J. Math. Kyoto Univ. (JMKYAZ)
21-1 (1981) 105–123

Optimal stopping rules for jump processes

By

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(Communicated by Prof. S. Watanabe, Oct. 12, 1979)

1. Introduction

Let X_t, $t \in R^+$, be a stochastic process (which we shall call a reward process) defined on a probability space $(\Omega, \mathscr{F}, P)$ and $\{\mathscr{F}_t\}$ be an increasing family of sub-σ-fields of $\mathscr{F}$. The set of all finite $\mathscr{F}_t$-stopping times τ for which at least one of X_τ^+ and X_τ^- is integrable will be denoted by $\mathscr{M}$. Assuming that X_t is adapted, i.e., X_t is $\mathscr{F}_t$-measurable for each t, and is measurable, an optimal stopping problem is to find a stopping time $\tau^* \in \mathscr{M}$ such that

$$\sup_{\tau \in \mathscr{M}} EX_\tau = EX_{\tau^*}.$$

The purpose of this paper is to study the above optimal stopping problem when the reward process has a special structure. Suppose that the reward process X_t has the following properties:

(A1) There exists an $\mathscr{F}_t$-adapted measurable process $f_t(\omega)$ such that, for any $T \in \mathscr{M}$,

$$EX_T = EX_0 + E\int_0^T f_s \mathrm{d}s$$

and $EX_0 < \infty$.

(A2) There exists a stopping time τ^* such that $f_t \geq 0$ for $t \leq \tau^*$ and $f_t \leq 0$ for $t > \tau^*$.

Then the optimal stopping problem reduces to the problem of finding a stopping time in $\mathscr{M}$ to attain the value

$$\sup_{\tau \in \mathscr{M}} E\int_0^T f_s \mathrm{d}s,$$

and τ^* defined in (A2) is an optimal stopping time if τ^* is finite.

For a jump process X_t with a local description $\{n(A, t, \omega), \Lambda(t)\}$ in the sense defined in Section 2 such that $\Lambda(t)$ is absolutely continuous, that is,

$$\Lambda(t) = \int_0^t \lambda(s, \omega)\mathrm{d}s,$$

it is known that

$$EX_T = EX_0 + E\int_0^T\int_R (x - X_{s-})n(\mathrm{d}x, s, \omega)\lambda(s, \omega)\mathrm{d}s$$

for any stopping time $T \in \mathscr{M}$ under appropriate assumptions. This implies that a jump process X_t can be expressed in the form as in (A1) using its local description and hence (A1) is usually satisfied. Our main concern in this paper is, therefore, to discuss some sufficient conditions for (A2) in terms of characteristics of X_t.

The situation considered here is a continuous-time version of Chow and Robbins [2] where discrete-time parameter reward processes were studied. In contrast to the discrete-time parameter case, the process f_t in (A1) is not directly obtained for general continuous-time parameter processes. For jump processes, once we are given their local description which is the usual situation in applications f_t is directly obtained as above and this is the reason why we treat jump type reward processes. The case of continuous-time parameter Markov processes was discussed in Ross [3].

In Section 2 a summary of results on the "local description" of jump processes in connection with the condition (A1) will be presented. This section is based on some recent results in martingale approach to jump processes.

In applications, "local characteristics" are usually given as data to describe the evolution of jump processes. Then our concern is to obtain local descriptions for jump processes from such data since we must know local descriptions of jump processes to get the process f_t in (A1). This problem is discussed in Section 3.

In Section 4 we shall discuss some conditions under which the reward process of jump type satisfies (A2). Several examples will be given in Section 5.

2. Preliminary facts

In this section we shall present a summary of results required in the sequel.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with an increasing family of sub-σ-fields $\{\mathscr{F}_t, t \in [0, \infty]\}$ of $\mathscr{F}$ which satisfies the usual condition, i.e., $\{\mathscr{F}_t\}$ is right continuous and $\mathscr{F}_0$ contains all the negligible sets of $\mathscr{F}$.

An $\mathscr{F}_t$-adapted stochastic process X_t defined on $(\Omega, \mathscr{F}, P)$ and taking values in R is called a jump process if, for each $\omega \in \Omega$, the sample path $t \to X_t(\omega)$ is piecewise constant, right continuous and has only a finite number of jumps in every finite interval.

Let $\mu(\omega, dt, dx)$ be a random measure on $[0, \infty) \times R$ defined by

$$\mu(\omega; (0, t], B) = \sum_{s \leq t} I_{\{X_{s-} \neq X_s\}} I_{\{X_s \in B\}}, \quad B \in \mathscr{B}(R). \tag{1}$$

Let $\mathscr{P}$ be the σ-field of predictable sets of $\Omega \times [0, \infty)$ generated by the applications $(\omega, t) \to Y_t(\omega)$ which are $\mathscr{F}_t$-measurable in ω and left-continuous in t. A real-valued process X_t is predictable if the application $(\omega, t) \to X_t(\omega)$ is $\mathscr{P}$-measurable. A random measure $\nu(\omega; dt, dx)$ on $[0, \infty) \times R$ is called a (dual) *predictable projection* of the random measure μ if, for any nonnegative $\mathscr{P} \times \mathscr{B}(R)$-measurable function $X(\omega, t, x)$, the process defined by

$$(\nu X)_t(\omega)=\int_0^t\int_R X(\omega, s, x)\nu(\omega; \mathrm{d}s, \mathrm{d}x)$$

is predictable and

$$E\int_0^\infty\int_R X(\omega, s, x)\mu(\omega; \mathrm{d}s, \mathrm{d}x)=E\int_0^\infty\int_R X(\omega, s, x)\nu(\omega; \mathrm{d}s, \mathrm{d}x). \tag{2}$$

According to Theorem 2.1 in Jacod [4], there exists one and only one (up to $\mathscr{P}$-equivalence) such predictable projection ν of the random measure μ defined by (1). For any $A\in\mathscr{B}(R)$, the measure $\nu(\omega; \mathrm{d}t, A)$ is absolutely continuous with respect to $\nu(\omega; \mathrm{d}t, R)=\Lambda(\omega, \mathrm{d}t)$ and hence there exists a measurable function $n(A, t, \omega)$ such that

$$\nu(\omega; [0, t], A)=\int_0^t n(A, s, \omega)\Lambda(\omega, \mathrm{d}t).$$

Then by the usual discussion we can choose $n(A, t, \omega)$ in such a way that (i) for each $A\in\mathscr{B}(R)$, $n(A, \cdot, \cdot)$ is measurable, and (ii) for each $\omega\in\Omega$, $n(\cdot, t, \omega)$ is a probability measure on $(R, \mathscr{B}(R))$ for all t except a set of $\mathrm{d}\Lambda$-measure 0. The pair of stochastic measures $(n(\mathrm{d}x, t, \omega), \Lambda(\omega, \mathrm{d}t)\}$ is called a *local description* of jump process X_t or a Lévy system.

Now using these notions we have the following proposition which is more or less a known result (see Yor [6], for example).

Proposition 1. *Let $f(\cdot, \cdot, \cdot)\colon \Omega\times[0, \infty)\times R\to R$ be a $\mathscr{P}\times\mathscr{B}(R)$-measurable function. Then under the condition that either*

$$E\sum_{s<\infty}(f(\omega, s, X_s)-f(\omega, s, X_{s-}))^+<\infty$$

or

$$E\sum_{s<\infty}(f(\omega, s, X_s)-f(\omega, s, X_{s-}))^-<\infty\,, \tag{3}$$

where $a^\pm$ denote the positive and negative parts of a respectively, for any $\mathscr{F}_t$-stopping time T we have

$$\begin{aligned}E&\sum_{s\le T}(f(\omega, s, X_s)-f(\omega, s, X_{s-}))\\ &=E\int_0^T\int_R(f(\omega, s, x)-f(\omega, s, X_{s-}))\nu(\omega; \mathrm{d}s, \mathrm{d}x)\\ &=E\int_0^T\int_R(f(\omega, s, x)-f(\omega, s, X_{s-}))n(\mathrm{d}x, s, \omega)\Lambda(\omega, \mathrm{d}s).\end{aligned}$$

For the proof of this proposition it is enough to note the following facts: First for any stopping time T and for any non-negative $\mathscr{P}\times\mathscr{B}(R)$-measurable function $Y(\omega, t, x)$, $\tilde{Y}(\omega, t, x)=Y(\omega, t, x)I_{[0,T]}(\omega, t)$ is also $\mathscr{P}\times\mathscr{B}(R)$-measurable and hence from (2)

$$E\int_0^T\int_R Y(\omega, s, x)\nu(\omega; \mathrm{d}s, \mathrm{d}x)=E\int_0^T\int_R Y(\omega, s, x)\mu(\omega; \mathrm{d}s, \mathrm{d}x).$$

We also note that

$$\sum_{s\leq t}(f(\omega, s, X_s)-f(\omega, s, X_{s-}))$$
$$=\int_0^t\int_R(f(\omega, s, x)-f(\omega, s, X_{s-}))\mu(\omega; \mathrm{d}s, \mathrm{d}x)$$

and that condition (3) implies either

$$E\int_0^T\int_R(f(\omega, s, x)-f(\omega, s, X_{s-}))^+\mu(\omega; \mathrm{d}s, \mathrm{d}x)<\infty$$

or

$$E\int_0^T\int_R(f(\omega, s, x)-f(\omega, s, X_{s-}))^-\mu(\omega; \mathrm{d}s, \mathrm{d}x)<\infty.$$

As a direct consequence of the above proposition, if $f(\cdot, \cdot, \cdot)$ is differentiable with respect to the second argument, then under the same condition (3) in Proposition 1 and the condition $E|f(\omega, 0, X_0)|+E\left|\int_0^T f'(\omega, s, X_{s-})\mathrm{d}s\right|<\infty$ we have the following result: for any stopping time T

$$Ef(\omega, T, X_T)=Ef(\omega, 0, X_0)+E\int_0^T f'(\omega, s, X_{s-})\mathrm{d}s$$
$$+E\int_0^T\int_R(f(\omega, s, x)-f(\omega, s, X_{s-}))n(\mathrm{d}x, s, \omega)\Lambda(\mathrm{d}s) \tag{4}$$

where $f'(\omega, s, x)=\frac{\partial}{\partial s}f(s, \omega, x)$.

Let Y_t be another $\mathscr{F}_t$-adapted jump process with the local description $\{n'(\mathrm{d}x, t, \omega), \Lambda'(\mathrm{d}t)\}$. Then we have the following proposition.

Proposition 2. *Suppose that X_t and Y_t are both increasing and have no common discontinuities. Also suppose that a measurable function $f(\cdot, \cdot)$: $R\times R\to R$ satisfies the condition that $f(\cdot, y)$ is increasing for each y and $f(x, \cdot)$ is decreasing for each x. Then under the condition that either*

$$E\sum_{s<\infty}(f(X_s, Y_s)-f(X_{s-}, Y_s))<\infty$$

or

$$E\sum_{s<\infty}(f(X_s, Y_s)-f(X_s, Y_{s-}))>-\infty,$$

we have, for any $\mathscr{F}_t$-stopping time T,

$$Ef(X_T, Y_T)=Ef(X_0, Y_0)+E\int_0^T\int_R(f(x, Y_{s-})-f(X_{s-}, Y_{s-}))n(\mathrm{d}x, s, \omega)\Lambda(\mathrm{d}s)$$
$$+E\int_0^T\int_R(f(X_{s-}, y)-f(X_{s-}, Y_{s-}))n'(\mathrm{d}y, s, \omega)\Lambda'(\mathrm{d}s)$$

where we assume $Ef(X_0, Y_0)<\infty$.

Proof. The proof is direct if we note that

$$f(X_t, Y_t)=f(X_0, Y_0)+\sum_{s\leq t}(f(X_s, Y_s)-f(X_{s-}, Y_s))+\sum_{s\leq t}(f(X_s, Y_s)-f(X_s, Y_{s-}))$$

since X_t and Y_t have no common discontinuities. Then we have, for any stopping time T,

$$E\sum_{s\leq T}(f(X_s, Y_s)-f(X_{s-}, Y_s))=E\int_0^T\int_R(f(x, Y_s)-f(X_{s-}, Y_s))n(\mathrm{d}x, s, \omega)\Lambda(\mathrm{d}s)$$

and

$$E\sum_{s\leq T}(f(X_s, Y_s)-f(X_s, Y_{s-}))=E\int_0^T\int_R(f(X_s, y)-f(X_s, Y_{s-}))n'(\mathrm{d}y, s, \omega)\Lambda'(\mathrm{d}s)$$

since

$$\sum_{s\leq t}(f(X_s, Y_s)-f(X_{s-}, Y_s)) \quad \text{and} \quad \sum_{s\leq t}(f(X_s, Y_s)-f(X_s, Y_{s-}))$$

are increasing and decreasing processes respectively. Either one of the above expectations is finite and the conclusion follows. Q.E.D.

3. Local characterization of jump processes and local description

Given a jump process X_t, there exists a local description $\{n(\mathrm{d}x, t, \omega), \Lambda(\omega, \mathrm{d}t)\}$. In applications, however, a local characterization of jump processes in the form of (C1) and (C2) below is usually given a priori as data to describe the evolution of jump processes. Our concern then is to obtain local descriptions of jump processes from such data of local characterization.

Here we shall assume

(C0) $\quad \mathscr{F}_t=\sigma(X_s, s\leq t)$, σ-field generated by X_s, $s\leq t$.

For a given jump process X_t, let $N_t=\mu(\omega; [0, t], R)$, i.e., N_t is the number of jumps up to time t. We shall suppose that X_t has the following two properties:

(C1)
$$P(N_{t+h}-N_t=1\mid\mathscr{F}_t)=\lambda(t, \omega)h+o(h)$$
$$P(N_{t+h}-N_t=0\mid\mathscr{F}_t)=1-\lambda(t, \omega)h+o(h)$$

where $\lambda(t, \omega)$ is a positive $\mathscr{F}_t$-adapted predictable process and we assume that there exists a constant K such that

$$\frac{1}{h}P(N_{t+h}-N_t\geq 1\mid\mathscr{F}_t)\leq K \tag{C1}^*$$

(C2)
$$\lim_{h\downarrow 0}\frac{P(X_{t+h}\in A, N_{t+h}-N_t=1|\mathscr{F}_t)}{P(N_{t+h}-N_t=1|\mathscr{F}_t)}=n'(A, t, \omega), \qquad A\in\mathscr{B}(R)$$

where $n'(A, t, \omega)$ is a predictable process for each $A\in\mathscr{B}(R)$.

Remark. The condition (C1)* is technical and is needed only to apply the dominated convergence theorem in the proof of Lemmas 1 and 2. Hence it could be replaced by a more general but more complicated one.

From a probabilistic interpretation of local description we expect that $\{n'(\mathrm{d}x, t, \omega), \Lambda(\omega, \mathrm{d}t)\}$, where

$$\Lambda(\omega, [0, t]) = \int_0^t \lambda(s, \omega)\mathrm{d}s,$$

should be a local description of X_t, and indeed this is true as is shown in Theorem 1. Several lemmas will be required for this.

Let us define $\lambda_t(s, \omega)$ by

$$\lambda_t(s, \omega) = \lim_{h\to 0} \frac{1}{h}\left(1 - \frac{P(N_t = N_{s+h}|\mathscr{F}_t)}{P(N_t = N_s|\mathscr{F}_t)}\right), \qquad s \geq t,$$

when $P(N_t = N_s|\mathscr{F}_t)(\omega) \neq 0$. $\lambda_t(s, \omega)$ may be defined arbitrarily if $P(N_t = N_s|\mathscr{F}_t)(\omega) = 0$. Due to assumption (C1) we have

$$\lambda_t(s, \omega) = \frac{E(I_{\{N_t=N_s\}}(\omega)\lambda(s, \omega)|\mathscr{F}_t)}{E(I_{\{N_t=N_s\}}(\omega)|\mathscr{F}_t)}$$

where $I_A(\omega)$ is an indicator function of set A. For $s<t$ we shall define $\lambda_t(s, \omega)$ by $\lambda_t(s, \omega) = \lambda(s, \omega)$.

Then we have

Lemma 1. *For any $\mathscr{F}_t$-stopping time T*

$$P(N_{T+h} = N_T|\mathscr{F}_T) = \exp\left\{-\int_T^{T+h} \lambda_T(s, \omega)\mathrm{d}s\right\}, \qquad h \geq 0.$$

Proof. The proof can be done in the same way as in the proof of the same fact for Poisson processes. See Rubin [9] where the similar result is given without proof.

(i) First we shall suppose that T is a constant time, i.e., $T=t$. Put, for $s \geq t$,

$$f(s, \omega) = P(N_t = N_s|\mathscr{F}_t).$$

Then, if $f(s, \omega) \neq 0$, we have

$$\frac{1}{h}(f(s+h, \omega) - f(s, \omega)) = -\frac{1}{h}\left(1 - \frac{P(N_t = N_{s+h}|\mathscr{F}_t)}{P(N_t = N_s|\mathscr{F}_t)}\right)f(s, \omega),$$

and, by letting h tend to zero, we get

$$f'(s, \omega) = -\lambda_t(s, \omega)f(s, \omega) \quad \text{a.s.} \tag{5}$$

If $f(s, \omega) = 0$, then clearly we have $f(s+h, \omega) = 0$ for $h>0$, and hence the right derivative of $f(\cdot, \omega)$ at s is 0. Also if $f(s, \omega) = 0$ and $h<0$, then

$$\frac{1}{h}|f(s+h, \omega) - f(s, \omega)| = \frac{1}{h}E\{I_{\{N_t=N_{s+h}\}}E\{I_{\{N_{s+h}=N_s\}^C}|\mathscr{F}_{s+h}\}|\mathscr{F}_t\}$$

$$\leq KE\{I_{\{N_t=N_{s+h}\}}|\mathscr{F}_t\} \longrightarrow KE\{I_{\{N_t=N_{s-}\}}|\mathscr{F}_t\} = 0$$

since $P\{N_s = N_{s-}+1 \mid \mathscr{F}_t\} = 0$ and $P(N_t = N_s \mid \mathscr{F}_t) = 0$. Therefore the left derivative of $f(\cdot, \omega)$ at s is also 0. Thus if $f(s, \omega) = 0$, then $f'(s, \omega) = 0$ and (5) holds a.s. . In any way from the above discussion we have the conclusion that for almost all ω (5) holds for each s. Noting that $f(s, \omega) > 0$ for s sufficiently near to t because of (C1), we obtain

$$f(s, \omega) = \exp\left\{-\int_t^s \lambda_t(u, \omega)\mathrm{d}u\right\}, \qquad s \geq t$$

as a unique solution of (5) i.e.,

$$P(N_{t+h} = N_t \mid \mathscr{F}_t) = \exp\left\{-\int_t^{t+h} \lambda_t(u, \omega)\mathrm{d}u\right\}, \qquad h \geq 0.$$

(ii) Next we supose that T is countably-valued, i.e., T takes its value in $\{t_1, t_2, \ldots\}$. In this case it is sufficient to show that

$$\int_A I_{\{N_T = N_{T+h}\}}\mathrm{d}P = \int_A \exp\left\{-\int_T^{T+h} \lambda_T(s, \omega)\mathrm{d}s\right\}\mathrm{d}P \tag{6}$$

for any $A \in \mathscr{F}_T$, i.e.,

$$\sum_{k=1}^{\infty} \int_{A \cap \{T=t_k\}} I_{\{N_{t_k} = N_{t_k+h}\}}\mathrm{d}P = \sum_{k=1}^{\infty} \int_{A \cap \{T=t_k\}} \exp\left\{-\int_{t_k}^{t_k+h} \lambda_{t_k}(s, \omega)\mathrm{d}s\right\}\mathrm{d}P$$

But, since $A \cap \{T=t_k\} \in \mathscr{F}_{t_k}$, from the result of (i)

$$\int_{A \cap \{T=t_k\}} I_{\{N_{t_k} = N_{t_k+h}\}}\mathrm{d}P = \int_{A \cap \{T=t_k\}} \exp\left\{-\int_{t_k}^{t_k+h} \lambda_{t_k}(s, \omega)\mathrm{d}s\right\}\mathrm{d}P.$$

Hence (6) holds.

(iii) Finally let T be an arbitrary $\mathscr{F}_t$-stopping time. Then there exists a sequence of countably-valued stopping times T_n $(n=1, 2, \ldots)$ such that $T_n \downarrow T$. To see that (6) holds note that $A \in \mathscr{F}_{T_n}$ (since $T_n \geq T$), and hence by the result of (ii)

$$\int_A I_{\{N_{T_n} = N_{T_n+h}\}}\mathrm{d}P = \int_A \exp\left\{-\int_{T_n}^{T_n+h} \lambda_{T_n}(s, \omega)\mathrm{d}s\right\}\mathrm{d}P.$$

Then since N_t is right-continuous and piecewise-constant

$$\int_A I_{\{N_{T_n} = N_{T_n+h}\}}\mathrm{d}P \longrightarrow \int_A I_{\{N_T = N_{T+h}\}}\mathrm{d}P \qquad (n \to \infty).$$

On the other hand, since $\lambda_t(s, \omega)$ is right-continuous in t as we see in the next step (iv) and since $\lambda_{T_n}(s, \omega) \leq K$ (see condition (C1)*) we have

$$\int_A \exp\left\{-\int_{T_n}^{T_n+h} \lambda_{T_n}(s, \omega)\mathrm{d}s\right\}\mathrm{d}P \longrightarrow \int_A \exp\left\{-\int_T^{T+h} \lambda_T(s, \omega)\mathrm{d}s\right\}\mathrm{d}P$$

and hence (6) follows.

(iv) Let us show that we can choose a right-continuous version of $\lambda_t(s, \omega)$ as a function of t for an arbitrarily fixed s. We note that $Z(t) \equiv E[I_{\{N_t = N_s\}}(\omega)\lambda(s, \omega) \mid \mathscr{F}_t]$ and $\tilde{Z}(t) \equiv E[I_{\{N_t = N_s\}}(\omega) \mid \mathscr{F}_t]$ are both submartingales on the interval $[0, s]$ and admit right-continuous modification. Indeed, for $u<t\leq s$,

$$E[Z(t)\mid \mathscr{F}_u]=E[I_{\{N_t=N_s\}}(\omega)\lambda(s,\omega)\mid \mathscr{F}_u]$$
$$\geq E[I_{\{N_u=N_s\}}(\omega)\lambda(s,\omega)\mid \mathscr{F}_u]$$
$$=Z(u).$$

Furthermore the function $t\to EZ(t)$ is right-continuous. Thus $Z(t)$ is a submartingale and admits a right-continuous modification on $[0, s]$. Similarly the same fact holds for $\tilde{Z}(t)$. On the other hand, on $[s,\infty)$ $Z(t)=I_{\{N_t=N_s\}}(\omega)\lambda(s,\omega)$ and $\tilde{Z}(t)=I_{\{N_t=N_s\}}(\omega)$ and they are certainly right-continuous on (s,∞). From these since $\lambda_t(s,\omega)=Z(t)/\tilde{Z}(t)$ the desired result follows. Q.E.D.

Let T_n be the n-th jump time of X_t and put $S_n=T_n-T_{n-1}$. Then as a corollary of the above lemma we have

Corollary. *The conditional distribution of S_{n+1} given $\mathscr{F}_{T_n}$ is*

$$P\{S_{n+1}\leq y\mid \mathscr{F}_{T_n}\}=1-\exp\left\{-\int_{T_n}^{T_n+y}\lambda_{T_n}(s,\omega)ds\right\},$$

and its conditional density $f_n(x)$ given $\mathscr{F}_{T_n}$ is

$$f_n(x)=\lambda_{T_n}(T_n+x,\omega)\exp\left\{-\int_{T_n}^{T_n+x}\lambda_{T_n}(s,\omega)ds\right\}.$$

Lemma 2. *For an arbitrary stopping time T*

$$\lambda_T(s,\omega)=\frac{E[I_{\{N_T=N_s\}}(\omega)\lambda(s,\omega)|\mathscr{F}_T]}{E[I_{\{N_T=N_s\}}(\omega)|\mathscr{F}_T]}$$

Proof. As in Lemma 1, first we consider the case where T is a constant time, then the case where T is countably-valued, and finally the general case. The formal procedure is very much like that of Lemma 1 and hence we shall omit the proof.

Lemma 3. *Given an $\mathscr{F}_t$-predictable process $Y_t(\omega)$, there exists, for each n, a process $Y_t^n(\omega)$ which is $\mathscr{F}_{T_n}$-measurable for each t and*

$$Y_t(\omega)=Y_t^n(\omega)\quad on\quad \{T_n\leq t<T_{n+1}\}.$$

Proof. See Yacod [4, Lemma 3.3].

Corollary. $\lambda_{T_n}(s,\omega)=\lambda(s,\omega)$ *on* $\{T_n\leq s<T_{n+1}\}$.

Proof. From Lemma 2

$$\lambda_{T_n}(s,\omega)=\frac{E[I_{\{N_{T_n}=N_s\}}(\omega)\lambda(s,\omega)|\mathscr{F}_{T_n}]}{E[I_{\{N_{T_n}=N_s\}}(\omega)|\mathscr{F}_{T_n}]}.$$

But since $\lambda(s,\omega)$ is $\mathscr{F}_{T_n}$-measurable on $\{T_n\leq s<T_{n+1}\}$ by Lemma 3, we have

$$\lambda_{T_n}(s,\omega)=\lambda(s,\omega)\quad\text{on}\quad\{T_n\leq s<T_{n+1}\}.$$ Q.E.D.

In the same manner, let us define

$$n_t(B, s, \omega) = \lim_{h\to 0} \frac{P(N_t = N_s,\ N_{s+h} - N_s = 1,\ X_{s+h} \in B | \mathscr{F}_t)}{P(N_t = N_s,\ N_{s+h} - N_s = 1 | \mathscr{F}_t)}.$$

Then we have

$$n_t(B, s, \omega) = \frac{E\{I_{\{N_t = N_s\}}(\omega)\cdot n'(B, s, \omega)\lambda(s, \omega)|\mathscr{F}_t\}}{E\{I_{\{N_t = N_s\}}(\omega)\lambda(s, \omega)|\mathscr{F}_t\}}.$$

Using this relation we can establish the following Lemma 4 which corresponds to Corollary of Lemma 3 and whose proof is similar to the above.

Lemma 4. $n_{T_n}(B, s, \omega) = n'(B, s, \omega)$ *on* $\{T_n \leq s < T_{n+1}\}$.

Now we have

Theorem 1. *Under conditions* (C0–2) *a local description of* $\{n(\mathrm{d}x, t, \omega), \Lambda(\omega, \mathrm{d}t)\}$ *of* X_t *is given by*

$$n(\mathrm{d}x, t, \omega) = n'(\mathrm{d}x, t, \omega)$$

$$\Lambda(\omega, \mathrm{d}t) = \lambda(t, \omega)\mathrm{d}t.$$

Proof. Let $G_n(\omega, \mathrm{d}t, \mathrm{d}x)$ be a regular version of the conditional law of $(S_{n+1}, X_{T_{n+1}})$ given $\mathscr{F}_{T_n}$. Let $H_n(\omega; \mathrm{d}t) = G_n(\omega; \mathrm{d}t, R)$. Then by Jacod [4] predictable projection of μ is given by

$$\nu((0, t] \times B) = \sum_{n\geq 0} I_{\{T_n < t \leq T_{n+1}\}}\left(\sum_{p=0}^{n-1}\int_0^{S_{p+1}} \frac{G_p(\mathrm{d}s, B)}{H_p([s, \infty))} + \int_0^{t-T_n} \frac{G_n(\mathrm{d}s, B)}{H_n([s, \infty))}\right),$$

On the other hand since

$$G_p([0, s], R) = H_p([0, s]) = 1 - \exp\left\{-\int_{T_p}^{T_p+s} \lambda_{T_p}(u, \omega)\mathrm{d}u\right\}$$

by Corollary of Lemma 1, we have

$$\int_0^{S_{p+1}} \frac{G_p(\mathrm{d}s, R)}{H_p([s, \infty))} = \int_0^{S_{p+1}} \lambda_{T_p}(T_p + s, \omega)\mathrm{d}s = \int_0^{S_{p+1}} \lambda(T_p + s, \omega)\mathrm{d}s$$

by Corollary of Lemma 3. Thus

$$\begin{aligned} &\Lambda(\omega, [0, t]) = \nu([0, t], R) \\ &= \sum_{n>0} I_{\{T_n < t \leq T_{n+1}\}}\left(\sum_{p=0}^{n-1}\int_0^{S_{p+1}} \lambda(T_p + s, \omega)\mathrm{d}s + \int_0^{t-T_n} \lambda(T_n + s, \omega)\mathrm{d}s\right) \\ &= \int_0^t \lambda(s, \omega)\mathrm{d}s. \end{aligned}$$

Next we show that $n'(B, t, \omega)$ is a version of local description $n(B, t, \omega)$. To see this we note that the measure $G_p(\cdot, B)$ is absolutely continuous with respect to $G_p(\cdot, R)$ i.e., there exists a measurable function $\tilde{n}_p(B, s, \omega)$ such that

$$G_p((0, t], B)=\int_0^t \tilde{n}_p(B, s, \omega)G_p(\mathrm{d}s, R).$$

Then by defining $\tilde{n}(B, s, \omega)=\tilde{n}_p(B, s-T_p, \omega)$ on $T_p \leq s < T_{p+1}$, we can say that $\tilde{n}(B, t, \omega)$ is a version of local description $n(B, t, \omega)$. Indeed,

$$\begin{aligned} \nu((0, t]\times B) &= \sum_{n>0} I_{\{T_n<t\leq T_{n+1}\}}\left(\sum_{p=0}^{n-1}\int_0^{S_{p+1}} \frac{\tilde{n}_p(B, s, \omega)}{H_p([s, \infty))}\cdot G_p(\mathrm{d}s, R)\right. \\ &\quad \left.+\int_0^{t-T_n} \frac{\tilde{n}_n(B, s, \omega)}{H_n([s, \infty))} G_n(\mathrm{d}s, R)\right) \\ &= \int_0^t \tilde{n}(B, s, \omega)\lambda(s, \omega)\mathrm{d}s \\ &= \int_0^t \tilde{n}(B, s, \omega)\Lambda(\mathrm{d}s). \end{aligned}$$

Now we show that $\tilde{n}(B, t, \omega)$ is a version of $n'(B, t, \omega)$, and from this we reach the conclusion that $n'(B, t, \omega)$ is a local description corresponding to $n(B, t, \omega)$. Since

$$G_p((0, t], B)=\int_0^t \tilde{n}_p(B, s, \omega)\exp\left\{-\int_{T_p}^{T_p+s} \lambda_{T_p}(u, \omega)\mathrm{d}u\right\}\lambda_{T_p}(T_p+s, \omega)\mathrm{d}s,$$

for almost all t we have

$$\frac{\mathrm{d}}{\mathrm{d}t}G_p((0, t], B)=\tilde{n}_p(B, t, \omega)\exp\left\{-\int_{T_p}^{T_p+t} \lambda_{T_p}(u, \omega)\mathrm{d}u\right\}\lambda_{T_p}(T_p+t, \omega).$$

The lefthand side of the above equation is calculated as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}G_p((0, t], B) &= \lim_{h\to 0}\frac{1}{h}(G_p((0, t+h], B)-G_p((0, t], B)) \\ &= \lim_{h\to 0}\frac{1}{h}P\{t<S_{p+1}\leq t+h,\ X_{T_{p+1}}\in B\,|\,\mathscr{F}_{T_p}\} \\ &= \lim_{h\to 0}\frac{1}{h}P\{t<S_{p+1}\leq t+h|\mathscr{F}_{T_p}\}\frac{P\{t<S_{p+1}\leq t+h,\ X_{T_{p+1}}\in B|\mathscr{F}_{T_p}\}}{P\{t<S_{p+1}\leq t+h|\mathscr{F}_{T_p}\}} \\ &= \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} \lambda_{T_p}(T_p+s, \omega)\exp\left\{-\int_{T_p}^{T_p+s}\lambda_{T_p}(u, \omega)\mathrm{d}u\right\}\mathrm{d}s\cdot n_{T_p}(B, T_p+t, \omega) \\ &= \lambda_{T_p}(T_p+t, \omega)\exp\left\{-\int_{T_p}^{T_p+t}\lambda_{T_p}(u, \omega)\mathrm{d}u\right\}n_{T_p}(B, T_p+t, \omega) \end{aligned}$$

for almost all t, and hence

$$\tilde{n}_p(B, t, \omega)=n_{T_p}(B, T_p+t, \omega) \qquad \text{for almost all } t.$$

Now on $\{T_p\leq s<T_{p+1}\}$

$$\tilde{n}(B, s, \omega)=\tilde{n}_p(B, s-T_p, \omega)=n_{T_p}(B, s, \omega)=n'(B, s, \omega), \qquad \text{(Lemma 4)},$$

i.e., $\tilde{n}(B, s, \omega)=\tilde{n}'(B, s, \omega)$ for almost all s and this implies the conclusion. Q. E. D.

4. Sufficient conditions for monotonicity

Let X_t be a jump process as was defined in Section 2. Suppose that in the local description $\{n(A, t, \omega), \Lambda\}$ of X_t $\Lambda(\omega, [0, t])$ is absolutely continuous, i.e.,

$$\Lambda(\omega, [0, t]) = \int_0^t \lambda(s, \omega)\mathrm{d}s$$

for an $\mathscr{F}_t$-adapted measurable process $\lambda(t, \omega)$. Then from Section 2 we know that under some integrability conditions for the process $f(t, X_t)$ we have, for any stopping time T,

$$Ef(T, X_T) = Ef(0, X_0) + E\int_0^T f'(s, X_s)\mathrm{d}s + E\int_0^T\int_R (f(s, x) - f(s, X_{s-}))$$

$$\cdot\, n(\mathrm{d}x, s, \omega)\lambda(s, \omega)\mathrm{d}s,$$

and condition (A1) in Section 1 is satisfied. Hence in this section we shall investigate sufficient conditions for (A2) in terms of the local description.

Following the discrete-time parameter case in Chow and Robbins [2], a real valued $\mathscr{F}_t$-adapted stochastic process X_t is said to be monotone if it satisfies conditions (A1) and if either X_t or $-X_t$ satisfies the condition (A2).

Note that X_t is monotone if, in particular, when X_t satisfies (A1) and f_t appearing in (A1) is increasing or decreasing. In the former case X_t will be called *convex* and the latter case *concave*.

Given a jump process X_t with the local description $\{n(A, t, \omega), \Lambda\}$ where

$$\Lambda(\omega; [0, t]) = \int_0^t \lambda(s, \omega)\mathrm{d}s\,,$$

let us define a stochastic measure $q(\cdot, t, \omega)$ by

$$q(A, t, \omega) = \lambda(t, \omega)n(X_{t-} + A, t, \omega), \quad A \in \mathscr{B}(R) \tag{7}$$

where

$$X_{t-} + A = \{X_{t-} + a,\ a \in A\}\,.$$

The measure $q(\cdot, t, \omega)$ is said to be *monotone decreasing* (*increasing*) if for any $y \geq 0$ $q([y, \infty), t, \omega)$ is decreasing (increasing) a.s. and $q((-\infty, -y], t, \omega)$ is increasing (decreasing) a.s. with respect to t. Then we have

Theorem 2. *Suppose that*

$$\textit{either}\quad E\sum_{s<\infty}(X_s - X_{s-})^+ < \infty \quad \textit{or} \quad E\sum_{s<\infty}(X_s - X_{s-})^- < \infty$$

and that $EX_0 < \infty$. If the measure $q(\cdot, t, \omega)$ defined by (7) *is monotone decreasing (increasing), then X_t is concave (convex).*

Proof. We shall give the proof only for the concave case. By Proposition 1 we have for any stopping time T

$$EX_T = EX_0 + E\int_0^T \int_R (x - X_{s-}) n(\mathrm{d}x, s, \omega) \lambda(s, \omega) \mathrm{d}s .$$

Then

$$\begin{aligned} f_s &= \int_R (x - X_{s-}) \lambda(s, \omega) n(\mathrm{d}x, s, \omega) \\ &= \int_R y q(\mathrm{d}y, s, \omega) \\ &= \int_{[0,\infty)} y q(\mathrm{d}y, s, \omega) + \int_{(-\infty,0)} y q(\mathrm{d}y, s, \omega) . \end{aligned}$$

By the integration by parts, we have

$$\int_{[0,\infty)} y q(\mathrm{d}y, s, \omega) = \int_{[0,\infty)} q([y, \infty), s, \omega) \mathrm{d}y$$

and

$$\int_{(-\infty,0)} y q(\mathrm{d}y, s, \omega) = \int_{(0,\infty)} (-q((-\infty, -y], s, \omega)) \mathrm{d}y .$$

By assumption $q([y, \infty), s, \omega)$ and $-q((-\infty, -y], s, \omega)$ are both decreasing and hence f_s is decreasing i.e., X_t is concave. Q. E. D.

Before proceeding to the next theorem, we need a lemma. As was remarked in the previous section, $n(A, t, \omega)$ has the interpretation that $n(A, t, \omega)$ is the chance that $X_t \in A$ given $\mathscr{F}_t$ and given that a jump occurs at t. Thus if X_t is increasing, then $n(A, t, \omega)$ must be zero when $A \subset [X_{t-}, \infty)^c$. The following lemma assures this.

Lemma 2. *Let X_t be increasing with the local description $\{n(A, t, \omega), \Lambda\}$. Then, for any Borel set A such that $A \subset [X_{t-}, \infty)^c$, we have that with probability one*

$$n(A, t, \omega) = 0 \qquad a.s. \quad \Lambda(\mathrm{d}t) .$$

Proof. Let A be a Borel set such that $A \subset [X_{t-}, \infty)^c$. Then since X_t is increasing,

$$\mu((0, t+h], A) - \mu((0, t], A) = 0$$

for any $h \geq 0$. Hence

$$\begin{aligned} &E(\nu((0, t+h], A) - \nu((0, t], A)) \\ &= E(\mu((0, t+h], A) - \mu((0, t], A)) \\ &= 0 \end{aligned}$$

and since $\nu((0, t], A)$ is increasing, with probability one

$$\nu((0, t+h], A) - \nu((0, t], A) = 0,$$

that is, for any $h \geq 0$

$$\int_t^{t+h} n(A, s, \omega)\Lambda(\mathrm{d}s) = 0$$

with probability one. From this we can easily show that with probability one

$$\int_t^{t+h} n(A, s, \omega)\Lambda(\mathrm{d}s) = 0$$

for all $h \geq 0$. Hence with probability one

$$n(A, t, \omega) = 0, \qquad \text{a.s.} \quad \Lambda(\mathrm{d}t). \qquad\qquad \text{Q.E.D.}$$

Let f be a measurable function from $[0, \infty) \times R \to R$ and be differentiable with respect to the first argument. In the next theorem conditions will be given under which the process $f(t, X_t)$ is concave.

Theorem 3. *Let X_t be increasing with monotone decreasing stochastic measure $q(\cdot, t, \omega)$ and let $f(\cdot, \cdot)$ satisfy the following conditions:*
(i) $f(\cdot, x)$ is strictly decreasing and concave for an arbitrarily fixed x, and $f'(t, X_t)$ is decreasing with respect to t.
(ii) $f(t, \cdot)$ is increasing and concave for any fixed t.
(iii) $Ef(0, X_0) < \infty$ and, for any stopping time T, $E\int_0^T f'(s, X_s)\mathrm{d}s$ exists and either $E\int_0^T f'(s, X_s)\mathrm{d}s$ or $E\int_0^T\int_R (f(s, x) - f(s, X_{s-}))n(\mathrm{d}x, s, \omega)\lambda(s, \omega)\mathrm{d}s$ is finite.
Then $f(t, X_t)$ is concave.

Proof. From (4), for any stopping time T,

$$Ef(T, X_T) = Ef(0, X_0) + E\int_0^T [f'(s, X_s) + \int_R (f(s, x) - f(s, X_{s-}))n(\mathrm{d}x, s, \omega)\lambda(s, \omega)]\mathrm{d}s.$$

By assumption $f'(s, X_s)$ is decreasing, and hence we shall show that

$$g_s = \int_R (f(s, x) - f(s, X_{s-}))\lambda(s, \omega)n(\mathrm{d}x, s, \omega)$$

is decreasing. Let us define stochastic measures by

$$\tilde{n}(A, s, \omega) = n(f_s^{-1}(A), s, \omega)\lambda(s, \omega)$$

$$\tilde{q}(A, s, \omega) = \tilde{n}(f(s, X_{s-}) + A, s, \omega)$$

where $f_s^{-1}(A) = \{x, f(s, x) \in A\}$. Then

$$g_s=\int_{[X_{s-},\infty)}(f(s, x)-f(s, X_{s-}))\lambda(s, \omega)n(\mathrm{d}x, s, \omega)$$
$$=\int_{[f(s,X_{s-}),f(s,\infty))}(y-f(s, X_{s-}))\tilde{n}(\mathrm{d}y, s, \infty)$$
$$=\int_{[0,f(s,\infty)-f(s,X_{s-}))}z\tilde{q}(\mathrm{d}z, s, \omega)$$

where $f(s, \infty)=\lim_{x\to\infty}f(s, x)$ and the first equality is due to Lemma 2. Then noting that

$$\tilde{q}([f(s, \infty)-f(s, X_{s-}), \infty), s, \omega)$$
$$=\lambda(s, \omega)n(f_s^{-1}[f(s, \infty), \infty), s, \omega)$$
$$=0,$$

we have

$$g_s=\int_{[0,\infty)}z\tilde{q}(\mathrm{d}z, s, \omega)=\int_{[0,\infty)}\tilde{q}([z, \infty), s, \omega)\mathrm{d}z\,.$$

Now

$$\tilde{q}([z, \infty), s, \omega)=\tilde{n}([f(s, X_{s-})+z, \infty), s, \omega)$$
$$=\lambda(s, \omega)n(f_s^{-1}[f(s, X_{s-})+z, \infty), s, \omega)$$
$$=\lambda(s, \omega)n([X_{s-}+x(s), \infty), s, \omega)$$
$$=q([x(s), \infty), s, \omega)$$

where $x(s)$ is the minimal point such that

$$f(s, X_{s-})+z=f(s, X_{s-}+x(s))$$

and $x(s)=\infty$ if no such point exists.

Then we can show that $x(s)\leq x(t)$ $(s<t)$.

As a matter of fact, if we let x' be the minimal point such that

$$f(t, X_{s-})+z=f(t, X_{s-}+x'),$$

then

$$f(t, X_{s-}+x')-f(t, X_{s-})=f(t, X_{t-}+x(t))-f(t, X_{t-})$$

and since $f(t, \cdot)$ is increasing and concave we see that $x'\leq x(t)$. We can also show that $x(s)\leq x'$. Indeed suppose the contrary: $x(s)>x'$. Then

$$f(s, X_{s-}+x(s))>f(t, X_{s-}+x(s))\geq f(t, X_{s-}+x')\,.$$

On the other hand

$$f(s, X_{s-}+x(s))-f(t, X_{s-}+x')=f(s, X_{s-})-f(t, X_{s-})<0$$

and this is contradiction. Thus we have shown that $x(s) \leq x(t)$ $(s<t)$.

Now

$$
\begin{aligned}
& q([x(s), \infty), s, \omega) \\
& \quad \geq q([x(s), \infty), t, \omega) \qquad (q(\cdot, t, \omega) \text{ is monotone decreasing}). \\
& \quad \geq q([x(t), \infty), t, \omega).
\end{aligned}
$$

Hence $\tilde{q}([z, \infty), s, \omega)$ is decreasing with respect to s and so is g_s. Q.E.D.

In the following propositions examples are given where reward processes are monotone but not concave.

Proposition 3. *Let X_t be increasing with monotone decreasing stochastic measure $q(A, t, \omega)$ and f: $R \to R$ be an increasing concave function such that*

$$Ef(X_0) < \infty \quad \text{and} \quad E\int_0^\infty |f(X_s)| e^{-\alpha s} ds < \infty .$$

Then the process $Z_t = e^{-\alpha t} f(X_t)$ $(\alpha > 0)$ is monotone.

Proof. From (4), for any stopping time T we have

$$EZ_T = EZ_0 + E\int_0^T e^{-\alpha s}\Big(\int_R (f(x) - f(X_{s-})) n(dx, s, \omega) \lambda(s, \omega) - \alpha f(X_s)\Big) ds$$

with

$$f_s = e^{-\alpha s}\Big(\int_R (f(x) - f(X_{s-})) n(dx, s, \omega) \lambda(s, \omega) - \alpha f(X_s)\Big).$$

By our assumption the term in parentheses is decreasing with respect to s and hence condition (A2) is satisfied. Q.E.D.

Proposition 4. *Let X_t and Y_t be increasing jump processes with no common discontinuities. Suppose that X_t has the monotone decreasing stochastic measure $q_X(\cdot, t, \omega)$ and Y_t monotone increasing $q_Y(\cdot, t, \omega)$ (so X_t is concave and Y_t is convex). Let f: $R \to R$ be bounded, positive, increasing and concave. Then the reward process $Z_t = e^{-\alpha Y_t} f(X_t)$ $(\alpha > 0)$ is monotone.*

Proof. By Proposition 2, for any stopping time T we have

$$
\begin{aligned}
& Ee^{-\alpha Y_T} f(X_T) \\
& \quad = Ef(X_0) + E\int_0^T e^{-\alpha Y_{s-}} \Big[\int_R (e^{-\alpha(y - Y_{s-})} - 1) f(X_{s-}) \lambda_Y(s, \omega) n_Y(dy, s, \omega) \\
& \qquad + \int_R (f(x) - f(X_{s-})) \lambda_X(s, \omega) n_X(dx, s, \omega)\Big] ds
\end{aligned}
$$

where $\{n_X(dx, s, \omega), \lambda_X(s, \omega)\}$ and $\{n_Y(dy, s, \omega), \lambda_Y(s, \omega)\}$ are local descriptions of X_t and Y_t respectively. By Theorem 3

$$\int_R (f(x)-f(X_{s-}))\lambda_X(s, \omega)n_X(\mathrm{d}x, s, \omega)$$

is decreasing under our assumption. Hence to see that Z_t is monotone it is sufficient to show that

$$\begin{aligned} h_s &= \int_R (1-e^{-\alpha(y-Y_{s-})})\lambda_Y(s, \omega)n_Y(\mathrm{d}y, s, \omega) \\ &= \int_{[Y_{s-},\infty)} (1-e^{-\alpha(y-Y_{s-})})\lambda_Y(s, \omega)n_Y(\mathrm{d}y, s, \omega) \end{aligned}$$

is increasing since it is non-negative and $f(X_{s-})$ is positive and increasing. Now we have

$$\begin{aligned} h_s &= \int_{[0,\infty)} (1-e^{-\alpha z})q_Y(\mathrm{d}z, s, \omega) \\ &= \int_{[0,1)} u\tilde{q}_Y(\mathrm{d}u, s, \omega) \end{aligned}$$

where $\tilde{q}_Y(A, s, \omega)=q_Y(h^{-1}(1-A), s, \omega)$, $h(x)=e^{-\alpha x}$ and $1-A=\{1-a, a\in A\}$. Then we have

$$\begin{aligned} &\int_{[0,1)} u\tilde{q}_Y(\mathrm{d}u, s, \omega)=\int_{[0,1)} \tilde{q}_Y([u, 1), s, \omega)\mathrm{d}u \\ &=\int_{[0,1)} q_Y(h^{-1}(1-[u, 1)), s, \omega)\mathrm{d}u \\ &=\int_{[0,1)} q_Y([u^*(u), \infty), s, \omega)\mathrm{d}u, \qquad (e^{-\alpha u^*(u)}=1-u) \\ &\leq\int_{[0,1)} q_Y([u^*(u), \infty), t, \omega)\mathrm{d}u \qquad (q_Y(\cdot, t, \omega)\text{: monotone-increasing}) \\ &=h_t. \end{aligned}$$

Thus h_s is increasing. Q.E.D.

5. Example

1. Let N_t be an $\mathscr{F}_t$-adapted jump process with positive jump size 1, that is, N_t is a counting process. We shall assume that N_t is characterized by its conditional jump rate $\lambda(t, \omega)$ in the following way:

(D1)
$$\begin{aligned} &P(N_{t+h}-N_t=1 \mid \mathscr{F}_t)=\lambda(t, \omega)h+o(h) \\ &P(N_{t+h}-N_t=0 \mid \mathscr{F}_t)=1-\lambda(t, \omega)h+o(h) \end{aligned}$$

where $\lambda(t, \omega)$ is an $\mathscr{F}_t$-adapted measurable process. Also we assume that there exists a constant K such that

$$\frac{1}{h}P(N_{t+h}-N_t\geq 1\mid\mathscr{F}_t)\leq K\,.$$

Here we also assume that $\mathscr{F}_t=\mathscr{G}_t$ where $\mathscr{G}_t$ is the σ-field generated by all the processes which define costs we are going to consider.

Let $\{Y_n,\ n=1,2,\dots\}$ be a sequence of positive, independent and identically distributed integrable random variables with a common distribution function F which are also independent of the process N_t. Let

$$X_t=Y_0+\cdots+Y_{N_t}\qquad\text{with}\quad Y_0=0.$$

Then putting $N'_t=\sum_{s\leq t}I_{\{X_s\neq X_{s-}\}}$, we have

$$\begin{aligned}&\lim_{h\downarrow 0}\frac{1}{h}P(N'_{t+h}-N'_t=1\mid\mathscr{F}_t)\\&=\lim_{h\downarrow 0}\frac{1}{h}P(N_{t+h}-N_t=1\mid\mathscr{F}_t)=\lambda(t,\omega)\end{aligned}$$

and

$$\begin{aligned}n'(A,t,\omega)&=\lim_{h\downarrow 0}\frac{P(X_{t+h}\in A,\ N'_{t+h}-N'_t=1|\mathscr{F}_t)}{P(N'_{t+h}-N'_t=1|\mathscr{F}_t)}\\&=\int_{A-X_{t-}}\mathrm{d}F(x),\quad\text{where}\quad A-X_{t-}=\{a-X_{t-},\ a\in A\}\,.\end{aligned}$$

Therefore these are a local description of X_t by Theorem 1.

Let now

$$X_t=\max(Y_0,\ Y_1,\dots,\ Y_{N_t})\,.$$

Then we have

$$\begin{aligned}&\lim_{h\downarrow 0}\frac{1}{h}P(N'_{t+h}-N'_t=1\mid\mathscr{F}_t)\\&=\lim_{h\downarrow 0}\frac{1}{h}P(Y_{N_t+1}>X_{t-},\ N_{t+h}-N_t=1\mid\mathscr{F}_t)\\&=(1-F(X_{t-}))\lambda(t,\omega)\end{aligned}$$

and

$$\begin{aligned}n'(A,t,\omega)&=\lim_{h\downarrow 0}\frac{P(X_{t+h}\in A,\ N'_{t+h}-N'_t=1|\mathscr{F}_t)}{P(N'_{t+h}-N'_t=1|\mathscr{F}_t)}\\&=\lim_{h\downarrow 0}\frac{P(Y_{N_t+1}>X_{t-},\ Y_{N_t+1}\in A-X_{t-}|\mathscr{F}_t)}{1-F(X_{t-})}\\&=\frac{1}{1-F(X_{t-})}\int_{(A-X_{t-})\cap(X_{t-},\infty)}\mathrm{d}F(y)\,,\end{aligned}$$

and these are a local description of $X_t=\max(Y_0,\dots,\ Y_{N_t})$. We have

$$q([y,\infty),t,\omega)=\lambda(t,\omega)(1-F(X_{t-}+y))$$

and hence if $\lambda(t,\omega)$ is decreasing in t then $q([y,\infty),t,\omega)$ is decreasing in t.

2. Using the above results, let us find an optimal stopping rule for a reward process R_t:

$$R_t=e^{-Y_t}f(X_t)$$

where

$$Y_t=W_0+\cdots+W_{M_t}$$

$$X_t=\max(Z_0,\dots,Z_{N_t}).$$

Here M_t and N_t are counting processes with no common discontinuities which satisfy the condition (D1) with jump rates $\lambda_M(t,\omega)$ and $\lambda_N(t,\omega)$ respectively. $\{W_n\}$ and $\{Z_n\}$ are sequences of positive, independent, and identically distributed integrable random variables which are also independent of M_t and N_t. For simplicity, let $f(\cdot)$ be bounded and $F(\mathrm{d}x)$ be the common distribution for $\{Z_n\}$.

Then from Proposition 2 and the results in 1 of this section we have, for any stopping time T,

$$ER_T=Ef(Z_0)+E\int_0^T e^{-Y_{s-}}[f(X_{s-})\lambda_M(s,\omega)(Ee^{-W_0}-1)$$

$$+\int_{X_{s-}}^{\infty}(f(x)-f(X_{s-}))\lambda_N(s,\omega)\mathrm{d}F(x)]\mathrm{d}s.$$

Also, under assumptions that (i) f is positive and increasing and (ii) $\lambda_M(s,\omega)$ is increasing and $\lambda_N(s,\omega)$ is decreasing, we see directly by the representation of R_t that R_t is monotone and a stopping time τ^* defined by

$$\tau^*=\inf\Big\{s;\,f(X_{s-})\lambda_M(s,\omega)(Ee^{-W_0}-1)$$

$$+\int_{X_{s-}}^{\infty}(f(x)-f(X_{s-}))\mathrm{d}F(x)\lambda_N(s,\omega)\leq 0\Big\}$$

is optimal as long as $\tau^*<\infty$ a.s..

Conditions that $f(\cdot)$ is continuous and $N_t\to\infty$ a.s. ($t\to\infty$) are sufficient for τ^* to be finite a.s.. This is a consequence of the following fact

$$\int_{X_{s-}}^{\infty}(f(x)-f(X_{s-})\mathrm{d}F(x)\longrightarrow 0 \quad \text{a.s.} \quad (s\to\infty)$$

which we shall now show.

Indeed, if we put

$$\sigma=\sup\{y,\ P(W_0\leq y)<1\},$$

X_s increases a.s. to σ as $s\to\infty$ since $X_s\leq y$ a.s. if $y>\sigma$ and $P(X_s<y)\to 0$ as $s\to\infty$ if $y<\sigma$.

The latter comes from the fact that

$$P(X_s<y)=\sum_{n=0}^{\infty} P(W_0<y)^n P(N_s=n) \leq \sum_{n=0}^{\infty} P(W_0<y)^n<\infty$$

(since $P(W_0<y)<1$) and that $P(N_s=n)\to 0$ as $s\to\infty$).

On the other hand

$$\int_{X_{s-}}^{\infty}(f(x)-f(X_{s-}))\mathrm{d}F(x)=\int_{-\infty}^{\infty}(f(x)-f(X_{s-}))^{+}\mathrm{d}F(x)=g(f(X_{s-}))$$

$$(a^+=a \text{ if } a\geq 0 \text{ and } a^+=0 \text{ if } a<0)$$

where

$$g(a)=\int_{-\infty}^{\infty}(f(x)-a)^{+}\mathrm{d}F(x)=E(f(W_0)-a)^{+}.$$

When $a\to f(\sigma)$, then $g(a)\to 0$ since $(f(W_0)-a)^{+}\to 0$. Thus since $f(X_{s-})\to f(\sigma)$ a.s., $g(f(X_{s-}))\to 0$ a.s. (See Neveu [1] for a discrete version of this problem).

Acknowledgment

The author is deeply indebted to Professor S. Watanabe for pointing out serious errors in the original manuscript and for various suggestions toward improvement of the paper.

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