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On Rees algebras of ideals generated by a subsystem of parameters

By

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1. Introduction.

The purpose of this paper is to prove the following

Theorem 1.1. *Let A be a Noetherian local ring and let $0<r<\dim A$ be an integer. Then the following conditions are equivalent.*
(1) *A is a Cohen-Macaulay ring.*
(2) *The Rees algebra $R(q)=\bigoplus_{n\geq 0} q^n$ is a Cohen-Macaulay ring for every ideal q of A generated by a subsystem of parameters for A of length r.*

In case A is a Cohen-Macaulay local ring J. Barshay [1] showed that the Rees algebras of ideals generated by subsystems of parameters for A are always Cohen-Macaulay (cf. p. 93, Corollary), and it seems to be natural to ask whether the converse of his result is true. But unfortunately this does not hold in the case of the parameter ideals. In fact, recently S. Goto and the author [2] (cf. Theorem 1.1), heve proved that the Rees algebras of parameter ideals of certain Buchsbaum local rings are always Cohen-Macaulay. Neverthless the above theorem guarantees that the converse of Barshay's result is true if the length of subsystems of parameters considered is less than dim A.

The idea of the proof of Theorem 1.1 is essentially same as that of the proof of the main theorem of [2]. We will prove Theorem 1.1 in Section 3. Section 2 will be devoted to some preliminary results which we shall need for this purpose. Finally we will show with an example that the condition that *every* ideal q of A generated by a subsystem of parameters for A is not superfluous.

In this paper we denote by A a Noetherian local ring of dimension d with maximal ideal m.

2. Preliminary.

Let q be an ideal of A generated by a subsystem of parameters for A of length r. We put $q=(a_1, a_2, \cdots, a_r)$ and $R=R(q)$. Notice that the ring R is canonically identified with the A-subalgebra

$$A[a_1X, a_2X, \cdots, a_rX]$$

of $A[X]$, where X is an indeterminate over A. By M we denote the unique graded maximal ideal of R, i.e.,

$$M=(m, a_1X, a_2X, \cdots, a_rX).$$

Recall that

$$\dim R=\dim R_M=d+1$$

(cf. [6]). Let $a_{r+1}, \cdots, a_d$ be a system of elements of m such that $a_1, a_2, \cdots, a_r, a_{r+1}, \cdots, a_d$ forms a system of parameters for A and put

$$N=(a_1, a_2+a_1X, \cdots, a_r+a_{r-1}X, a_rX, a_{r+1}, \cdots, a_d).$$

We begin with

Lemma 2.1. *$M=\sqrt{N}$. In particular,*

$$a_1, a_2+a_1X, \cdots, a_r+a_{r-1}X, a_rX, a_{r+1}, \cdots, a_d$$

is a system of parameters for R_M.

Proof. Suppose that $a_iX\in\sqrt{N}$ for some i. Then $a_{i-1}X\in\sqrt{N}$, as

$$(a_{i-1}X)^2=(a_i+a_{i-1}X)\cdot a_{i-1}X-a_{i-1}\cdot a_iX.$$

Thus we have $a_iX\in\sqrt{N}$ for $1\leqq i\leqq r$ by induction on i, and so $q\subset\sqrt{N}$, as $a_i+a_{i-1}X\in N$ by definition. Hence we have $M\subset\sqrt{N}$, which implies $M=\sqrt{N}$.

Corollary 2.2. *R is a Cohen-Macaulay ring if and only if*

$$a_1, a_2+a_1X, \cdots, a_r+a_{r-1}X, a_rX, a_{r+1}, \cdots, a_d$$

is an R_M-sequence.

Proof. If $a_1, a_2+a_1X, \cdots, a_r+a_{r-1}X, a_rX, a_{r+1}, \cdots, a_d$ forms an R_M-sequence, R_M is a Cohen-Macaulay local ring by Lemma 2.1. Therefore R is a globally Cohen-Macaulay ring by [3], Theorem. The converse is obvious.

Lemma 2.3. *Let x be an element of*

$$(a_1, a_2, \cdots, a_{r-1}, a_{r+1}^2, \cdots, a_d^2): a_r^2$$

and suppose that R is a Cohen-Macaulay ring. Then

$$xa_r^t\in(a_1, a_2, \cdots, a_{r-1})q^{t-1}+(a_{r+1}^2, \cdots, a_d^2)q^t$$

for $t\geqq r-1$.

Proof. Let us express $xa_r^2=y+\sum_{j=1}^{r-1} y_ja_j$ ($y\in(a_{r+1}^2, \cdots, a_d^2)$, $y_j\in A$) and put

$$I=(a_1X, \{a_j-a_{j+1}X\}_{1\leq j\leq r-2}, a_{r+1}^2, \cdots, a_d^2).$$

Then

$$xa_r^{t+2}X^t = ya_r^tX^t + \sum_{j=1}^{r-1} y_ja_ja_r^tX^t$$

and $a_j \equiv a_{j+1}X \bmod I$ for every $1 \leqq j \leqq r-2$. Observe the equations

$$a_ja_r^tX^t \equiv a_{j-1}a_r^tX^{t-1} \equiv \cdots \equiv a_1a_r^tX^{t-j+1} \equiv a_1Xa_r^tX^{t-j} \equiv 0 \bmod I$$

$(1 \leqq j \leqq r-1)$, and we have $xa_r^{t+2}X^t \in I$. On the other hand, a_r, $a_{r-1}-a_rX, \cdots$, a_1-a_2X, a_1X, $a_{r+1}^2, \cdots, a_d^2$ is an R_M-sequence by Corollary 2.2. Thus $xa_r^tX^t \in IR_M$, i.e., $fxa_r^tX^t \in I$ for some $f \in R \backslash M$. Now let us express

$$(*) \qquad fxa_r^tX^t = ga_1X + \sum_{j=1}^{r-2} g^{(j)}(a_j - a_{j+1}X) + \sum_{i=r+1}^{d} h^{(i)}a_i^2,$$

where g, $g^{(j)}$ and $h^{(i)} \in R$. Let g_k, $g_k^{(j)}$ and $h_k^{(i)}$ denote the coefficient of the term X^k in g, $g^{(j)}$ and $h^{(i)}$, respectively. Then, comparing the term X^t in the equation $(*)$, we see

$$f_0xa_r^t = a_1g_{t-1} + \sum_{j=1}^{r-2} g_t^{(j)}a_j - \sum_{j=1}^{r-2} g_{t-1}^{(j)}a_{j+1} + \sum_{i=r+1}^{d} h_t^{(i)}a_i^2.$$

As f_0 is a unit of A, this equation gives the desired result.

The following result is well known.

Lemma 2.4. *Let C be a Noetherian local ring and suppose that*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is an exact sequence of finitely generated C-modules. Then either

a) depth $L \geqq$ depth $M =$ depth N

b) depth $M \geqq$ depth $L =$ depth $N+1$

c) depth $N >$ depth $L =$ depth M.

In particular if depth $L =$ depth N, *then*

$$\text{depth } L = \text{depth } M = \text{depth } N.$$

Definition 2.5. (cf. [5]) For an ideal q of A we put

$$\mathrm{Assh}(q) = \{p \in \mathrm{Ass}(q)\,;\, \dim A/p = \dim A/q\}.$$

We denote by $U(q)$ the intersection of all primary ideals of q, of which belonging prime ideals are contained in Assh (q).

Remark 2.6. Let a be an element of m such that $\dim A/(q, a) = \dim A/q - 1$, and we see that $a \notin \bigcup_{p \in \mathrm{Assh}(q)} p$ by definition and that a is an $A/U(q)$-regular element.

Definition 2.7. (cf. [5]) A system of elements $x_1, x_2, \cdots, x_k$ of m is called a weak regular sequence if

$$(x_1, x_2, \cdots, x_i):x_{i+1}=(x_1, x_2, \cdots, x_i):m$$

holds for every $0\leqq i<k$. A ring A is called a Buchsbaum ring if any system of parameters for A forms a weak regular sequence.

In the rest of this section we assume that A is a Buchsbaum ring. Now we put $q_i=(a_1, a_2, \cdots, a_i)$. We state the following results without proof.

Lemma 2.8. ([2] Lemma 4.2) *$U(q_i)\cap q^n=q_iq^{n-1}$ for every integer $n>0$ and every $1\leqq i\leqq r$.*

Corollary 2.9. ([2] Corollary 4.3) *$U(a_1A)\cap q^n=a_1q^{n-1}$ for every integer $n>0$.*

Lemma 2.10. ([2] Proposition 4.4) *There is an exact sequence*

$$0 \longrightarrow {}_hU(a_1A) \longrightarrow R/(a_1X) \longrightarrow R(q+U(a_1A)/U(a_1A)) \longrightarrow 0$$

of graded R-modules. Here we denote $U(a_1A)$ by ${}_hU(a_1A)$ when we consider it via h an R-module, where h is the canonical projection $R\rightarrow A$.

Lemma 2.11. *Suppose that* depth $A>0$. *Then a_1X is a nonzero divisor of R.*

3. Proof of Theorem 1.1.

First, we state the following

Proposition 3.1. *Suppose that A is a Buchsbaum ring and that* depth $A\geqq 2$. *Let q be an ideal of A generated by a subsystem of parameters $a_1, a_2, \cdots, a_r$ for A. Then* depth $R(q)_M=$depth $A+1$.

Proof. We put $R=R(q)$. At first, notice that $U(a_1A)\cong a_1A:m\cong a_1A\cong A$ as depth $A\geqq 2$. Then by Lemma 2.10 we see that there is an exact sequence

$$(*)\qquad 0 \longrightarrow {}_hA \longrightarrow R/(a_1X) \longrightarrow R(q/a_1A) \longrightarrow 0$$

of graded R-modules. We put depth $A=s$. We will prove the assertion by induction on s.

The assertion is trivial in case $r=1$. Thus we may assume that $r\geqq 2$. Suppose that $s=2$. Notice that a_1, a_2 forms an A-sequence because depth $A\geqq 2$ and A is a Buchsbaum ring.

($r=2$) As $R(q/a_1A)\cong A/a_1A[X]$ we see that depth $R(q/a_1A)_M=$depth $A/a_1A+1=2$. Thus depth $(R/(a_1X))_M=2$ by Lemma 2.4 and so we have depth $R_M=3$ by Lemma 2.11.

($r\geqq 3$) We put $\bar{A}=A/a_1A$ and $\bar{q}=q/a_1A$. For an element a of A let us denote by $\bar{a}$ an element of $\bar{A}$. Notice that $\bar{a}_2X$ is a nonzero divisor of $R(\bar{q})$ by Lemma 2.11 as depth $\bar{A}>0$. We show that $\bar{a}_2X$, $\bar{a}_3$ forms an $R(\bar{q})$-sequence. In fact, let f be an element of $(\bar{a}_2X):\bar{a}_3$. As we may assume that f is homogeneous, we can express $f=\bar{b}X^n(b\in q^n)$. Observe the equation

$$\bar{b}X^n\cdot\bar{a}_3=\bar{a}_2X\cdot\bar{c}X^{n-1}$$

for some $c\in q^{n-1}$. Then we have $ba_3-ca_2\in a_1A$ and so $ba_3\in U(q_2)$. As a_3 is $A/U(q_2)$-regular by Remark 2.6, we see $b\in U(q_2)$. Thus $b\in U(q_2)\cap q^n=q_2q^{n-1}$ by Lemma 2.8. Now we can express $b=a_1y_1+a_2y_2$ for some $y_1, y_2\in q^{n-1}$. Hence

$$\bar{b}X^n=\bar{a}_2X\cdot\bar{y}_2X^{n-1}$$

and the above claim is proved. Therefore we see that depth $R(\bar{q})_M\geqq 2$ and depth $(R/(a_1X))_M\geqq 2$ by Lemma 2.4. Hence we have depth $R_M=3$ by Lemma 2.11.

Suppose that $s\geqq 3$. By induction hypothesis we see that depth $R(\bar{q})_{\bar{M}}$ $=$depth $\bar{A}+1=(s-1)+1=s$. Hence depth $(R/(a_1X))_M=s$ by Lemma 2.4 and so depth $R_M=s+1$ by Lemma 2.11.

Corollary 3.2. *Suppose that A is a Buchsbaum ring and that* depth $A\geqq 2$. *Let $0<r\leqq$*dim *A be an integer. Then A is a Cohen-Macaulay ring if and only if the Rees algebra $R(q)$ is a Cohen-Macaulay ring for an ideal q of A generated by a subsystem of parameters for A of length r.*

Remark 3.3. Corollary 3.2 is known in case $r=d=$dim A (cf. [2] Theorem 4.1).

Proof of Theorem 1.1. (2)$\Rightarrow$(1). By virtue of Corollary (3.2) this follows from the following proposition.

Proposition 3.4. *Suppose that the Rees algebra $R(q)$ is a Cohen-Macaulay ring for every ideal q of A generated by a subsystem of parameters for A of length r, then A is a Buchsbaum ring and* depth $A\geqq 2$ *if* dim $A\geqq 2$.

Proof. We put $q=(a_1, a_2, \cdots, a_r)$ and $R=R(q)$. Let $a_{r+1}, \cdots, a_d$ be a system of elements of m such that $a_1, a_2, \cdots, a_{r+1}, \cdots, a_d$ forms a system of parameters for A. Then $a_1, a_2+a_1X, \cdots, a_r+a_{r-1}X, a_rX, a_{r+1}, \cdots, a_d$ forms an R_M-sequence by Corollary 2.2. In particular, a_1, a_{r+1} is an R_M-sequence. We show that a_1, a_{r+1} is also an A-sequence. In deed, let x be an element of A such that $xa_1=0$. As a_1 is R_M-regular, we see $x=0$ in R_M, i.e.,

$$fx=0$$

for some $f\in R\backslash M$. Comparing the constant term in the above equation, we have (unit)$\cdot x=0$ and so $x=0$ in A. Now let y be an element of A such that ya_{r+1} $=za_1$. Since a_1, a_{r+1} is an R_M-sequence, we see $y\in(a_1)R_M$, i.e.,

$$fy=sa_1$$

for some $f\in R\backslash M$ and $s\in R$. Then, comparing the constant term in the above equation, we have $y\in a_1A$. Hence the above claim follows. Therefore we have depth $A\geqq 2$.

Now let $a_1, a_2, \cdots, a_r, a_{r+1}, \cdots, a_d$ be a system of parameters for A and fix $a_{r+1}, \cdots, a_d$. Since $R'=R((a_1, a_2, \cdots, a_r))$ is a Cohen-Macaulay ring by the assumption, $a_{r+1}, \cdots, a_d$ forms an R'_M-sequence by Corollary 2.2. Then $a_{r+1}, \cdots, a_d$ is an A-sequence, using the same proof as the above. We put $B=A/(a_{r+1}^2, \cdots, a_d^2)$. In order to show that A is a Buchsbaum ring it suffices to show that B is a Buchsbaum ring by [7], Theorem. Notice that $\dim B=r$ as $a_{r+1}^2, \cdots, a_d^2$ is a subsystem of parameters for A. For any element b of A let us denote by $\bar{b}$ the image of b in B.

Assume that the equality

$$(*) \qquad (\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{r-1}) : \bar{b}_r^2 = (\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{r-1}) : \bar{b}_r$$

holds for every system of parameters $\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_r$ for B. Let $\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{r-1}, \bar{b}$ and $\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{r-1}, \bar{c}$ be two system of parameters for B. In order to prove that B is a Buchsbaum ring it suffices to show that

$$(\underline{\bar{b}}) : \bar{b} = (\underline{\bar{b}}) : \bar{c}$$

where $(\underline{\bar{b}})=(\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{r-1})$ (cf. [5] Satz 5). Of course by the symmetry between $\bar{b}$ and $\bar{c}$, we have only to prove $(\underline{\bar{b}}) : \bar{b} \subset (\underline{\bar{b}}) : \bar{c}$. Let $n>0$ be an integer such that $\bar{c}^n \in (\underline{\bar{b}}) + \bar{b}B$ and express $\bar{c}^n = \sum_{i=1}^{r-1} \bar{b}_i \bar{x}_i + \bar{b}\bar{x}$ with $\bar{x}_i, \bar{x} \in B$.

Now let $\bar{y}$ be an element of B such that $\bar{y}\bar{b} \in (\underline{\bar{b}})$. Then we see $\bar{y} \in (\underline{\bar{b}}) : \bar{c}^n$ as $\bar{y}\bar{c}^n = \sum_{i=1}^{r-1} \bar{y}\bar{b}_i\bar{x}_i + \bar{y}\bar{b}\bar{x}$. Hence we have $\bar{y} \in (\underline{\bar{b}}) : \bar{c}$ by the assumption $(*)$. Thus, $(\underline{\bar{b}}) : \bar{b} \subset (\underline{\bar{b}}) : \bar{c}$, as desired.

Therefore in order to conclude the proof of Proposition 3.4 we have to prove the assumption $(*)$. This follows the next lemma.

Lemma 3.5. *Under the same situation as in the Proposition* 3.4. *Let* $B=A/(a_{r+1}^2, \cdots, a_d^2)$ *and let* $\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_r$ *be a system of parameters for* B. *Then,*

$$(\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{r-1}) : \bar{b}_r^2 = (\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{r-1}) : \bar{b}_r .$$

Proof. Let $\bar{x}$ be an element of $(\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_{r-1}) : \bar{b}_r^2$. As $R=R((b_1, b_2, \cdots, b_r))$ is a Cohen-Macaulay ring by the assumption, we see $xb_r^t \in (b_1, b_2, \cdots, b_{r-1})q_1^{t-1} + (a_{r+1}^2, \cdots, a_d^2)q_1^t$ for $t \geqq r-1$ by Lemma 2.3, where $q_1=(b_1, b_2, \cdots, b_r)$. Hence we can express

$$xb_r^t = \sum_{j=1}^{r-1} b_j z_j + z$$

for some $z_j \in q_1^{t-1}$ and $z \in (a_{r+1}^2, \cdots, a_d^2)q_1^t$. We put

$$J=(b_1, b_2-b_1X, \cdots, b_{r-1}-b_{r-2}X, b_rX, a_{r+1}^2, \cdots, a_d^2).$$

At first, notice that

$$xb_r(b_r-b_{r-1}X)^t = xb_r \sum_{i=1}^{t} (-1)^i \binom{t}{i} \cdot b_r^{t-i}(b_{r-1}X)^i$$

$$= xb_r^{t+1} + xb_r \sum_{i=1}^{t} (-1)^i \cdot \binom{t}{i} \cdot b_r^{t-i}(b_{r-1}X)^i$$

$$\equiv x b_r^{t+1} \equiv \sum_{j=1}^{r-1} b_r b_j z_j \bmod J.$$

On the other hand, as $b_r b_j z_j \equiv b_{j-1} X b_r z_j \equiv \cdots \equiv b_1 X b_r z_j X^{j-2} \bmod J$ $(1 \leqq j \leqq r-1)$, we see $x b_r^{t+1} \equiv 0 \bmod J$. Thus we have $x b_r (b_r - b_{r-1} X)^t \in J$, and so $x b_r \in J R_M$ because $b_r - b_{r-1} X$ is $R_M / J R_M$-regular by Corollary 2.2. Hence $f \cdot x b \in J$ for some $f \in R \backslash M$. Comparing the constant term similary as in the proof of Lemma 2.3, we see that

$$x b_r \in (b_1, b_2, \cdots, b_{r-1}, a_{r+1}^2, \cdots, a_d^2),$$

which implies $\tilde{x} \in (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_{r-1}) : \tilde{b}_r$.

Example 3.6. Even if $R(q)$ is a Cohen-Macaulay ring for and ideal q of A generated by a subsystem of parameters for A of length r with $1 < r < d$, A is not necessarily a Cohen-Macaulay ring. For example, let k be a field and let U, V and W be indeterminates over k. We put

$$A = k[[U^4, U^3 V, U V^3, V^4, W]].$$

Let $a_1 = U^4$, $a_2 = V^4$, $a_3 = W$ and $q = (a_1, a_2) A$. Then we have the equality

$$(a_3) R = (a_3) A[X] \cap R$$

where $R = R(q)$. In fact, let f be an element of $(a_3) A[X] \cap R$ and we will show that $f \in (a_3) R$. Of course we may assume that f is homogeneous. Let us express $f = c X^n$ $(c \in q^n)$, and we see $c \in q^n \cap (a_3) A$. Thus we have $c \in a_3 q^n$ as a_3 is obviously a nonzero divisor of A / q^n for every integer $n > 0$. Hence $f \in (a_3) R$, which implies $(a_3) A[X] \cap R \subset (a_3) R$. The oppesite inclusion is trivial. Recall that $R / (a_3) A[X] \cap R$ is a Cohen-Macaulay ring by virtue of Theorem 1.1 in [2] because $R / (a_3) A[X] \cap R = R(q + a_3 A / a_3 A)$ and $A / a_3 A$ is a two dimensional Buchsbaum ring. Thus $R / (a_3) R$ is a Cohen-Macaulay ring by the above claim and we have that R is Cohen-Macaulay.

Of course A is not a Cohen-Macaulay ring. (A is not even a Buchsbaum ring (cf. [4] Bemerkung 4.6)).

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References

[1] J. Barshay, Graded Algebras of Powers of Ideals Generated by A-Sequences, J. Alg., **25** (1973), 90-99.

[2] S. Goto and Y. Shimoda, On Rees algebras over Buchsbaum rings, J. Math. Kyoto Univ., **20** (1980), 691-708.

[3] J. Matijevic and P. Roberts, A conjecture of Nagata on graded Cohen-Macaulay rings, J. Math. Kyoto Univ., **14** (1974), 125-128.

[4] P. Schenzel, N. Trung and N. Cuong, Verallgemeinerte Cohen-Macaulay-Moduln, Math. Nachr., **85** (1978), 57-73.

[5] J. Stückrad and W. Vogel, Eine Verallgemeinerung der Cohen-Macaulay Ringe und Anwendungen auf ein Problem der Multiplitätstheorie, J. Math. Kyoto Univ., **13** (1973), 513-528.

[6] G. Valla, Certain graded algebras are always Cohen-Macaulay, J. Alg., **42** (1976), 537-548.
[7] W. Vogel, A non-zero-divisor characterization of Buchsbaum modules, preprint.