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Diffusion processes in population genetics

By

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§ 1. Introduction

In population genetics theory we often encounter diffusion processes on the compact domain $K=\{(x_1,\dots, x_d)\in R^d;\ x_1\geq 0,\dots, x_d\geq 0,\ 1-x_1-\cdots-x_d\geq 0\}$. In order to construct such diffusion processes, we will consider a martingale problem on K.

Let A be a second order differential operator on K

(1.1) $$A=\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

with domain $D(A)=C^2(K)$,[1)] where $\{a_{ij}(x)\}_{1\leq i,j\leq d}$ is a real symmetric and non-negative definite matrix defined on K and $\{b_i(x)\}_{1\leq i\leq d}$ is an R^d-valued measurable function defined on K.

We assume that $\{a_{ij}(x)\}$ and $\{b_i(x)\}$ are continuous on K. Let $\Omega=C([0, \infty): K)$ be the space of all K-valued continuous functions defined on $[0, \infty)$. For each $\omega\in\Omega$ and each $t\geq 0$, we denote $x(t:\omega)=\omega(t)$. Let $\mathscr{F}_t$ and $\mathscr{F}$ be the σ-fields generated by $\{x(s);\ 0\leq s\leq t\}$ and $\{x(s);\ s\geq 0\}$ respectively.

Let $x\in K$. A probability measure P on $(\Omega, \mathscr{F})$ is called a solution of the (K, A, x)-martingale problem if it satisfies the following conditions,

(1.2) $P[\omega;\ x(0:\omega)=x]=1$, and

(1.3) denoting $M_f(t)=f(x(t))-\int_0^t Af(x(s))ds$, $(M_f(t), \mathscr{F}_t)$ is a P-martingale for each $f\in C^2(K)$.

It is known that if a solution of the (K, A, x)-martingale problem exists, the following conditions must be satisfied, (cf. Okada [9]).

(1.4) $$a_{ii}(x)=0 \quad\text{if}\quad x_i=0,\quad\text{and}\quad \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)=0 \quad\text{if}\quad \sum_{i=1}^{d} x_i=1,$$

and

1) Each element of $C^2(K)$ is a C^2- function defined on an open set containing K.

(1.5) $\quad b_i(x)\geq 0$ if $x_i=0$, and $\sum_{i=1}^{d} b_i(x)\leq 0$ if $\sum_{i=1}^{d} x_i=1$.

Conversely, if $\{a_{ij}(x)\}$ and $\{b_i(x)\}$ are continuous on K satisfying (1.4) and (1.5), then a solution of the (K, A, x)-martingale problem always exists.

However, the uniqueness of solutions of the (K, A, x)-martingale problem has not been generally established. The difficulty is due to the fact that $\{a_{ij}(x)\}$ can not always be extended to be smooth and non-negative definite on the whole space R^d, even if $\{a_{ij}(x)\}$ is sufficiently smooth on K.

For this problem, Ethier [2] proved that if $\{a_{ij}(x)\}=\{x_i(\delta_{ij}-x_j)\}$[2] and $\{b_i(x)\}$ are C^4-functions satisfying (1.5), then the uniqueness of the (K, A, x)-martingale problem holds. Also, Okada [10] recently showed that the uniqueness holds for a rather general class in two dimension.

In the present paper we will first discuss the uniqueness problem of the (K, A, x)-martingale problem. Although we impose a rather restrictive condition, our result covers the Ethier's case when $\{b_i(x)\}$ are assumed to be polynomials. Our method consists in using the notion of dual processes which has proved useful in the theory of infinite interacting systems. Secondly, we will study some ergodic behaviors of the diffusion processes on K under a genetical assumption. In the final section we will discuss a problem whether or not the diffusion process hits on subsets of the boundary ∂K with positive probability.

§ 2. Uniqueness of solutions of the (K, A, x)-martingale problem

Let $J=Z_d^+$ be the d products of the set of non-negative integers. For α and $\beta\in J$, $\alpha\pm\beta$ is defined componentwise. If $\alpha_i=1$ and $\alpha_j=0$ for $j\neq i$, α is denoted by ε_i. For each $\alpha\in J$ and $x\in K$, we denote $f_\alpha(x)=\prod_{i=1}^{d} x^{\alpha_i}$, and $f_0(x)=1$. Set $|\alpha|=\alpha_1+\cdots+\alpha_d$ for each $\alpha\in J$.

In this section we shall consider the uniqueness problem of the (K, A, x)-martingale problem under the following condition.

Condition I.

[I-1] *$a_{ij}(x)$ are polynomials on K and satisfy* (1.4). *Furthermore, denoting $a_{ij}(x)=\sum_{\beta\in J}\hat{a}_{ij}(\beta)f_\beta(x)$, there exists a constant $0<c<1$ such that*

$$\sum_{\beta\neq\varepsilon_i+\varepsilon_j}|\hat{a}_{ij}(\beta)|c^{2-|\beta|}+\hat{a}_{ij}(\varepsilon_i+\varepsilon_j)<0 \qquad \textit{for all} \quad 1\leq i, j\leq d, \tag{2.1}$$

and

$$\sum_{j=1}^{d}\sum_{|\beta|>2}|\hat{a}_{ij}(\beta)|c^{1-|\beta|}(|\beta|-2)<\hat{a}_{ii}(\varepsilon_i) \qquad \textit{for all} \quad 1\leq i, j\leq d. \tag{2.2}$$

[I-2] *$b_i(x)$ are polynomials on K and satisfy* (1.5).

Then, we have

2) δ_{ij} is the Kronecker's symbol, i.e. $\delta_{ij}=0$ $(i\neq j)$ and $\delta_{ii}=1$

Theorem 2.1. *Suppose that the condition* I *is fulfilled. Then, for any* $x\in K$ *the uniqueness of solutions of the* (K, A, x)*-martingale problem holds.*

Suppose that the condition [I–2] holds. Then as consequences of this theorem we have the following.

Corollary 2.1. $\{a_{ij}(x)=\sigma^2 x_i(\delta_{ij}-x_j)\}$ *satisfies the condition* [I–1]. *Thus, our result gives another proof of Ethier's theorem* [2] *in the case that* $b_i(x)$ *are polynomials.*

Corollary 2.2. *Let* $a_{ij}(x)=x_ix_j(\sum_{k=0}^{d}\beta_kx_k-\beta_i-\beta_j)+\delta_{ij}\beta_ix_i$, *where* $x_0=1-x_1-\cdots-x_d$ *and* $\beta_k\geq 0$, $k=0, 1,\dots, d$. *(cf. Sato* [13]*). If* $\max_{0\leq i\leq d}\beta_i<\left(1+\frac{1}{2d^2}\right)\min_{0\leq i\leq d}\beta_i$, *then* [I–1] *is satisfied. Hence the uniqueness of the martingale problem holds.*

Proof. Without loss of generality, we may assume $\beta_0=\min_{d\leq i\leq d}\beta_i$. Then, [I–1] is satisfied with $c=\frac{1}{2}$.

Corollary 2.3. *Let* $a_{ij}(x)$ *be any polynomials satisfying* (1.1) *and* (1.4). *Then, there exists a positive constant* ε_0 *such that for any* $x\in K$ *and any* ε *with* $0<\varepsilon<\varepsilon_0$, *the* (K, A^ε, x)*-martingale problem has a unique solution,*

where
$$A^\varepsilon=\sum_{i,j=1}^{d}(x_i(\delta_{ij}-x_j)+\varepsilon a_{ij}(x))\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i(x)\frac{\partial}{\partial x_i},$$

Remark 2.1. Suppose that the uniqueness of the (K, A, x)-martingale problem holds for any $x\in K$. Then the A-diffusion process is uniquely determined and its transition probability induces a Feller semi-group on $C(K)$.[3)]

Remark 2.2. If $a_{ij}(x)=\sum_{\beta\in J}\hat{a}_{ij}(\beta)f_\beta(x)$ satisfies (1.4), then it follows automatically that

$$\hat{a}_{ij}(\beta)=0 \text{ for all } \beta \text{ with } |\beta|\leq 2 \text{ except } \beta=\varepsilon_i+\varepsilon_j \text{ if } i\neq j,$$

$$\hat{a}_{ii}(\beta)=0 \text{ for all } \beta \text{ with } |\beta|\leq 1 \text{ except } \beta=\varepsilon_i, \text{ and}$$

$$\hat{a}_{ii}(\varepsilon_k+\varepsilon_m)=0 \text{ if } k\neq i \text{ and } m\neq i.$$

For each $\alpha\in J$, we associate the symbol $\bar{\alpha}$ and set $|\bar{\alpha}|=|\alpha|$. Let $\bar{J}=\{\bar{\alpha};\ \alpha\in J\}$ and $J^*=J\cup\bar{J}$. For each $\alpha\in J$ we denote $\phi_\alpha(x)=c^{|\alpha|}f_\alpha(x)$ and $\phi_{\bar{\alpha}}(x)=-\phi_\alpha(x)$.

Let $b_i(x)=\sum_{\beta\in J}\hat{b}_i(\beta)f_\beta(x)$. Then, by a simple calculation we have

Lemma 2.1.

$$A\phi_\alpha(x)=\sum_{\beta\in J^*}Q_{\alpha,\beta}(\phi_\beta(x)-\phi_\alpha(x))+h_\alpha\phi_\alpha(x),\qquad \alpha\in J, \tag{2.3}$$

where

3) $C(K)$ denotes the set of all continuous functions on K.

$$h_\alpha=\sum_{i=1}^{d}\alpha_i(\alpha_i-1)\Big(\sum_{\substack{\beta\neq 2\varepsilon_i\\ \beta\in J}}|\hat{a}_{ii}(\beta)|c^{2-|\beta|}+\hat{a}_{ii}(2\varepsilon_i)\Big)$$

$$+\sum_{\substack{i=1\\(i\neq j)}}^{d}\sum_{j=1}^{d}\alpha_i\alpha_j\Big(\sum_{\substack{\beta\neq\varepsilon_i+\varepsilon_j\\ \beta\in J}}|\hat{a}_{ij}(\beta)|c^{2-|\beta|}+\hat{a}_{ij}(\varepsilon_i+\varepsilon_j)\Big)$$

$$+\sum_{i=1}^{d}\alpha_i\Big(\sum_{\substack{\beta\neq\varepsilon_i\\ \beta\in J}}|\hat{b}_i(\beta)|c^{1-|\beta|}+\hat{b}_i(\varepsilon_i)\Big),$$

and $\{Q_{\alpha,\beta}\}_{\alpha\in J,\beta\in J^*}$ *are defined by*

$$\sum_{\beta\in J^*}Q_{\alpha,\beta}(\phi_\beta-\phi_\alpha)=\sum_{i=1}^{d}\alpha_i(\alpha_i-1)\Big(\sum_{\substack{\beta\neq 2\varepsilon_i\\ \beta\in J}}c^{2-|\beta|}(\hat{a}_{ii}^{+}(\beta)(\phi_{\alpha-2\varepsilon_i+\beta}-\phi_\alpha)$$

$$+\hat{a}_{ii}^{-}(\beta)(\phi_{\overline{\alpha-2\varepsilon_i+\beta}}-\phi_\alpha))\Big)+\sum_{\substack{i=1\\(i\neq j)}}^{d}\sum_{j=1}^{d}\alpha_i\alpha_j\Big(\sum_{\substack{\beta\neq\varepsilon_i+\varepsilon_j\\ \beta\in J,}}c^{2-|\beta|}(\hat{a}_{ij}^{+}(\beta)(\phi_{\alpha-\varepsilon_i-\varepsilon_j+\beta}-\phi_\alpha)$$

$$+\hat{a}_{ij}^{-}(\beta)(\phi_{\overline{\alpha-\varepsilon_i-\varepsilon_j+\beta}}-\phi_\alpha))\Big)+\sum_{i=1}^{d}\alpha_i\Big(\sum_{\substack{\beta\neq\varepsilon_i\\ \beta\in J}}c^{1-|\beta|}(\hat{b}_i^{+}(\beta)(\phi_{\alpha-\varepsilon_i+\beta}-\phi_\alpha)$$

$$+\hat{b}_i^{-}(\beta)(\phi_{\overline{\alpha-\varepsilon_i+\beta}}-\phi_\alpha))\Big).^{4)}$$

Let us define an infinitesimal matrix $\boldsymbol{Q}=\{Q_{\alpha,\beta}\}$ on $J^*\times J^*$ by for each $\alpha\in J$, $Q_{\alpha,\alpha}=-\sum_{\substack{\beta\in J^*\\ \beta\neq\alpha}}Q_{\alpha,\beta}$ and $Q_{\bar{\alpha},\bar{\beta}}=Q_{\alpha,\beta}$, where $(\bar{\beta})=\beta$ for each $\beta\in J$.

Then we have the following.

Lemma 2.2.

(i) *h_α is bounded above on J^*.*

(ii) *Let $f(\alpha)=|\alpha|$. Then, $\boldsymbol{Q}f(\alpha)=\sum_{\beta\in J^*}Q_{\alpha,\beta}(f(\beta)-f(\alpha))$ is bounded above on J^*.*

(iii) *The minimal Markov chain $(\alpha_t,\boldsymbol{P}_\alpha)_{\alpha\in J^*}$, generated by $\boldsymbol{Q}=\{Q_{\alpha,\beta}\}$, is conservative.*

Proof. (i) and (ii) follows from (2.1) and (2.2) respectively. As for (iii), let $(\alpha_t,\boldsymbol{P}_\alpha)$ be the minimal Markov chain generated by $\boldsymbol{Q}$, (cf. Chung [1], II § 18). For each $N>0$, define $\tau_N=\inf\{t>0;\ |\alpha_t|\geq N\}$. By the Dynkin formula, we see

(2.5) $\boldsymbol{E}_\alpha[f(\alpha_t\wedge\tau_N)]=f(\alpha)+\boldsymbol{E}_\alpha\left[\int_0^{t\wedge\tau_N}\boldsymbol{Q}f(\alpha_s)ds\right]$, for any finitely supported function f on J^*.

Noting that $\{a_{ij}(x)\}$ and $\{b_i(x)\}$ are polynomials, we can choose a finitely supported function f_N on J^* such that $f_N(\alpha)=|\alpha|$ if $|\alpha|<N$, and $\boldsymbol{Q}f_N(\alpha)=\sum_{\beta\in J^*}Q_{\alpha,\beta}(|\beta|-|\alpha|)$ if $|\alpha|<N$. Then it follows from (ii) that there exists a constant $K>0$ satisfying

$$\max_{\substack{|\alpha|<N\\ \alpha\in J^*}}\boldsymbol{Q}f_N(\alpha)<K\qquad\text{for any}\quad N>0. \tag{2.6}$$

4) $a^+=\max\{a,0\}$, and $a^-=-\min\{a,0\}$.

Thus, by (2.5) and (2.6) we have

(2.7) $$E_\alpha[|\alpha_{t\wedge\tau_N}|]<|\alpha|+Kt \qquad \text{for any} \quad N>0.$$

So, letting $N\to\infty$,

$$P_\alpha[\lim_{N\to\infty}\tau_N>t]=1 \quad \text{for any finite} \quad t>0,$$

namely,

(2.8) $$P_\alpha[\lim_{N\to\infty}\tau_N=+\infty]=1.$$

Hence, (α_t, P_α) is conservative, because it does not explode in finite time.

Lemma 2.3 (Kac's formula). *Let $h(\alpha)$ be any function bounded above on J^*. For any bounded function $f(\alpha)$ on J^* and $\lambda>\|h^+\|$, let us consider the following equation for u.*

(2.9) $$(\lambda-Q-h)u=f$$

with the subsidiary condition

(2.10) $R^\lambda(|hu|)$ *is bounded on* J^*.[5)]

Then, there exists one and only one bounded solution of (2.9) *with* (2.10), *and it is given by*

(2.11) $$u(\alpha)=E_\alpha\left[\int_0^\infty \exp\left(-\lambda t+\int_0^t h(\alpha_u)du\right)f(\alpha_t)\,dt\right], \qquad \alpha\in J^*.$$

Proof. Without loss of generality, we may assume that $h\le 0$.

1°. Let f be a function on J^* such that $R^\lambda(|f|)$ is bounded on J^*. Then we shall show that

$$u(\alpha)=E_\alpha\left[\int_0^\infty \exp\left(-\lambda t+\int_0^t h(\alpha_u)du\right)f(\alpha_t)dt\right] \text{ satisfies (2.9) and (2.10)}.$$

We may assume $f\ge 0$. Then it is easy to see that

(2.12) $$R^\lambda f(\alpha)-u(\alpha)=R^\lambda((-h)u)(\alpha), \qquad \text{(cf. Ito [6])}.$$

Hence $R^\lambda(|h|u)(\alpha)$ is bounded. Since (α_t, P_α) is generated by Q, it holds that

(2.13) $$\lambda R^\lambda_{\alpha,\beta}-\sum_{\gamma\in J^*}Q_{\alpha,\gamma}R^\lambda_{\gamma,\beta}=\delta_{\alpha,\beta} \quad \text{for all} \quad \alpha \text{ and } \beta\in J^*.$$

So, by (2.12) and (2.13) we have

(2,14) $$f(\alpha)-(\lambda-Q)u(\alpha)=-h(\alpha)u(\alpha), \qquad \alpha\in J^*.$$

Thus $u(\alpha)$ is a bounded solution of (2.9) and (2.10).

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5) $R^\lambda f(\alpha)=E_\alpha\left[\int_0^\infty e^{-\lambda t}f(\alpha_t)dt\right]$.

2°. Suppose that $h(\alpha)$ and $\boldsymbol{R}^{\lambda}(|f|)(\alpha)$ are bounded. Then, (2.9) has a unique bounded solution. For, let u_1 and u_2 be two bounded solutions of (2.9). Setting $u_1-u_2=w$, it holds that

$$(2.15)\qquad (\lambda-\boldsymbol{Q})w=hw.$$

Since the minimal Markov chain $(\alpha_t, \boldsymbol{P}_\alpha)$ is conservative, $(\lambda-\boldsymbol{Q})u=f$ has a unique bounded solution u for any bounded function f, which is given by $u=\boldsymbol{R}^{\lambda}f$. Thus, we have

$$(2.16)\qquad w=\boldsymbol{R}^{\lambda}(hw).$$

If $\mu>2\lambda$ and $\mu>\|h\|$, it follows from the resolvent equation that

$$w=\boldsymbol{R}^{\lambda}(hw)=\boldsymbol{R}^{\mu}(hw)+(\mu-\lambda)\boldsymbol{R}^{\mu}\boldsymbol{R}^{\lambda}(hw)=\boldsymbol{R}^{\mu}((h+\mu-\lambda)w).$$

So, $\|w\|\leqq\frac{1}{\mu}\|(h+\mu-\lambda)w\|\leqq\frac{\mu-\lambda}{\mu}\|w\|$, and this proves $\|w\|=0$.

3°. Let h be any non-positive function on J^*, f be any bounded function and u be a bounded solution of (2.9) and (2.10). Set $h_n=h\vee(-n)$.[6] Then, $(\lambda-\boldsymbol{Q}-h_n)u=f+(h-h_n)u$ holds. Since $\boldsymbol{R}^{\lambda}(|f+(h-h_n)u|)$ is bounded, we get by the step 1° and 2°

$$(2.17)\qquad u(\alpha)=\boldsymbol{E}_\alpha\left[\int_0^\infty \exp\left(-\lambda t+\int_0^t h_n(\alpha_u)du\right)(f+(h-h_n)u)(\alpha_t)dt\right].$$

Noting that $\boldsymbol{R}^{\lambda}(|hu|)(\alpha)$ is bounded on J^* and letting $n\to\infty$, we obtain (2.11).

Now, we shall prove *Theorem* 2.1. Let P_x be a solution of the (K, A, x)-martingale problem. Denote $u(t, \alpha, x)=E_x[\phi_\alpha(x_t)]$ for $\alpha\in J^*$. Then, by *Lemma* 2.1 we have

$$(2.18)\qquad u(t, \alpha, x)-\phi_\alpha(x)=\int_0^t E_x[A\phi_\alpha(x_s)]ds$$

$$=\int_0^t \Big(\sum_{\beta\in J^*} Q_{\alpha,\beta}(u(s, \beta, x)-u(s, \alpha, x))+h_\alpha u(s, \alpha, x)\Big)ds.$$

Setting $v_\lambda(\alpha, x)=\int_0^\infty e^{-\lambda t}u(t, \alpha, x)dt$ for each $\lambda>\|h^+\|$, it holds

$$(2.19)\qquad (\lambda-\boldsymbol{Q}-h)v_\lambda(\alpha, x)=\phi_\alpha(x).$$

Noting that $v_\lambda(\alpha, x)$ satisfies (2.10), we have by Lemma 2.3

$$(2.20)\qquad v_\lambda(\alpha, x)=\boldsymbol{E}_\alpha\left[\int_0^\infty \exp\left(-\lambda t+\int_0^t h_{\alpha_u}du\right)\phi_{\alpha_t}(x)dt\right].$$

Accordingly, we see that P_x is uniquely determined by using a standard argument.

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6) $a\vee b=\max\{a, b\}$.

§ 3. Limiting behaviors

In this section we shall discuss limiting behaviors of the following special type of diffusion processes on K:

$$A=\sum_{i=1}^{d}\sum_{j=1}^{d}x_i(\delta_{ij}-x_j)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i(x)\frac{\partial}{\partial x_i} \tag{3.1}$$

$$b_i(x)=\sum_{k=0}^{d}\lambda_{ki}x_k+x_i(\gamma_i(x)-\sum_{k=0}^{d}\gamma_k(x)x_k), \qquad i=0, 1,\dots, d, \tag{3.2}$$

where $x_0=1-x_1-\cdots-x_d$, $\{\lambda_{ki}\}_{0\le k,i\le d}$ is a constant matrix satisfying $\lambda_{ki}\ge 0$ $(k\ne i)$ and $\sum_{i=0}^{d}\lambda_{ki}=0$, and $\{\gamma_k(x)\}_{0\le k\le d}$ are C^4-functions on K.

Without loss of generality, we may assume that $\gamma_k(x)>0$ for any $x\in K$ and $0\le k\le d$ by adding some constant.

Genetically $\{\lambda_{ki}\}$ involves the intensity of mutation and $\{\gamma_k(x)\}$ is the intensity of selection.

By Ethier's results [2], the A-diffusion on K $(\Omega, \mathscr{F}, P_x, \{\mathscr{F}_t\}, x(t))$ is uniquely determined. Hereafter we shall often regard it as a diffusion process on $K^*=\{(x_0, x_1,\dots, x_d);\ x_0\ge 0, x_1\ge 0,\dots, x_d\ge 0, \sum_{i=0}^{d}x_i=1\}$ by setting $x_0(t)=1-\sum_{i=1}^{d}x_i(t)$.

Let us denote by $\{P_t(x, E)\}_{t\ge 0, x\in K^*, E\in\mathscr{B}(K^*)}$ the transition probability of the A-diffusion process on K^*.

Let $I=\{0, 1,\dots, d\}$. Let us define a relation on I by $\{\lambda_{ij}\}$. If $\lambda_{ij}>0$, we denote $i\rightrightarrows j$. For i and j, we denote $i\to j$ if there exists a chain $[i_1=i, i_2,\dots, i_p=j]$ such that $i_n\rightrightarrows i_{n+1}$ for any $1\le n\le p-1$. If either $i=j$, or $i\to j$ and $j\to i$ holds, we denote $i\leftrightarrow j$. Then, the relation "$\leftrightarrow$" defines an equivalent relation on I.

An equivalent class $C\subset I$ is called recurrent if $i\to j$ does not hold for any $i\in C$ and $j\notin C$. For each recurrent class R, let us define

$K(R)=\{x\in K^*;\ \sum_{i\in R}x_i=1\}$ and $\xi_R=\inf\{t\ge 0;\ x(t)\in K(R)\}$, or $+\infty$ if $\{\ \}=\phi$.

Then, we have

Theorem 3.1. *Let $R_1, R_2,\dots, R_r$ be all recurrent classes. Then,*
(i) *For each R_i, there exists a unique stationary probability measure ν_i such that*

$$\lim_{t\to\infty}\|P_t(x,\ \cdot)-\nu_i\|_{\text{var}}=0 \qquad \textit{for any} \quad x\in K(R_i),$$

(ii) $P_x[\xi_{R_i}<+\infty$ *for some* $1\le i\le r]=1$ *for any* $x\in K^*$,

and

(iii) $\lim_{t\to\infty}\|P_t(x,\ \cdot)-\sum_{i=1}^{r}c_i(x)\nu_i\|_{\text{var}}=0$[7] *for any* $x\in K^*$, *where*

7) $\|\ \|_{var}$ stands for the total variation norm.

$$c_i(x)=P_x[\xi_{R_i}<+\infty].$$

First, we shall prove it in case that $\{\lambda_{ij}\}$ is an irreducible matrix. Generally we shall introduce the following condition. Let $\partial K_i^*=\{x=(x_0, x_1,\dots, x_d)\in K^*; x_i=0\}$ $(0\le i\le d)$, $\partial K^*=\bigcup_{i=0}^{d}\partial K_i^*$, and $\mathring{K}^*=K^*\backslash\partial K^*$.

Condition II. *For any* $x\in\partial K^*$, $\sum_{i=0}^{d} I_{\{0\}}(x_i)b_i(x)>0$ *holds.*
This condition means that the drift coefficient $\{b_i(x)\}$ never degenerates on the boundary ∂K^*. In particular, if $\{\lambda_{ij}\}$ is irreducible, $\{b_i(x)\}$ of (3.2) satisfies the *condition II.*
Then, we can show the following.

Theorem 3.2. *Assume the condition II. Then,*
(i) *For each* $(t, x)\in(0, \infty)\times K^*$, $P_t(x, \cdot)$ *is mutually absolutely continuous to the Lebesgue measure* λ *on* K^*.
(ii) *Denoting by* $\{p_t(x, y)\}$ *the density functions of* $\{p_t(x, \cdot)\}$ *w.r.t.* λ, *there exists a positive probability density function* $\mu(x)$ *on* K^* *such that* $\mu(x)$ *is smooth in* $\mathring{K}^*$, *and*

$$\lim_{t\to\infty}\int_{K^*}|p_t(x, y)-\mu(y)|\lambda(dy)=0 \qquad \textit{for any} \quad x\in K^*.$$

Remark 3.1. Generally, Theorem 3.2 is valid if $\{a_{ij}(x)\}$ is smooth on K and positive definite in the interior $\mathring{K}$ of K, and if the (K, A, x)-martingale problem is uniquely solvable.

For the proof of *Theorem* 3.2, we apply the following general fact which is essentially due to Harris [4] and Orey [11].

Lemma 3.1. *Let* $(X, \mathscr{B})$ *be a measurable space and* $\{P_t(x, E)\}_{t\ge0,x\in X,E\in\mathscr{B}}$ *be a conservative transition probability on* $(X, \mathscr{B})$. *Suppose that there exists a stationary probability measure* ν *of* $\{P_t(x, E)\}$, *which satisfies the following condition,*

(3.3) *for some* $t_0>0$, *it holds that* $P_{t_0}(x, \cdot)$ *is mutually absolutely continuous with respect to* ν *for each* $x\in X$.

Then,

$$\lim_{t\to\infty}\|P_t(x, \cdot)-\nu\|_{\mathrm{var}}=0 \qquad \textit{for each} \quad x\in X. \tag{3.4}$$

Proof. We may assume $t_0=1$. First we shall show that any P_1-invariant function is a constant function.

Let $h(x)$ be any P_1-invariant function and set $E=\{x\in X; h(x)>0\}$. Then, $\langle\nu, h^+\rangle=\langle\nu, I_E h\rangle=\langle\nu, I_E P_1 h\rangle\le\langle\nu, I_E P_1 h^+\rangle\le\langle\nu, P_1 h^+\rangle$. Since $\langle\nu, h^+\rangle=\langle\nu, P_1 h^+\rangle$, because of the stationarity of ν, we see

$$I_E P_1 h^+=P_1 h^+ \quad \nu\text{-a.e.}.$$

Thus, it follows that $I_{E^c}P_1 I_E=0$ ν-a.e., and this implies

(3.5) $$P_1 I_E \leq I_E \quad \nu\text{-a.e.}.$$

Noting $\langle \nu, P_1 I_E\rangle = \langle \nu, I_E\rangle$, it holds $\nu(E)=0$ or 1. In the same way we see that $\nu(\{x\in X;\ h(x)>\alpha\})=0$ or 1 holds for any $\alpha\in R^1$. Accordingly, there exists a constant a such that $h(x)=a$ ν-a.e.. Moreover, the P_1-invariance of h and (3.3) imply $h(x)=a$ for all $x\in X$.

Next, we shall show that the Markov chain $(\Omega, \mathscr{F}, P_x, \{\mathscr{F}_n\}, x(n))$, associated with $\{P_n(x, E)\}_{n\in Z^+, x\in X, E\in\mathscr{B}}$, satisfies the Harris recurrent condition.

(H) *For any* $E\in\mathscr{B}$ *with* $\nu(E)>0$, $P_x[\sum_{n=1}^{\infty} I_E(x(n))=+\infty]=1$ *holds for every* $x\in X$.

Set $\Gamma=[\sum_{n=1}^{\infty} I_E(x(n))=+\infty]$. Then, since $P_x[\Gamma]$ is P_1-invariant, it is a constant function. Let $P_x[\Gamma]=a$. Applying the individual ergodic theorem for the stationary process $\{x(n), P_\nu\}$, we see

(3.6) $P_\nu\left[\lim_{t\to\infty}\frac{1}{n}\sum_{m=1}^{n} I_E(x(m))=f^* \text{ exists}\right]=1$, where f^* is a random variable and satisfies $E_\nu[f^*]=\nu(E)>0$.

This implies $P_\nu[\Gamma]=a>0$.

On the other hand, it follows that

$$I_\Gamma(\omega)=\lim_{n\to\infty} P_\nu[\Gamma\mid\mathscr{F}_n](\omega)=\lim_{n\to\infty} P_{x(n:\omega)}[\Gamma]=a, \quad P_\nu\text{-a.s.}.$$

Hence we have $a=1$.

Finally, it is clear that $\{P_n(x, E)\}$ is aperiodic. So, by using *Theorem* 3 in [11], p. 816, we have

(3.7) $$\lim_{n\to\infty}\|P_n(x,\cdot)-\nu\|_{\text{var}}=0 \qquad \text{for all} \quad x\in X.$$

Moreover, noting that $\|P_t(x,\cdot)-\nu\|_{\text{var}}$ is non-increasing in $t>0$, we complete the proof of this lemma.

Lemma 3.2. $P_x\left[\int_0^\infty I_{\partial K}(x(t))dt=0\right]=1$ *for any* $x\in K$.

Proof. Let define a sequence of C^2-functions $\{\psi_n\}$ on $[0,\infty)$ by

$$\psi_n''(u)=\frac{n^2}{2}\left(u-\frac{2}{n}\right)\wedge 0,\ \psi_n'(u)=\int_0^u\psi_n''(v)dv+1, \text{ and}$$

$$\psi_n(u)=\int_0^u\psi_n'(v)\,dv.$$

Then it holds that $\lim_{n\to\infty}\|\psi_n\|=0$, $\lim_{n\to\infty}\psi_n'(u)=I_{\{0\}}(u)$ boundedly, $\lim_{n\to\infty}|u\psi_n''(u)|=0$ boundedly, and $\operatorname{supp}[\psi_n'']\subset\left[0, \frac{2}{n}\right]$. Let $0\leq i\leq d$. Since $\left\{f(x(t))-\int_0^t Af(x(s))ds, \mathscr{F}_t\right\}$ is a P_x-martingale for $f(x)=\psi_n(x_i)$, we have

$$E_x[\psi_n(x_i(t))]-\psi_n(x_i)=E_x\left[\int_0^t Af(x(s))ds\right]$$
$$=E_x\left[\int_0^t x_i(s)(1-x_i(s))\psi_n''(x_i(s))ds\right]+E_x\left[\int_0^t\psi_n'(x_i(s))b_i(x(s))ds\right].$$

Letting $n\to\infty$, we get

$$E_x\left[\int_0^t I_{\{0\}}(x_i(s))b_i(x(s))ds\right]=0 \qquad (0\leq i\leq d). \tag{3.8}$$

Consequently,

$$\int_0^t E_x[\sum_{i=0}^d I_{\{0\}}(x_i(s))b_i(x(s))]ds=0. \tag{3.9}$$

Thus, noting the integrand of (3.9) is non-negative by (1.5), it follows from the *condition II* that

$$P_x\left[\int_0^\infty I_{\partial K}(x(t))dt=0\right]=1 \text{ holds for each } x\in X.$$

Now, we shall prove *Theorem* 3.2. First, we note that there exists a stationary probability measure ν of the diffusion process $(\Omega, \mathscr{F}, P_x, \{\mathscr{F}_t\}, x(t))$ since it induces a Feller semigroup on $C(K)$ and K is compact. Also, if $P_t(x, \cdot)\simeq\lambda$ (the Lebesgue measure on K)[8] for any $t>0$ and $x\in K$, then $\lambda\simeq\nu$ holds and *Lemma* 3.1 is applicable. So, we shall show $P_t(x, \cdot)\simeq\lambda$ for any $t>0$ and $x\in K$.

Let $\{P_t^0(x, E)\}_{t\geq 0, x\in\mathring{K}, E\in\mathscr{B}(\mathring{K})}$ be transition probability of the minimal process, which are obtained by killing the A-diffusion process on K at the boundary ∂K. Since the diffusion coefficients of A is not degenerated in $\mathring{K}$, it follows that $P_t^0(x, \cdot)\simeq\lambda$ for any $x\in\mathring{K}$, (cf. [3]). Thus, $P_t(x, \cdot)\gg P_t^0(x, \cdot)\simeq\lambda$ for any $t>0$ and $x\in\mathring{K}$. Also, for any $x\in\partial K$ it is clear that $P_t(x, \cdot)\gg\lambda$ by *Lemma* 3.2.

Conversely, let $\lambda(E)=0$ for some $E\in\mathscr{B}(K)$. For any fixed $x\in\mathring{K}$, we choose $\delta>0$ such that $x\in K_\delta=\{x\in K;\ x_1\geq\delta,\cdots, x_d\geq\delta,\ 1-\sum_{i=1}^d x_i\geq\delta\}$. Let σ and τ be the first hitting times for ∂K and K_δ respectively. $\{\sigma_n\}$ and $\{\tau_n\}$ are defined inductively by

$$\sigma_1=\sigma,\ \tau_1=\sigma_1+\tau(\theta_{\sigma_1}),\cdots,\ \sigma_n=\tau_{n-1}+\sigma(\theta_{\tau_{n-1}}),^{9)} \text{ and } \tau_n=\sigma_n+\tau(\theta_{\sigma_n}).$$

Noting $P_t^0(x, E_\delta)=P_x[x(t)\in E_\delta,\ t<\sigma]=0$ for $E_\delta=E\cap K_\delta$, we see

$$\begin{aligned}P_t(x, E_\delta)&=P_x[x(t)\in E_\delta,\ \tau_n<t\leq\sigma_n \text{ for some } n]\\ &=P_t^0(x, E_\delta)+\sum_{n=1}^\infty E_x[P_{t-\tau_n}^0(x_{\tau_n}, E_\delta);\ \tau_n<t]=0.\end{aligned}$$

Thus, for any $x\in\mathring{K}$

$$P_t(x, E)=P_t(x, E\cap\mathring{K})=\lim_{\delta\to 0}P_t(x, E_\delta)=0 \text{ holds by } \textit{Lemma } 3.2.$$

Furthermore, since it is evident that $P_t(x, E)=0$ holds for any $x\in\partial K$, we see $P_t(x, \cdot)\ll\lambda$ for any $t>0$ and $x\in K$. Therefore, using *Lemma* 3.1 we complete *Theorem* 3.2.

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8) If μ is absolutely continuous with respect to ν we denote $\mu\ll\nu$, and if $\mu\ll\nu$ and $\nu\ll\mu$ we denote $\mu\simeq\nu$.

9) $\{\theta_t\}$ stands for the shift of $(\Omega, \mathscr{F}, P_x, \{\mathscr{F}_t\}, x(t))$.

In order to prove Theorem 3.1, we shall prepare some lemmas.

Lemma 3.3. *Let* $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}_x, \{\tilde{\mathscr{F}}_t\}, z(t))$ *be the diffusion process on* $[0, 1]$, *which is defined by the following generator,*

$$B = x(1-x)\frac{d^2}{dx^2} + \gamma x(1-x)\frac{d}{dx}, \quad \gamma \in R^1. \tag{3.10}$$

Then, $\tilde{P}_z[\tilde{\sigma}_0 < +\infty] = h(\gamma: z)$, *and* $\tilde{P}_z[\tilde{\sigma}_1 < +\infty] = 1 - h(\gamma: z)$, *where* $\tilde{\sigma}_0$ *and* $\tilde{\sigma}_1$ *are the first hitting times for* $\{0\}$ *and* $\{1\}$, *respectively, and*

$$h(\gamma: z) = \begin{cases} (\exp(-\gamma z) - \exp(-\gamma))/(1 - \exp(-\gamma)) & \text{if } \gamma \neq 0 \\ 1 - z & \text{if } \gamma = 0. \end{cases}$$

Proof. See Ito-McKean [7].

The following lemma is due to Ikeda-Watanabe [5], which is a modification of the comparison theorem by Yamada [15].

Lemma 3.4. *Suppose that we are given the following,*

(i) *a real measurable function* $a(t, x)$ *defined on* $[0, \infty) \times R$ *such that*

$$|a(t, x) - a(t, y)| \leq \rho(|x-y|), \qquad x, y \in R, \ t \geq 0, \tag{3.11}$$

where $\rho(u)$ *is an increasing function on* $[0, \infty)$ *such that* $\rho(0) = 0$ *and*

$$\int_{0+} \rho(u)^{-2} du = +\infty, \tag{3.12}$$

(ii) *real measurable functions* $b_1(t, x)$ *and* $b_2(t, x)$ *defined on* $[0, \infty) \times R$ *such that for* $i = 1$, *or* $i = 2$,

$$|b_i(t, x) - b_i(t, y)| \leq \kappa(|x-y|), \qquad x, y \in R, \ t \geq 0, \tag{3.13}$$

where $\kappa(u)$ *is an increasing concave function on* $[0, \infty)$ *such that* $\kappa(0) = 0$ *and*

$$\int_{0+} \kappa(u)^{-1} du = +\infty. \tag{3.13'}$$

Let $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\})$ *be a complete probability space with right continuous increasing family* $\{\mathscr{F}_t\}_{t \geq 0}$ *of sub-fields of* $\mathscr{F}$, *each containing P-null sets and suppose that we are given the following stochastic processes defined on it,*

(iii) *two real* $\{\mathscr{F}_t\}$*-adapted continuous processes* $x_1(t, \omega)$ *and* $x_2(t, \omega)$,

(iv) *a one-dimensional* $\{\mathscr{F}_t\}$*-Brownian motion* $B(t, \omega)$ *with* $B(0) = 0$,

(v) *two real* $\{\mathscr{F}_t\}$*-adapted well-measurable processes* $c_1(t, \omega)$ *and* $c_2(t, \omega)$.

We assume that they satisfy the following conditions with probability one.

$$x_i(t) - x_i(0) = \int_0^t a(s, x_i(s)) dB(s) + \int_0^t (b_i(s, x_i(s)) + c_i(s)) ds, \qquad i = 1, 2, \tag{3.14}$$

(3.15) $x_1(0) \leq x_2(0)$,

(3.16) $b_1(t, x) \leq b_2(t, x), \qquad x \in R, \ t \geq 0,$

(3.17) $c_1(t, \omega) \leq c_2(t, \omega), \qquad t \geq 0.$

Then, with probability one, we have

(3.18) $x_1(t, \omega) \leq x_2(t, \omega)$ *for all* $t \geq 0$.

Here, we will present a simple proof. Let us assume (3.13) holds for $b_1(t, x)$. Let $\psi_n(u)$ be a non-negative continuous function such that its support is contained in (a_n, a_{n-1}), $\int_{a_n}^{a_{n-1}} \psi_n(u)du = 1$, and $\psi_n(u) \leq \frac{2}{n}\rho(u)^{-2}$, where the sequence $a_0 = 1 > a_1 > \cdots > a_n > \to 0$ is defined by $\int_{a_n}^{a_{n-1}} \rho(u)^{-2}du = n$. Let $\phi_n(u) = \int_0^u dy \int_0^y \psi_n(z)dz$ if $u > 0$, and $\phi_n(u) = 0$ otherwise. Then, $\phi_n \in C^2(R)$, $\phi_n(u) \nearrow u^+$ as $n \to \infty$, and $0 \leq \phi_n'(u) \leq I_{R_+}(u)$. Using Ito's formula, we have

(3.19) $\phi_n(x_1(t) - x_2(t))$

$$= \text{a martingale} + \frac{1}{2}\int_0^t \phi_n''(x_1(s) - x_2(s))(a(s, x_1(s)) - a(s, x_2(s))^2 ds$$

$$+ \int_0^t \phi_n'(x_1(s) - x_2(s))(b_1(s, x_1(s)) - b_2(s, x_2(s)) + c_1(s) - c_2(s))ds.$$

Notice that $\phi_n''(x_1 - x_2)(a(s, x_1) - a(s, x_2))^2 \leq \frac{2}{n}$, and

$$\phi_n'(x_1 - x_2)(b_1(s, x_1) - b_2(s, x_2)) \leq \kappa((x_1 - x_2)^+).$$

Taking the expectation and letting $n \to \infty$ in (3.19), we have

$$(3.20) \quad E[(x_1(t) - x_2(t))^+] \leq \int_0^t E[\kappa((x_1(s) - x_2(s))^+]ds \leq \int_0^t \kappa(E[(x_1(s) - x_2(s))^+])ds.$$

Hence, it follows easily that $E[(x_1(t) - x_2(t))^+] = 0$ for all $t \geqq 0$ and this implies (3.18).

Lemma 3.5. *Let $C \subset I$ be a non-recurrent equivalent class. Then,*

$$P_x[\zeta_C < +\infty \quad \textit{and} \quad \sum_{i \in C} x_i(t) = 0 \quad \textit{for any} \quad t \geq \zeta_C] = 1,$$

where $\zeta_C = \inf\{t > 0; \sum_{i \in C} x_i(t) = 0\}$ or $+\infty$ if $\{\ \} = \phi$.

Proof. It is sufficient to show this for any non-recurrent class C such that $k \to i$ does not hold for any $i \in C$ and any $k \notin C$. Let $y(t) = \sum_{i \in C} x_i(t)$. Applying the martingale relation to $f(\sum_{i \in C} x_i)$ with $f \in C^2(R)$, we have

$$f(y(t)) - \int_0^t y(s)(1 - y(s))f''(y(s))ds - \int_0^t \sum_{i \in C} b_i(x(s))f'(y(s))ds$$

is a P_x-martingale. This implies that there exists a one-dimensional $\{\mathscr{F}_t\}$-Brownian motion $\{B(t; \omega)\}$ such that

$$(3.21) \qquad y(t) - y(0) = \int_0^t \sqrt{2y(s)(1 - y(s))}dB(s) + \int_0^t \sum_{i \in C} b_i(x(s))ds.$$

Here we note that

$$\text{(3.22)} \qquad \sum_{i\in C} b_i(x) = \sum_{k\in C} (\sum_{i\in C} \lambda_{ki}) x_k + \sum_{i\in C} x_i(\gamma_i(x) - \sum_{k=0}^{d} \gamma_k(x) x_k)$$

$$\leq \bar{\gamma} y(1-y) \quad \text{where} \quad y = \sum_{i\in C} x_i \quad \text{and} \quad \bar{\gamma} = \max_{i\in C} \max_{x\in K} \gamma_i(x).$$

Let us consider the diffusion process on [0, 1] defined by the following stochastic differential equation

$$\text{(3.23)} \qquad z(t) - y(0) = \int_0^t \sqrt{2z(s)(1-z(s))}\, dB(s) + \int_0^t \bar{\gamma} z(s)(1-z(s)) ds.$$

Then, applying *Lemma* 3.4 to (3.21) and (3.23), we get

$$P_x[y(t) \leq z(t) \quad \text{for all} \quad t \geq 0] = 1 \quad \text{with} \quad y(0) = \sum_{i\in C} x_i.$$

Hence, it follows from *Lemma* 3.3

$$\text{(3.24)} \qquad P_x[\zeta_C < +\infty] \geq h(\bar{\gamma}, \sum_{i\in C} x_i) > 0 \quad \text{if} \quad \sum_{i\in C} x_i < 1.$$

Next, we shall show that denoting $\tau = \inf\{t \geq 0; \sum_{i\in C} x_i(t) < 1\}$,

$$\text{(3.25)} \qquad P_x[\tau = 0] = 1 \quad \text{for any } x \text{ with} \quad \sum_{i\in C} x_i = 1.$$

It follows from (3.21)

$$\text{(3.26)} \qquad E_x\left[\int_0^{t\wedge\tau} \sum_{i\in C} b_i(x(s)) ds\right] = 0 \quad \text{if} \quad \sum_{i\in C} x_i = 1.$$

Noting (3.22), we see

$$\text{(3.27)} \qquad E_x\left[\int_0^{t\wedge\tau} \sum_{k\in C} (\sum_{i\in C} \lambda_{ki}) x_k(s) ds\right] = 0.$$

Since C is non-recurrent, there exists a $k_0 \in C$ such that $\sum_{i\in C} \lambda_{k_0 i} < 0$. So, (3.27) implies

$$\text{(3.28)} \qquad x_{k_0}(s) = 0 \quad \text{for any} \quad s \leq \tau \ P_x\text{-a.s. on } [\tau > 0].$$

Also, noting

$$\text{(3.29)} \qquad E_x[x_{k_0}(t\wedge\tau)] - x_{k_0} = E_x\left[\int_0^{t\wedge\tau} b_{k_0}(x(s)) ds\right],$$

$$\text{(3.30)} \qquad E_x\left[\int_0^{t\wedge\tau} \sum_{k\in C} \lambda_{kk_0} x_k(s) ds\right] = 0 \text{ holds.}$$

Thus, if $k_1 \rightrightarrows k_0$, $x_{k_1}(s) = 0$ for all $s \leq \tau$ P_x-a.s. on $[\tau > 0]$. Since C is an equivalent class, it holds that for any $i \in C$

$$\text{(3.31)} \qquad x_i(s) = 0 \quad \text{for all} \quad s \leq \tau \ P_x\text{-a.s.} \quad \text{on} \quad [\tau > 0].$$

Therefore we obtain (3.25).
It follows from (3.24) and (3.25) that

(3.32) $$P_x[\zeta_C<+\infty]>0 \quad \text{for any} \quad x\in K^*.$$

Moreover, we can show

(3.33) $$\inf_{x\in K^*} P_x[\zeta_C<+\infty]>0.$$

For, let $f(x)$ be a non-negative continuous function such that $f(x)>0$ for any $x\in K^*$ with $\sum_{i\in C} x_i<1$.
Then (3.25) implies

(3.34) $$T_t f(x)=E_x[f(x(t))]>0 \quad \text{for any} \quad x\in K^* \quad \text{and} \quad t>0.$$

Noting that K^* is compact and $\{T_t\}$ is a Feller semigroup, it follows that Dini's theorem that there exist ε $(0<\varepsilon<1)$ and a continuous function $f_\varepsilon(x)$ defined on K^* such that $1\geq f_\varepsilon(x)>0$ if $\sum_{i\in C} x_i<1-\varepsilon$, and $f_\varepsilon(x)=0$ otherwise, and $T_t f_\varepsilon(x)>0$ for any $x\in K^*$ and $t>0$, $(\varepsilon>0)$. So, we have

$$\inf_{x\in K^*} P_x[\tau_\varepsilon<+\infty]\geq \inf_{x\in K^*} P_x[\sum_{i\in C} x_i(t)<1-\varepsilon]\geq \min_{x\in K^*} T_t f_\varepsilon(x)>0$$

where $\tau_\varepsilon=\inf\{t\geq 0;\ \sum_{i\in C} x_i(t)<1-\varepsilon\}$. (3.33) follows from this and (3.24). Also, it is clear from (3.33) that

(3.35) $$P_x[\zeta_C<+\infty]=1 \text{ holds for any } x\in K^*.$$

Furthermore, it is easy to see from (3.21) that

$$P_x[\sum_{i\in C} x_i(t)=0 \quad \text{for all} \quad t\geq\zeta_C]=1 \quad \text{for any} \quad x\in K^*.$$

Thus, we complete the proof of *Lemma* 3.5.

Now, we shall prove *Theorem* 3.1. Let R be a recurrent class. First, we note that $P_x[x(t)\in K(R)$ for all $t\geq 0]=1$ if $x\in K(R)$. Let $y(t)=\sum_{i\in R} x_i(t)$. Noting $\sum_{i\in R}\lambda_{ki}=0$ for any $k\in R$, we see

$$|E_x[y(t)]-1|=\left|\int_0^t E_x[\sum_{i\in R} x_i(s)(\gamma_i(x(s))-\sum_{k=0}^{d}\gamma_k(x(s))x_k(s)]ds\right|$$

$$\leq\bar{\gamma}\int_0^t |E_x[y(s)-1]|ds \quad \text{where} \quad \bar{\gamma}=\max_{0\leq k\leq d}\max_{x\in K^*}|\gamma_k(x)|.$$

Hence, $P_x[y(t)=1$ for all $t\geq 0]=1$ for any $x\in K(R)$.

Also, if we restrict our consideration on $K(R)$, (i) follows from *Theorem* 3.2, since $\{b_i(x)\}_{i\in R, x\in K(R)}$ satisfies the *condition II* by the recurrence of R.

Let $K_r=\{x\in K^*;\ \sum_{i=1}^{r}\sum_{j\in R_i} x_j=1\}$ and denote by σ_r the first hitting time for K_r. Then, by *Lemma* 3.5,

$$P_x[\sigma_r<+\infty \quad \text{and} \quad x(t)\in K_r \quad \text{for all} \quad t\geq\sigma_r]=1 \quad \text{for any} \quad x\in K^*.$$

Let $x\in K_r$ and set $y_i(t)=\sum_{j\in R_i} x_j(t)$. Then, noting $\sum_{k\in R_i}\sum_{j\in R_i}\lambda_{kj}=0$, $y_i(t)$ is represented in the same way as (3.21) by a one-dimensional $\mathscr{F}_t$-Brownian motion $\{B(t, \omega)\}$,

(3.36) $y_i(t)-y_i(0)$

$$=\int_0^t \sqrt{2y_i(s)(1-y_i(s))}\,dB(s)+\int_0^t \sum_{j\in R_i} x_j(s)(\gamma_j(x(s))-\sum_{k=0}^d \gamma_k(x(s))x_k(s))ds.$$

Notice that $\sum_{j\in R_i} x_j(\gamma_j(x)-\sum_{k=0}^d \gamma_k(x)x_k)\geq \hat{\gamma}_i y_i(1-y_i)$ for some $\hat{\gamma}_i\in R^1$. Applying *Lemma* 3.3 and *Lemma* 3.4, we obtain

$$P_x[\xi_{R_i}<+\infty]\geq 1-h(\hat{\gamma}_i, y_i),\quad y_i=\sum_{j\in R_i} x_j. \tag{3.37}$$

So, we have

$$P_x[\xi_{R_i}<+\infty \text{ for some } 1\leq i\leq r]\geq(1-h(\hat{\gamma}_1, y_1))\vee\cdots\vee(1-h(\hat{\gamma}_r, y_r)),$$

and this implies

$$\inf_{x\in K^*} P_x[\xi_{R_i}<+\infty \quad \text{for some} \quad 1\leq i\leq r]>0. \tag{3.38}$$

Accordingly, (ii) follows from (3.38). Also, (iii) is evident by (i) and (ii).

§ 4. Hitting on subsets of the boundary ∂K^*

For each subset H of $I=\{0, 1,\dots, d\}$, let us denote $V_H=\{x\in K^*;\ \sum_{i\in H} x_i=1\}$. In this section we shall obtain a necessary and sufficient condition for the A-diffusion process on K^* $(\Omega, \mathscr{F}, P_x, \{\mathscr{F}_t\}, x(t))$ to hit on any relatively open subset of V_H with positive probability. Let

$$A=\sum_{i=1}^d \sum_{j=1}^d x_i(\delta_{ij}-x_j)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}. \tag{4.1}$$

From now on, we suppose that $\{b_i(x)\}_{1\leq i\leq d}$ satisfy (1.5) and Lipschitz continuous on K and that the (K, A, x)-martingale problem is uniquely solvable for any $x\in K$.

Let $b_0(x)=-\sum_{i=1}^d b_i(x)$. For each measurable subset $E\subset K^*$, $\sigma(E)$ denotes the first hitting time on E. Then, we have the following.

Theorem 4.1. *Let $\phi\neq H\subsetneqq I$ and let U be any open subset of K^* with $U\cap V_H\neq\phi$.*

(i) *If* $\sup\{\sum_{i\in H} b_i(x);\ x\in U\cap V_H\}>-1$, *then* $P_x[\sigma(U\cap V_H)<+\infty]>0$ *for any* $x\in \mathring{K}$.

(ii) *Conversely, if* $\sup\{\sum_{i\in H} b_i(x);\ x\in U\cap V_H\}\leq -1$, *then* $P_x[\sigma(U\cap V_H)=+\infty]=1$ *for any* $x\in \mathring{K}^*$.

For the proof we shall use the following fact.

Lemma 4.1. *Let $(\tilde{\Omega}, \tilde{F}, \tilde{P}_z, \{\tilde{F}_t\}, z(t))$ be the diffusion process on R^1 defined by the following generator,* (4.2) $$B=|z(1-z)|\frac{d^2}{dz^2}+(\alpha+\beta(1-z))\frac{d}{dz}, \quad \alpha, \beta\in R^1.$$

Then, if $\alpha>-1$, $\tilde{P}_z[\sigma_{\{1\}}<+\infty]>0$ *for any* $0<z<1$, *and* $\lim_{z\uparrow 1} \tilde{P}_z[\sigma_{\{1\}}<+\infty]=1$. *Conversely, if* $\alpha\leq -1$, $\tilde{P}_z[\sigma_{\{1\}}=+\infty]=1$ *for any* $0<z<1$, *where* $\sigma_{\{1\}}$ *stands for the first hitting time on* $\{1\}$.

Proof. It is immediate from the theory of one-dimensional diffusion processes. (cf. [6]).

Proof of Theorem 4.1.

First, we assume $\sup\{\sum_{i\in H} b_i(x);\, x\in U\cap V_H\}>-1$. Then, there exist $\beta>-1$, $\gamma>0$, and $\bar{x}\in U\cap V_H$ such that $\{x\in K^*;\, |x-\bar{x}|<\gamma\}\subset U$ and

(4.3) $$\sum_{i\in H} b_i(x)>\beta \quad \text{for any} \quad x\in K^* \quad \text{with} \quad |x-\bar{x}|<\gamma.$$

Let $\beta>\eta$ and $0>\eta>-1$, and set

(4.4) $$w(x)=\int_{\sum_{i\in H} x_i}^1 \exp\left(\int_{\frac{1}{2}}^u \frac{-\eta}{v(1-v)}\,dv\right)du+|x-\bar{x}|^2.$$

It is easy to see that $w(x)\in C^2(K^*\backslash V_H)$ and for some $0<\delta<\gamma$,

(4.5) $$Aw(x)\leq 0 \quad \text{for any} \quad x\in K^*\backslash V_H \quad \text{with} \quad |x-\bar{x}|<\delta, \quad \text{and} \quad w(\bar{x})=0.$$

Let us denote by τ_δ the first leaving time from $\{x\in K^*;\, |x-\bar{x}|<\delta\}$, and set $\tau=\tau_\delta\wedge\sigma(V_H)$. Then, it follows from (4.4) and (4.5)

(4.6) $$E_x[|x(t\wedge\tau)-\bar{x}|^2]\leq E_x[w(x(t\wedge\tau))]\leq w(x).$$

So, using Chebyshev's inequality and (4.6), there exists a constant δ_1 such that $0<\delta_1<\delta$ and

(4.7) $$P_x\left[|x(t\wedge\tau)-\bar{x}|>\frac{\delta}{2}\right]\leq\frac{1}{2} \text{ for any } x\in K^*\backslash V_H \text{ with } |x-\bar{x}|<\delta_1 \text{ and any } t>0.$$

Also,

$$\begin{aligned}\frac{1}{2}&\leq P_x\left[|x(t\wedge\tau)-\bar{x}|\leq\frac{\delta}{2}\right]\\ &\leq P_x\left[|x(t)-\bar{x}|\leq\frac{\delta}{2},\, t<\tau\right]+P_x\left[|x(\tau)-\bar{x}|\leq\frac{\delta}{2},\, \tau\leq t\right]\\ &\leq P_x[t<\tau_\delta]+P_x[\sigma(V_H)<\tau_\delta].\end{aligned}$$

Letting $t\to\infty$, we see

(4.8) $$P_x[\tau_\delta=+\infty]+P_x[\sigma(V_H)<\tau_\delta]\geq\frac{1}{2} \text{ for any } x\in K^*\backslash V_H \text{ with } |x-\bar{x}|<\delta_1.$$

Let $y(t)=\sum_{i\in H} x_i(t)$. Then, there exists a one-dimensional $\{\mathscr{F}_t\}$-Brownian motion $\{B(t;\omega)\}$ such that

(4.9) $$y(t)-y(0)=\int_0^t \sqrt{2y(s)(1-y(s))}\,dB(s)+\int_0^t \sum_{i\in H} b_i(x(s))\,ds$$
$$=\int_0^t \sqrt{2y(s)(1-y(s))}\,dB(s)+\int_0^t (\beta+c(s))\,ds,$$

where $c(s)=\sum_{i\in H} b_i(x(s))-\beta>0$ P_x-a.s. on $[t<\tau_\delta]$.

On the other hand, let us consider the diffusion process on R^1 defined by the following stochastic differential equation,

(4.10) $$z(t)-y(0)=\int_0^t \sqrt{2z(s)(1-z(s))}\,dB(s)+\beta t.$$

By *Lemma* 3.4,

(4.11) $$P_x[y(s)\geq z(s) \quad \text{for all} \quad s\in[0, \tau_\delta)]=1.$$

Also,

(4.12) $$\begin{aligned} P_x[\sigma(V_H)<\tau_\delta]&=P_x[y(s)=1 \quad \text{for some} \quad s\in[0, \tau_\delta)] \\ &\geq P_x[z(s)=1 \quad \text{for some} \quad s\in[0, \tau_\delta)] \\ &\geq P_x[z(s)=1 \quad \text{for some} \quad s\in[0, \infty)]-P_x[\tau_\delta<+\infty]. \end{aligned}$$

Combining (4.8) and (4.12), we have

(4.13) $$2P_x[\sigma(V_H)<\tau_\delta]\geq P_x[z(s)=1 \quad \text{for some} \quad s\in[0, \infty)]-\frac{1}{2}.$$

By Lemma 4.1 we can find δ_2 such that $0<\delta_2<\delta_1$ and for any $x\in K^*$ with $y(0)=\sum_{i\in H} x_i>1-\delta_2$

(4.14) $$P_x[z(s)=1 \quad \text{for some} \quad 0\leq s<+\infty]>\frac{1}{2},$$

Then, we have

(4.15) $P_x[\sigma(V_H)<\tau_\delta]>0$ for any $x\in K^*$ satisfying $|x-\bar{x}|<\delta_1$ and $\sum_{i\in H} x_i>1-\delta_2$.

Noting that the A-diffusion process on K^* starting at any interior point of K^* hits on non-empty open subset of K^* with positive probability, (4.15) implies that

(4.16) $P_x[|x(t)-\bar{x}|<\delta$ and $x(t)\in V_H$ for some $t>0]>0$ for any $x\in \mathring{K}^*$.

Thus, we complete the proof of (i).

Next, in order to prove (ii) we suppose $\sup\{\sum_{i\in H} b_i(x); x\in U\cap V_H\}\leq -1$. We may assume that $U=\{x\in K^*; \sum_{i=0}^{d}|x_i-\bar{x}_i|<\delta\}$ for some $\delta>0$ and $\bar{x}=(\bar{x}_0, \bar{x}_1,\dots, \bar{x}_d)\in V_H$, since for any open subset U with $U\cap V_H\neq\phi$, there exists a countably family of such subsets $\{U_n\}$, satisfying $U\cap V_H=\bigcup_{n\geq 1} U_n\cap V_H$.

First, we note that there exists a constant $\beta\in R^1$ such that

(4.17) $$\sum_{i\in H} b_i(x) \leq -1+\beta(1-\sum_{i\in H} x_i) \quad \text{for any} \quad x\in U.$$

For any $x \in U$ define $x' \in V_H$ by

(4.18) $$x'_j = x_j / \sum_{i\in H} x_i \ (j\in H) \quad \text{and} \quad x'_j = 0 \ (j \notin H).$$

Then, it is easy to see $x' \in U$, and by the Lipschitz continuity of $\{b_i(x)\}$ there are some constant $\beta \in R^1$ such that

$$\sum_{i\in H} b_i(x) \leq \sum_{i\in H} b_i(x') + \frac{\beta}{2}\sum_{i=1}^{d} |x_i - x'_i| \leq -1+\beta(1-\sum_{i\in H} x_i).$$

Recall that $y(t)=\sum_{i\in H} x_i(t)$ can be represented by using a $\{\mathscr{F}_t\}$-Brownian motion $\{B(t;\omega)\}$ as follows,

(4.19) $$y(t)-y(0)=\int_0^t \sqrt{2y(s)(1-y(s))}\, dB(s) + \int_0^t \sum_{i\in H} b_i(x(s))ds.$$

Also, let us consider the solution $\{z(t)\}$ of the following equation,

(4.20) $$z(t)-y(0)=\int_0^t \sqrt{2z(s)(1-z(s))}\, dB(s) + \int_0^t (-1+\beta(1-z(s))ds.$$

Then, by (4.17), (4.19), (4.20) and *Lemma* 3.4, we have

(4.21) $$P_x[y(s) \leq z(s) \quad \text{for all} \quad s\in[0,\tau_U)]=1 \qquad \text{for any} \quad x\in U,$$

where τ_U stands for the first leaving time from U. Since *Lemma* 4.1 implies that

(4.22) $$P_x[z(s)<1 \quad \text{for all} \quad s\geq 0]=1 \qquad \text{for any} \quad x\in K^*\backslash V_H,$$

it follows

$$P_x[y(s)<1 \quad \text{for all} \quad s\in[0,\tau_U)]=1 \qquad \text{for any} \quad x\in U\backslash V_H,$$

namely,

(4.23) $$P_x[x(s)\notin V_H \quad \text{for all} \quad s\in[0,\tau_U)]=1 \qquad \text{for any} \quad x\in U\backslash V_H.$$

On the other hand applying Ito's formula for $1-y(t)$ and $\{\psi_n\}$ in the proof of *Lemma* 3.2, we can easily see

(4.24) $$P_x\left[\int_0^\infty I_{V_H\cap U}(x(s))ds=0\right]=1 \qquad \text{for any} \quad x\in K^*.$$

Let W be any open set such that $\overline{W}\subset U$. Since (4.23) and (4.24) imply that $x(t)$ goes out of U before it attains $W\cap V_H$, it follows that

(4.25) $$P_x[\sigma(W\cap V_H)=+\infty]=1 \text{ holds for any } x\in \mathring{K}^*,$$

and after all this implies that $P_x[\sigma(U\cap V_H)=+\infty]=1$ holds for any $x\in\mathring{K}^*$.

Corollary 4.1. *Suppose that* $\{b_i(x)\}$ *satisfies* (3.2). *Then,* $P_x[\sigma(V_H)<+\infty]>0$

for any $x \in \mathring{K}^*$, *if and only if* $\max_{k\in H} \sum_{i\in H} \lambda_{ki} > -1$. *In particular, denoting by* e_i *the i-th vertex of* K^*, *i.e.* $(e_i)_j = \delta_{ij}$, $P_x[\sigma(\{e_i\}) < +\infty] > 0$ *holds for any* $x \in \mathring{K}^*$, *if and only if* $\lambda_{ii} > -1$.

Remark 4.1. Suppose $b_i(x) = \beta_i - (\sum_{j=0}^{d} \beta_j) x_i$ $(0 \le i \le d)$ with $\beta_i \ge 0$. In this case Shimakura constructed the Poisson kernel associated with A. (cf. [14], §8). His results imply that if $E \subset \partial K^*$ is a null set with respect to the volume element of ∂K^*, (4.26) $P_x[x(\sigma) \in E, \sigma < +\infty] = 0$ holds for any $x \in \mathring{K}^*$, where $\sigma = \sigma(\partial K^*)$. Accordingly, by *Theorem* 4.1 we can see that if $|H| \le d-1$,

(4.27) $P_x[x(t) \in \partial K^*$ infinitely often as $t \uparrow \sigma(V_H) \mid \sigma(V_H) < +\infty] = 1$ holds for any $x \in \mathring{K}^*$.

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