak{b}_n\cap A$ for any $n>0$ (cf. (0.2.3), (0.2.4)). We first claim

(3.1) $\mathfrak{a}_n\hat{A}=\mathfrak{b}_n$ for any $n>0$.

Proof of (3.1). Let $\mathfrak{b}_n=Q_1\cap\cdots\cap Q_s$ be a primary decomposition of $\mathfrak{b}_n$ (with P_i-primary ideal Q_i $(i=1, 2,\dots, s)$). Set $\mathfrak{q}_i=Q_i\cap A$ and $\mathfrak{p}_i=P_i\cap A$. Then $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary and $\mathfrak{a}_n=\mathfrak{q}_i\cap\cdots\cap\mathfrak{q}_s$ (which may not be an irredundant decomposition). We show

(3.2) If P_i is maximal, then $\mathfrak{q}_i\hat{A}=Q_i$ (this is clear).

(3.3) If $\mathfrak{p}$ is a non-maximal prime of A which contains I, then $\mathfrak{a}_n\hat{A}_T=(\mathfrak{b}_n)_T$, where $T=A-\mathfrak{p}$.

Proof of (3.3). Let $A_{\mathfrak{p}}^*$ be the $I_{\mathfrak{p}}$-adic completion of $A_{\mathfrak{p}}$ and $(\hat{A}_T)^*$ the $I\hat{A}_T$-adic completion of $\hat{A}_T$. Let ρ (or τ) be the canonical map of A to $\hat{A}$ (or of $\hat{A}_T$ to $(\hat{A}_T)^*$, resp.) and let ρ_T (or ρ_T^*) be the induced map of $A_{\mathfrak{p}}$ to $\hat{A}_T$ (or of $A_{\mathfrak{p}}^*$ to $(\hat{A}_T)^*$, resp.). We have the following commutative diagram:

$$\begin{array}{ccccc} \hat{A} & \longrightarrow & \hat{A}_T & \xrightarrow{\tau} & (\hat{A}_T)^* \\ {\scriptstyle\rho}\uparrow & & \uparrow{\scriptstyle\rho_T} & & \uparrow{\scriptstyle\rho_T^*} \\ A & \longrightarrow & A_{\mathfrak{p}} & \longrightarrow & A_{\mathfrak{p}}^* \end{array}$$

Since the induced map $\bar{\rho}_T^*(=\rho_T^*\otimes(A_{\mathfrak{p}}^*/IA_{\mathfrak{p}}^*))$ of $A_{\mathfrak{p}}^*/IA_{\mathfrak{p}}^*(=(A/I)_{\mathfrak{p}})$ to $(\hat{A}_T)^*/I(\hat{A}_T)^*$ $(=(A/I)_T^{\wedge})$ is regular (or normal, reduced, resp.) by assumption b) and $A_{\mathfrak{p}}^*$ is a G-ring (or a Z-ring, an N-ring, resp.) by assumption c), ρ_T^* is regular (or normal, reduced, resp.) (cf. (2.4), (2.5)).

Let $\mathfrak{c}=\mathrm{sing}(A_{\mathfrak{p}}^*)$ (or $\mathfrak{c}=\text{non-nor}(A_{\mathfrak{p}}^*)$, $\mathfrak{c}=\mathrm{nil}(A_{\mathfrak{p}}^*)$, resp.) and let $\mathfrak{c}_n=\mathfrak{c}+I^nA_{\mathfrak{p}}^*$ (cf. (0.2.4)). Then, as ρ_T^* is regular (or normal, reduced, resp.) and τ is regular, we have $\mathfrak{b}(\hat{A}_T)^*=\mathfrak{c}(\hat{A}_T)^*$ and, consequently, $\mathfrak{b}_n(\hat{A}_T)^*=\mathfrak{c}_n(\hat{A}_T)^*$ (for any $n>0$) (cf. [7, (31.D), Theorem 70], [7, (33.B), Lemma 2]). Hence $(\mathfrak{a}_n)_{\mathfrak{p}}=\mathfrak{c}_n\cap A_{\mathfrak{p}}$.

On the other hand, since $\mathfrak{a}_n$ contains I^n, we see $\mathfrak{a}_nA_{\mathfrak{p}}^*=\mathfrak{c}_n$. Hence, $\mathfrak{a}_n(\hat{A}_T)^*$ $=\mathfrak{b}_n(\hat{A}_T)^*$. Therefore, $\mathfrak{a}_n\hat{A}_T=(\mathfrak{b}_n)_T$ (Thus (3.3) is proved and this also completes the proof of (3.1)).

Final step of the proof. By (3.1), we have $\mathfrak{b}=(\mathfrak{b}\cap A)\hat{A}$ (cf. (1.2)). Hence ${}^a\rho^{-1}(\mathrm{Reg}(A))=\mathrm{Reg}(\hat{A})$ (or ${}^a\rho^{-1}(\mathrm{Nor}(A))=\mathrm{Nor}(\hat{A})$, $\mathrm{nil}(A)\hat{A}=\mathrm{nil}(\hat{A})$, resp.), for $\mathfrak{b}$ is semi-prime. Therefore, $\hat{A}\otimes_A K$ is regular (or normal, reduced, resp.). q.e.d.

4. Some application of the theorem

Proposition. (4.1) *Let A be a noetherian ring and I an ideal of A. Suppose*

a) *A is complete and separated in the I-adic topology, and*

b) *A/I is a G-ring (or a Z-ring, an N-ring).*

Then the following are equivalent to each other:

(4.1.1) *A is a G-ring (or a Z-ring, an N-ring, resp.).*

(4.1.2) *For any maximal ideal $\mathfrak{m}$ of A, the canonical map $\rho_{\mathfrak{m}}$ of a local ring $A_{\mathfrak{m}}$ to the $I_{\mathfrak{m}}$-adic completion $A_{\mathfrak{m}}^*$ of $A_{\mathfrak{m}}$ is regular (or normal, reduced, resp.).*

(4.1.3) *There exists a faithfully flat ring-homomorphism ψ of A to an A-algebra B, which is a G-ring (or a Z-ring, an N-ring, resp.), such that the induced map $\bar{\psi}(=\psi\otimes A/I)$ of A/I to B/IB is regular (or normal, reduced, resp.).*

(4.2) (cf. [2, (0_{I} 7.6.15), (0_{I} 7.6.18)]) Let A be a noetherian ring, J an ideal of A and S a multiplicatively closed set of A such that $J\cap S=\phi$. Let A_S^* be the J_S-adic completion of A_S. We denote by $A_{\{S\}}$ the inductive limit $\varinjlim_{f\in S} A_{\{f\}}$, where $A_{\{f\}}$ is the J_f-adic completion of A_f. Then

(4.2.1) A_S^* is faithfully flat over $A_{\{S\}}$, and

(4.2.2) $A_{\{S\}}/J^{\nu}A_{\{S\}} \xrightarrow{\sim} A_S^*/J^{\nu}A_S^*$ for any $\nu>0$.

Hence

(4.2.3) $A_{\{S\}}$ is noetherian,

(4.2.4) $JA_{\{S\}}$ is contained in the (Jacobson) radical of $A_{\{S\}}$, and

(4.2.5) the $JA_{\{S\}}$-adic completion of $A_{\{S\}}$ is identified with A_S^*.

When $S=A-\mathfrak{p}$ with prime ideal $\mathfrak{p}$ of A, we adopt the notation $A_{\{\mathfrak{p}\}}$ instead of $A_{\{S\}}$.

Proposition. (4.3) *With notation as above, let A be a noetherian ring and I an ideal of A. Suppose*

a) *A is complete and separated in the I-adic topology,*

b) *A/I is a G-ring (or a Z-ring, an N-ring), and*

c) *for any maximal ideal $\mathfrak{m}$ of A and for any ideal J of $A_{\mathfrak{m}}$ which contains $IA_{\mathfrak{m}}$, $A_{\{\mathfrak{m}\}}$ (with respect to the J-adic topology) is a G-ring (or a Z-ring, an N-ring, resp.).*

Then A is also a G-ring (or a Z-ring, an N-ring, resp.).

5. Examples

(5.1) (cf. Hochster [4, Proposition 1, Example 1]) Let k be a field of characteristic 2. Letting X_i be indeterminates, we set $R_i=k[X_i^2, X_i^3]$ with (fixed) maximal ideal $\mathfrak{p}_i=(X_i^2, X_i^3)$ $(i=1, 2,\ldots)$. Put $R'=\bigotimes_{k\,i} R_i$ and $S=R'-\bigcup_i \mathfrak{p}_i R'$. Let $R=R'_S$. Then

(5.1.0) maximal ideals $\mathfrak{q}_i$ of R are in one-to-one correspondence with N via $i\mapsto \mathfrak{q}_i=\mathfrak{p}_i R$,

(5.1.1) $R_{\mathfrak{q}_i}=(K_i\otimes_k R_i)_{\tilde{\mathfrak{q}}_i}$, where K_i is an extension field of k and $\tilde{\mathfrak{q}}_i=\mathfrak{q}_i(K_i\otimes_k R_i)$ (Note that $\mathfrak{p}_i$ is an absolutely prime ideal), and

(5.1.2) any non-zero element of R is contained in only a finite number of maximal ideals.

Hence

(5.1.3) R is a one-dimensional noetherian domain with field of quotients $K=k(X_1, X_2,\ldots)$, and

(5.1.4) R is a G-ring (cf. (5.1.1)).

Let $\bar{R}$ be the derived normal ring of R. Then

(5.1.5) $\bar{R}=R[X_1, X_2,\ldots]$, and

(5.1.6) the set $\bar{R}^2=\{r^2 \mid r\in \bar{R}\}$ is contained in R.

Moreover, for any non-zero prime $\mathfrak{q}_i$ of R,

(5.1.7) $R_{\mathfrak{q}_i}$ is not normal, but its derived normal ring $(\overline{R_{\mathfrak{q}_i}})$ is a finite $R_{\mathfrak{q}_i}$-module (cf. (5.1.1), (5.1.4)).

(5.2) Letting T and W be two indeterminates, we set $B=R[[T]]$, $C=\bar{R}[[T]]$,

$\omega=\sum_{j=1}^{\infty} X_jT^j \in C, f(W)=W^2+\omega^2 \in B[W]$ and $g(W)=W+\omega \in C[W]$. Then

(5.2.1) $f(W)=g(W)^2$ in $C[W]$ and the set $C[W]^2=\{h^2 \mid h \in C[W]\}$ is contained in $B[W]$ (cf. (5.1.6)).

We claim

(5.2.2) $f(W)$ is a prime element in $B[W]$.

Proof of (5.2.2). Since $C \xrightarrow{\sim} C[W]/gC[W]$ (this is clear), $g(W)$ is a prime element in $C[W]$. Let $Q=gC[W] \cap B[W]$ (a prime ideal of $B[W]$). Then the set $\{q^2 \mid q \in Q\}$ is contained in $fB[W]$ (cf. (5.2.1)). Hence, $Q^{\nu} \subset fB[W]$ for some sufficiently large $\nu>0$. Consequently, Q is the only one minimal prime ideal of the principal ideal $fB[W]$. Moreover, as $B[W]$ is (locally) Cohen-Macaulay, Q is the only one associated prime ideal of $fB[W]$ (cf. (5.1.3), [8, (25.6)]). Therefore, to get the claim, it is sufficient to prove

(5.2.3) $f(W)$ is a prime element in $B[W]_Q$.

Let L be the field of quotients of B. Then, as $Q \cap B=(0)$, (5.2.3) is equivalent to

(5.2.4) $f(W)$ is irreducible in $L[W]$.

Proof of (5.2.4). Suppose $f(W)$ is reducible in $L[W]$, i.e. $f(W)=(W+\omega)^2$ in $L[W]$. There exist α, $\beta \in B$ such that

(5.2.5) $$\omega=\frac{\beta}{\alpha}, \quad \text{i.e.} \quad \alpha\omega=\beta.$$

Let $\alpha=\sum_{m=0}^{\infty} a_mT^m$ and $\beta=\sum_{n=0}^{\infty} b_nT^n$, where a_m, $b_n \in R$. We compare the coefficients of T^n in (5.2.5). Then

(5.2.6) $$\sum_{m+j=n} a_mX_j=b_n \qquad \text{for any} \quad n>0.$$

Hence, letting $a=a_{m_0}$ where $m_0=\min\{m \mid a_m \neq 0\}$, we have

(5.2.7) $$a^jX_j \in R \qquad \text{for any} \quad j>0.$$

Consequently

(5.2.8) $$\bar{R}=R[X_1, X_2,\dots, X_j,\dots] \subset R_a.$$

This is a contradiction (cf. (5.1.2), (5.1.7)). Thus (5.2.4) is proved and this completes the proof of (5.2.2).

Example. (5.3) With notation as above, let $A=R[T]$ and $I=TA$. Then

(5.3.1) A is a two-dimensional G-ring and B is the I-adic completion of A (cf. [7, (33.G), Theorem 77]).

We claim

(5.3.2) B is not an N-ring.

Proof of (5.3.2). Take a maximal ideal $\mathfrak{m}$ of B and let $\mathfrak{q}=\mathfrak{m}\cap R$ (a maximal ideal of R, say $\mathfrak{q}_i$). Let $B_{\mathfrak{m}}^*$ be the $IB_{\mathfrak{m}}$-adic completion of $B_{\mathfrak{m}}$. Then $B_{\mathfrak{m}}^*=R_{\mathfrak{q}}[[T]]$. Let $\mathfrak{n}$ be the corresponding maximal ideal of C and $C_{\mathfrak{n}}^*$ the $IC_{\mathfrak{n}}$-adic completion of $C_{\mathfrak{n}}$. Then $C_{\mathfrak{n}}^*=\bar{R}_{\mathfrak{q}}[[T]]=B_{\mathfrak{m}}^*\otimes_R\bar{R}$, for $\bar{R}_{\mathfrak{q}}$ is a finite $R_{\mathfrak{q}}$-module. Hence

(5.3.3) $$C_{\mathfrak{n}}^*\otimes_R K=B_{\mathfrak{m}}^*\otimes_R K.$$

Put $B[w]=B[W]/fB[W]\hookrightarrow C$ and $M=\mathfrak{n}\cap B[w]$. Then the $IB[w]_M$-adic completion $(B[w]_M)^*$ of a local domain $B[w]_M$ is isomorphic to $B_{\mathfrak{m}}^*[W]/fB_{\mathfrak{m}}^*[W]$. Hence $(B[w]_M)^*\otimes_R K=(C_{\mathfrak{n}}^*\otimes_R K)[W]/f(C_{\mathfrak{n}}^*\otimes_R K)[W]$. Moreover, we have already seen that $f(W)=g(W)^2$ in $C[W]$. Therefore, $(B[w]_M)^*$ is not reduced. q.e.d.

Remark. (5.4) With notation as above, we see

a) B is complete and separated in the IB-adic topology,

b) B/IB is a G-ring, and

c) B is a two-dimensional noetherian domain.

Hence $B/\mathfrak{b}$ is a G-ring for any non-zero ideal $\mathfrak{b}$ of B.

Remark. (5.5) With notation as above, the argument in (4.3) shows that if $B_{\{\mathfrak{m}\}}$ (with respect to the I-adic topology) is a G-ring (or a Z-ring, an N-ring), then B is also a G-ring (or a Z-ring, an N-ring, resp.). Therefore, A gives also a negative answer to [3, (7.4.8), C]. Moreover, we see that the following question is not affirmative in general:

Question. Let B be a noetherian ring, I an ideal of B and S a multiplicatively closed set of B such that $I\cap S=\phi$. Let B_S^* be the I_S-adic completion of B_S. Suppose

a) B is complete and separated in the I-adic topology, and

b) B/I is a G-ring (or a Z-ring, an N-ring).

Then, are the canonical maps $\rho_S\colon B_S\to B_S^*$ and $\tilde{\rho}_S\colon B_{\{S\}}\to B_S^*$ regular (or normal, reduced, resp.)?

(5.6) With notation as above, let a be a non-zero element of $MB[w]_M$ and $(B[w]_M)^{**}$ the $aB[w]_M$-adic completion of $B[w]_M$. Then

(5.6.1) $(B[w]_M)^{**}$ is not reduced.

Proof of (5.6.1). Suppose $(B[w]_M)^{**}$ is reduced. Then, as $B[w]_M/aB[w]_M$ is a G-ring, $(B[w]_M)^{**}$ is also a (reduced) G-ring by Theorem (cf. (5.4)). Hence, $B[w]_M$ is analytically unramified (cf. [3, (7.3.17)]). Contradiction (cf. (5.3.2)). q.e.d.

Let P be a height-one prime ideal of $B[w]_M$. Since the regular local ring $C_{\mathfrak{n}}$ is integral over $B[w]_M$, there exists a prime element π of $C_{\mathfrak{n}}$ such that $\pi C_{\mathfrak{n}}\cap B[w]_M=P$. Let $a=\pi^2\in B[w]_M$ and $(B[w]_M)^{**}$ the $aB[w]_M$-adic completion of $B[w]_M$. Then, as $aB[w]_M$ is P-primary, we have a canonical injection $(B[w]_M)^{**}\hookrightarrow(B[w]_P)^\wedge$. Hence

(5.6.2) $(B[w]_P)^{\wedge}$ is not reduced (cf. (5.6.1)).

Consequently, the derived normal ring $(\overline{B[w]_P})$ of the one-dimensional local domain $B[w]_P$ is not a finite $B[w]_P$-module.

(5.7) With notation as above, let $(\overline{B[w]_M})$ be the derived normal ring of $B[w]_M$ and $(\overline{B[w]_M})^*$ the $I(\overline{B[w]_M})$-adic completion of $(\overline{B[w]_M})$. Then

(5.7.1) $$(\overline{B[w]_M})^* = C_{\mathfrak{n}}^*.$$

Hence $(\overline{B[w]_M})^*$ is a regular local ring.

Proof of (5.7.1). First we note that, as $B[w]_M$ is two-dimensional, $(\overline{B[w]_M})$ is noetherian (cf. [8, (33.12)]). Since $C_{\mathfrak{n}}$ is integral over the normal domain $(\overline{B[w]_M})$, we see

(5.7.2) $$T^{\nu}C_{\mathfrak{n}} \cap (\overline{B[w]_M}) = T^{\nu}(\overline{B[w]_M}) \quad \text{for any} \quad \nu>0.$$

Moreover, there exist canonical injections

(5.7,3) $$\bar{R}_{\mathfrak{q}} \hookrightarrow (\overline{B[w]_M})/T(\overline{B[w]_M}) \hookrightarrow C_{\mathfrak{n}}/TC_{\mathfrak{n}} = \bar{R}_{\mathfrak{q}}.$$

Thus we have an isomorphism

(5.7.4) $(\overline{B[w]_M})/T^{\nu}(\overline{B[w]_M}) \xrightarrow{\sim} C_{\mathfrak{n}}/T^{\nu}C_{\mathfrak{n}}$ for any $\nu>0$. (This completes the proof of (5.7.1))

Example. (5.8) (cf. Nagata [8, p. 207, Example 4]) With notation as above, let a be a non-zero element of $MB[w]_M$ and $D=(\overline{B[w]_M}) \cap (B[w]_M)[1/a]$ (=the integral closure of $B[w]_M$ in $(B[w]_M)[1/a]$). Then

(5.8.1) D is not noetherian.

This gives an example of a two-dimensional local domain which has (an infinite number of) non-noetherian (quasi-local) over-rings between the domain and its derived normal ring.

Proof of (5.8.1). Suppose D is noetherian. Since $a^{\nu}(\overline{B[w]_M}) \cap D = a^{\nu}D$, we have a canonical injection

(5.8.2) $$D/a^{\nu}D \hookrightarrow (\overline{B[w]_M})/a^{\nu}(\overline{B[w]_M}) \quad \text{for any} \quad \nu>0.$$

Then the aD-adic completion D^{**} of D is reduced (cf. (5.7)).

Let Q' be a prime ideal of D which contains aD, $\bar{Q}$ the prime ideal of $(\overline{B[w]_M})$ such that $\bar{Q} \cap D = Q'$, and let $Q = Q' \cap B[w]_M$. Note that $k(\bar{Q})$ is a finite (algebraic) extension of $k(Q)$ (cf. [8, (33.10)]). Then D/Q' is a finite $(B[w]_M/Q)$-module, for $B[w]_M/Q$ is a G-ring (cf. (5.4)). Hence D/Q' is also a G-ring (cf. (0.2.1)). This means that D/aD is a G-ring. Therefore, D^{**} is a (reduced) G-ring by Theorem. Hence D is analytically unramified. Consequently, for any prime ideal P of $D[1/a]$ $(=(B[w]_M)[1/a])$, D_P $(=B[w]_P)$ is also analytically unramified (cf. [8, (36.8)]). This is a contradiction (cf. (5.6.2)). q.e.d.

6. Further examples

(6.1) We make a minor change of notation. Let R be the same as in (5.1). We use Y_i, Z_j for X_{2i-1}, X_{2j} $(i, j=1, 2,\ldots)$. Letting T, U and W be three indeterminates, we set $B=R[[T, U]]$, $C=\bar{R}[[T, U]]$, $\omega_1=\sum_{i=1}^{\infty} Y_i T^i \in C$, $\omega_2=\sum_{j=1}^{\infty} Z_j U^j \in C$, $\omega=\omega_1+\omega_2$. Then

(6.1.0) B is a three-dimensional noetherian domain and C is a (noetherian) regular domain, where the set $C^2=\{y^2 \mid y\in C\}$ is contained in B.

Let L be the field of quotients of B. Then a similar argument as in the proof of (5.2.4) shows

(6.1.1) $$\omega \notin L.$$

Let $P=(T, U)B$ (a prime ideal of B) and $B_P^{\wedge}$ the completion of B_P. Then

(6.1.2) B_P is a two-dimensional regular local ring, and

(6.1.3) $B_P^{\wedge}=K[[T, U]]$, where K $(=k(Y_1, Z_1, Y_2, Z_2,\ldots))$ is the field of quotients of R.

Example. (6.2) (cf. Nagata [8, p. 208, Example 6]) With notation as above, let $B[w]=B[W]/(W^2+\omega^2)$ and let Q be the prime ideal of $B[w]$ such that $Q\cap B=P$. Then

(6.2.1) $B[w]_Q$ $(=B_P[w])$ is normal.

This gives an example of a two-dimensional normal local ring which is analytically ramified.

Proof of (6.2.1). Let α be an element of $L(w)$. Then α can be expressed as $\beta+\gamma w$, where $\beta, \gamma\in L$. The element α is integral over $B_P[w]$ if and only if

(6.2.2) $$\beta^2+\gamma^2 w^2=\beta^2+\gamma^2\omega^2\in B_P.$$

Let $\beta=\frac{b}{a}$ and $\gamma=\frac{c}{a}$, where $a, b, c\in B_P$. Then (6.2,2) is expressed as

(6.2.3) $$\frac{b^2+c^2\omega^2}{a^2}=d\in B_P, \quad \text{i.e.} \quad b^2+c^2\omega^2=da^2.$$

We first claim

(6.2.4) $$c\in aB_P.$$

Before we prove (6.2.4), we make some remarks, fixing notation. First we note that B_P is the ring of quotients of B with respect to the multiplicatively closed set $S=\left\{\sum_{i,j}^{\infty} r_{ij}T^iU^j\in R[[T, U]] \mid r_{00}\neq 0\right\}$. Then, for any element x of B_P, we can find a non-zero element s of R such that $x\in R_s[[T, U]]$. Hence

(6.2.5) there exists a non-zero element s of R such that $a, b, c, d \in R_s[[T, U]]$ and that a, b, c have no common divisor in B_P.

Therefore, we can express a, b, c, d as formal power series in T and U with coefficients in R_s:

(6.2.6) $$a=\sum_{i,j}^{\infty} a_{ij}T^iU^j,\quad b=\sum_{i,j}^{\infty} b_{ij}T^iU^j,\quad c=\sum_{i,j}^{\infty} c_{ij}T^iU^j,\quad d=\sum_{i,j}^{\infty} d_{ij}T^{2i}U^{2j},$$

where $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in R_s$ $(i, j=0, 1, 2,\dots)$.

Then, comparing the coefficients of $T^{2i}U^{2j}$ in (6.2.3), we have

(6.2.7) $$b_{ij}^2+\sum_{m_1+m_2=i} c_{m_1 j}^2 Y_{m_2}^2+\sum_{n_1+n_2=j} c_{in_1}^2 Z_{n_2}^2=\sum_{\substack{k_1+k_2=i\\ l_1+l_2=j}} d_{k_1 l_1} a_{k_2 l_2}^2$$

for any non-negative integers i, j.

If $a \notin PB_P$, our claim (6.2.4) is clear. Hence we may assume

(6.2.8) $$a \in PB_P, \quad \text{i.e.} \quad a_{00}=0.$$

Moreover, we remark

(6.2.9) there exists an $i>0$ such that $a_{i0} \neq 0$.

Proof of (6.2.9). Suppose $a_{i0}=0$ for any $i \geq 0$. Consider the relation (6.2.3) modulo $UR_s[[T, U]]$. Then, denoting by $\bar{x}$ the class of an element x of $R_s[[T, U]]$ modulo $UR_s[[T, U]]$, we have $\bar{b}^2+\bar{c}^2\omega^2 \equiv 0$. Consequently, as $\omega^2 \equiv \omega_1^2$ and $\bar{b}\bar{c} \not\equiv 0$ (cf. (6.2.5)), ω_1 is contained in the field of quotients of $R_s[[T]]$. A similar reasoning as in the proof of (5.2.4) shows this is impossible. (Thus (6.2.9) is proved)

Finally, as $B_P^{\wedge}$ $(=K[[T, U]])$ is faithfully flat over B_P, we note that, to get our claim, it is sufficient to show

(6.2.10) $c \in aR_s[[T, U]]$ for some non-zero element s of R.

Proof of (6.2.4). Suppose $c \notin aB_P$. Let $i_0=\min\{i \mid a_{i0} \neq 0\}$ (cf. (6.2.9)). By the above remark (6.2.10), we may assume

(6.2.11) $$a_{i_0 0}=1.$$

Let $j_0=\min\{j \mid c_{ij} \neq 0 \text{ for some } i \geq 0\}$ and $i_1=\min\{i \mid c_{ij_0} \neq 0\}$. By adding a suitable multiple of a to c if necessary, we may assume

(6.2.12) $$i_1<i_0 \quad \text{and} \quad c_{i_1 j_0}=1 \quad (\text{cf. } (6.2.10)).$$

From now on, we fix a non-zero element s of R which ensures the assumptions (6.2.5), (6.2.11), (6.2.12). Under these assumptions, we can show

(6.2.13) $$Z_j \in R_s[Y] \quad (=R_s[Y_1, Y_2,\dots]) \qquad \text{for any} \quad j>0.$$

Proof of (6.2.13). First we note two preliminary steps.

Step 1. Let $n \geq -1$. Suppose

(6.2.14) d_{ij} is contained in the set $R_s[Y]^2 (=\{f^2 \mid f \in R_s[Y]\})$ for any $j \leq n$ and for any $i \geq 0$, and

(6.2.15) $Z_k \in R_s[Y]$ for any $k \leq n - j_0 + 1$.

Under these assumptions, we compare the coefficients of $T^{2(i_0+i)}U^{2(n+1)}$ in (6.2.3). Then

$$b^2_{(i_0+i)(n+1)} + \sum_{m_1+m_2=i_0+i} c^2_{m_1(n+1)} Y^2_{m_2} + \sum_{\substack{n_1+n_2=n+1 \\ (n_2 \leq n-j_0+1)}} c^2_{(i_0+i)n_1} Z^2_{n_2} \tag{6.2.16}$$

$$= d_{i(n+1)} + \sum_{\substack{k_1+k_2=i_0+i \\ l_1+l_2=n+1 \\ (k_1<i \text{ or } l_1<n+1)}} d_{k_1 l_1} a^2_{k_2 l_2} \quad \text{(cf. (6.2.7), (6.2.11), (6.2.12))}.$$

Hence, by induction on $i \geq 0$

(6.2.17) $d_{i(n+1)}$ is contained in the set $R_s[Y]^2$ for any $i \geq 0$.

Step 2. Let n be a non-negative integer. Suppose

(6.2.18) d_{ij} is contained in the set $R_s[Y]^2$ for any $j \leq n$ and for any $i \geq 0$, and

(6.2.19) $Z_k \in R_s[Y]$ for any $k \leq n - j_0$.

Under these assumptions, we compare the coefficients of $T^{2i_1}U^{2(n+1)}$ in (6.2.3). Then

$$b^2_{i_1(n+1)} + \sum_{m_1+m_2=i_1} c^2_{m_1(n+1)} Y^2_{m_2} + \sum_{\substack{n_1+n_2=n+1 \\ (n_2 \leq n-j_0)}} c^2_{i_1 n_1} Z^2_{n_2} + Z^2_{(n-j_0+1)} \tag{6.2.20}$$

$$= \sum_{\substack{k_1+k_2=i_1 \\ l_1+l_2=n+1 \\ (l_2>0)}} d_{k_1 l_1} a^2_{k_2 l_2} \quad \text{(cf. (6.2.7), (6.2.12))}.$$

Hence

$$Z_{(n-j_0+1)} \in R_s[Y]. \tag{6.2.21}$$

We prove (6.2.13) by induction: Let m be an non-negative integer. Suppose

$$Z_k \in R_s[Y] \quad \text{for any} \quad k \leq m, \quad \text{where we let} \quad Z_0 = 0. \tag{6.2.22}$$

Then, by double induction on i and j

(6.2.23) d_{ij} is contained in the set $R_s[Y]^2$ for any $j \leq m + j_0$ and for any $i \geq 0$ (cf. Step 1).

Hence, the assumptions (6.2.18), (6.2.19) in Step 2 for $n = m + j_0$ are fulfilled. Therefore

(6.2.24) $Z_{(m+1)} \in R_s[Y]$. (This completes the proof of (6.2.13))

Final step of the proof of (6.2.4). By (6.2.13), we have

(6.2.25) $$\bar{R}=R[Y_1, Z_1, Y_2, Z_2, \dots]\subset R_s[Y_1, Y_2, \dots].$$

This is a contradiction (cf. (5.1.1), (5.1.2)). (The proof of (6.2.4) is finished)

Final step of the proof of (6.2.1). We have shown that $\gamma \in B_P$ (cf. (6.2.4)). Then β is also integral over B_P. Hence $\beta \in B_P$, for B_P is regular. Therefore, $\alpha = \beta + \gamma w \in B_P[w]$. q.e.d.

Example. (6.3) (cf. Nagata [8, p. 207, Example 5]) With notation as above, let M be a maximal ideal of $B[w]$. Then

(6.3.0) $B[w]_M$ is a three-dimensional (noetherian) local domain.

Let $(\overline{B[w]_M})$ be the derived normal (quasi-local) ring of $B[w]_M$. Then

(6.3.1) $(\overline{B[w]_M})$ is not noetherian.

Proof of (6.3.1). Suppose $(\overline{B[w]_M})$ is noetherian. Let $\mathfrak{n}$ be the maximal ideal of C such that $\mathfrak{n} \cap B[w]=M$ and let $M \cap R=\mathfrak{q}$ (say $\mathfrak{q}_{2j}$). Then, as $C_\mathfrak{n}$ is integral over the normal domain $(\overline{B[w]_M})$, we have

(6.3.2) $$Z_j^\nu C_\mathfrak{n} \cap (\overline{B[w]_M})=Z_j^\nu(\overline{B[w]_M}) \quad \text{for any} \quad \nu>0.$$

Moreover, there exist canonical injections

(6.3.3) $$K_{2j}[[T, U]]=B/\mathfrak{q}_{2j}B \hookrightarrow (\overline{B[w]_M})/Z_j(\overline{B[w]_M}) \hookrightarrow C_\mathfrak{n}/Z_jC_\mathfrak{n}=K_{2j}[[T, U]].$$

Hence, we have an isomorphism

(6.3.4) $$(\overline{B[w]_M})/Z_j^\nu(\overline{B[w]_M}) \xrightarrow{\sim} C_\mathfrak{n}/Z_j^\nu C_\mathfrak{n} \quad \text{for any} \quad \nu>0.$$

Let $(\overline{B[w]_M})^{**}$ be the $Z_j(\overline{B[w]_M})$-adic completion of $(\overline{B[w]_M})$ and $C_\mathfrak{n}^{**}$ the $Z_jC_\mathfrak{n}$-adic completion of $C_\mathfrak{n}$. Then

(6.3.5) $$(\overline{B[w]_M})^{**}=C_\mathfrak{n}^{**} \quad \text{(cf. (6.3.4))}.$$

Consequently, $(\overline{B[w]_M})$ is regular (cf. (6.1.0)). Therefore, as $(\overline{B[w]_M})_Q=B[w]_Q$, $B[w]_Q$ is analytically unramified (cf. (6.2.1)). Contradiction. q.e.d.

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