

Degeneration of Abelian integrals on an open Riemann surface to elliptic ones

By

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An Abelian integral

$$\Phi(x, y) = \int F(x, y) dx$$

on a closed Riemann surface defined by an irreducible polynomial $P(x, y) = 0$ is said to be *reducible* to an elliptic integral

$$E(\xi, \eta) = \int G(\xi, \eta) d\xi$$

if we can find two rational functions

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

such that

$$\Phi(x, y) = E(\xi(x, y), \eta(x, y))$$

or

$$F(x, y) = G(\xi(x, y), \eta(x, y)) \frac{\frac{\partial \xi}{\partial x} \frac{\partial P}{\partial y} - \frac{\partial \xi}{\partial y} \frac{\partial P}{\partial x}}{\frac{\partial P}{\partial y}}$$

(see Appell-Goursat [2], Baker [3] and Krazer [6]). The study of reducible Abelian integrals has as long history as the theory of hyperelliptic integrals; its origin goes back to the age of Jacobi and Legendre (cf. [3], [6], [13] etc.). But a general fundamental result was first announced by Weierstrass (see his letter to Königsberger [12]; cf. also Kowalevski [5]). He gave a necessary and sufficient condition for an Abelian integral to be reducible (to an elliptic one) in terms of its periodicity module.

Besides the above theorem Weierstrass also obtained some interesting results. In the present paper, however, we shall be concerned simply with the previous theorem. We first note that it can be easily generalized for open surfaces (see

Theorem 1). But such an immediate generalization seems to yield no remarkable consequence. Therefore we shall restrict the case and give theorems which correspond to the classical case (see Theorems 1' and 5) — it is indeed such a restricted kind of theorems that we shall need in a subsequent paper, where we shall prove another theorem of Weierstrass (and Poincaré) and give, for open surfaces of genus one, an extension of Koebe's generalized uniformization theorem for planar surfaces. In Theorems 2 and 3 we shall give some criteria for an Abelian integral to be reducible. The precise definition of "reducible" Abelian integrals, together with some other tools, will be given in the preliminary part of the paper.

Concerning the classical definition of reducibility, a few remarks should be included here. First we note that the problem of finding a reducible integral on a given surface is obviously equivalent to the problem of deciding whether there exists an analytic mapping of the surface onto a torus. Secondly, we note that Severi [9] adopts another definition; he studied not Abelian integrals themselves but their periods. Inasmuch we shall consider only the reduction to *elliptic* integrals, it does not matter which definition we take.

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I. Preliminaries

1. Throughout this paper R denotes an open Riemann surface of genus g ($\leq \infty$). Let ∂R denote the Kerékjártó-Stoïlow ideal boundary of R .¹⁾ Set $J = \{1, 2, \dots, g\}$. By a canonical homology basis of $R \bmod \partial R$ we understand a collection of 1-cycles A_j, B_j ($j \in J$) on R which have the following properties:

- (i) $A_j \times B_k = \delta_{jk}^{2), 3)}$, $A_j \times A_k = B_j \times B_k = 0$, $j, k \in J$,
- (ii) $A_j \cap A_k = B_j \cap B_k = \emptyset$; $A_j \cap B_k = \emptyset$, $j \neq k$, while $A_j \cap B_j$ consists of a single point,
- (iii) there is a canonical exhaustion $\{R_n\}_{n=1}^{\infty}$ of R such that if $0 \leq m < n$ then $\{A_j, B_j\}_{j \in J_n \setminus J_m}$ is a homology basis of $R_n \setminus \bar{R}_m$ modulo the border,⁴⁾ where $J_n = \{1, 2, \dots, g_n\}$ and g_n is the genus of R_n .

We add an appropriate number of dividing cycles D_i ($i \in I$) to $\{A_j, B_j\}_{j \in J}$ to obtain a canonical homology basis of R . Then every (finite) 1-cycle γ on R can be uniquely represented as

$$\gamma \approx \sum_{j \in J} (m_j A_j + n_j B_j) + \sum_{i \in I} p_i D_i \quad (\text{a finite sum})$$

1) Part of the results below can be easily modified to be valid for closed surfaces as well, if we interpret then $\partial R = \emptyset$.

2) The definition of the intersection number in the present paper follows the classical one and hence is different from those in Ahlfors-Sario [1] and Shiba [11] etc. .

3) $\delta_{jk} = 1$ for $j = k$ and $= 0$ otherwise.

4) For simplicity, we set $R_0 = \emptyset$ and $J_0 = \emptyset$.

with some integers m_j, n_j and p_i , where the symbol \approx reads "is homologous to". We can take D_i to be a contour of some R_n and hence $D_i \cap (A_j \cup B_j) = \emptyset$ for every $(i, j) \in I \times J$. We may also assume that each of A_j and B_j is an analytic Jordan curve.

2. Let $A = A(R)$ be the *real* Hilbert space of square integrable *complex* differentials on R (see [10] and [11]) with the inner product

$$\langle \lambda_1, \lambda_2 \rangle = \operatorname{Re} \iint_R \lambda_1 \wedge \bar{\lambda}_2^*.$$

(We denote by $\bar{\lambda}$ the complex conjugate of λ and by λ^* the conjugate differential of λ . See [1], p. 266 and p. 271.) We denote by $\|\lambda\|$ the norm of $\lambda \in A$: $\|\lambda\| = \sqrt{\langle \lambda, \lambda \rangle}$. Let $A_h = A_h(R)$ be the closed subspace of A consisting of harmonic differentials on R . Also we set

$$A_{e_o}^{(1)} = A_{e_o}^{(1)}(R) = \left\{ \lambda \in A \left| \begin{array}{l} \exists f \in C^2(R), \exists f_n \in C_o^2(R) \text{ such} \\ \text{that } \lambda = df \text{ and } \lim_{n \rightarrow \infty} \|df - df_n\| = 0. \end{array} \right. \right\}.$$

Hereafter we shall denote by \mathbf{C} the set of all complex numbers (and its equivalent: Gaussian plane). For brevity, we call a straight line L passing through the coordinate origin of \mathbf{C} simply a line. Every line L is determined by a $\zeta \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ so that we shall write as

$$L = L(\zeta) = \{z \in \mathbf{C} \mid z = t\zeta, t \in \mathbf{R}\},$$

where \mathbf{R} is the real number system. We also use the notation $z \equiv 0 \pmod L$ to represent $z \in L$.

A closed subspace A_o of A_h is called a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ associated with a family $\{L_j\}_{j \in J}$ of lines, if

- (i) every $\lambda_o \in A_o$ is semiexact,⁵⁾
- (ii) $A_h = A_o \oplus iA_o^*$ (direct sum), $i^2 = -1$, and
- (iii) for each $\lambda_o \in A_o$ $\int_{A_j} \lambda_o \equiv \int_{B_j} \lambda_o \equiv 0 \pmod L_j, j \in J$.

Note that we consider the family $\{L_j\}_{j \in J}$ to be an ordered set. Thus if π is a permutation on J , then $\{L_j\}_{j \in J} \neq \{L_{\pi(j)}\}_{j \in J}$ in general.

Given a behavior space A_o associated with $\{L_j\}_{j \in J}$, we can find a sequence $\{\zeta_k\}_{k=1}^{\infty}, \zeta_k \in \mathbf{C}^* (1 \leq k \leq \infty)$ such that (i) $\arg \zeta_i \not\equiv \arg \zeta_k \pmod{\pi}, i \neq k$ and (ii) for any $j \in J$ there is a (unique) $k = k(j), 1 \leq k \leq \infty$ with $L_j = L(\zeta_k)$. The uniquely determined number κ will be called the *index* of A_o ; we shall use the notation $\kappa = \operatorname{ind} A_o$.

We recall now that a behavior space \tilde{A}_o is called equivalent to A_o if they define

5) Namely, $\int_d \lambda_o = 0$ for any dividing cycle d on R .

the same boundary behavior (see [10]). We know that \tilde{A}_o is equivalent to A_o if and only if any one of \tilde{A}_o and A_o can be obtained from the other through a transformation (see [10], Theorem 3). A transformation is, roughly speaking, an exchange of a finite number of lines for another family. We set

$$\text{ess. ind } A_o = \min \{ \text{ind } \tilde{A}_o \mid \tilde{A}_o \text{ is equivalent to } A_o \},$$

and call it the *essential index* of A_o .

3. A regular analytic (=holomorphic) differential ψ on R is called an *Abelian differential* on R . The integral Ψ of ψ is called an *Abelian integral* on R . Note that ψ is not supposed to be square integrable on R .

Let A_o be a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ associated with $L = \{L_j\}_{j \in J}$. Then a A_o -Abelian differential on R is, by definition, an Abelian differential ψ on R such that

$$\int_{A_j} \psi \equiv \int_{B_j} \psi \equiv 0 \pmod{L_j}$$

for all sufficiently large $j \in J$ (cf. [11]).

We shall say that a A_o -Abelian differential ψ is of the *first kind* if it has A_o -behavior (see [10], for instance), *i.e.*, if there are $\lambda_o \in A_o$, $\lambda_{e_o} \in A_{e_o}^{(1)}$ such that

$$\psi = \lambda_o + \lambda_{e_o}$$

outside some compact set on R . If a A_o -Abelian differential is semiexact, it is called of the *second kind*; otherwise it is called of the *third kind*. Such a classification of A_o -Abelian differentials obviously generalizes the classical terminology. Note that a A_o -Abelian differential of the first kind is necessarily square integrable, while A_o -Abelian differentials of the second or third kind are not always square integrable.

Let P be a regular partition of ∂R (see [1], p. 87 ff.). For every $n=1, 2, \dots$ we set $(P)A_L^n$ to be the class of all (P) semiexact⁶⁾ holomorphic differential λ on $R \setminus \bar{R}_n$ such that $\int_{A_j} \lambda \equiv \int_{B_j} \lambda \equiv 0 \pmod{L_j}$, $j \in J \setminus J_n$. With natural identification of its elements $(P)A_L = \bigcup_{n=1}^{\infty} (P)A_L^n$ has a real vector space structure. Each equivalence class of $(P)A_L$ modulo A_o -behavior is called a $(P)A_o$ -singularity (cf. [10]). In this paper we shall be mainly concerned with the identity partition I of ∂R . For brevity, we shall omit the prefix $(I)A_o$ whenever it is easily seen from the context that we speak of $(I)A_o$ -singularities. For example, if ψ is known to be a A_o -Abelian differential, then the singularity of ψ always means the $(I)A_o$ -singularity.

4. Suppose that A_o is a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ associated with $\{L(\zeta_j)\}_{j \in J}$, $\zeta_j \in \mathbb{C}^*$. Then we can find $2g$ A_o -Abelian differentials $\phi(A_j)$, $\phi(B_j)$ of the first kind on R such that

6) This means that $\int_d \varphi = 0$ for every (P) dividing cycle d on R [which is contained in $R \setminus \bar{R}_n$]. (As usual, (Q) semiexactness is often referred to simply as semiexactness, Q being the canonical partition of ∂R . Cf. [1]; see also footnote 5.)

$$\int_{A_k} \phi(A_j) \equiv \int_{B_k} \phi(B_j) \equiv 0 \pmod{L(\zeta_k)}, \quad k \in J.$$

$$\int_{B_k} \phi(A_j) \equiv - \int_{A_k} \phi(B_j) \equiv 2\pi i \delta_{jk} / \zeta_k$$

What is more, $\phi(A_j)$ and $\phi(B_j)$ are uniquely determined by the sequence $\{\zeta_j\}_{j \in J}$. For the details, see [10]. We refer to these differentials as the *elementary* [\mathcal{A}_σ -Abelian] differentials of the first kind determined by the sequence $\{\zeta_j\}_{j \in J}$.

Next let ψ be a \mathcal{A}_σ -Abelian differential which is not of the first kind. Then ψ is called *elementary* if

$$\int_{A_j} \psi \equiv \int_{B_j} \psi \equiv 0 \pmod{L(\zeta_j)}, \quad j \in J.$$

We recall the following facts ([10], [11]):

- 1° For any $(I)\mathcal{A}_\sigma$ -singularity $\sigma \neq 0$ there is a unique elementary \mathcal{A}_σ -Abelian differential ψ_σ ⁷⁾ which has the singularity σ .
- 2° A \mathcal{A}_σ -Abelian differential ψ can be uniquely decomposed into the sum of two \mathcal{A}_σ -Abelian differentials ψ_o and ψ_s ; ψ_o is the elementary \mathcal{A}_σ -Abelian differential with the same singularity σ as ψ and ψ_s is a \mathcal{A}_σ -Abelian differential of the first kind.

Henceforth we shall call ψ_o (resp. ψ_s) the *regular* (resp. *singular*) component of ψ .

5. Every behavior space \mathcal{A}_σ has a (unique) dual behavior space (with respect to the real line \mathbf{R}) which we denote by \mathcal{A}'_σ . For the precise definition, see [10]. In the present paper we only need the following facts: $\mathcal{A}'_\sigma = \{\lambda \in \mathcal{A}_h \mid \bar{\lambda} \in \mathcal{A}_\sigma\}$ and it is a behavior space on $(\mathbf{R}, \{A_j, B_j\}_{j \in J})$ associated with the family $\{L'_j\}_{j \in J}$, where $L'_j = \{z \in \mathbf{C} \mid \bar{z} \in L_j\}$. A characteristic property of \mathcal{A}'_σ is that $\langle \lambda'_o, i\lambda_o^* \rangle = 0$ for all $(\lambda'_o, \lambda'_o) \in \mathcal{A}_\sigma \times \mathcal{A}'_\sigma$ (cf. [10]).

Let \mathcal{A}_σ be a behavior space and \mathcal{A}'_σ the dual behavior space (with respect to \mathbf{R}). Then for any \mathcal{A}'_σ -Abelian integral Φ' of the first kind and any \mathcal{A}_σ -Abelian differential ψ the limit

$$\lim_{n \rightarrow \infty} \operatorname{Re} \left[-\frac{1}{2\pi i} \int_{\partial R_n} \Phi' \psi \right]$$

exists and is finite, which we denote by

$$\operatorname{Res}_{\partial R} \Phi' \psi.$$

For each $(I)\mathcal{A}_\sigma$ -singularity σ

$$\operatorname{Res}_{\partial R} \Phi' \sigma$$

is similarly defined by means of a representative of σ . Indeed, we can easily verify that the value does not depend on the particular choice of a representative differential

7) The ψ_σ is of the second kind if and only if σ is semiexact.

of σ . The quantity $\text{Res}_{\frac{\partial R}{\partial R}} \Phi' \psi$ (resp. $\text{Res}_{\frac{\partial R}{\partial R}} \Phi' \sigma$) is called the residue of $\Phi' \psi$ (resp. $\Phi' \sigma$) at the ideal boundary ∂R . For the details, see [11], for example.

The following proposition is a generalization of Riemann's bilinear relation and is proved by a similar method to the classical case (cf. *e.g.*, Kusunoki [7]).

Proposition 1. *Let Φ' , ψ be as above and σ be the singularity of ψ .⁸⁾ Then*

$$\text{Res}_{\frac{\partial R}{\partial R}} \Phi' \psi = \text{Res}_{\frac{\partial R}{\partial R}} \Phi' \sigma = \frac{1}{2\pi} \sum_{j \in J} \text{Im} \left(\int_{A_j} d\Phi' \int_{B_j} \psi - \int_{B_j} d\Phi' \int_{A_j} \psi \right).$$

We also have (cf. [10], Lemma 1)

Proposition 1'. *If Φ is a Λ_o -Abelian integral of the first kind, then*

$$\|d\Phi\|^2 = -2 \sum_{j \in J} \text{Im} \left(\int_{A_j} d\Phi \int_{B_j} d\bar{\Phi} \right).$$

The following corollary of Proposition 1 will be often used later.

Corollary. *Let Λ_o be a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ associated with a family $\{L(\zeta_j)\}_{j \in J}$, $\zeta_j \in \mathbf{C}^*$, and Λ'_o be the dual behavior space (with respect to \mathbf{R}). Let σ be an $(I)\Lambda_o$ -singularity and ψ_σ be the elementary Λ_o -Abelian differential with the singularity σ . Let Φ'_{A_j} and Φ'_{B_j} be the elementary Λ'_o -Abelian integrals of the first kind determined by the sequence $\{\zeta_j\}_{j \in J}$. Then we have*

$$\begin{aligned} \int_{A_j} \psi_\sigma &= -\zeta_j \text{Res}_{\frac{\partial R}{\partial R}} \Phi'_{A_j} \sigma \\ \int_{B_j} \psi_\sigma &= -\zeta_j \text{Res}_{\frac{\partial R}{\partial R}} \Phi'_{B_j} \sigma \end{aligned} \quad j \in J.$$

For the proof we only need to apply Proposition 1 to Φ'_{A_j} (resp. Φ'_{B_j}) and ψ_σ .

6. Let $\Omega = (\omega_1, \omega_2) \in \mathbf{C}^* \times \mathbf{C}^*$, where ω_1, ω_2 are linearly independent over reals. We can and do always assume that the indices are so arranged that $\text{Im}(\omega_2/\omega_1) > 0$. Denoting by \mathbf{Z} the set of all integers, we obtain a \mathbf{Z} -module or a lattice

$$[\omega_1, \omega_2]_{\mathbf{Z}} = \{z \in \mathbf{C} \mid \exists m, n \in \mathbf{Z}, z = m\omega_1 + n\omega_2\}.$$

In order to save the notations we let the same letter Ω denote the module $[\omega_1, \omega_2]_{\mathbf{Z}}$ as well as the 2-vector (ω_1, ω_2) . Thus when we refer to a lattice it is tacitly presupposed that a basis $\{\omega_1, \omega_2\}$ is fixed so that $\text{Im}(\omega_2/\omega_1) > 0$.

A *lineation* on R is, by definition, a triple $(\{A_j, B_j\}_{j \in J}, \varepsilon, \Omega)$, where $\{A_j, B_j\}_{j \in J}$ is a canonical homology basis of $R \bmod \partial R$, ε is a mapping of J into the set $\{1, 2\}$ and Ω is a lattice. We shall say that a behavior space Λ_o is *subordinate* to a lineation $(\{A_j, B_j\}_{j \in J}, \varepsilon, \Omega)$ if Λ_o is a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ associated with the family $\{L(\omega_{\varepsilon(j)})\}_{j \in J}$.⁹⁾ Then clearly $\text{ind } \Lambda_o \leq 2$.

Given a lattice $\Omega = [\omega_1, \omega_2]_{\mathbf{Z}}$, there exists a torus $T = \mathbf{C}/\Omega$. In such a situation

8) Here the case $\sigma = 0$ is not excluded.

9) Such a behavior space was called, in [11], an ε -allowable behavior space belonging to the lattice Ω .

a canonical homology basis $\{C_1, C_2\}$ of T is always taken so that C_k corresponds to the ω_k , $k=1, 2$, by means of the natural projection mapping.

A behavior space A_o is said to be *compatible* with a torus T if we can choose a lattice Ω so that (i) $T=\mathbf{C}/\Omega$ (biholomorphically homeomorphic), and (ii) A_o is associated with $\{L(\omega_{\varepsilon(j)})\}_{j \in J}$ with some $\varepsilon: J \rightarrow \{1, 2\}$. If a marking of T is pre-assigned, we furthermore require that the marking coincides with that of \mathbf{C}/Ω .

Let $\Omega=[\omega_1, \omega_2]_{\mathbf{Z}}$ be a lattice. Then a singularity σ on R is said to be Ω -admissible ([11]) if

$$\int_{\gamma} \sigma \equiv 0 \pmod{\Omega}$$

for every dividing cycle γ which lies outside a sufficiently large compact set on R . Here, of course, the singularity σ is an $(I)A_o$ -singularity with some behavior space A_o which need not to be compatible with the torus \mathbf{C}/Ω , and the period $\int_{\gamma} \sigma$ is computed by means of a representative differential of σ — any choice yields the same period.

7. We shall later make use of the following proposition, which is a version of a result due to K. Matsui.

Proposition 2. *There always exists a behavior space which is subordinate to a prescribed lineation. In other words: Let a canonical homology basis $\{A_j, B_j\}_{j \in J}$ of $R \bmod \partial R$, a mapping $\varepsilon: J \rightarrow \{1, 2\}$, and a 2-vector $\Omega=(\omega_1, \omega_2)$ be given, $\omega_1, \omega_2 \in \mathbf{C}^*$, $\text{Im}(\omega_2/\omega_1) > 0$. Then we can find a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ associated with the family $\{L(\omega_{\varepsilon(j)})\}_{j \in J}$.*

For the proof let $\Gamma_h = \Gamma_h(R)$ be the Hilbert space of square integrable *real* harmonic differentials on R and Γ_{hse} its subspace which consists of semiexact differentials. Note that $A_h = \Gamma_h \oplus i\Gamma_h$ (direct sum) and $\Gamma_h = \{\lambda \in A_h \mid \lambda \text{ is real}\}$. Set

$$J_k = \{j \in J \mid \varepsilon(j) = k\}, \quad k=1, 2;$$

then $J_1 \cap J_2 = \emptyset$ and $J_1 \cup J_2 = J$. Let Γ_1, Γ_2 be two subspaces of Γ_{hse} which satisfy the following conditions:

- (1) If $\tau \in \Gamma_1$ and $j \in J_1$, then $\int_{A_j} \tau = \int_{B_j} \tau = 0$.
- (2) If $\tau \in \Gamma_2$ and $j \in J_2$, then $\int_{A_j} \tau = \int_{B_j} \tau = 0$.
- (3) $\Gamma_h = \Gamma_1 \oplus \Gamma_2^* = \Gamma_1^* \oplus \Gamma_2$.

As for the existence of such subspaces, consult “*On the Riemann-Roch theorem on open Riemann surfaces*” by the present author (J. Math. Kyoto Univ., Vol. 11 (1971)), pp. 522–524.

10) Cf. K. Matsui’s forthcoming paper. The author thanks Prof. Matsui for the generosity of permitting him to include a modified proof in this paper.

To prove the proposition¹⁰⁾ it now suffices to consider the case $\omega_1=1$ and $\omega_2=e^{it}$, $0<t<\pi$. We set

$$A_o = e^{it}\Gamma_1 \oplus \Gamma_2$$

and show that A_o is a behavior space subordinate to the given lineation. Because each element λ_o of A_o is clearly semiexact and satisfies the period condition

$$\int_{A_j} \lambda_o \equiv \int_{B_j} \lambda_o \equiv 0 \pmod{L(\omega_{\varepsilon(j)})}, \quad j \in J,$$

we only have to demonstrate the orthogonal decomposition $A_h = A_o \oplus iA_o^*$.

Let $e^{it}\theta_1 + \theta_2$, $e^{it}\tau_1 + \tau_2$ be any two elements of A_o , where θ_k and τ_k are elements of Γ_k , $k=1, 2$. Then we have

$$\begin{aligned} & \langle e^{it}\theta_1 + \theta_2, i(e^{it}\tau_1 + \tau_2)^* \rangle \\ &= \sin t \{ \langle \theta_1, \tau_2^* \rangle - \langle \theta_2, \tau_1^* \rangle \} = 0, \end{aligned}$$

so that A_o is orthogonal to iA_o^* . Conversely, let $\lambda \in A_h$ satisfy the equation $\langle \lambda, i\lambda_o^* \rangle = 0$ for every $\lambda_o = e^{it}\theta_1 + \theta_2 \in A_o$ ($\theta_k \in \Gamma_k$, $k=1, 2$). On setting

$$\begin{aligned} \theta &= \frac{\lambda - \bar{\lambda}}{2i \sin t} = \frac{1}{\sin t} \operatorname{Im} \lambda \\ \tau &= \frac{e^{it}\bar{\lambda} - e^{-it}\lambda}{2i \sin t} = \frac{1}{\sin t} \operatorname{Im} (e^{it}\bar{\lambda}), \end{aligned}$$

we have

$$\lambda = e^{it}\theta + \tau; \quad \theta, \tau \in \Gamma_h.$$

We see that $0 = \langle \lambda, i\lambda_o^* \rangle = \sin t \{ \langle \theta, \theta_2^* \rangle - \langle \tau, \theta_1^* \rangle \}$ holds for any pair $(\theta_1, \theta_2) \in \Gamma_1 \times \Gamma_2$. Since $\Gamma_h = \Gamma_1 \oplus \Gamma_2^* = \Gamma_1^* \oplus \Gamma_2$, it follows that $\theta \in \Gamma_1$ and $\tau \in \Gamma_2$. Hence we have proved the proposition.

II. Results and proofs

8. We shall begin with the following

Definition 1. An Abelian integral Ψ on R is called *quasi-reducible* if there are a torus T , a holomorphic integral E on T , and an analytic mapping f of R into T such that

$$\Psi = E \circ f^{11}).$$

Note that we retain the term ‘‘reducible’’ for a more restricted situation (see Definition 2 below), since we can easily prove

11) Namely, Ψ is quasi-reducible if and only if $d\Psi$ can be represented as the pull-back of a holomorphic differential dE on a torus T via an analytic mapping $f: d\Psi = f^*(dE)$.

Theorem 1. *Let $\{A_j, B_j; D_i\}_{j \in J, i \in I}$ be an arbitrarily fixed canonical homology basis of R . Then an Abelian integral Ψ on R is quasi-reducible if and only if we can find a lattice $\Omega = [\omega_1, \omega_2]_{\mathbb{Z}}$ and integers m_{jk}, n_{jk}, p_{ik} ($j \in J, i \in I; k = 1, 2$) such that*

$$\begin{aligned} \int_{A_j} d\Psi &= m_{j1}\omega_1 + m_{j2}\omega_2 \\ \int_{B_j} d\Psi &= n_{j1}\omega_1 + n_{j2}\omega_2 \quad j \in J, i \in I. \\ \int_{D_i} d\Psi &= p_{i1}\omega_1 + p_{i2}\omega_2 \end{aligned}$$

If this is the case, Ψ is quasi-reducible to a holomorphic integral on the torus \mathbb{C}/Ω .

The proof is similar to that of the compact case (see Appell-Goursat [2] and Krazer [6]):

First of all, let Ψ be a quasi-reducible Abelian integral on R and E be an holomorphic integral on a torus T , to which Ψ is quasi-reduced by an analytic mapping $f: R \rightarrow T$. Taking a canonical homology basis $\{C_1, C_2\}$ of T , we set

$$\omega_k = \int_{C_k} dE, \quad k = 1, 2.$$

Since we have $\int_{\gamma} d\Psi = \int_{f(\gamma)} dE$ for every cycle γ on R , we can find integers m_{jk}, n_{jk}, p_{ik} ($j \in J, i \in I; k = 1, 2$) such that

$$(*) \quad \begin{cases} \int_{A_j} d\Psi = m_{j1}\omega_1 + m_{j2}\omega_2 \\ \int_{B_j} d\Psi = n_{j1}\omega_1 + n_{j2}\omega_2 \quad j \in J, i \in I. \\ \int_{D_i} d\Psi = p_{i1}\omega_1 + p_{i2}\omega_2 \end{cases}$$

Suppose, conversely, we are given a lattice $\Omega = [\omega_1, \omega_2]_{\mathbb{Z}}$ and an Abelian integral Ψ which satisfy equations (*). Denoting by T the torus \mathbb{C}/Ω and by ρ the natural projection mapping of \mathbb{C} onto T (a holomorphic universal covering mapping), we can easily verify that $f = \rho \circ \Psi$ is a well-defined analytic mapping of R into T . The inverse ρ^{-1} of ρ is locally single-valued and $d\rho^{-1}$ is a holomorphic differential on T , which we will denote by dE . Then we can write as $d\Psi = f^*(dE)$ so that Ψ is quasi-reducible.

If $\partial R \neq \emptyset$ actually, a famous theorem due to Behnke and Stein (see Behnke-Sommer [4], Satz 10, S. 451) together with the above Theorem 1 implies that there are a plenty of quasi-reducible Abelian integrals on R . We also notice that the existence of a quasi-reducible Abelian integral on an open Riemann surface induces, in contrast to the classical case, no special characteristic property of the surface.

It will still make sense to ask whether a given Λ_o -Abelian integral is quasi-reducible or not. Concerning this problem, we have the following proposition, which can be easily deduced from Theorem 1 and Corollary to Proposition 1.

Proposition 3. *Let Λ_o be a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ associated with a family $\{L(\zeta_j)\}_{j \in J}$, $\zeta_j \in \mathbf{C}^*$, and Λ'_o be the dual behavior space. Let Φ'_{A_j} and Φ'_{B_j} be the elementary Λ'_o -Abelian integrals of the first kind determined by the sequence $\{\bar{\zeta}_j\}_{j \in J}$. Finally, let Ω be any lattice. Then a Λ_o -Abelian integral $\Psi = \Psi_o + \Psi_\sigma$ is quasi-reducible to a holomorphic integral on the torus \mathbf{C}/Ω if and only if*

(i) σ is Ω -admissible, and

$$(ii) \quad \int_{A_j} d\Psi_o \equiv \zeta_j \operatorname{Res}_{\partial R} \Phi'_{A_j} \sigma \pmod{\Omega}, \quad j \in J.$$

$$\int_{B_j} d\Psi_o \equiv \zeta_j \operatorname{Res}_{\partial R} \Phi'_{B_j} \sigma$$

9. Our problem is now to characterize, in connection with behavior spaces, an Abelian integral which is the pull-back of a holomorphic integral on a torus via an analytic mapping of more restricted nature. The restriction which we shall impose on the mapping is described in terms of the homological property of the mapping.

To be more precise, let R be as before with a fixed canonical basis $\Xi(R) = \{A_j, B_j\}_{j \in J} \pmod{\partial R}$. Let T be a torus and $\Xi(T) = \{C_1, C_2\}$ a canonical homology basis of T . A continuous mapping $f: R \rightarrow T$ is called of finite (resp. null) type relative to $(\Xi(R), \Xi(T))$ if

$$\sum_{j \in J} \prod_{v=1,2} \{(f(A_j) \times C_v)^2 + (f(B_j) \times C_v)^2\}$$

is finite (resp. is equal to zero). Cf. [10].

Using these terms, we have now another

Definition 2. If, in Definition 1, the mapping f can be chosen so as to be of finite (resp. null) type relative to some pair of canonical bases $(\Xi(R), \Xi(T))$, then Ψ is called *reducible* (resp. *strongly reducible*). If it is necessary to refer to the basis $\Xi(R)$ used, we say that Ψ is reducible (resp. strongly reducible) with respect to $\Xi(R)$.

From the definitions we have immediately

- (a) *A reducible Abelian integral is quasi-reducible.*
- (a') *In case $g < \infty$, the converse of (a) is also true.*
- (b) *A strongly reducible Abelian integral is reducible.*

On the analogy of Theorem 1 we can prove the following theorem (cf. [2], [3], [5], [6] and [12]).

Theorem 1'. *Let $\{A_j, B_j; D_i\}_{j \in J, i \in I}$ be a canonical homology basis of R . Then an Abelian integral Ψ on R is reducible with respect to $\{A_j, B_j\}_{j \in J}$ if and only if there exist integers m_{jk}, n_{jk} and p_{ik} ($j \in J, i \in I; k = 1, 2$) and two complex*

numbers v_1, v_2 such that

- (i) $\int_{A_j} d\Psi = m_{j1}v_1 + m_{j2}v_2$
 $\int_{B_j} d\Psi = n_{j1}v_1 + n_{j2}v_2$ $j \in J$,
- (i') $\int_{D_i} d\Psi = p_{i1}v_1 + p_{i2}v_2$ $i \in I$,
- (ii) $\sum_{j \in J} (m_{j1}^2 + n_{j1}^2)(m_{j2}^2 + n_{j2}^2) < \infty$,
- (iii) $v_1, v_2 \neq 0, \text{Im}(v_2/v_1) \neq 0$.

10. The following proposition is an easy consequence of Theorem 1' and Proposition 2 (cf. the proof of Proposition 6).

Proposition 4. *If Ψ is a [nonconstant] Abelian integral on R which is reducible with respect to a basis $\{A_j, B_j\}_{j \in J} \bmod \partial R$, then there exists a behavior space Λ_o on $(R, \{A_j, B_j\}_{j \in J})$ such that Ψ is Λ_o -Abelian.*

Thus every reducible Abelian integral on R is Λ_o -Abelian with a suitably chosen behavior space Λ_o and hence we can speak of the singularity of a reducible Abelian integral. Also, it makes sense to characterize the possible singularities of reducible Abelian integrals. We shall discuss this problem later; see Theorems 2, 3 and 4 below.

We also have

Proposition 5. *Let T be a torus and Λ_o a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ compatible with T . Then a Λ_o -Abelian integral on R which is quasi-reducible to a holomorphic integral on T is reducible.*

Proof. Suppose that Ψ is a Λ_o -Abelian integral and is quasi-reducible to a holomorphic integral on T . Since Λ_o is compatible with T , we can find a lattice $\Omega = [\omega_1, \omega_2]_{\mathbf{Z}}$ and a mapping $\varepsilon: J \rightarrow \{1, 2\}$ such that

- (i) $T = \mathbf{C}/\Omega$, and
- (ii) Λ_o is associated with the family $\{L(\omega_{\varepsilon(j)})\}_{j \in J}$.

If we set $\varepsilon^*(j) = 3 - \varepsilon(j)$, $j \in J$, ε^* is also a mapping of J into the set $\{1, 2\}$, and $\varepsilon^*(j) \neq \varepsilon(j)$ for all $j \in J$. By Theorem 1 we have

$$\begin{aligned} \int_{A_j} d\Psi &= m_{j\varepsilon(j)}\omega_{\varepsilon(j)} + m_{j\varepsilon^*(j)}\omega_{\varepsilon^*(j)} \\ \int_{B_j} d\Psi &= n_{j\varepsilon(j)}\omega_{\varepsilon(j)} + n_{j\varepsilon^*(j)}\omega_{\varepsilon^*(j)} \quad j \in J, i \in I, \\ \int_{D_i} d\Psi &= p_{i1}\omega_1 + p_{i2}\omega_2 \end{aligned}$$

with some integers m_{jk}, n_{jk}, p_{ik} ($j \in J, i \in I; k=1, 2$). Because Ψ is a Λ_o -Abelian integral and $\text{Im}(\omega_2/\omega_1) > 0$, we see that

$$m_{j\epsilon^*(j)} = n_{j\epsilon^*(j)} = 0$$

except for a finite number of j . It now follows from Theorem 1' that Ψ is reducible. q. e. d.

11. In connection with Proposition 4 we shall prove the following

Proposition 6. *Let Ψ be a strongly reducible non-constant Abelian integral on R . Then there is a behavior space Λ_o (on R) such that Ψ is a Λ_o -Abelian integral which is not of the first kind.*

Proof. Since Ψ is strongly reducible, there are a torus T , a holomorphic integral E on T , and an analytic mapping $f: R \rightarrow T$ which have the following properties:

(i) $d\Psi = f^*(dE)$,

(ii) there are canonical homology bases $\{A_j, B_j\}_{j \in J} \pmod{\partial R}$ and $\{C_1, C_2\}$ on R and T , respectively, such that f is of null type relative to $(\{A_j, B_j\}_{j \in J}, \{C_1, C_2\})$.

Set $\omega_k = \int_{C_k} dE$, $k=1, 2$, and $\Omega = [\omega_1, \omega_2]_{\mathbf{Z}}$. We have then (cf. the proof of Theorem 1)

$$\begin{aligned} \int_{A_j} d\Psi &= m_{j1}\omega_1 + m_{j2}\omega_2 \\ \int_{B_j} d\Psi &= n_{j1}\omega_1 + n_{j2}\omega_2 \end{aligned} \quad j \in J,$$

where m_{jk}, n_{jk} are appropriate integers satisfying

$$\sum_{j \in J} (m_{j1}^2 + n_{j1}^2)(m_{j2}^2 + n_{j2}^2) = 0.$$

If we set

$$\epsilon(j) = \begin{cases} 1 & \text{if } m_{j2} = n_{j2} = 0 \\ 2 & \text{otherwise,} \end{cases}$$

then ϵ is a well-defined mapping of J into the set $\{1, 2\}$. Proposition 2 ensures that there is a behavior space Λ_o which is subordinate to the lineation $(\{A_j, B_j\}_{j \in J}, \epsilon, \Omega)$. The integral Ψ is obviously Λ_o -Abelian. (The discussion so far constitutes, with a few necessary modifications, a proof of Proposition 4.)

If Ψ were of the first kind, Ψ would be constant; for then by Proposition 1' we would have

$$\begin{aligned} \|d\Psi\|^2 &= -2 \sum_{j \in J} \text{Im} \left(\int_{A_j} d\Psi \int_{B_j} d\bar{\Psi} \right) \\ &= -2 \text{Im}(\omega_1 \bar{\omega}_2) \sum_{j \in J} (m_{j1}n_{j2} - m_{j2}n_{j1}) = 0, \end{aligned}$$

a contradiction.

q. e. d.

We can also prove

Proposition 7. *Let Λ_o be a behavior space on $(R, \{A_j, B_j\}_{j \in J})$. Then there is no non-constant Λ_o -Abelian integral of the first kind which is strongly reducible with respect to the basis $\{A_j, B_j\}_{j \in J}$.*

The proof is omitted (cf. [11]). It should be noted that Λ_o may be quite arbitrary; in particular, the (essential) index of Λ_o is not necessarily less than or equal to two.

12. In the remaining of the paper we shall give some conditions for an Abelian integral on R to be reducible. We shall first prove a refinement of Proposition 4.

Theorem 2. *Suppose that Ψ is a reducible Abelian integral on R . Then we can find a lineation $(\{A_j, B_j\}_{j \in J}, \varepsilon, (\omega_1, \omega_2))$ which has the following properties: For any behavior space Λ_o which is subordinate to the lineation*

- (A) Ψ is a Λ_o -Abelian integral.
- (B) The singularity σ of $d\Psi$ is $[\omega_1, \omega_2]_{\mathbf{Z}}$ -admissible.
- (C) The regular component $d\Psi_o$ of $d\Psi$ satisfies

$$\int_{A_j} d\Psi_o \equiv \omega_{\varepsilon(j)} \operatorname{Res}_{\partial R} \Phi'_{A_j} \sigma \pmod{[\omega_1, \omega_2]_{\mathbf{Z}}, j \in J.}$$

$$\int_{B_j} d\Psi_o \equiv \omega_{\varepsilon(j)} \operatorname{Res}_{\partial R} \Phi'_{B_j} \sigma$$

Here, Φ'_{A_j} and Φ'_{B_j} are the elementary Λ'_o -Abelian integrals of the first kind determined by the sequence $\{\bar{\omega}_{\varepsilon(j)}\}_{j \in J}$, Λ'_o being the dual behavior space of Λ_o .

Proof. Since Ψ is reducible, there are a torus T , a holomorphic integral E on T , and an analytic mapping $f: R \rightarrow T$ of finite type such that $\Psi = E \circ f$. Let $(\{A_j, B_j\}_{j \in J}, \{C_1, C_2\})$ be a pair of canonical homology bases of $R \pmod{\partial R}$ and T relative to which f is of finite type. Namely, there is a finite subset J_o of J such that

$$\begin{aligned} f(A_j) &\approx m_j C_{v_j} & j \in J \setminus J_o, \\ f(B_j) &\approx n_j C_{v_j} \end{aligned}$$

where m_j, n_j are integers and v_j is 1 or 2 depending on j . Setting

$$\varepsilon(j) = \begin{cases} v_j & \text{for } j \in J \setminus J_o \\ 1 & \text{for } j \in J_o \end{cases}$$

and

$$\omega_k = \int_{C_k} dE, \quad k = 1, 2,$$

we obtain a mapping ε of J into $\{1, 2\}$ and a lattice $\Omega = [\omega_1, \omega_2]_{\mathbf{Z}}$. We claim that $(\{A_j, B_j\}_{j \in J}, \varepsilon, \Omega)$ is a lineation for which we are looking.

Take an arbitrary behavior space Λ_o which is subordinate to $(\{A_j, B_j\}_{j \in J}, \varepsilon, \Omega)$.

Such a behavior space Λ_o surely exists (Proposition 2). Then Ψ is a Λ_o -Abelian integral, for Λ_o is associated with the family $\{L(\omega_{\varepsilon(j)})\}_{j \in J}$ and the differential $d\Psi$ satisfies

$$\begin{aligned} \int_{A_j} d\Psi &= \int_{f(A_j)} dE = m_j \omega_{\varepsilon(j)} \equiv 0 \\ &\quad \text{mod } L(\omega_{\varepsilon(j)}), \quad j \in J \setminus J_o. \\ \int_{B_j} d\Psi &= \int_{f(B_j)} dE = n_j \omega_{\varepsilon(j)} \equiv 0 \end{aligned}$$

Since Ψ is reducible and hence quasi-reducible to E which is a holomorphic integral on the torus $T = \mathbf{C}/\Omega$, we have by Proposition 3

$$\begin{aligned} \int_{A_j} d\Psi_o &\equiv \omega_{\varepsilon(j)} \operatorname{Res}_{\partial R} \Phi'_{A_j} \sigma \\ &\quad \text{mod } \Omega, \quad j \in J. \\ \int_{B_j} d\Psi_o &\equiv \omega_{\varepsilon(j)} \operatorname{Res}_{\partial R} \Phi'_{B_j} \sigma \end{aligned}$$

(Compare the proof of Theorem 1 in Shiba [11].) The remaining assertion (B) is obvious by Theorem 1'. q. e. d.

13. The following theorem gives a sort of converse of Theorem 2.

Theorem 3. *Let Ψ be an Abelian integral on R . Suppose there exist a lineation and a behavior space Λ_o subordinate to the lineation such that (A), (B) and (C) are fulfilled. Then Ψ is reducible.*

We shall only give a sketch of the proof. In the first place, Proposition 3 implies that Ψ is quasi-reducible to a holomorphic integral on the torus \mathbf{C}/Ω . Since Λ_o is clearly compatible with \mathbf{C}/Ω and Ψ is Λ_o -Abelian (condition (A)), we see by Proposition 5 that Ψ is reducible.

For convenience' sake we reformulate Theorems 2 and 3 in the following

Corollary. *An Abelian integral Ψ on R is reducible if and only if there exist a lineation $(\{A_j, B_j\}_{j \in J}, \varepsilon, \Omega)$ and a behavior space Λ_o subordinate to the lineation such that (A), (B) and (C) are satisfied.*

Note that the use of Proposition 5 in the proof of above Theorem 3 is dispensable. We then observe that Theorem 3 together with Proposition 3 gives another proof of Proposition 5. To see this, let T and Λ_o be the same as in Proposition 5. Let $(\{A_j, B_j\}_{j \in J}, \varepsilon, \Omega)$ be a lineation, $\Omega = (\omega_1, \omega_2)$, such that

- (i) $T = \mathbf{C}/\Omega$, and
- (ii) Λ_o is subordinate to the lineation.

Let Ψ be a Λ_o -Abelian integral which is quasi-reducible to a holomorphic integral on T and σ be the singularity of $d\Psi$. Decompose $d\Psi$ into its regular and singular components: $d\Psi = d\Psi_o + d\Psi_\sigma$. Then Proposition 3 implies that σ is Ω -admissible and that

$$\begin{aligned} \int_{A_j} d\Psi_o &\equiv \omega_{\varepsilon(j)} \operatorname{Res}_{\partial R} \Phi'_{A_j} \sigma \\ &\quad \text{mod } \Omega, \quad j \in J, \\ \int_{B_j} d\Psi_o &\equiv \omega_{\varepsilon(j)} \operatorname{Res}_{\partial R} \Phi'_{B_j} \sigma \end{aligned}$$

Φ'_{A_j} and Φ'_{B_j} being the elementary Λ'_o -Abelian integrals of the first kind determined by the sequence $\{\bar{\omega}_{\varepsilon(j)}\}_{j \in J}$. Thus conditions (B) and (C) are also fulfilled. We can now use Theorem 3 to conclude that Ψ is reducible. So we have proved Proposition 5.

14. We will call an $(I)\Lambda_o$ -singularity σ *reducible* if there is a reducible Λ_o -Abelian integral with the singularity σ . Then we can prove

Theorem 4. *Let Λ_o be a behavior space subordinate to the lineation $(\{A_j, B_j\}_{j \in J}, \varepsilon, \Omega)$, $\Omega = (\omega_1, \omega_2)$. Then an $(I)\Lambda_o$ -singularity σ is reducible if and only if*

(D) σ is Ω -admissible;

(E) There exists a Λ_o -Abelian differential ϕ of the first kind such that

$$\begin{aligned} \int_{A_j} \phi &\equiv \omega_{\varepsilon(j)} \operatorname{Res}_{\partial R} \Phi'_{A_j} \sigma \\ &\quad \text{mod } \Omega, \quad j \in J. \\ \int_{B_j} \phi &\equiv \omega_{\varepsilon(j)} \operatorname{Res}_{\partial R} \Phi'_{B_j} \sigma \end{aligned}$$

Proof. Assume, first of all, (D) and (E). (We could make use of the reasoning used in the proof of Theorem 3, but here we apply Abel's theorem (see [11]) in order to point out the explicit interrelation of these two topics.) Denote, as usual, by ψ_σ the (unique) elementary Λ_o -Abelian differential with the singularity σ (cf. section 4, 1°). Let T be the torus \mathcal{C}/Ω and $\{C_1, C_2\}$ be its canonical homology basis which is induced by the lattice Ω (see section 6). Let E_o be the normalized holomorphic integral on T with respect to $\{C_1, C_2\}$, i.e., suppose $\int_{C_1} dE_o = 1$. Then $\int_{C_2} dE_o = \omega_2/\omega_1$. Setting $E = \omega_1 E_o$, we know by Theorem 1 in [11] that conditions (D) and (E) imply the existence of an analytic mapping $f: R \rightarrow T$ such that

(i) f is of finite type relative to $(\{A_j, B_j\}_{j \in J}, \{C_1, C_2\})$, and

(ii) $f^*(dE) - \psi_\sigma$ is of the first kind.

Therefore the integral of $f^*(dE) = (f^*(dE) - \psi_\sigma) + \psi_\sigma$ is a reducible Λ_o -Abelian integral with the singularity σ . Hence σ is reducible. The converse follows easily from the same Theorem 1 in [11], so that we have completed the proof.

Let Λ_o be an arbitrary behavior space on R . Then an $(I)\Lambda_o$ -singularity σ will be called *quasi-reducible* if there is a quasi-reducible Λ_o -Abelian integral on R which has the singularity σ . The following proposition is an immediate consequence of Proposition 3. Compare the preceding theorem, too.

Proposition 8. *Let Λ_o be a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ associated with the family $\{L(\zeta_j)\}_{j \in J}$. Then an $(I)\Lambda_o$ -singularity σ is quasi-reducible if and only*

if there are a lattice Ω and a Λ_o -Abelian differential ϕ of the first kind such that σ is Ω -admissible and

$$\begin{aligned} \int_{A_j} \phi &\equiv \zeta_j \operatorname{Res}_{\partial R} \Phi'_{A_j} \sigma \\ \int_{B_j} \phi &\equiv \zeta_j \operatorname{Res}_{\partial R} \Phi'_{B_j} \sigma \end{aligned} \quad \text{mod } \Omega, \quad j \in J.$$

Here, of course, Φ'_{A_j} and Φ'_{B_j} stand for the elementary Λ'_o -integrals of the first kind determined by the sequence $\{\zeta_j\}_{j \in J}$.

15. In this section we shall be concerned with Abelian integrals of the first kind. Notice that the term “of the first kind” is meaningless unless we fix a behavior space on R .

First we shall prove the following generalization of Weierstrass’s theorem (see [5], [12]; cf. also [2], [3] and [6]).

Theorem 5. *Let Λ_o be a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ and Φ be a non-constant Λ_o -Abelian integral of the first kind. Then Φ is reducible with respect to $\{A_j, B_j\}_{j \in J}$ if and only if there are $v_k \in \mathbf{C}$; $m_{jk}, n_{jk} \in \mathbf{Z}$ ($k=1, 2$; $j \in J$) such that*

$$\begin{aligned} \text{(i)} \quad & \int_{A_j} d\Phi = m_{j1}v_1 + m_{j2}v_2 \\ & \int_{B_j} d\Phi = n_{j1}v_1 + n_{j2}v_2 \\ & \qquad \qquad \qquad j \in J, \\ \text{(ii)} \quad & \sum_{j \in J} (m_{j1}^2 + n_{j1}^2)(m_{j2}^2 + n_{j2}^2) < \infty. \end{aligned}$$

Proof. The only if part is proved in Theorem 1’, where we have actually shown that the numbers v_1, v_2 are subject to a more restricted condition (see (iii) in Theorem 1’). To prove the if part it suffices to verify that v_1, v_2 satisfy condition (iii) in Theorem 1’. But this is a simple consequence of Proposition 1’. In fact, we have

$$\begin{aligned} 0 < \|d\Phi\|^2 &= -2 \sum_{j \in J} \operatorname{Im} \left(\int_{A_j} d\Phi \int_{B_j} d\bar{\Phi} \right) \\ &= -2 \operatorname{Im} (v_1 \bar{v}_2) \sum_{j \in J} (m_{j1}n_{j2} - m_{j2}n_{j1}), \end{aligned}$$

so that neither v_1 nor v_2 vanishes and $\operatorname{Im} (v_1 \bar{v}_2) \neq 0$. q. e. d.

Now we recall that two behavior spaces on R are said to be equivalent if they define the same boundary behavior (see [10]). We have then

Theorem 6. *Let Λ_o be a behavior space on $(R, \{A_j, B_j\}_{j \in J})$. Suppose that, among the Λ_o -Abelian integrals of the first kind, there is a reducible Φ with respect to $\{A_j, B_j\}_{j \in J}$ such that $\# \left\{ j \in J \mid \int_{A_j} d\Phi = \int_{B_j} d\Phi = 0 \right\} < \infty$. Then Λ_o is equivalent to a behavior space $\tilde{\Lambda}_o$ whose index is at most two. (Hence Φ is a $\tilde{\Lambda}_o$ -Abelian inte-*

gral of the first kind. Also, $\text{ess. ind } \Lambda_0 \leq 2$.)

Proof. Evidently, it suffices to consider the case $g = \infty$; so we assume that R is of infinite genus. Let $\{L_j\}_{j \in J}$ be the family of lines with which Λ_0 is associated. Then we can find a positive integer j_1 such that

$$\int_{A_j} d\Phi \equiv \int_{B_j} d\Phi \equiv 0 \pmod{L_j}, \quad j > j_1,$$

for Φ is a Λ_0 -Abelian integral. On the other hand, since Φ is reducible with respect to $\{A_j, B_j\}_{j \in J}$, we can find $m_{jk}, n_{jk} \in \mathbf{Z}$ and $v_1, v_2 \in \mathbf{C}^*$, $\text{Im}(v_2/v_1) > 0$ (cf. Theorems 1' and 5) such that

$$\begin{aligned} \int_{A_j} d\Phi &= m_{j1}v_1 + m_{j2}v_2 \\ &\qquad\qquad\qquad j \in J \\ \int_{B_j} d\Phi &= n_{j1}v_1 + n_{j2}v_2 \end{aligned}$$

and

$$\sum_{j \in J} (m_{j1}^2 + n_{j1}^2)(m_{j2}^2 + n_{j2}^2) < \infty.$$

Hence there exists a positive integer j_2 such that if $j > j_2$ then either $m_{j1} = n_{j1} = 0$ or $m_{j2} = n_{j2} = 0$.

For $j \in J$ we set $\varepsilon(j) = 1$ if $m_{j2} = n_{j2} = 0$ and $\varepsilon(j) = 2$ otherwise. If we set furthermore $L_k^* = L(v_k)$, $k = 1, 2$, we have

$$\int_{A_j} d\Phi \equiv \int_{B_j} d\Phi \equiv 0 \pmod{L_{\varepsilon(j)}^*}, \quad j > j_2.$$

By assumption, if j is sufficiently large, say, $j > j_3$, then $\left| \int_{A_j} d\Phi \right| + \left| \int_{B_j} d\Phi \right| \neq 0$; and hence we see

$$L_j = L_{\varepsilon(j)}^* \quad \text{for } j > j_3,$$

j_0 being the maximum of j_1, j_2 and j_3 .

Let $J^* = \{1, 2, \dots, j_0\}$ and set

$$\tilde{L}_j = \begin{cases} L_j & \text{if } j \in J \setminus J^* \\ L_{\varepsilon(j)}^* & \text{if } j \in J^*. \end{cases}$$

In [10] we proved that $\tilde{\Lambda}_0 = \{\lambda \in \Lambda_h \mid \lambda \text{ has } \Lambda_0\text{-behavior and } \int_{A_j} \lambda \equiv \int_{B_j} \lambda \equiv 0 \pmod{\tilde{L}_j}, j \in J\}$ is also a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ and that $\tilde{\Lambda}_0$ is equivalent to the Λ_0 . The $\tilde{\Lambda}_0$ thus obtained is subordinate to the lineation $(\{A_j, B_j\}_{j \in J}, \varepsilon, (v_1, v_2))$ and so $\text{ind } \tilde{\Lambda}_0 \leq 2$. Finally, by Proposition 8 in [10], the integral Φ may well be regarded as a $\tilde{\Lambda}_0$ -Abelian integral of the first kind. q. e. d.

Corollary. *Let R be an open Riemann surface of infinite genus. If Λ_0 is a behavior space on $(R, \{A_j, B_j\}_{j \in J})$ such that $\text{ess. ind } \Lambda_0 > 2$, then any Λ_0 -Abelian*

integral of the first kind which is reducible with respect to $\{A_j, B_j\}_{j \in J}$ is single-valued on a subregion of infinite genus.

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