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Continuous time multi-allelic stepping stone models in population genetics

By

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§ 1. Introduction

The stepping stone model was first proposed by M. Kimura in 1953 for the purpose of investigation of local differentiation in geographically structured population [6]. Since then, many biologists have extensively studied this model. They have mainly discussed the genetic correlation of gene frequencies between colonies, the probability of identity and the rate of convergence to the stationary state. We refer to Sawyer [14] for mathematical treatment. But most studies have been made for the discrete time model. On the other hand the continuous time model was defined as an infinite dimensional diffusion process, which is more tractable for analysis of stationary states and limiting behaviors [15][16].

From a viewpoint of probability theory we are interested in the continuous time model since it provides a concrete and analyzable example of infinite dimesional diffusion processes. Also we can regard the continuous time stepping stone model as a diffusion-type model in the theory of infinitely interacting systems.

Let us consider a multi-allelic locus with $A_1, \cdots, A_d$ where d is a positive integer $\geqq 2$. Let S be a countable set. Each element k of S corresponds to a subpopulation, which is called a colony. Denote by $(x_k^1, \cdots, x_k^d)$ the gene frequencies of the $A_1, \cdots, A_d$ at colony k, that is $x_k^1 \geqq 0, \cdots, x_k^d \geqq 0, x_k^1 + \cdots + x_k^d = 1$. Usually we suppose that the change of gene frequencies is caused by random sampling drift, mutation, selection and migration among colonies.

Let $X_d = \{\boldsymbol{x} = \{x_k^p\}\ ;\ x_k^p \geqq 0,\ x_k^1 + \cdots + x_k^d = 1 \text{ for all } k \in S\}$, which is equipped with the product topology. We consider a time evolution of gene frequencies as a diffusion process on X_d.

Let

$$(1.1) \qquad A^d = \sum_{i\in S} \sum_{p=1}^{d} \sum_{q=1}^{d} x_i^p(\delta_{pq} - x_i^q) D_{i,p} D_{i,q} + \sum_{i\in S} \sum_{p=1}^{d} \Big(\sum_{q=1}^{d} \lambda_{qp} x_i^q + x_i^p \big(s_p - \sum_{q=1}^{d} s_q x_i^q\big) + \sum_{j\in S} q_{ji} x_j^p \Big) D_{i,p}$$

where $D_{i,p} = \dfrac{\partial}{\partial x_i^p}$, $\{\lambda_{qp}\}_{1\leq s,\, p\leq d}$ is a $d\times d$-matrix satisfying $\lambda_{qp} \geqq 0$ $(q \neq p)$ and

$\sum_{p=1}^{d}\lambda_{qp}=0$ for all $1\leqq q\leqq d$, $\{s_p\}_{1\leq p\leq d}$ is a real d-vector, and $\{q_{ji}\}_{j,i\in S}$ is a matrix on $S\times S$ satisfying $q_{ji}\geqq 0\,(j\neq i)$ and $\sum_{j\in S}q_{ji}=0$ for all $i\in S$.

Genetically $\{\lambda_{qp}\}$, $\{s_p\}$ and $\{q_{ji}\}$ stand for the intensities of mutation, selection and migration respectively.

Let $C(X_d)$ be the set of all continuous functions on X_d which is a Banach space with the uniform norm, and let $C_f^2(X_d)$ be the set of all such $f\in C(X_d)$ that depend only on finitely many coordinates and are twice continuously differentiable.

We assume

(1.2) $$\sup_{i\in S}|q_{ii}|<+\infty\,.$$

Then it will be shown in § 2 that there exists a unique strongly continuous contraction semi-group $\{T_t^d\}$ on $C(X_d)$ such that

(1.3) $$T_t^d f\geqq 0 \quad \text{for any } f\in C(X_d) \quad \text{with } f\geqq 0, \text{ and } T_t^d 1=1\,,$$

and

(1.4) $$T_t^d f-f=\int_0^t T_s^d A^d f ds \qquad \text{for any} \quad f\in C_f^2(X_d)\,.$$

Furthermore $\{T_t^d\}$ defines a diffusion process $(\Omega,\mathcal{F},P_x,\{\mathcal{F}_t\};\boldsymbol{x}(t))$ on X_d which we call *a continuous time stepping stone model with d alleles.*

Let $\mathcal{P}(X_d)$ be the set of all probability measures on X_d equipped with the topology of weak convergence. Since X_d is compact $\mathcal{P}(X_d)$ also is compact. Denote by $\{T_t^{d*}\}$ the adjoint semi-group on $\mathcal{P}(X_d)$ induced by $\{T_t^d\}$ and denote by $\mathcal{S}_d$ the set of all stationary states, i. e. $\mathcal{S}_d=\{\mu\in\mathcal{P}(X_d);\ T_t^{d*}\mu=\mu$ for all $t\geqq 0\}$. Then $\mathcal{S}_d$ is a non-empty compact and convex set. $(\mathcal{S}_d)_{ext}$ denotes the set of all extremal elements of $\mathcal{S}_d$.

In the previous paper [15] we studied diallelic models. In particular we obtained a complete description of extremal stationary states and some ergodic theorems. In the present paper we shall be concerned with multi-allelic models.

In § 2 we shall construct a class of infinite dimensional diffusion processes including infinite-allelic stepping stone models. It should be noted that Ethier also constructed such processes by making use of the semi-group method [1].

In § 3 results on diallelic models will be summarized for the subsequent need. In § 4 we shall present a complete description of extremal stationary states for multi-allelic models with mutation.

In the last two sections we shall study the scaling limit of the fluctuation processes of stepping stone models. Let $(\Omega,\mathcal{F},P,\{\mathcal{F}_t\};\boldsymbol{x}(t)=\{(x_k^1(t),\cdots,x_k^d(t))\})$ be a multi-allelic stepping stone model without mutation and selection. Let $S=Z^r$: the r-dimensional integer lattice space. Then $\boldsymbol{x}(t)$ is regarded as a vector-valued measure process on R^r such as

(1.5) $$\sum_{k\in Z^r}(x_k^1(t)-E[x_k^1(t)],\ \cdots,\ x_k^d(t)-E[x_k^d(t)])\cdot\delta_{\{k\}}$$

where $\delta_{\{x\}}$ stands for the point-mass at $x\in R^r$. In §5 it will be shown that taking a scaling limit

(1.6) $$\lambda^{-(r+2)/2}\sum_{k\in Z^r}(x_k^1(\lambda^2t)-E[x_k^1(\lambda^2t)],\ \cdots,\ x_k^d(\lambda^2t)-E[x_k^d(\lambda^2t)])\cdot\delta_{\{k/\lambda\}}$$

it converges as $\lambda\to+\infty$ to a $\overset{d}{\otimes}\mathcal{S}'(R^r)$-valued Ornstein-Uhlenbeck process $N_t=(N_t^1,\cdots,N_t^d)$ defined by

(1.7) $$dN_t^p=\sum_{q=1}^d \alpha_{qp}dW_t^q+LN_t^p dt \qquad p=1,\cdots,d,$$

with a suitably chosen initial condition N_0, where $(W_t^1,\cdots,W_t^d)$ is a $\overset{d}{\otimes}\mathcal{S}'(R^r)$-valued standard Wiener process, $\{\alpha_{qp}\}_{1\le q,p\le d}$ is a constant $d\times d$-matrix and L is an elliptic differential operator determined by $\{q_{ji}\}$. In §6 we shall present some variations of scaling limits.

For such problem of scaling limits we refer to Holley-Stroock [4], [5], who discussed on infinitely many branching Brownian particles and various kinds of infinitely ineracting systems. We also refer to a recent work by H. Tanaka [19], who presented a rigorous proof on a scaling limit of the fluctuation process for Kac's one-dimensional model of Maxwellian molecules in statistical mechanics.

§2. Construction of stepping stone models

In this section we shall construct infinite-allelic models. Let $\bar{X}_\infty=\{\boldsymbol{x}=\{x_i^p\}_{i\in S,p\ge1};\ x_i^p\geqq0$ and $\sum_{p=1}^\infty x_i^p\leqq1$ for each $i\in S\}$, and let $X_\infty=\{\boldsymbol{x}\in\bar{X}_\infty;\ \sum_{p=1}^\infty x_i^p=1$ for each $i\in S\}$, which are equipped with the topology of the component-wise convergence. Then $\bar{X}_\infty$ is compact but X_∞ is not so.

Let us consider the following differential operator on $\bar{X}_\infty$,

(2.1) $$A^\infty=\sum_{i\in S}\sum_{p=1}^\infty\sum_{q=1}^\infty x_i^p(\delta_{pq}-x_i^q)D_{i,p}D_{i,q}+\sum_{i\in S}\sum_{p=1}^\infty b_i^p(\boldsymbol{x})D_{i,p} \qquad D(A^\infty)=C_f^2(\bar{X}_\infty),$$

where $D_{i,p}=\dfrac{\partial}{\partial x_i^p}$ and $C_f^2(\bar{X}_\infty)$ denotes the set of all C^2-functions on $\bar{X}_\infty$ depending only on finitely many coordinates.

In order to construct a A^∞-diffusion process on X_∞ we shall consider a martingale problem. Let $\Omega=C([0,\infty),X_\infty)$ be the set of all X_∞-valued continuous functions defined on $[0,\infty)$. For each $t\geqq0$ we define $\boldsymbol{x}(t);\ \Omega\to X_\infty$ by $\boldsymbol{x}(t:\omega)=\omega(t)$, and let $\mathcal{F}$ $(\mathcal{F}_t)$ be the σ-field generated by $\{\boldsymbol{x}(s:\omega);\ s\geqq0\}$ $(\{\boldsymbol{x}(s:\omega);\ 0\leqq s\leqq t\})$.

Let $\boldsymbol{x}\in X_\infty$. A probability measure P on $(\Omega,\mathcal{F})$ is called a solution of the $(X_\infty, A_\infty, \boldsymbol{x})$-martingale problem, if

(2.2) $$P[\boldsymbol{x}(0)=\boldsymbol{x}]=1, \quad \text{and}$$

(2.3) $(f(\boldsymbol{x}(t))-\int_0^t A^\infty f(\boldsymbol{x}(s))ds,\ \{\mathcal{F}_t\})$ is a P-martingale for any $f\in C_f^2(\bar{X}_\infty)$.

Condition [A]

(2.4) $$\sup_{i\in S}\sup_{x\in X_\infty}\sum_{p=1}^{\infty}|b_i^p(\boldsymbol{x})|<+\infty,$$

(2.5) $$b_i^p(\boldsymbol{x})\geqq 0 \quad \textit{if} \quad \boldsymbol{x}\in\bar{X}_\infty \quad \textit{and} \quad x_i^p=0 \quad (i\in S,\ p\geqq 1),$$

(2.6) *there exist positive constants* $\{C_i\}$ *satisfying*

$$\left|\sum_{p=1}^{\infty}b_i^p(\boldsymbol{x})\right|\leqq C_i\left|1-\sum_{p=1}^{\infty}x_i^p\right| \quad \textit{for any } \boldsymbol{x}\in\bar{X}_\infty \textit{ and } i\in S.$$

Let $J=\{\alpha=\{\alpha_i^p\}\ ;\ \alpha_i^p\in Z_+,\ |\alpha|=\sum_{i\in S}\sum_{p=1}^{\infty}\alpha_i^p<+\infty\}$, and for each $\alpha\in J$ we denote $f_\alpha(\boldsymbol{x})=\prod_{i\in S}\prod_{p=1}^{\infty}(x_i^p)^{\alpha_i^p}$.

Condition [B]

$$b_i^p(\boldsymbol{x})=\sum_{\beta\in J}\hat{b}_i^p(\beta)f_\beta(\boldsymbol{x}) \qquad (\hat{b}_i^p(\beta)\in R_1),\ \textit{which satisfy}$$

(2.7) $$\sup_{i\in S}\sup_{p\geqq 1}\sum_{\beta\in J}|\hat{b}_i^p(\beta)|<+\infty,$$

and there exists a positive integer κ *such that*

(2.8) $$\hat{b}_i^p(\beta)=0 \quad \textit{for any } \beta\in J \textit{ with } |\beta|\geqq\kappa.$$

Then, we obtain the following.

Theorem 2.1. *Let* $\boldsymbol{x}\in X_\infty$. *Under the conditions* [A] *and* [B] *the* $(X_\infty, A^\infty, \boldsymbol{x})$-*martingale problem has a unique solution.*

As to existence of a solution it suffices to show that there exists a solution of the following stochastic differential equation,

(2.9) $$dx_i^p(t)=\sum_{q=1}^{p}\alpha_{pq}(x_i(t))dB_i^q(t)+b_i^p(\boldsymbol{x}(t))dt$$

$$x_i^p(0)=x_i^p,$$

with a subsidiary condition

(2.10) $$x_i^p(t)\geqq 0 \quad \text{and} \quad \sum_{p=1}^{\infty}x_i^p(t)=1 \quad (i\in S,\ p\geqq 1),$$

where $\{B_i^p(t)\}_{i\in S,\,p\geqq 1}$ is an independent system of one-dimensional standard Brownian motions and $\{\alpha_{pq}(x_i)\}$ are continuous functions defined on $\{(x_i^1,\cdots,x_i^p);\ x_i^1\geqq 0,\cdots,x_i^p\geqq 0,\ x_i^1+\cdots+x_i^p\leqq 1\}$ satisfying

(2.11) $$\sum_{r=1}^{p\wedge q}\alpha_{pr}(x_i)\alpha_{qr}(x_i)=x_i^p(\delta_{pq}-x_i^q) \qquad (p,\ q\geqq 1).$$

However by the same argument as Theorem 3.1 of [16] we can show that (2.9) has a solution satisfying

(2.12) $$P[\boldsymbol{x}(t)\in\bar{X}_\infty \quad \text{for all} \quad t\geqq 0]=1.$$

So, it is sufficient to see

$$P[\boldsymbol{x}(t)\in X_\infty \quad \text{for all} \quad t\geqq 0]=1. \tag{2.13}$$

For this, let $z_i(t)=\sum_{p=1}^{\infty} x_i^p(t)$. It follows from (2.6) and (2.9) that

$$E[1-z_i(t)]\leqq C_i\int_0^t E[1-z_i(s)]ds \qquad (i\in S). \tag{2.14}$$

Also, since it is easy to check that $z_i(t)$ is continuous in t P-*a.e.*, we obtain

$$P[z_i(t)=1 \text{ for all } t\geqq 0]=1 \tag{2.15}$$

For the proof of uniqueness we modify the Feynman-Kac theorem.

Lemma 2.1. *Let I be a countable set, $Q=\{q_{ij}\}_{i,j\in I}$ be a matrix on $I\times I$ satisfying $q_{ij}\geqq 0\,(i\neq j)$ and $\sum_{j\in I} q_{ij}=0$ for any $i\in I$, $h(i)$ be a function on I, and $u(t, i)$ be a function defined on $[0, \infty)\times I$. Suppose that*

(i) *$u(t, i)$, $Qu(t, i)=\sum_{j\in I} q_{ij}u(t, j)$ and $h(i)u(t, i)$ are bounded on $[0, \infty)\times I$,*

(ii) *the minimal Markov chain on I generated by Q is conservative which is denoted by $(\Omega, \mathcal{B}, P_i\,;\,\xi_t)_{i\in I}$,*

(iii) *there exists a positive number t_0 such that*

$$E_i\left[\exp\left(\int_0^{t_0} h^+(\xi_u)\,du\right)\right]<+\infty \qquad \textit{for any } i\in I, \textit{ and} \tag{2.16}$$

(iv) *$u(t, i)$ is C^1-function of $t\in[0, \infty)$ for each $i\in I$ and satisfies the following equation,*

$$\frac{d}{dt}u(t, i)=Qu(t, i)+h(i)u(t, i), \qquad i\in I. \tag{2.17}$$

Then for any $0\leqq t\leqq t_0$ and any $t_1\geqq 0$

$$u(t+t_1, i)=E_i\left[u(t_1, \xi_t)\exp\left(\int_0^t h(\xi_u)du\right)\right] \tag{2.18}$$

holds for any $i\in I$. Thus, $\{u(t, i)\}$ is uniquely determined for given Q, h, and $u(0, \cdot)$.

Proof. It suffices to show (2.18) for $t_1=0$. Setting $u_\lambda(i)=\int_0^\infty e^{-\lambda t}u(t, i)dt$ for each $\lambda>0$, it follows from (2.17) that

$$\lambda u_\lambda(i)-u(0, i)=\sum_{j\in I} q_{ij}u_\lambda(j)+h(i)u_\lambda(i), \qquad i\in I. \tag{2.19}$$

Let $\{I_n\}$ be a sequence of finite subsets of I satisfying $I_n\nearrow I$ and set

$$h_n(i)=\begin{cases} h(i) & \text{if } \; i\in I_n, \\ 0 & \text{otherwise}. \end{cases} \tag{2.20}$$

Then for any $\lambda>0$

1) $h^+(i)=\max\{h(i), 0\}$.

(2.21) $$(\lambda-Q-h_n)u_\lambda(i)=u(0,\, i)+w_\lambda^n(i)\,,$$

where $w_\lambda^n(i)=(h-h_n)(i)u_\lambda(i)$. Since $u(0,\, i)$, $h_n(i)$ and $w_\lambda^n(i)$ are bounded on I, it holds by the Feynman-Kac theorem (cf. Lemma 2.3 in [17]) that for any $\lambda>\|h_n^+\|_\infty$

(2.22) $$\begin{aligned} u_\lambda(i)&=\int_0^\infty e^{-\lambda t}E_i\Big[(u(0,\,\xi_t)+w_\lambda^n(\xi_t))\exp\Big(\int_0^t h_n(\xi_n)du\Big)\Big]dt \\ &=\int_0^\infty e^{-\lambda t}E_i\Big[u(0,\,\xi_t)\exp\Big(\int_0^t h_n(\xi_n)du\Big)\Big]dt \\ &\quad+\int_0^\infty e^{-\lambda t}\Big(\int_0^t E_i\Big[w_n(t-s,\,\xi_s)\exp\Big(\int_0^t h_n(\xi_u)du\Big)\Big]ds\Big)dt\,, \end{aligned}$$

where $w_n(t,\, i)=(h-h_n)(i)u(t,\, i)$. Accordingly we have by the uniqueness of the Laplace transformation

(2.23) $$\begin{aligned} u(t,\, i)=&E_i\Big[u(0,\,\xi_t)\exp\Big(\int_0^t h_n(\xi_u)du\Big)\Big] \\ &+\int_0^t E_i\Big[w_n(t-s,\,\xi_s)\exp\Big(\int_0^s h_n(\xi_n)du\Big)\Big]ds \end{aligned}$$

for any $i\in I$ and $t>0$. Noting that $\{w_n(t,\, i)\}$ are uniformly bounded and $\lim_{n\to\infty} w_n(t,\, i)=0$ for each $(t,\, i)\in[0,\, \infty)\times I$, it follows from the assumption (iii) and Lebesgue's convergence theorem that

(2.24) $$\lim_{n\to\infty}\int_0^t E_i\Big[w_n(t-s,\,\xi_s)\exp\Big(\int_0^s h_n(\xi_u)du\Big)\Big]ds=0$$

for any $t\leqq t_0$ and $i\in I$. Hence, we get

(2.25) $$u(t,\, i)=E_i\Big[u(0,\,\xi_t)\exp\Big(\int_0^t h(\xi_u)du\Big)\Big] \qquad \text{for any} \quad t\leqq t_0 \text{ and } i\in I\,.$$

Therefore the proof of Lemma 2.1 is completed.

For each $\alpha\in J$ let us denote by $\bar{\alpha}$ a copy of α with $|\bar{\alpha}|=|\alpha|$, and define $\bar{(\bar{\alpha})}=\alpha$. Let $\bar{J}=\{\bar{\alpha};\, \alpha\in J\}$ and $J^*=\bar{J}\cup J$. For $\bar{\alpha}\in\bar{J}$ we define $f_{\bar{\alpha}}(\boldsymbol{x})=-f_\alpha(\boldsymbol{x})$. If $\alpha_i^q=0$ $((j,\, q)\neq(i,\, p))$ and $\alpha_i^p=1$ we denote α by ε_i^p.

Let $c\,(0<c<1)$ be fixed and set $\phi_\alpha(\boldsymbol{x})=c^{|\alpha|}f_\alpha(\boldsymbol{x})$ for each $\alpha\in J^*$. Then we can easily see the following.

Lemma 2.2.
(i) $A^\infty\phi_\alpha(x)=\sum_{\beta\in J^*}Q_{\alpha,\beta}\phi_\beta(x)+h(\alpha)\phi_\alpha(x)$ *for each* $\alpha\in J^*$,
where for $\alpha\in J$

2) $\|h\|_\infty=\sup_i|h(i)|$.

$$Q_{\alpha,\beta}=\begin{cases} c\alpha_i^p(\alpha_i^p-1), & \text{if } \beta=\alpha-\varepsilon_i^p, \\ \alpha_i^p c^{1-|\gamma|}\hat{b}_i^{p+}(\gamma), & \text{if } \beta=\alpha-\varepsilon_i^p+\gamma, \\ \alpha_i^p c^{1-|\gamma|}\hat{b}_i^{p-}(\gamma), & \text{if } \beta=\overline{\alpha-\varepsilon_i^p+\gamma}, \\ -\sum_{i\in S}\sum_{p=1}^{\infty} c\alpha_i^p(\alpha_i^p-1)-\sum_{i=S}\sum_{p=1}^{\infty}\alpha_i^p\sum_{\gamma\in J}c^{1-|\gamma|}|\hat{b}_i^p(\gamma)|, & \text{if } \beta=\alpha \\ 0, \text{ otherwise}, \end{cases}$$

$$h(\alpha)=-\sum_{i\in S}\Big(\sum_{p=1}^{\infty}\alpha_i^p\Big)\Big(\Big(\sum_{p=1}^{\infty}\alpha_i^p\Big)-1\Big)+c\sum_{i\in S}\sum_{p=1}^{\infty}\alpha_i^p(\alpha_i^p-1)$$

$$+\sum_{i\in S}\sum_{p=1}^{\infty}\alpha_i^p\sum_{\gamma\in J}c^{1-|\gamma|}|\hat{b}_i^p(\gamma)|,$$

and for $\alpha\in\bar{J}$ $Q_{\bar{\alpha},\beta}=Q_{\alpha,\bar{\beta}}$ *and* $h(\bar{\alpha})=h(\alpha)$.

(ii) *There exists a positive constant C satisfying* $h(\alpha)\leqq C|\alpha|$ *for any* $\alpha\in J^*$.

Let $(\Omega, \mathscr{B}, P_\alpha; \alpha_t, t<\zeta)$ be the minimal Markov chain on J^* generated by $Q=\{Q_{\alpha,\beta}\}$, where ζ is the explosion time. Then, it is not hard to see that $P_\alpha[\zeta=+\infty]=1$ holds for any $\alpha\in J^*$. Furthermore we have

Lemma 2.3. *There exists a positive number* t_0 *such that*

(2.26) $$E_\alpha\Big[\exp\Big(\int_0^{t_0}h^+(\alpha_u)du\Big]<+\infty \qquad \textit{for any} \quad \alpha\in J^*.$$

Proof. Let us introduce a conservative Markov chain $(\bar{\Omega}, \mathscr{B}, \bar{P}_n; N_t)$ on $\mathcal{N}=\{1, 2, 3, \cdots\}$ generated by the following infinitesimal matrix

(2.27) $$R_{n,m}=\begin{cases} nL & \text{if } m=n+\kappa, \\ -nL & \text{if } m=n, \\ 0 & \text{otherwise}, \end{cases}$$

where $L=\sup_{i\in S}\sup_{p\geqq 1}\sum_{\gamma\in J}c^{1-|\gamma|}|\hat{b}_i^p(\gamma)|$. Then, by making use of a coupling process and Lemma 2.2 we see

(2.28) $$E_\alpha\Big[\exp\Big(\int_0^t h^+(\alpha_u)du\Big)\Big]\leqq E_\alpha\Big[\exp\Big(C\int_0^t|\alpha_u|du\Big)\Big]\leqq \bar{E}_{|\alpha|}[e^{CtN_t}].$$

On the other hand it is easy to calculate the transition matrix of this Markov chain.

(2.29) $$\bar{P}_n[N_t=r]=\frac{n(n+\kappa)\cdots(n+(m-1)\kappa)}{m!\,\kappa^m}e^{-nLt}(1-e^{-\kappa Lt})^m$$

$$\text{if } \quad r=n+m\kappa,$$

$$=0 \quad \text{otherwise}.$$

Hence for sufficiently small $t_0>0$ we obtain

(2.30) $$\bar{E}_\kappa[\exp(Ct_0\cdot N_{t_0})]<+\infty.$$

3) $b^+=\max\{b, 0\}$, $b^-=-\min\{b, 0\}$.

Noting that $(N_t, \bar{P}_n)$ is a continuous time branching process, it holds that

(2.31) $\qquad \bar{E}_n[\exp Ct_0\cdot N_{t_0})]<+\infty$ holds for any $n\in \boldsymbol{N}$.

Thus we complete the proof of Lemma 2.3.

Proof of Theorem 2.1.

Let P be a solution of the $(X_\infty, A^\infty, \boldsymbol{x})$-martingale problem. For the proof it is sufficient to show that $\{E^P[\phi_\alpha(\boldsymbol{x}(t))]\}_{\alpha\in J}$ is uniquely determined. Let $u(t, \alpha)=E^P[\phi_\alpha(\boldsymbol{x}(t))]$ for each $\alpha\in J^*$. Since $(\phi_\alpha(\boldsymbol{x}(t))-\int_0^t A^\infty\phi_\alpha(\boldsymbol{x}(s))ds, \{\mathcal{F}_t\})$ is a P-martingale, it follows from Lemma 2.2 that

(2.32) $$\frac{d}{dt}u(t, \alpha)=\sum_{\beta\in J^*}Q_{\alpha,\beta}u(t, \beta)+h(\alpha)u(t, \alpha) \quad \text{for any} \quad \alpha\in J^*.$$

Also, noting that $u(t, \alpha)$, $\sum_{\beta\in J^*}Q_{\alpha,\beta}u(t, \beta)$ and $h(\alpha)u(t, \alpha)$ are bounded on $[0, \infty)\times J^*$, the assumptions of Lemma 2.1 are verified, and $\{u(t, \alpha)\}$ is uniquely determined. Therefore we complete the proof of Theorem 2.1.

Example. Let $b_i^p(\boldsymbol{x})=\sum_{q=1}^\infty \lambda_{qp}x_i^q+x_i^p(s_p-\sum_{q=1}^\infty s_q x_i^q)+\sum_{j\in S}q_{ji}x_j^p$, where $\{\lambda_{qp}\}_{q,p\geq 1}$ is a real matrix on $\boldsymbol{N}\times\boldsymbol{N}$ satisfying $\lambda_{qp}\geqq 0\,(q\neq p)$, $\sum_{p=1}^\infty\lambda_{qp}=0$ and $\sup_{p\geq 1}\sum_{q=1}^\infty|\lambda_{qp}|<+\infty$, $\{s_p\}_{p\geq 1}$ is a real vector on $\boldsymbol{N}$ satisfying $\sum_{p=1}^\infty|s_p|<+\infty$, and $\{q_{ji}\}_{j,i\in S}$ is a matrix on $S\times S$ satisfying $q_{ji}\geqq 0\,(j\neq i)$, $\sum_{j\in S}q_{ji}=0$ and $\sup_{j\in S}|q_{ii}|<+\infty$. Then, the conditions [A] and [B] are satisfied.

Genetically, this model is an infinite allelic stepping stone model with mutation, selection, and migration.

Remark. Let d be an integer $\geqq 2$. Suppose that $\lambda_{qp}=0$ if $q\leqq d$ and $p>d$. Then, for any $\boldsymbol{x}\in X_d=\{\boldsymbol{x}\in X_\infty;\ \sum_{p=1}^d x_i^p=1$ for all $i\in S\}$, the solution P of the $(X_\infty, A^\infty, \boldsymbol{x})$-martingale problem satisfies $P[\boldsymbol{x}(t)\in X_d$ for all $t\geqq 0]=1$. Then the diffusion process associated with the solution P is a d-allelic stepping stone model.

Corollary 2.1. *Assume the same condition as Theorem 2.1. Then there exists a unique strongly continuous contraction semigroup $\{T_t^\infty\}$ on $C_b(X_\infty)$ such that*

(i) $T_t^\infty f\geqq 0$ *for any* $f\in C_b(X_\infty)$ *with* $f\geqq 0$, *and* $T_t 1=1$, *and*

(ii) $T_t^\infty f-f=\int_0^t T_s^\infty A^\infty f ds$ *for any* $f\in C_f^2(\bar{X}_\infty)$.

Proof. For any $\boldsymbol{x}\in X_\infty$ denote by P_x the unique solution of the $(X_\infty, A^\infty, \boldsymbol{x})$-martingale problem. Then, $\{P_x\}$ is weakly continuous in $\boldsymbol{x}\in X_\infty$. In fact, assume that $\lim_{n\to\infty}\boldsymbol{x}_n=\boldsymbol{x}$ $(\boldsymbol{x}_n, \boldsymbol{x}\in X_\infty)$. Then we can easily see that $\{P_{x_n}\}$ is tight. Fur-

4) $C_b(X_\infty)$ denotes the Banach space of all bounded continuous functions defined on X_∞ with the uniform norm.

theremore it follows from the uniqueness of the $(X_\infty, A^\infty, \boldsymbol{x})$-martingale problem that $\lim_{n\to\infty} P_{x_n}=P_x$. Hence setting $T_t^\infty f(x)=E^{P_x}[f(x(t))]$ for each $f\in C_b(X_\infty)$, we have $T_t^\infty f\in C_b(X_\infty)$. Also it is obvious that $\{T_t^\infty\}$ satisfies (i) and (ii).

Finally uniqueness of semi-group follows immediately from the uniqueness of the $(X_\infty, A^\infty, \boldsymbol{x})$-martingale problem.

§ 3. Diallelic models

For the subsequent sections we shall summarize the results on diallelic stepping stone models which was obtained in [15].

Let S be a countable set that is the set of colonies. Assuming that there are two alleles A_1 and A_2 at each colony, we denote by x_i and $1-x_i$ the gene frequencies of the A_1-allele and the A_2-allele at colony $i\in S$ respectively.

Let $X=[0, 1]^S=\{\boldsymbol{x}=\{x_i\}_{i\in S}; 0\leqq x_i\leqq 1 \text{ for all } i\in S\}$. Let us consider the following differential operator,

$$A=\sum_{i\in S} x_i(1-x_i)D_i^2+\sum_{i\in S}(\sum_{j\in S} q_{ji}x_j)D_i \tag{3.1}$$

where $D_i=\frac{\partial}{\partial x_i}$, and it is assumed that $\{q_{ji}\}$ satisfies the conditions of (1.1) and (1.2). Then it is known that there exists a unique strongly continuous contraction semi-group $\{T_t\}$ on $C(X)$ such that

$$T_tf\geqq 0 \quad \text{for any } f\in C(X) \text{ with } f\geqq 0, \text{ and } T_t1=1, \tag{3.2}$$

and

$$T_tf-f=\int_0^t T_sAfds \quad \text{for any } f\in C_f^2(X), \tag{3.3}$$

where $C(X)$ is the Banach space of all continuous functions on X with the uniform norm and $C_f^2(X)$ denotes the set of all C^2-functions on X depending only on finitely many coordinates.

A diffusion process on X, which is associated with $\{T_t\}$, is called a continuous time diallelic stepping stone model without mutation and selection.

Let $\mathcal{P}(X)$ be the set of all probability measures on X equipped with the topology of weak convergence. Let $\{T_t^*\}$ be the adjoint semi-group on $\mathcal{P}(X)$ induced by $\{T_t\}$. We denote by $\mathcal{S}$ the set all stationary states of $\{T_t\}$, i.e. $\mathcal{S}=\{\mu\in\mathcal{P}(X); T_t^*\mu=\mu \text{ for all } t\geqq 0\}$. $\mathcal{S}$ is a non-empty, compact and convex set and we denote by $\mathcal{S}_{ext}$ the set of all extremal elements.

For $Q=\{q_{ji}\}$, $P_t=e^{tQ^*}$ is well-defined for all $t\geqq 0$ and it is a transition probability of a continuous time Markov chain on S.

Throughout this paper we shall assume that $Q=\{q_{ji}\}$ is irreducible. Let $(X_t=(X_t^1, X_t^2), P_i)_{i\in S\times S}$ be the continuous time irreducible Markov chain on $S\times S$ which is defined by

5) Q^* is the transposed matrix of Q.

(3.4) $$P_t\otimes P_t(\vec{i}, \vec{j})=P_t(i_1, j_1)P_t(i_2, i_2) \quad \text{for each } \vec{i}=(i_1, i_2) \text{ and } \vec{j}=(j_1, j_2)\in S\times S.$$

In order to describe $\mathcal{S}_{ext}$ let us introduce the space of Q^*-harmonic functions $\mathcal{H}$ and a sub-class $\mathcal{H}^*$ of $\mathcal{H}$.

(3.5) $$\mathcal{H}=\{h\,;\ \text{defined on } S,\ 0\leqq h\leqq 1 \text{ and } Q^*h=0\},$$

(3.6) $$\mathcal{H}^*=\{h\in\mathcal{H}\,;\ \lim_{t\to\infty} h(X_t^1)=\lim_{t\to\infty} h(X_t^2)=0 \text{ or } 1\ P_{\vec{i}}\text{-a.s. on } \Omega^{(1)} \text{ for any } \vec{i}\in S\times S\}, \text{ where}$$

$$\Omega^{(1)}=\left[\int_0^\infty I_{\Delta_2}(X_t)dt=+\infty\right] \text{ and } \Delta_2=\{\vec{i}=(i_1, i_2)\in S\times S\,;\ i_1=i_2\}.$$

We regard each $h\in\mathcal{H}$ as an element of X. Then we obtained

Theorem 3.1 ([15])
(i) *For each $h\in\mathcal{H}$ there exists a $\nu_h\in\mathcal{P}(X)$ satisfying that $\lim_{t\to\infty} T_t^*\delta_h=\nu_h$ exists, where δ_h stands for the point mass at h.*

(ii) $\int_X x_i\nu_h(d\boldsymbol{x})=h(i)$ *for any $h\in\mathcal{H}$ and $i\in S$.*

(iii) $\mathcal{S}_{ext}=\{\nu_h\,;\ h\in\mathcal{H}^*\}$,

Theorem 3.2 ([15]) *Let $\mu\in\mathcal{P}(X)$ and $h\in\mathcal{H}^*$. Then $\lim_{t\to\infty} T_t^*\mu=\nu_h$ if and only if*

(3.7) $$\lim_{t\in\infty}\int_X\Big(\sum_{j\in S} P_t(i, j)x_j-h(i)\Big)^2\mu(d\boldsymbol{x})=0 \qquad \textit{for all } i\in S.$$

Let us consider the following classification by the migration rate $\{q_{ji}\}$.

Case I $P_{\vec{i}}[\Omega^{(1)}]=1$ *for all* $\vec{i}\in S\times S$.
Case II $P_{\vec{i}}[\Omega^{(1)}]=0$ *for all* $\vec{i}\in S\times S$.
Case III $0<P_{\vec{i}}[\Omega^{(1)}]<1$ *for all* $\vec{i}\in S\times S$.

Since Q is irreducible these three cases exhaust all posibilities. It follows from Theorem 3.1 that $\mathcal{S}_{ext}=\{\delta_0, \delta_1\}$ holds for Case I and $\mathcal{S}_{ext}=\{\nu_h\,;\ h\in\mathcal{H}\}$ holds for Case II. For Case III we notice that $\mathcal{H}_{ext}\subseteqq\mathcal{H}^*\subsetneqq\mathcal{H}$ holds, where $\mathcal{H}_{ext}$ denotes the set of all extemal elements of $\mathcal{H}$.

Further we obtained

Theorem 3.3 ([15]) *Assume Case I. Let $\mu\in\mathcal{P}(X)$. Then, $\lim_{t\to\infty} T_t^*\mu$ exists if and only if $\lim_{t\to\infty}\sum_{j\in S} P_t(i, j)\int_X x_j\mu(d\boldsymbol{x})$ exists for each $i\in S$. Moreover, if this condition is satisfied $\lim_{t\to\infty}\sum_{j\in S} P_t(i, j)\int_X x_j\mu(d\boldsymbol{x})=\lambda$ is independent of $i\in S$, and*

(3.8) $$\lim_{t\to\infty} T_t^*\mu=\lambda\delta_1+(1-\lambda)\delta_0\,.$$

Let us introduce a mapping ρ_t from X onto itself defined by

(3.9) $$(\rho_t \boldsymbol{x})_i = \sum_{j \in S} P_t(i, j) x_j \qquad \text{for each} \quad i \in S.$$

We denote by $\rho_t \mu$ the image measure of $\mu \in \mathcal{P}(X)$, i. e.

(3.10) $$\langle \rho_t \mu, f \rangle = \langle \mu, f \circ \rho_t \rangle \qquad \text{for any} \quad f \in C(X).$$

Then by modifying the proof of the above theorems we obtain

Theorem 3.4. *Assume Case II. Let $\mu \in \mathcal{P}(X)$. $T_t^* \mu$ converges as $t \to \infty$ if and only if $\rho_t \mu$ converges as $t \to \infty$. Moreover, if this condition is satisfied, setting $\mu^\infty = \lim_{t \to \infty} \rho_t \mu$, it holds that*

(3.11) $$\mu^\infty[\mathcal{H}] = 1, \quad \textit{and}$$

(3.12) $$\lim_{t \to \infty} T_t^* \mu = \int_{\mathcal{H}} \nu_h \mu^\infty(dh).$$

§ 4. Multi-allelic models

This section will be devoted to a description of extremal stationary states of multi-allelic stepping stone model with mutation.

Let $d \geqq 2$ be a positive integer and $X_d = \{\boldsymbol{x} = \{x_i^p\}\ ;\ x_i^p \geqq 0,\ x_i^1 + \cdots + x_i^d = 1$ for all $i \in S\}$.

A continuous time multi-allelic stepping stone model with mutation is a diffusion process on X_d with the following infinitesimal generator,

(4.1) $$A^d = \sum_{i \in S} \sum_{p=1}^{d} \sum_{q=1}^{d} x_i^p (\delta_{pq} - x_i^q) D_{i,p} D_{i,q} + \sum_{i \in S} \sum_{p=1}^{d} \Big(\sum_{q=1}^{d} \lambda_{qp} x_i^q + \sum_{j \in S} q_{ji} x_j^p \Big) D_{i,p}$$

where $\{\lambda_{qp}\}$ and $\{q_{ji}\}$ satisfy the conditions of (1.1) and (1.2). Then it follows from Theorem 2.1 that for any $\boldsymbol{x} \in X_d$ the $(X_d, A^d, \boldsymbol{x})$-martingale problem has a unique solution. Accordingly there exists a unique strongly continuous contraction semi-group $\{T_t^{d\bullet}\}$ on $C(X_d)$ satisfying that

(4.2) $$T_t^d f \geqq 0 \qquad \text{for any} \quad f \in C(X_d) \text{ with } f \geqq 0, \text{ and } T_t^d 1 = 1,$$

and

(4.3) $$T_t^d f - f = \int_0^t T_s^d A^d f ds \qquad \text{for any} \quad f \in C_f^2(X_d).$$

Denote by $(\Omega, \mathcal{F}, P_{\boldsymbol{x}}\ ;\ \boldsymbol{x}(t))_{\boldsymbol{x} \in X_d}$ the diffusion process on X_d associated with $\{T_t^d\}$ and we use the same notations $\mathcal{P}(X_d)$, $\{T_t^{d*}\}$, $\mathcal{S}_d$ and $(\mathcal{S}_d)_{ext}$ as § 1.

Let us introduce a classification of $I = \{1, 2, \cdots, d\}$ according to the mutation rate $\{\lambda_{qp}\}$. If $\lambda_{qp} > 0$ we denote $q \Rightarrow p$. If there exists a chain $[p_0 = q, p_1, \cdots, p_r = p]$ of I satisfying $p_{k-1} \Rightarrow p_k$ for any $1 \leqq k \leqq r$, we denote $q \to p$. In particular, if either $q = p$ or both $q \to p$ and $p \to q$ hold we denote $p \leftrightarrow q$. Then

"$\leftrightarrow$" defines an equivalence relation on I. An equivalence class R is said recurrent if $q \rightarrow p$ does not hold for any $q \in R$ and for any $p \notin R$.

Setting $\Lambda = \{\lambda_{qp}\}$ and $\Lambda_t = e^{t\Lambda}$, it is well-known from the theory of finite Markov chains that for any recurrent class R there exists a strictly positive vector $\{\pi_q^{(R)}\}$ such that for any $q \in R$

$$\lim_{t\to\infty} \Lambda_t(q, p) = \begin{cases} \pi_p^{(R)} & \text{if } p \in R \\ 0 & \text{otherwise}, \end{cases} \tag{4.4}$$

and for any non-recurrent class C

$$\lim_{t\to\infty} \Lambda_t(q, p) = 0 \quad \text{for any } q \in I \text{ and any } p \in C. \tag{4.5}$$

Denote by $R_1, \cdots, R_r$ all recurrent classes.

Let $\mathcal{H}_r^* = \{\boldsymbol{h} = (h_1, \cdots, h_r);\ h_a \in \mathcal{H}^* \text{ for } 1 \leqq a \leqq r,\ h_1 + \cdots + h_r = 1\}$. For $\boldsymbol{h} \in \mathcal{H}_r^*$, $\rho(\boldsymbol{h}) \in X_d$ is defined by

$$\rho(\boldsymbol{h})_i^p = \begin{cases} \pi_p^{(a)} h_a(i) & \text{if } p \in R_a \text{ for some } 1 \leqq a \leqq r, \\ 0 & \text{otherwise, where } \pi_p^{(a)} = \pi_p^{(R_a)}. \end{cases} \tag{4.6}$$

Our main result in this section is

Theorem 4.1.

(i) *For each* $\boldsymbol{h} = (h_1, \cdots, h_r) \in \mathcal{H}_r^*$ *there exists a* $\nu_{\boldsymbol{h}} \in \mathcal{P}(X_d)$ *such that* $\lim_{t\to\infty} T_t^{d*} \delta_{\rho(\boldsymbol{h})} = \nu_{\boldsymbol{h}}$.

(ii) $\int_X x_i^p \nu_{\boldsymbol{h}}(d\boldsymbol{x}) = \rho(\boldsymbol{h})_i^p$ *for all* $i \in S$ *and* $1 \leqq p \leqq d$.

(iii) $(\mathcal{S}_d)_{ext} = \{\nu_{\boldsymbol{h}};\ \boldsymbol{h} \in \mathcal{H}_r^*\}$.

We assume $R_1 \cup R_2 \cup \cdots \cup R_r = \{1, 2, \cdots, e\} \subset I$. Let J be the set of all non-negative integer-valued functions $\alpha = \{\alpha_i^p\}$ defined on $S \times \{1, 2, \cdots, e\}$ satisfying $|\alpha| = \sum_{i\in S} \sum_{p=1}^{e} \alpha_i^p < +\infty$. If $\alpha_j^q = 0$ for $(j, q) \neq (i, p)$ and $\alpha_i^p = 1$, α is denoted by ε_i^p. Define $\{\pi_p\}_{1\leq p\leq e}$ by

$$\pi_p = \pi_p^{(a)} \quad \text{for } p \in R_a. \tag{4.7}$$

For each $\alpha \in J$, set $\phi_\alpha(\boldsymbol{x}) = c^{|\alpha|} \prod_{i\in S} \prod_{p=1}^{e} x_i^p / \prod_{p=1}^{e} (\pi_p)^{|\alpha^p|}$, where c is a fixed constant satisfying $0 < c < \min_{1\leq p\leq e} \pi_p$, and $|\alpha^p| = \sum_{i\in S} \alpha_i^p$.

Let $X^{(R)} = \{x \in X_d;\ \sum_{a=1}^{r} \sum_{p\in R_a} x_i^p = 1 \text{ for all } i \in S\}$. Then we see

Lemma 4.1. *For any* $\boldsymbol{x} \in X^{(R)}$ *and* $\alpha \in J$,

$$A^d \phi_\alpha(\boldsymbol{x}) = \sum_{\beta\in J} R_{\alpha,\beta} \phi_\beta(\boldsymbol{x}) - \langle\alpha\rangle \phi_\alpha(\boldsymbol{x}) \tag{4.7}$$

where

$$(4.8)\qquad R_{\alpha,\beta}=\begin{cases}\dfrac{c}{\pi_p}\alpha_i^p(\alpha_i^p-1) & \text{if}\quad \beta=\alpha-\varepsilon_i^p\,,\\ \alpha_i^p\bar{\lambda}_{pq} & \text{if}\quad \beta=\alpha-\varepsilon_i^p+\varepsilon_i^q\ (p\neq q)\,,\\ \alpha_i^p q_{ji} & \text{if}\quad \beta=\alpha-\varepsilon_i^p+\varepsilon_j^p\ (i\neq j)\,,\\ -\sum_{i\in S}\sum_{p=1}^{e}\dfrac{c}{\pi_p}\alpha_i^p(\alpha_i^p-1)+\sum_{p=1}^{e}|\alpha^p|\bar{\lambda}_{pp}+\sum_{i\in S}|\alpha_i|q_{ii} & \\ \text{if}\ \beta=\alpha\,, & \\ 0\ \text{otherwise}\,, & \end{cases}$$

$$(4.9)\qquad \langle\alpha\rangle=\sum_{i\in S}|\alpha_i|(|\alpha_i|-1)-\sum_{i\in S}\sum_{p=1}^{e}\frac{c}{\pi_p}\alpha_i^p(\alpha_i^p-1)\,,$$

$$\bar{\lambda}_{pq}=\pi_q\lambda_{qp}/\pi_p,\quad \text{and}\quad |\alpha_i|=\sum_{p=1}^{e}\alpha_i^p\,.$$

Let $R=\{R_{\alpha,\beta}\}_{\alpha,\beta\in J}$. Then e^{tR} is well-defined, which is a transition matrix on J. We denote by $(\Omega, \mathcal{B}, \boldsymbol{P}_\alpha\,;\ \alpha(t))_{\alpha\in J}$ the continuous time Markov chain on J associated with e^{tR}. Then by making use of the Feynman-Kac formula we obtain

Lemma 4.2. *For any* $\boldsymbol{x}\in X^{(R)}$,

$$(4.10)\qquad T_t^d\phi_\alpha(\boldsymbol{x})=\boldsymbol{E}_\alpha\Big[\phi_{\alpha(t)}(\boldsymbol{x})\exp\Big(-\int_0^t\langle\alpha(u)\rangle du\Big)\Big]\qquad \textit{for any}\quad \alpha\in J\,.$$

For each $1\leqq a\leqq r$ let us define a mapping $\psi_a\,;\ X_d\to X$ by

$$(4.11)\qquad (\psi_a\boldsymbol{x})_i=\sum_{p\in R_a}x_i^p\qquad \text{for each}\quad i\in S\,.$$

Then we have

Lemma 4.3. *For any* $f\in C(X)$ *and* $\boldsymbol{x}\in X^{(R)}$,

$$(4.12)\qquad T_t^d(f\circ\psi_a)(\boldsymbol{x})=T_tf(\psi_a\boldsymbol{x})\,.$$

Proof. It follows immediately from

$$(4.13)\qquad A^d(f\circ\psi_a)=(Af)\circ\psi_a\ \text{on}\ X^{(R)}\qquad \text{for any}\quad f\in C_f^2(X)\,.$$

Lemma 4.4.
(i) *For any* $\mu\in\mathcal{S}_d$, $\mu[X^{(R)}]=1$.
(ii) $\psi_a[\mathcal{S}_d]\subset\mathcal{S}$.
(iii) $\psi_a[(\mathcal{S}_d)_{ext}]\subset\mathcal{S}_{ext}$.

Proof. (i); For any fixed $\boldsymbol{x}\in X_d$, set $m_i^p(t)=E_x[x_i^p(t)]$. Then it follows from (4.3) that

$$(4.14)\qquad \frac{d}{dt}m_i^p(t)=\sum_{q=1}^{d}\lambda_{qp}m_i^q(t)+\sum_{j\in S}q_{ji}m_j^p(t)\,.$$

So we get

(4.15) $$m_i^p(t)=\sum_{j\in S}\sum_{q=1}^{d}\Lambda_t(q,\ p)P_t(i,\ j)m_j^q(0).$$

Since, noting (4.4), $\lim_{t\to\infty} m_i^p(t)=0$ holds for any $p\notin R_1\cup R_2\cup\cdots\cup R_r$ and $i\in S$, (i) follows from this. (ii) is trivial by (i) and Lemma 4.3. Let $\mu\in(\mathcal{S}_d)_{ext}$. Suppose that for some ν_1 and $\nu_2\in\mathcal{S}$

(4.16) $$\phi_a\mu=\frac{1}{2}(\nu_1+\nu_2).$$

Since ν_1 and ν_2 are absolutely continuus with respect to $\phi_a\mu$, we denote by $\xi_1(z)$ and $\xi_2(z)$ their densities. Define μ_1 and $\mu_2\in\mathcal{P}(X_d)$ by

(4.17) $$\mu_i(dx)=\xi_i(\phi_a x)\mu(dx) \qquad (i=1,\ 2).$$

Then we see

(4.18) $$\phi_a\mu_i=\nu_i \quad (i=1,\ 2) \quad \text{and} \quad \mu=\frac{1}{2}(\mu_1+\mu_2).$$

We claim that

(4.19) $$\lim_{t\to\infty}\frac{1}{t}\int_0^t T_s^{d*}\mu_i ds=\mu \qquad \text{for} \quad i=1,\ 2.$$

Suppose that $\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}T_s^{d*}\mu_i ds=\bar{\mu}_i\ (i=1,\ 2)$ exists for some sequence $\{t_n\}$ tending to $+\infty$. Noting that $\bar{\mu}_1\in\mathcal{S}_d$, $\bar{\mu}_2\in\mathcal{S}_d$, $\mu\in(\mathcal{S}_d)_{ext}$ and $\mu=\frac{1}{2}(\bar{\mu}_1+\bar{\mu}_2)$, we get $\bar{\mu}_1=\bar{\mu}_2=\mu$. Hence (4.19) holds. Finally by (4.18), (4.19) and Lemma 4.3 we obtain $\nu_1=\nu_2=\phi_a\mu$. Thus we see $\phi_a\mu\in\mathcal{S}_{ext}$.

Let us consider another continuous time Markov chain $(\tilde{\Omega},\ \tilde{\mathcal{B}},\ \tilde{\boldsymbol{P}}_\alpha\,;\ \alpha(t))$ on J, generated the following infinitesimal matrix $\tilde{R}=\{\tilde{R}_{\alpha,\beta}\}$

(4.20) $$\tilde{R}_{\alpha,\beta}=\begin{cases}\alpha_i^p\bar{\lambda}_{pq} & \text{if} \quad \beta=\alpha-\varepsilon_i^p+\varepsilon_i^q\ (p\neq q)\\ \alpha_i^p q_{ji} & \text{if} \quad \beta=\alpha-\varepsilon_i^p+\varepsilon_j^p\ (i\neq j)\\ \sum_{p=1}^{d}|\alpha^p|\bar{\lambda}_{pp}+\sum_{i\in S}|\alpha_i|q_{ii} & \text{if} \quad \beta=\alpha\\ 0 & \text{otherwise.}\end{cases}$$

Notice that this Markov chain is identified with the direct product Markov process of $|\alpha|$ number of $P_t\otimes\bar{\Lambda}_t$-Markov chains on $S\times\{1,\ 2,\ \cdots,\ e\}$.

Lemma 4.5. *For any $\mu\in(\mathcal{S}_d)_{ext}$ and $\alpha\in J$, set $g(\alpha)=\langle\mu,\ \phi_\alpha\rangle$. Then there exists a $\boldsymbol{h}=(h_1,\ \cdots,\ h_r)\in\mathcal{H}_r^*$ such that*

(i) $\int_{X_d}x_i^p\mu(d\boldsymbol{x})=\pi_p^{(a)}h_a(i)$ *if* $p\in R_a$ *with some* $1\leqq a\leqq r$,

and

(ii) $\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t))]=\phi_\alpha(\rho(\boldsymbol{h}))$ *holds for any* $\alpha\in J$.

Proof. Since $\phi_a\mu\in\mathcal{S}_{ext}$ holds by Lemma 4.4, it follows from Theorem 3.2

that there exists a $\boldsymbol{h}=(h_1, \cdots, h_r)\in\mathcal{H}_r^*$ such that for any $1\leqq a\leqq r$

$$\lim_{t\to\infty}\int_{X_p}(\sum_{j\in S}P_t(i,j)(\sum_{p\in R_a}x_j^p)-h_a(i))^2\mu(d\boldsymbol{x})=0. \tag{4.21}$$

Noting

$$\lim_{t\to\infty}\bar{\Lambda}_t(p,q)=\pi_q^{(a)} \quad \text{for any } p \text{ and } q\in R_a \quad (1\leqq a\leqq r), \tag{4.22}$$

we get

$$\sum_{j\in S}\sum_{q\in R_a}P_t(i,j)\bar{\Lambda}_t(p,q)\frac{x_j^q}{\pi_q}\longrightarrow h_a(i) \quad \text{as} \quad t\to+\infty \tag{4.23}$$

in probability with respect to μ for any $p\in R_a$ and $i\in S$. Hence

$$\boldsymbol{E}_\alpha[\phi_{\alpha(t)}(\boldsymbol{x})]\longrightarrow\phi_\alpha(\rho(\boldsymbol{h})) \qquad \text{in probability w.r.t. } \mu. \tag{4.24}$$

In particular the integration of (4.24) by μ yields (ii). If $\alpha=\varepsilon_i^p$ for some $p\in R_a$, $\boldsymbol{E}_{\varepsilon_i^p}[g(\alpha(t))]=\boldsymbol{E}_{\varepsilon_i^p}[g(\alpha(t))]=g(\varepsilon_i^p)$ holds. Therefore (i) follows immediately.

Lemma 4.6. *Let g be a function defined on J. Suppose that g is bounded on $J_N=\{\alpha\in J;\ |\alpha|\leqq N\}$ for each $N>0$, and $\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t))]$ exists (which we denote by $\bar{h}(\alpha)$) for each $\alpha\in J$. Then $\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t))\exp(-\int_0^t\langle\alpha(u)\rangle du)]$ exists for each $\alpha\in J$, and moreover this limit is determined by $\{\bar{h}(\alpha)\}$.*

Proof. Let $\Delta=\{\alpha\in J;\ |\alpha_i|\geqq 2$ for some $i\in S\}$. Let us define some stopping times. $\zeta=\inf\{t\geqq 0;\ |\alpha(t)|<|\alpha(0)|\}$, $\zeta_k=\inf\{t\geqq 0;\ |\alpha(t)|\leqq k\}$, $\sigma_1=\inf\{t\geqq 0;\ \alpha(t)\in\Delta\}$, $\tau_1=\inf\{t\geqq\sigma_1;\ \alpha(t)\notin\Delta$ and $|\alpha(t)|=|\alpha(0)|\}$, $\cdots$, $\sigma_n=\inf\{t\geqq\tau_{n-1};\ \alpha(t)\in\Delta\}$ and $\tau_n=\inf\{t\geqq\sigma_n;\ \alpha(t)\notin\Delta,\ |\alpha(t)|=|\alpha(0)|\}$. We note that for some constant $K>0$

$$K\sum_{i\in S}|\alpha_i|(|\alpha_i|-1)\leqq\langle\alpha\rangle\leqq\sum_{i\in S}|\alpha_i|(|\alpha_i|-1). \tag{4.25}$$

Let $\alpha\notin\Delta$. Since $(\alpha(t\wedge\sigma_1), \boldsymbol{P}_\alpha)$ and $(\alpha(t\wedge\sigma_1), \tilde{\boldsymbol{P}}_\alpha)$ have the identical probability law,

$$\begin{aligned}
&\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t))\exp(-\int_0^t\langle\alpha(u)\rangle du);\ \sigma_1=+\infty] \\
&=\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t));\ \sigma_1=+\infty]=\lim_{t\to\infty}\tilde{\boldsymbol{E}}_\alpha[g(\alpha(t));\ \sigma_1=+\infty] \\
&=\bar{h}(\alpha)-\boldsymbol{E}_\alpha[\bar{h}(\alpha(\sigma_1));\ \sigma_1<+\infty].
\end{aligned} \tag{4.26}$$

Noting $[\zeta=+\infty]\cap[\sigma_n<+\infty$ for all $n]\subset[\int_0^\infty\langle\alpha(u)\rangle du=+\infty]$ $\boldsymbol{P}_\alpha$—a.s. we see

$$\begin{aligned}
&\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t))\exp(-\int_0^t\langle\alpha(u)\rangle du);\ \zeta=+\infty] \\
&=\sum_{n=1}^\infty\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t))\exp(-\int_0^t\langle\alpha(u)\rangle du);\ \tau_n<+\infty,\ \sigma_n=+\infty] \\
&=\sum_{n=1}^\infty\boldsymbol{E}_\alpha[\bar{h}_1(\alpha(\tau_n))\exp(-\int_0^{\tau_n}\langle\alpha(u)\rangle du);\ \tau_n<+\infty]
\end{aligned} \tag{4.27}$$

where $\bar{h}_1(\alpha)=\bar{h}(\alpha)-\boldsymbol{E}_\alpha[\bar{h}(\alpha(\sigma_1))\,;\ \sigma_1<+\infty]$. So we get

$$\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t))\exp(-\int_0^t\langle\alpha(u)\rangle du)] \tag{4.28}$$

$$=\sum_{k=1}^{|\alpha|}\boldsymbol{E}_\alpha[\bar{h}_2(\alpha(\zeta_k))\exp(-\int_0^{\zeta_k}\langle\alpha(u)\rangle du)\,;\ \zeta_k<+\infty]$$

where $\bar{h}_2(\alpha)=\sum_{n=1}^{\infty}\boldsymbol{E}_\alpha[\bar{h}(\alpha(\tau_n))\exp(-\int_0^{\tau_n}\langle\alpha(u)\rangle du)\,;\ \tau_n<+\infty]$.

Theorem 4.2. *Let $\boldsymbol{h}=(h_1,\cdots,h_r)\in\mathcal{H}_r^*$. Suppose that $\mu\in\mathcal{P}(X_d)$ satisfies $\mu[X^{(R)}]=1$. Then $\lim_{t\to\infty}T_t^{d*}\mu=\nu_{\boldsymbol{h}}$ if and only if*

$$\lim_{t\to\infty}\int_{X_d}(\sum_{j\in S}P_t(i,j)(\sum_{p\in R_a}x_j^p)-h_a(i))^2\mu(d\boldsymbol{x})=0 \tag{4.29}$$

for any $1\leqq a\leqq r$ and $i\in S$.

Proof. Suppose that $\lim_{t\to\infty}T_t^{d*}\mu=\nu_{\boldsymbol{h}}$. For any $f\in C(X)$

$$\langle\psi_a\nu_{\boldsymbol{h}},f\rangle=\lim_{t\to\infty}T_t^d(f\circ\psi_a)(\rho(\boldsymbol{h}))=\lim_{t\to\infty}T_tf(\psi_a\rho(\boldsymbol{h})) \tag{4.30}$$

$$=\lim_{t\to\infty}T_tf(h_a)=\langle\nu_{h_a},f\rangle.$$

So by using Lemma 4.3 we have

$$\lim_{t\to\infty}T_t^*(\psi_a\mu)=\lim_{t\to\infty}\psi_a(T_t^{d*}\mu)=\psi_a\nu_{\boldsymbol{h}}=\nu_{h_a}. \tag{4.31}$$

Hence (4.29) follows from Theorem 3.2 and (4.31).

Conversely assume that (4.29) is fulfilled for any $1\leqq a\leqq r$ and $i\in S$. In the same way as the proof of Lemma 4.5, we obtain

$$\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t))]=\phi_\alpha(\rho(\boldsymbol{h}))\quad\text{where}\quad g(\alpha)=\langle\mu,\phi_\alpha\rangle. \tag{4.32}$$

Accordingly by Lemma 4.2 and Lemma 4.6,

$$\lim_{t\to\infty}\langle T_t^{d*}\mu,\phi_\alpha\rangle=\lim_{t\to\infty}\boldsymbol{E}_\alpha[g(\alpha(t))\exp(-\int_0^t\langle\alpha(u)\rangle du)] \tag{4.33}$$

$$=\lim_{t\to\infty}\boldsymbol{E}_\alpha[\phi_{\alpha(t)}(\rho(\boldsymbol{h}))\exp(-\int_0^t\langle\alpha(u)\rangle du)]=\lim_{t\to\infty}T_t^d\phi_\alpha(\rho(\boldsymbol{h}))$$

$$=\langle\nu_{\boldsymbol{h}},\phi_\alpha\rangle.$$

Also, it follows from Lemma 4.4 (i) that

$$\lim_{t\to\infty}\int_{X_d}x_i^pT_t^{d*}\mu(d\boldsymbol{x})=0\qquad\text{if } p\notin R_1\cup\cdots\cup R_r.\quad(i\in S) \tag{4.34}$$

(4.33) and (4.34) imply that $\lim_{t\to\infty}\langle T_t^{d*}\mu,f\rangle=\langle\nu_{\boldsymbol{h}},f\rangle$ holds for any $f\in C(X_d)$ and we conclude $\lim_{t\to\infty}T_t^{d*}\mu=\nu_{\boldsymbol{h}}$.

Proof of Theorem 4.1.

Let $\boldsymbol{h}=(h_1,\cdots,h_r)\in\mathcal{H}_r^*$. It is easy to see that

$$E_\alpha[\phi_{\alpha(t)}(\rho(\boldsymbol{h}))]=\phi_\alpha(\rho(\boldsymbol{h}))\,. \tag{4.35}$$

Hence (i) follows from Lemma 4.2 and Lemma 4.6. (ii) also is trivial. For (iii) let $\mu\in(\mathcal{S}_d)_{ext}$ and set $\langle\mu,\phi_\alpha\rangle=g(\alpha)$. By Lemma 4.5 and Lemma 4.6 there exists a $\boldsymbol{h}=(h_1,\cdots,h_r)\in\mathcal{H}_r^*$ satisfying

$$\lim_{t\to\infty}E_\alpha[g(\alpha(t))]=\phi_\alpha(\rho(\boldsymbol{h})) \qquad \text{for any} \quad \alpha\in J, \tag{4.36}$$

and

$$\begin{aligned} g(\alpha)&=E_\alpha[g(\alpha(t))\exp(-\int_0^t\langle\alpha(u)\rangle du)] \\ &=\lim_{t\to\infty}E_\alpha[\phi_{\alpha(t)}(\rho(\boldsymbol{h}))\exp(-\int_0^t\langle\alpha(u)\rangle du)] \\ &=\lim_{t\to\infty}\langle T_t^{d*}\delta_{\rho(\boldsymbol{h})},\phi_\alpha\rangle=\langle\nu_{\boldsymbol{h}},\phi_\alpha\rangle\,. \end{aligned} \tag{4.37}$$

Thus we see $\mu=\nu_{\boldsymbol{h}}$. Furthermore notice that the converse is an immediate result of Theorem 4.2. Therefore we complete the proof of Theorem 4.1.

Corollary 4.1. *Suppose that there is only one recurrent class. Then $\{T_t^d\}$ is ergodic in the sense that there exists a unique stationary state ν such that*

$$\lim_{t\to\infty}T_t^{d*}\mu=\nu \qquad \textit{for any} \quad \mu\in\mathcal{P}(X_d)\,. \tag{4.38}$$

Proof. Let R be the unique recurrent class. By the above theorms we have a unique stationary measure ν and moreover if $\mu\in\mathcal{P}(X_d)$ satisfies $\mu[X^{(R)}]=1$, then $\lim_{t\to\infty}T_t^{d*}\mu=\nu$.

Next, it holds by the proof of Lemma 4.4 (i) that there are some constants $K>0$ and $\gamma>0$ such that if $p\notin R$

$$E_x[x_i^p(t)]\leqq Ke^{-\gamma t} \qquad \text{for any} \quad x\in X_d,\ i\in S \text{ and } t>0\,. \tag{4.39}$$

Then any $\mu\in\mathcal{P}(X_d)$ be fixed. For each $\alpha\in J$, set $u(t,\alpha)=\langle T_t^{d*}\mu,\phi_\alpha\rangle$. Then it follows easily

$$\frac{d}{dt}u(t,\alpha)=\sum_{\beta\in J}R_{\alpha,\beta}u(t,\beta)-\langle\alpha\rangle u(t,\alpha)+w(t,\alpha) \tag{4.40}$$

where $w(t,\alpha)=\sum_{i\in S}\sum_{p\in R}\sum_{q\in R}\alpha_i^p\lambda_{qp}\dfrac{c}{\pi_p}E_x[x_i^q(t)\phi_{\alpha-\varepsilon_i^p}(\boldsymbol{x}(t))]\mu(d\boldsymbol{x})$. By (4.39) we have some constant $K_1>0$ satisfying

$$|w(t,\alpha)|\leqq K_1|\alpha|e^{-\gamma t}, \qquad \alpha\in J. \tag{4.41}$$

Also, since $\{u(t,\alpha)\}$ is a solution of (4.40), it is represented

$$\begin{aligned} u(t+t_0,\alpha)=&E_\alpha[u(t_0,\alpha(t))\exp(-\int_0^t\langle\alpha(u)\rangle du)] \\ &+\int_0^t E_\alpha[w(t_0+s,\alpha(t-s))\exp(-\int_0^{t-s}\langle\alpha(u)\rangle du)]ds\,. \end{aligned} \tag{4.42}$$

For each $\boldsymbol{x}\in X_d$ let us define $\bar{\boldsymbol{x}}\in X^{(R)}$ by

(4.43)
$$\bar{x}_i^p=\begin{cases} x_i^p+\dfrac{1}{|R|}\sum_{q\in R} x_i^q & \text{if} \quad p\in R \\ 0 & \text{otherwise}. \end{cases}$$

For $\mu\in\mathcal{P}(X_d)$, $\bar{\mu}$ denotes the image measure by the mapping $\boldsymbol{x}\to\bar{\boldsymbol{x}}$. Let $\bar{u}(t,\alpha)=\langle\overline{T_t^{d*}\mu},\phi_\alpha\rangle$. Then using (4.39) it is easy to check that for some $K_2>0$

(4.44)
$$|u(t,\alpha)-\bar{u}(t,\alpha)|\leqq K_2|\alpha|e^{-\gamma t}.$$

So, it follows from (4.42) that for some $K_3>0$

(4.45)
$$|u(t+t_0,\alpha)-\boldsymbol{E}_\alpha[\bar{u}(t_0,\alpha(t))\exp(-\int_0^t\langle\alpha(u)\rangle du)|\leqq K_3|\alpha|e^{-\gamma t_0}$$

Since $\overline{T_t^{d*}\mu}[X^{(R)}]=1$ we obtain

(4.46)
$$\overline{\lim_{t\to\infty}}|u(t+t_0,\alpha)-\langle\nu,\phi_\alpha\rangle|\leqq K_3|\alpha|e^{-\gamma t_0} \quad \text{for any} \quad \alpha\in J$$

and any $t_0>0$. Consequently this implies $\lim_{t\to\infty} T_t^{d*}\mu=\nu$.

Corollary 4.2. *For each $1\leqq a\leqq r$ there exists a $\nu^{(a)}\in(\mathcal{S}_d)_{ext}$ such that*

(4.47)
$$\lim_{t\to\infty} T_t^{d*}\mu=\nu^{(a)} \quad \textit{for any} \quad \mu\in\mathcal{P}(X_d) \textit{ satisfying}$$
$$\mu[x\in X_d\,;\, \sum_{p\in R_a} x_i^p=1 \textit{ for all } i\in S]=1.$$

Proof. It is immediate from Theorem 4.2.

Corollary 4.3. *Assume the condition of Case I of* § 3. *Let $\mu\in\mathcal{P}(X_d)$. Then $\lim_{t\to\infty} T_t^{d*}\mu$ exists if and only if $\lim_{t\to\infty}\sum_{j\in S}P_t(i,j)\int\sum_{p\in R_a}x_i^p\mu(d\boldsymbol{x})$ exists for any $1\leqq a\leqq r$. Moreover if this condition is fulfilled, $\lim_{t\to\infty}\sum_{j\in S}P_t(i,j)\int\sum_{p\in R_a}x_i^p\mu(d\boldsymbol{x})=\lambda_a$ is independent of $i\in S$, and*

(4.48)
$$\lim_{t\to\infty} T_t^{d*}\mu=\sum_{a=1}^r\lambda_a\nu^{(a)}.$$

Corollary 4.4. *Assume the condition of Case II. Let $\mu\in\mathcal{P}(X_d)$ and $\boldsymbol{h}\in\mathcal{H}_r^*$. Then $T_t^{d*}\mu$ converges to $\nu_{\boldsymbol{h}}$ as $t\to+\infty$ if and only if*

(4.49)
$$\lim_{t\to\infty}\int_{X_d}(\sum_{j\in S}P_t(i,j)(\sum_{p\in R_a}x_j^p+\sum_{q=e+1}^d x_j^q\pi_{(a)}^q)-h_a(i))^2\mu(d\boldsymbol{x})=0$$

for any $1\geqq a\leqq r$ and $i\in S$, where $\pi_{(a)}^q=\sum_{p\in R_a}\lim_{t\to\infty}A_t(q,p)$ for $e+1\leqq p\leqq d$.

We will omit the proof since it can be shown by using the above theorems and a similar argument to Corollary 4.1.

§ 5. Scaling limit (I)

Form now on we shall consider the case $S=Z^r$ (r-dimensional integer lattice space). Regarding the d-alleles stepping stone model $\boldsymbol{x}(t)=\{x_i^p(t)\}_{i\in Z^r,1\leq p\leq d}$ as a d-vector-measure-valued process on R^r ($\sum_{i\in S}(x_i^1(t), \cdots, x_i^d(t))\cdot\delta_{\{i\}}$), we shall discuss a scaling limit of the fluctuation process of this process,

$$N_t^\lambda=\lambda^{-(r+2)/2}\sum_{i\in S}(x_i^1(\lambda^2t)-E[x_i^1(\lambda^2t)], \cdots, x_i^d(\lambda^2t)E[x_i^d(\lambda^2t)])\cdot\delta_{\{i/\lambda\}}$$

as $\lambda\to+\infty$. Since the limiting process, if exists, no longer vector-measure-valued process, we shall discuss the convergence of $\mathcal{S}'$-processes.

After [4] we prepare some facts on $\mathcal{S}'$-processes. Let

$$e_k(x)=(\sqrt{2\pi}\,2^k k!)^{-1/2}(-1)^k e^{x/2}D_x^k e^{-x^2} \qquad (k=0, 1, \cdots, x\in R^1).$$

Setting $e_\alpha(x)=e_{\alpha_1}(x_1)\cdots e_{\alpha_r}(x_r)$ for each $\alpha\in Z_+^r$ and $x=(x_1, \cdots, x_d)\in R^r$, $\{e_\alpha\}$ is a complete orthonormal system of $L^2(R^r)$. e_α is called the Hermite function of index α.

Let $\mathcal{S}(R^r)$ be the space of all rapidly decreasing C^∞-functions on R^r, which is equipped with the usual topology, and let $\mathcal{S}'(R^r)$ be the space of tempered distributions. For $\phi\in\mathcal{S}(R^r)$, set $\|\phi\|_{(m)}^2=\sum_\alpha(2|\alpha|+r)^m(\phi, e_\alpha)_{L^2}^2$ for $m>0$. Denote by $\mathcal{S}_{(m)}(R^r)$ the completion of $\mathcal{S}(R^r)$ with respect to $\|\ \|_{(m)}$. Let

$$\|\phi\|_m=(\sum_{|\alpha|\leq m}\|D_x^\alpha\phi\|^2)^{1/2} \quad\text{and}\quad \|\phi\|=\|\phi\|_0$$

is the $L^2(R^r)$-norm. It is known that for some constant $\lambda_m>0$

(5.1) $$\|\phi\|_m\leqq\lambda_m\|\phi\|_{(m)}.$$

For each $N\in\mathcal{S}'(R^r)$, $\|N\|_{(-m)}$ and $\|\Pi_n^\perp N\|_{(-m)}$ are defined by

(5.2) $$\|N\|_{(-m)}^2=\sum_\alpha(2|\alpha|+r)^{-m}N(e_\alpha)^2$$

(5.3) $$\|\Pi_n^\perp N\|_{(-m)}^2=\sum_{|\alpha|>n}(2|\alpha|+r)^{-m}N(e_\alpha)^2.$$

Let $\mathcal{S}_{(-m)}(R^r)=\{N\in\mathcal{S}'(R^r);\ \|N\|_{(-m)}<+\infty\}$. It is obvious that $\mathcal{S}_{(-m)}(R^r)$ is a separable Hilbert space and it is imbedded continuously into $\mathcal{S}'(R^r)$.

For each integer $d\geqq 2$ we denote $\overset{d}{\otimes}\mathcal{S}(R^r)=\mathcal{S}(R^r)\times\cdots\times\mathcal{S}(R^r)$ and $\overset{d}{\otimes}\mathcal{S}'(R^r)=\mathcal{S}'(R^r)\times\cdots\times\mathcal{S}'(R^r)$.

Let $C([0, \infty), \mathcal{S}'(R^r))$ ($C([0, \infty), \overset{d}{\otimes}\mathcal{S}'(R^r))$, $C([0, \infty), R^1)$) be the spaces of $\mathcal{S}'(R^r)$-valued ($\overset{d}{\otimes}\mathcal{S}'(R^r)$-valued, R^1-valued) continuous functions defined on $[0, \infty)$, which is equipped with the compact uniform topology. For $\eta\in C([0, \infty), \mathcal{S}'(R^r))$, denote the t-coordinate by η_t. We shall use the following criterion of tightness on $C([0, \infty), \mathcal{S}'(R^r))$.

Lemma 5.1. *Let $\{P^\lambda\}_{\lambda\geq 1}$ be a family of probability measures on $C([0, \infty), \mathcal{S}'(R^r))$. Suppose that*

(i) *for any fixed $\phi\in\mathcal{S}(R^r)$ the family of probability distributions on $C([0, \infty), R^1)$ induced by $(\eta_t(\phi), P^\lambda)$ is tight, and that*

(ii) *for some positive integer n_0*

$$\lim_{m\to\infty}\sup_{\lambda} P^{\lambda}[\sup_{0\le t\le T}\|\Pi^{\perp}_{m}\eta_t\|_{(-n_0)}>\varepsilon]=0 \qquad \textit{for any } \varepsilon>0 \textit{ and } T>0.$$

Then $\{P^{\lambda}\}_{\lambda\ge1}$ *is tight.*

Proof. See [4] § 1 and appendix.

Remark. Suppose that the following condition is satisfied for some $n>0$. For any $T>0$ we have some constant $C_T>0$ satisfying that

$$E^{P^{\lambda}}[\sup_{0\le t\le T}(\eta_t(\phi))^2]\le C_T\|\phi\|_n^2 \qquad \text{for any } \phi\in\mathcal{S}(R^r) \text{ and } \lambda\ge1. \tag{5.4}$$

Then, the condition (ii) of Lemma 5.1 is verified by (5.1).

Here we define a $\mathcal{S}'(R^r)$-valued standard Wiener process. A sample continuous $\mathcal{S}'(R^r)$-valued process $W=\{W_t\}_{t\ge0}$ is called a standard Wiener process if the following conditions are satisfied.

(i) $W_0=0$,

(ii) W_t has independent increments, and

(iii) $E[e^{iW_t(\phi)}]=\exp\left(-\frac{t}{2}\|\phi\|^2\right)$ for all $\phi\in\mathcal{S}(R^r)$.

Let us consider a multi-allelic stepping stone model without mutation and selection $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}\,;\, \boldsymbol{x}(t)=\{x_i^p(t)\}_{i\in Z^r, 1\le p\le d})$ which is a diffusion process on X_d generated by

$$A^d=\sum_{i\in Z^r}\sum_{p=1}^{d}\sum_{q=1}^{d}x_i^p(\delta_{pq}-x_i^q)D_{i,p}D_{i,q} + \sum_{i\in Z^r}\sum_{p=1}^{d}\Big(\sum_{j\in Z^r}q_{ji}x_j^p\Big)D_{i,p}. \tag{5.5}$$

We assume the following.

Condition [C]

(i) $q_{ji}=q_{j-i,0}(=q_{j-i})$ *for any* i *and* $j\in Z^r$,

(ii) $q_i\ge0\ (i\ne0,\ \sum_{i\in Z^r}q_i=0,\ \sum_{i\in Z^r}q_i\cdot i=0$ *and* $\sum_{i\in Z^r}q_i|i|^2<+\infty$, *and*

(iii) *an additive group generated by* $\{i\in Z^r;\ q_i\ne0\}$ *coincides with* Z^r.

Let $Q=\{q_{ji}\}$ and set $P_t=e^{tQ^*}$ and $R_t=P_tP_t^*$. Then under the condition [C] R_t is a symmetric and spatially homogenuous transition probability on Z^r. We denote by $(\hat{\Omega}, \mathcal{B}, \hat{P}_i, z_t)_{i\in Z^r}$ the continuous time Markov chain on Z^r associated with R_t.

It is well-known that if $r\ge3$ this Markov chain is transient. Moreover, assuming that $\{q_i\}$ is finitely supported, then the potential matrix $G(i, j)=\int_0^\infty R_t(i, j)dt$ satisfies that for some constant $C>0$

$$G(i, j)\le C(1+|i-j|)^{-r+2} \qquad i, j\in Z^r. \tag{5.6}$$

(We find this estimate in [18] p. 339 for the descrete time case. But the continuous time case is easily reduced to the discrete one.) So it is easy to see that for some constant $C>0$

$$\hat{P}_i[\sigma_{\{0\}}<+\infty]\leqq C(1+|i|)^{-r+2} \qquad i\in Z^r, \tag{5.7}$$

where $\sigma_{\{0\}}$ stands for the hitting time for $\{0\}$.

For any $\lambda\geqq 1$, $1\leqq p\leqq d$ and $\phi\in\mathcal{S}(R^r)$ define

$$N_t^{p,\lambda}(\phi)=\lambda^{-(r+2)/2}\sum_{i\in Z^r}\phi\left(\frac{i}{\lambda}\right)(x_i^p(\lambda^2 t)-E[x_i^p(\lambda^2 t)]),\ p=1,\cdots,d. \tag{5.8}$$

Then $\boldsymbol{N}_t^\lambda=\{N_t^{1,\lambda},\cdots,N_t^{d,\lambda}\}$ is a $\overset{d}{\otimes}\mathcal{S}'(R^r)$-valued continuous process. Then we obtain

Theorem 5.1. *Let $r\geqq 3$. Suppose that the initial distribution μ_0 of $\{\boldsymbol{x}(t)\}$ is Z^r-shift invariant and satisfies*

$$\sum_{i\in Z^r}\left|\int\mu_0(dx)(x_0^p-m_p)(x_i^p-m_p)\right|<+\infty \qquad (1\leqq p\leqq d), \tag{5.9}$$

and

$$\int\mu_0(d\boldsymbol{x})x_0^p x_i^p x_j^q x_k^q \text{ converges to } m_p^2 m_q^2 \tag{5.10}$$

if all of $|i|$, $|j|$, $|k|$, $|i-j|$, $|j-k|$ and $|k-i|$ tend to $+\infty$ for all $1\leqq p,q\leqq d$ where $m_p=\int\mu_0(dx)x_0^p$. Then, $\boldsymbol{N}_t^\lambda$ converges to a $\overset{d}{\otimes}\mathcal{S}'(R^r)$-valued Ornstein-Uhlenbeck process $\boldsymbol{N}_t$ as $\lambda\to+\infty$, in the sense of the probability measures on the path space $C([0,\infty),\overset{d}{\otimes}\mathcal{S}'(R^r))$, where $\boldsymbol{N}_t$ is defined by the following stochastic integral equation,

$$N_t^p(\phi)=\sum_{q=1}^d\alpha_{pq}W_t^q(\phi)+\int_0^t N_s^p(L\phi)ds \qquad \phi\in\mathcal{S}(R^r),\ p=1,\cdots,d, \tag{5.10}$$

where

(i) $\boldsymbol{W}_t=\{W_t^1,\cdots,W_t^d\}$ *is an independent system of $\mathcal{S}'(R^r)$-valued standard Wiener processes*

(ii) $\{\alpha_{pq}\}_{1\leq p,q\leq d}$ *is a constant $d\times d$-matrix satisfying*

$$(\alpha\alpha^*)_{pq}=2\rho m_p(\delta_{pq}-m_q), \tag{5.12}$$

and

$$\rho=\hat{E}_0\left[\exp\left(-2\int_0^\infty I_{\{0\}}(z_u)du\right)\right]>0, \text{ where } I_{\{0\}}(i)=\begin{cases}1 & (i=0)\\ 0 & (i\neq 0).\end{cases}$$

and

$$L=\sum_{u=1}^r\sum_{v=1}^r a_{uv}D_uD_v \quad \text{with} \quad a_{uv}=\sum_{i\in Z^r}q_i i_u i_v. \tag{5.13}$$

Next, we will consider the case that the initial distribution is a stationary state of the stepping stone model associated with (5.5). If $r\geqq 3$, we know by Theorem 4.1.

(5.14) $$(\mathcal{S}_d)_{ext}=\{\nu_m;\ \boldsymbol{m}=(m_1,\cdots,m_d)\in R_+^d,\ m_1+\cdots+m_d=1\}.$$

Under an additional condition that $\{q_i\}$ is finitely supported we obtain

Theorem 5.2. *Let $r\geqq 3$ and $\nu_m\in(\mathcal{S}_d)_{ext}$. Suppose that the initial condition $\{\boldsymbol{x}(0)\}$ is ν_m-distributed. Then, any finite dimensional distribution of $\{\boldsymbol{N}_t^\lambda\}$ converges as $\lambda\to+\infty$ to that of a $\overset{d}{\otimes}\mathcal{S}'(R^r)$-valued stationary Ornstein-Uhlenbeck process $\boldsymbol{N}_t$ defined by the following stochastic integral equation,*

(5.15) $$N_t^p(\phi)=N_0^p(\phi)+\sum_{q=1}^d \alpha_{pq}W_t^q(\phi)+\int_0^t N_s^p(L\phi)ds \qquad \phi\in\mathcal{S}(R^r),$$

$p=1,\cdots,d$ where α, L and $\boldsymbol{W}_t$ are the same as Theorem 5.1, *and $\boldsymbol{N}_0$ is a $\overset{d}{\otimes}\mathcal{S}'(R^r)$-valued Gaussian random variable independent of $\boldsymbol{W}_t$ satisfying that for any $(\phi_1,\cdots,\phi_d)\in\overset{d}{\otimes}\mathcal{S}(R^r)$*

(5.16) $$E[\exp(iN_0^1(\phi_1)+\cdots+iN_0^d(\phi_d))]$$
$$=\exp\Big(-\frac{\rho}{2}\sum_{p=1}^d\sum_{q=1}^d m_p(\delta_{pq}-m_q)(G\phi_p,\phi_q)_{L^2}\Big),$$

where

(5.17) $$G(x)=\frac{\Gamma(r/2-1)}{2^{r/2+2}(\pi)^{r/2}|A|^{1/2}}\langle A^{-1}x,x\rangle \ \text{with}\ A=\{a_{uv}\}_{1\leq u,v\leq r},$$
$$|A|=\det A \ \text{and}\ G\phi(x)=\int_{R^r}G(y)\phi(x-y)dy.$$

Corollary 5.1. *Let $\nu_m\in(\mathcal{S}_d)_{ext}$, and let $\xi=\{(\xi_i^1,\cdots,\xi_i^d)\}_{i\in Z^r}$ be a ν_m-distributed random field. Define a generalized random field $\boldsymbol{N}^\lambda=(N^{1,\lambda},\cdots,N^{d,\lambda})$ by*

$$N^{p,\lambda}(\phi)=\lambda^{-(r+2)/2}\sum_{i\in Z^r}\phi\Big(\frac{i}{\lambda}\Big)(\xi_i^p-m_p) \qquad \text{for}\ \phi\in S(R^r),\ p=1,\cdots,d.$$

Then the distribution of $\boldsymbol{N}^\lambda$ converges as $\lambda\to+\infty$ to that of $\boldsymbol{N}_0$ defined by (5.16).

For simplicity we will prove the above theorems for a dialelic model. Let $(\Omega,\mathcal{F},P;\ \boldsymbol{x}(t)=\{x_i(t)\}_{i\in Z^r})$ be the diffusion process on $[0,1]^{Z^r}$ generated by

(5.18) $$A=\sum_{i\in Z^r}x_i(1-x_i)D_i^2+\sum_{i\in Z^r}\Big(\sum_{j\in Z^r}q_{ji}x_j\Big)D_i.$$

Here it is assumed that $\{q_{ji}\}$ satisfies the condition [C].

For any $\phi\in\mathcal{S}(R^r)$ and $\lambda\geqq 1$, set

(5.19) $$N_t^\lambda=\lambda^{-(r+2)/2}\sum_{i\in Z^r}\phi\Big(\frac{i}{\lambda}\Big)(x_i(\lambda^2t)-E[x_i(\lambda^2t)]).$$

Hereafter we will prove the following theorems instead of the above.

Theorem 5.1′. *Let $r\geqq 3$. Suppose that the initial distribution μ_0 of $\{\boldsymbol{x}(t)\}$ is Z^r-shift invariant and satisfies*

(5.9)′
$$\sum_{i\in Z^r}\left|\int\mu_0(d\boldsymbol{x})(x_0-m)(x_i-m)\right|<+\infty,$$

and

(5.10)′
$$\int\mu_0(d\boldsymbol{x})x_0^p x_i^p x_j^q x_k^q\longrightarrow m_p^2 m_q^2$$

if all of $|i|$, $|j|$, $|k|$, $|i-j|$, $|j-k|$ *and* $|k-i|$ *tend to* $+\infty$ *for* p, $q=1$, 2, *where* $x_i^1=x_i$, $x_i^2=1-x_i$, $m=\int\mu_0(dx)x_0$, $m_1=m$ *and* $m_2=1-m$. *Then,* $\{N_t^\lambda\}$ *converges to a* $\mathcal{S}'(R^r)$*-valued Ornstein-Uhlenbeck process* N_t *as* $\lambda\to+\infty$, *in the sense of the probability distributions on the path space* $C[0,\infty)$, $\mathcal{S}'(R^r)$), *where* N_t *is defined by the following stochastic integral equation,*

(5.11)′
$$N_t(\phi)=\sqrt{2\rho m(1-m)}\,W_t(\phi)+\int_0^t N_s(L\phi)ds\qquad \phi\in\mathcal{S}(R^r),$$

where W_t *is a* $\mathcal{S}'(R^r)$*-valued standard Wiener process, and* L *and* ρ *are the same as Theorem* 5.1.

Theorem 5.2′. *Let* $r\geqq 3$ *and* $\nu_m\in\mathcal{S}_{ext}$. *Suppose that the initial condition* $\{\boldsymbol{x}(0)\}$ *is* ν_m*-distributed. Then, any finite dimensional distribution of* N_t^λ *converges as* $\lambda\to+\infty$ *to that of a* $\mathcal{S}'(R^r)$*-valued stationary Ornstein-Uhlenbeck process* N_t *defined by the following stochastic integral equation,*

(5.15)′
$$N_t(\phi)=N_0(\phi)+\sqrt{2\rho m(1-m)}\,W_t(\phi)+\int_0^t N_s(L\phi)ds\qquad \phi\in\mathcal{S}(R^r),$$

where W_t, L *and* ρ *are the same as Theorem* 5.1′, *and* N_0 *is a* $\mathcal{S}'(R^r)$*-valued Gaussian random variable independent of* W_t *satisfying that for any* $\phi\in\mathcal{S}(R^r)$

(5.16)′
$$E[e^{iN_0(\phi)}]=\exp\left(-\frac{\rho}{2}m(1-m)(G\phi,\phi)_{L^2}\right),$$

where G *is of* (5.17).

For the proof of the above theorems we list a series of lemmas. For $\lambda\geqq 1$ and $\phi\in\mathcal{S}(R^r)$, we denote $\phi_\lambda(i)=\phi(i/\lambda)$ and $Q_\lambda\phi(i)=\lambda^2\sum_{i\in Z^r}q_{i-j}\phi_\lambda(j)$. $C_0^2(R^m)$ denotes the set of all C^2-functions with compact support defined on R^m.

Let $M_t^\lambda(\phi)=N_t^\lambda(\phi)-\int_0^t\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(\lambda^2 u)du$. Then, we have

Lemma 5.2. *For any* $f\in C_0^2(R^1)$

(i) $f(\sum_{i\in Z^r}\phi(i)(x_i(t)-m))-\int_0^t\sum_{i\in Z^r}(\sum_{j\in Z^r}q_{ij}\phi(j))x_i(u)f'(\sum_{i\in Z^r}\phi(i)(x_i(u)-m))du-\int_0^t\sum_{i\in Z^r}\phi(i)^2x_i(u)(1-x_i(u))f''(\sum_{i\in Z^r}\phi(i)(x_i(u)-m))du$ *is a martingale,*

(ii) $f(N_t^\lambda(\phi))-\int_0^t\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(\lambda^2u)f'(N_u^\lambda(\phi))du-\int_0^t\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2x_i(\lambda^2u)(1-x_i(\lambda^2u))f'(N_u^\lambda(\phi))du$ *is a martingale, and*

(iii) $f(M_t^\lambda(\phi))-\int_0^t\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2x_i(\lambda^2u)(1-x_i(\lambda^2u))f''(M_u^\lambda(\phi))du$ *also is a martingale.*

Proof. These are immediate results from the fact that the distribution of $(\Omega, \mathcal{F}, P; \boldsymbol{x}(t))$ is a solution of the (X, A)-martingale problem with the initial condition $\boldsymbol{x}(0)$.

Lemma 5.3. *For some constant $C_r>0$, it holds that*

(i) $\lambda^{-r}\sum_{k\in Z^r}\phi_\lambda(k)^2\leqq C_r(\|\phi\|^2+\lambda^{-2}\|\phi\|_r^2)$,

(ii) $\lambda^{-r}\sum_{k\in Z^r}(Q_\lambda\phi(k))^2\leqq C_r(\|\phi\|_2^2+\lambda^{-2}\|\phi\|_{r+2}^2)$, *and*

(iii) $\lim_{\lambda\to\infty}\lambda^{-r}\sum_{k\in Z^r}(Q_\lambda\phi(k)-(L\phi)_\lambda(k))^2=0$.

Proof. (i) is easy. For (ii), we introduce $\{u_k\}$ by $u_k=1$ for $|k|=1$, $u_0=-2r$, and $u_k=0$ otherwise. If $\{q_k\}=\{u_k\}$ (ii) is easy. For a general $\{q_k\}$, denoting $\hat{\phi}(\eta)=\sum_{k\in Z^r}e^{i<k,\eta>}\phi(k)$ for a summable function ϕ, it is not hard to see that for some $C>0$

$$|\hat{q}(\eta)|\leqq C|\hat{u}(\eta)| \qquad \text{for any} \quad \eta\in R^r. \tag{5.20}$$

So, using Parseval's equality we get

$$\begin{aligned}\lambda^{-r}\sum_{k\in Z^r}(Q_\lambda\phi(k))^2&=\lambda^{-r+4}\sum_{k\in Z^r}(q*\phi_\lambda(k))^2\\ &=\text{const}\cdot\lambda^{-r+4}\int_{[-\pi,\pi]^r}|\hat{q}(\eta)\hat{\phi}_\lambda(\eta)|^2d\eta\\ &\leqq\text{const}\cdot\lambda^{-r+4}\int_{[-\pi,\pi]^r}|\hat{u}(\eta)\hat{\phi}_\lambda(\eta)|^2d\eta\,.\end{aligned}$$

Thus, we can reduce it to the case of $\{q_k\}=\{u_k\}$. For (iii), we use the Poisson formula,

$$\hat{\phi}_\lambda(\eta)=\lambda^r\sum_{k\in Z^r}\mathcal{F}\phi(\lambda\eta+2\pi\lambda\cdot k) \tag{5.21}$$

where $\mathcal{F}\phi(\eta)=\int_{R^r}e^{i<x,\eta>}\phi(x)dx$. Then it follows that for any $c>0$ and $\phi\in\mathcal{S}(R^r)$

$$\lim_{\lambda\to\infty}\lambda^c\int_{[-\pi,\pi]^r}|\hat{\phi}_\lambda(\eta)-\lambda^r\mathcal{F}\phi(\lambda\eta)|^2d\eta=0\,. \tag{5.22}$$

Also, we note that for any $\varepsilon>0$ there exists a positive number δ satisfying

$$|\hat{q}(\eta)-\sum_{u=1}^r\sum_{v=1}^r a_{uv}\eta_u\eta_v|\leqq\varepsilon|\eta|^2 \qquad \text{if} \quad |\eta|\leqq\delta\,. \tag{5.23}$$

Hence it follows from (5.22) and (5.23) that

$$\begin{aligned}&\overline{\lim_{\lambda\to\infty}}\lambda^{-r}\sum_{k\in Z^r}(Q_\lambda\phi(k)-(L\phi)_\lambda(k))^2\\ &=\overline{\lim_{\lambda\to\infty}}\lambda^{-r}(2\pi)^{-r}\int_{[-\pi,\pi]^r}|\lambda^2\hat{q}\hat{\phi}_\lambda(\eta)-\sum_{u,v=1}^r a_{uv}(D_uD_v\phi)_\lambda(\eta)|^2d\eta\end{aligned} \tag{5.24}$$

$$=\overline{\lim_{\lambda\to\infty}}\lambda^{-r}(2\pi)^{-r}\int_{[-\pi,\pi]^r}\lambda^{2r+4}|\hat{q}(\eta)-\sum_{u,v=1}^{r}a_{uv}\eta_u\eta_v|^2|\mathcal{F}\phi(\lambda\eta)|^2d\eta$$

$$\leqq\varepsilon^2\overline{\lim_{\lambda\to\infty}}\lambda^{r+4}\int_{R^r}|\eta|^4|\mathcal{F}\phi(\lambda\eta)|^2d\eta$$

$$+\text{const.}\overline{\lim_{\lambda\to\infty}}\lambda^{r+4}\int_{|\eta|\geqq\delta}|\mathcal{F}\phi(\lambda\eta)|^2d\eta\leqq\varepsilon^2\|\phi\|_2^2 .$$

Thus we obtain (iii).

Lemma 5.4.

(i) $E[x_i(t)]=m$ *for all* $t\geqq0$ *and* $i\in S$.

(ii) $\sum_{j\in Z^r}|E[(x_0(t)-m)(x_j(t)-m)]|\leqq v+t$,

where $v=\sum_{j\in Z^r}\left|\int\mu_0(dx)(x_0-m)(x_j-m)\right|$.

Proof. (i) is trivial. Setting $E[(x_i(t)-m)(x_j(t)-m)]=h_t(i,j)$, it follows from Lemma 5.2 that

$$\frac{d}{dt}h_t(i,j)=\sum_{k\in Z^r}q_{kj}h_t(i,k)+\sum_{k\in Z^r}q_{ki}h_t(k,j)+\delta_{ij}a(t) \tag{5.25}$$

where $a(t)=2E[x_0(t)(1-x_0(t))]$. Hence, using the transition matrix $P_t=e^{tQ^*}$, $h_t(i,j)$ is represented such as

$$h_t(i,j)=\sum_{k\in Z^r}\sum_{m\in Z^r}P_t(i,k)P_t(j,m)h_0(k,m) \tag{5.26}$$
$$+\int_0^t\sum_{k\in Z^r}P_{t-s}(i,k)P_{t-s}(j,k)a(s)ds .$$

Noting that $\{P_t(i,j)\}$ is spatially homogeneous, (ii) follows immediately from this.

Lemma 5.5. *For any* $T>0$ *there exists a constant* $C_T>0$ *such that*

(i) $E[(\lambda^{-(r+2)/2}\sum_{i\in Z^r}\phi(i)(x_i(\lambda^2t)-m))^2]\leqq\lambda^{-r}\sum_{i\in Z^r}\phi(i)^2\left(\frac{v}{\lambda^2}+t\right)$,

(ii) $E[\sup_{0\leqq t\leqq T}(N_t^\lambda(\phi))^2]\leqq C_T(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2+\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)$,

and

(iii) $E[(N_t^\lambda(\phi)-N_s^\lambda(\phi))^2]\leqq C_T(\|\phi\|_2^2+\lambda^{-2}\|\phi\|_{r+2}^2)|t-s|\qquad 0\leqq s,t\leqq T$.

Proof. (i); $E[(\lambda^{-(r+2)/2}\sum_{i\in Z^r}\phi(i)(x_i(\lambda^2t)-m))^2]\leqq$

$$\lambda^{-r-2}\sum_{i\in Z^r}\phi(i)^2\sum_{j\in Z^r}|h_{\lambda^2t}(i,j)|\leqq\lambda^{-r}\sum_{i\in Z^r}\phi(i)^2\left(\frac{v}{\lambda^2}+t\right).$$

(ii); Using Lemma 5.2 and a maximal inequality for martingales we see

$$E[\sup_{0\leqq t\leqq T}(N_t^\lambda(\phi))^2]\leqq2E[\sup_{0\leqq t\leqq T}(M_t^\lambda(\phi))^2] \tag{5.27}$$
$$+2E[\sup_{0\leqq t\leqq T}(\lambda^{-(r+2)/2}\int_0^t\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(\lambda^2s)ds)^2]$$

$$\leqq 8E[(M_T^\lambda(\phi))^2]+2T\int_0^T E[(\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)(x_i(\lambda^2 s)-m))^2]ds$$

$$\leqq 16E[(N_T^\lambda(\phi))^2]+18T\int_0^T E[(\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)(x_i(\lambda^2 s)-m))^2]ds.$$

Thus, (ii) follows from (i). (iii); Applying Lemma 5.2 for $f(x)=x^2$, we have

$$E[(N_t^\lambda(\phi)-N_s^\lambda(\phi))^2]=2E\left[\int_s^t \lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)(x_i(\lambda^2 u)-m)(N_u^\lambda(\phi) -N_s^\lambda(\phi))du\right]+2E\left[\int_s^t \lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 u)(1-x_i(\lambda^2 u))du\right]. \tag{5.28}$$

Hence (iii) follows easily from (i), (5.28) and Lemma 5.3.

Lemma 5.6. *For any* $t>0$,

(i) $\lim_{\lambda\to\infty} E[\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 t)(1-x_i(\lambda^2 t))]=\rho\|\phi\|^2$,

(ii) $\lim_{\lambda\to\infty} E[(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 t)(1-x_i(\lambda^2 t))-\rho\|\phi\|^2)^2]=0$,

where $\rho=\hat{E}_0\left[\exp\left(-2\int_0^\infty I_{\{0\}}(z_u)du\right)\right]>0$.

In order to show this lemma we introduce auxiliary Markov chains which were proved to be useful in [15].

Let I be the set of all non-negative integer-valued summable functions on Z^r, i.e. $I=\{\alpha=\{\alpha_i\}_{i\in Z^r};\ \alpha_i\in Z_+,\ |\alpha|=\sum_{i\in Z^r}\alpha_i<+\infty\}$. If $\alpha_i=1$ and $\alpha_j=0\ (j\neq i)$, $\alpha\in I$ denoted by $\alpha=\varepsilon^i$. Set $f_\alpha(\boldsymbol{x})=\prod_{i\in Z^r}x_i^{\alpha_i}$ for each $\alpha\in I$, and $f_0=1$. Let us define two infinitesimal matrices $R=\{R_{\alpha,\beta}\}$ and $\tilde{R}=\{\tilde{R}_{\alpha,\beta}\}$ on $I\times I$ by

$$R_{\alpha,\beta}=\begin{cases}\alpha_i\beta_{ji} & \text{if } \beta=\alpha-\varepsilon^i+\varepsilon^j\in I\quad (i\neq j),\\ \alpha_i(\alpha_i-1) & \text{if } \beta=\alpha-\varepsilon^i\in I,\\ \sum_{i\in Z^r}\alpha_i q_{ii}-\sum_{i\in Z^r}\alpha_i(\alpha_i-1) & \text{if } \beta=\alpha,\\ 0 & \text{otherwise,}\end{cases} \tag{5.29}$$

and

$$\tilde{R}_{\alpha,\beta}=\begin{cases}\alpha_i q_{ji} & \text{if } \beta=\alpha-\varepsilon^i+\varepsilon^j\in I\quad (i\neq j),\\ \sum_{i\in Z^r}\alpha_i q_{ii} & \text{if } \beta=\alpha,\\ 0 & \text{otherwise.}\end{cases} \tag{5.30}$$

Denote by $\{\alpha_t, \boldsymbol{P}_\alpha\}_{\alpha\in I}$ and $\{\alpha_t, \tilde{\boldsymbol{P}}_\alpha\}_{\alpha\in I}$ the continuous time Markov chains on I generated by R and $\tilde{R}$. Then we have by Lemma 3.1 of [15]

$$E[f_\alpha(x(t))]=\boldsymbol{E}_\alpha[\langle\mu_0, f_{\alpha_t}\rangle]\qquad \text{for any}\quad \alpha\in I. \tag{5.31}$$

Let us introduce some stopping times. Let $\Delta=\{\alpha\in I;\ \alpha_i\geqq 2 \text{ for some } i\in Z^r\}$.

6) $\langle\mu, f\rangle=\int f(\boldsymbol{x})\mu(d\boldsymbol{x})$.

$\zeta=\inf\{t\geqq 0\,;\ |\alpha_t|<|\alpha_0|\}$, $\zeta_k=\inf\{t\geqq 0\,;\ |\alpha_t|\leqq k\}$ for $k\geqq 0$, $\tau_1=\inf\{t\geqq 0\,;\ \alpha_t\in I\backslash\Delta$ and $|\alpha_t|=|\alpha_0|\}$, $\sigma_1=\inf\{t\geqq\tau_1\,;\ \alpha_t\in\Delta\}$ and $\{\tau_n,\sigma_n\}$ are defined inductively by $\tau_n=\sigma_{n-1}+\tau_1(\theta_{\sigma_{n-1}})$ and $\sigma_n=\tau_n+\sigma_1(\theta_{\tau_n})$, where $\{\theta_t\}$ is the shift operator.

We may assume that there exists another I-valued process $\{\tilde{\alpha}_t\}$ defined on the same probaility space as $\{\alpha_t, \boldsymbol{P}_\alpha\}$, such that $\{\tilde{\alpha}_t, \boldsymbol{P}_\alpha\}$ has the identical probability law with $\{\alpha_t, \boldsymbol{P}_\alpha\}$ and

(5.32) $$\alpha_t\leqq\tilde{\alpha}_t,\ \text{ and }\ \alpha_t=\tilde{\alpha}_t \text{ for } t\leqq\zeta\ P_\alpha\text{-a. e.}\qquad \text{for any }\ \alpha\in I$$

where $\alpha\leqq\tilde{\alpha}$ means that $\alpha_i\leqq\tilde{\alpha}_i$ for all $i\in Z^r$. For Lemma 5.5, it suffices to show that

(5.33) $$\lim_{\lambda\to\infty} E[(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 t))^2]=\|\phi\|^4 m^2\,,$$

(5.34) $$\lim_{\lambda\to\infty} E[\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 t)^2]=\|\phi\|^2(m-m(1-m)\rho)\,,$$

and

(5.35) $$\lim_{\lambda\to\infty} E[(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 t)^2)^2]=\|\phi\|^4(m-m(1-m)\rho)^2\,.$$

It follows from the assumption of Theorem 5.1′ that there exists a constant $M>0$ satisfying

(5.36) $$|\langle\mu_0, f_\alpha\rangle-m^{|\alpha|}|<\varepsilon\qquad \text{if } |\alpha|\leqq 4 \text{ and } d(\alpha)>M\,,$$

where $d(\alpha)=\begin{cases} 0 & \text{if}\quad \alpha\in\Delta\,,\\ \min\{|i-j|\,;\ \alpha_i>0,\ \alpha_j>0,\ i\neq j\} & \text{otherwise}\,.\end{cases}$

By (5.31)

(5.37) $$\begin{aligned}&|E[(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 t))^2]-m^2(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2)^2|\\ &\leqq\lambda^{-2r}\sum_{i\in Z^r}\sum_{j\in Z^r}\phi_\lambda(i)^2\phi_\lambda(j)^2|E_{\varepsilon i+\varepsilon j}[\langle\mu_0, f_{\alpha_{\lambda^2 t}}\rangle-m^2]|\,.\end{aligned}$$

Noting (5.36) and

$$\begin{aligned}&|\boldsymbol{E}_{\varepsilon i+\varepsilon j}[\langle\mu_0, f_{\alpha_t}\rangle]-m^2|\\ &\qquad\leqq\boldsymbol{E}_{\varepsilon i+\varepsilon j}[|\langle\mu_0, f_{\alpha_t}\rangle-m^2|\,;\ d(\alpha_t)\leqq M]+\varepsilon+\boldsymbol{P}_{\varepsilon i+\varepsilon j}[\sigma_1<+\infty]\\ &\qquad\leqq\sum_{|k-m|\leqq M}P_t(i,k)P_t(j,m)+\varepsilon+\hat{P}_{i-j}[\sigma_{\{0\}}<+\infty]\to\varepsilon\quad\text{as}\quad |i-j|\to+\infty\,,\end{aligned}$$

we obtain

(5.38) $$\lim_{\lambda\to\infty}\lambda^{-2r}\sum_{i\in Z^r}\sum_{j\in Z^r}\phi_\lambda(i)^2\phi_\lambda(j)^2|\boldsymbol{E}_{\varepsilon i+\varepsilon j}[\langle\mu_0, f_{\alpha_{\lambda^2 t}}\rangle-m^2]|\leqq\varepsilon\|\phi\|^4\,.$$

Thus we obtain (5.33). Next, we have by Theorem 3.2.

(5.39) $$\lim_{t\to\infty}T_t^*\mu_0=\nu_m\,.$$

So,

(5.40) $$\begin{aligned}\lim_{t\to\infty}E[x_0(t)^2]&=\int x_0^2\nu_m(d\boldsymbol{x})=E_{2\varepsilon 0}[m^{|\alpha_\infty|}]\\ &=m\boldsymbol{P}_{2\varepsilon 0}[\zeta<+\infty]+m^2\boldsymbol{P}_{2\varepsilon 0}[\zeta=+\infty]\,.\end{aligned}$$

Also, we can show

$$(5.41)\qquad \boldsymbol{P}_{2\varepsilon 0}[\zeta=+\infty]=\hat{E}_0\left[\exp\left(-2\int_0^\infty I_{\{0\}}(z_u)\,du\right)\right]=\rho\,.$$

We will omit the proof of this equality since it needs an elementary but tedious calculation. Thus we have $\lim_{t\to\infty} E[x_0(t)^2]=m-m(1-m)\rho$ and (5.34) holds. Finally we shall show (3.35). Note that

$$(5.42)\qquad |E[(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 t)^2)^2]-(E[\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 t)^2])^2|$$
$$\leqq\lambda^{-2r}\sum_{i\in Z^r}\sum_{j\in Z^r}(\phi_\lambda(i)^2\phi_\lambda(j)^2|\boldsymbol{E}_{2\varepsilon i+2\varepsilon j}[\langle\mu_0, f_{\alpha_{\lambda^2 t}}\rangle]$$
$$-\boldsymbol{E}_{2\varepsilon i}[\langle\mu_0, f_{\alpha_{\lambda^2 t}}\rangle]|\boldsymbol{E}_{2\varepsilon j}[\langle\mu_0, f_{\alpha_{\lambda^2 t}}\rangle]|,$$

$$(5.43)\qquad |\boldsymbol{E}_{2\varepsilon i+2\varepsilon j}[\langle\mu_0, f_{\alpha_t}\rangle]-\boldsymbol{E}_{2\varepsilon i}\otimes\boldsymbol{E}_{2\varepsilon j}[\langle\mu_0, f_{\alpha_t+\beta_t}\rangle]|$$
$$\leqq 4\boldsymbol{P}_{\varepsilon i+\varepsilon j}[\sigma_1<+\infty]=4\hat{P}_{i-j}[\sigma_{\{0\}}<+\infty]\,,$$

and by (5.36) and (5.32)

$$(5.44)\qquad |\boldsymbol{E}_{2\varepsilon i}\otimes\boldsymbol{E}_{2\varepsilon j}[\langle\mu_0, f_{\alpha_t+\beta_t}\rangle]-\boldsymbol{E}_{2\varepsilon i}\otimes\boldsymbol{E}_{2\varepsilon j}[\langle\mu_0, f_{\alpha_t}\rangle\langle\mu_0, f_{\beta_t}\rangle]|$$
$$\leqq 3\varepsilon+\boldsymbol{P}_{2\varepsilon i}\otimes\boldsymbol{P}_{2\varepsilon j}[d(\alpha_t+\beta_t)\leqq M]$$
$$\leqq 3\varepsilon+\boldsymbol{P}_{2\varepsilon i+2\varepsilon j}[d(\alpha_t)\leqq M]$$
$$\leqq 3\varepsilon+2\sum_{|m|\leqq M}R_t(0, m)+4\sum_{|m|\leqq M}R_t(i, j+m)\,.$$

From these estimates it follows

$$(5.45)\qquad \varlimsup_{\lambda\to\infty}\lambda^{-2r}\sum_{i\in Z^r}\sum_{j\in Z^r}\phi_\lambda(i)^2\phi_\lambda(j)^2|\boldsymbol{E}_{2\varepsilon i+2\varepsilon j}[\langle\mu_0, f_{\alpha_{\lambda^2 t}}\rangle]$$
$$-\boldsymbol{E}_{2\varepsilon i}[\langle\mu_0, f_{\alpha_{\lambda^2 t}}\rangle]\boldsymbol{E}_{2\varepsilon j}[\langle\mu_0, f_{\alpha_{\lambda^2 t}}\rangle]|\leqq 3\varepsilon\|\phi\|^4\,.$$

Therefore (5.35) holds, and we complete Lemma 5.6.

Lemma 5.7. *Let* $0<\delta<2/(r+2)$ *be fixed. For any* $T>0$ *there exists a constant* $C_T>0$ *satisfying that*

$$(5.46)\qquad E[|N_t^\lambda(\phi)-N_s^\lambda(\phi)|^{2+2\delta}]\leqq C_T|||\phi|||_\lambda\cdot|t-s|^{1+\delta}$$

holds for any $\phi\in\mathcal{S}(R^r)$, $\lambda\geqq 1$ *and* $0\leqq s, t\leqq T$, *where*

$$|||\phi|||_\lambda=(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2)^{1+\delta}+(\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)^{1+\delta}$$
$$+(\lambda^{-r}\sum_{i\in Z^r}|\phi_\lambda(i)|)^{2\delta}(\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)\,.$$

Proof. 1°. First we claim that for some constant $C_T>0$

$$(5.47)\qquad E[|\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(\lambda^2 t)|^{2+2\delta}]$$
$$\leqq C_T((\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)^{1+\delta}+(\lambda^{-r}\sum_{i\in Z^r}|\phi_\lambda(i)|)^{2\delta}(\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)\,.$$

Set $\phi_t(i)=\sum_{j\in Z^r}P_t(j,i)\phi(j)$ for each $t\geqq 0$ and define $L_t^T=\sum_{i\in Z^r}\phi_{T-t}(i)(x_i(t)-m)$ for $0\leqq t\leqq T$. Using Lemma 5.2, it is easy to see that for any $f\in C_0^2(R^1)$ and $0\leqq t\leqq T$ $f(L_t^T)-\int_0^t\sum_{i\in Z^r}\phi_{T-u}(i)x_i(u)(1-x_i(u))f''(L_u^T)du$ is a martingale. Accordingly, applying for $f(x)=x^2$, and $f(x)=|x|^{2+2\delta}$, we have

$$E[(L_t^T)^2]\leqq E[(L_0^T)^2]+t\sum_{i\in Z^r}\phi(i)^2\leqq(v+t)\sum_{i\in Z^r}\phi(i)^2, \tag{5.48}$$

and

$$\begin{aligned}E[|L_t^T|^{2+2\delta}]&=E[|L_0^T|^{2+2\delta}]+C\int_0^t\sum_i\phi_{T-u}(i)^2E[x_i(u)(1-x_i(u))\\ &|L_u^T|^{2\delta}]du\leqq(\sum_{i\in Z^r}|\phi_T(i)|)^{2\delta}E[(L_0^T)^2]+C\int_0^t(\sum_{i\in Z^r}\phi_{T-u}(i)^2)E[(L_u^T)^2]^{\delta}du\,.\end{aligned}\tag{5.49}$$

Setting $T=t$, we get

$$\begin{aligned}&E[|\sum_{i\in Z^r}\phi(i)(x_i(t)-m|^{2+2\delta}]\\ &\leqq v(\sum_{i\in Z^r}|\phi(i)|)^{2\delta}(\sum_{i\in Z^r}\phi(i)^2)+C(v+t)(\sum_{i\in Z^r}\phi(i)^2)^{1+\delta}\,.\end{aligned}\tag{5.50}$$

Here we used the following inequality;

$$\sum_{i\in Z^r}|\phi_t(i)|^p\leqq\sum_{i\in Z^r}|\phi(i)|^p \qquad \text{for any}\quad p\geqq 1\,. \tag{5.51}$$

Hence

$$\begin{aligned}&E[|\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(\lambda^2t)|^{2+2\delta}]\\ &=E[|\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)(x_i(\lambda^2t)-m)|^{2+2\delta}]\\ &\leqq\lambda^{-(r+2)(1+\delta)}v(\sum_{i\in Z^r}|Q_\lambda\phi(i)|)^{2\delta}(\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)\\ &+C\Big(\frac{v}{\lambda^2}+t\Big)^{1+\delta}(\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)^{1+\delta}\,.\end{aligned}\tag{5.52}$$

The first term $\leqq\lambda^{-(r+2)(1+\delta)}\lambda^{4\delta}v\|q\|^{2\delta}(\sum_{i\in Z^r}|\phi_\lambda(i)|)^{2\delta}(\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)$

$$=\text{const. }\lambda^{-\gamma}(\lambda^{-r}\sum_{i\in Z^r}|\phi_\lambda(i)|)^{2\delta}(\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)\,,$$

where $\gamma=(r+2)(1+\delta)-4\delta-2r\delta-r>0$ and $\|q\|=\sum_{i\in Z^r}|q_i|$. Thus, we obtain (5.47).

2°. It follows from Lemma 5.2 (iii) that for any $f\in C_0^2(R^1)$

$$f(M_t^\lambda(\phi)-M_s^\lambda(\phi))-\int_s^t\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2x_i(\lambda^2u)(1-x_i(\lambda^2u))f''(M_u^\lambda(\phi)-M_s^\lambda(\phi))du$$

is a martingale for $t\geqq s$. Taking $f(x)=x^2$ and $f(x)=x^4$, we have

$$E[(M_t^\lambda(\phi)-M_s^\lambda(\phi))^2]\leqq\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2|t-s|,$$

and

$$E[(M_t^\lambda(\phi)-M_s^\lambda(\phi))^4]\leqq 2(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2)^2(t-s)^2 .$$

Accordingly by Hölder's inequality we get

$$E[|M_t^\lambda(\phi)-M_s^\lambda(\phi)|^{2+2\delta}]\leqq \text{const.}(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2)^{1+\delta}|t-s|^{1+\delta} . \tag{5.53}$$

Also, it follows from (5.47)

$$\begin{aligned} &E\Big[\Big|\int_s^t \lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(\lambda^2 u)du\Big|^{2+2\delta}\Big] \\ &\leqq |t-s|^{1+2\delta}\int_s^t E[|\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(\lambda^2 u)|^{2+2\delta}]du \\ &\leqq \text{const.}|t-s|^{2+2\delta}(\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i))^2)^{1+\delta} \\ &\quad+\text{const.}(\lambda^{-r}\sum_{i\in Z^r}|\phi_\lambda(i)|)^{2\delta}(\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i))^2) . \end{aligned} \tag{5.54}$$

Therefore, combining these estimates we obtain (5.46).

Now, we are in position to prove Theorem 5.1′. If $m=1$ or 0 the proof is trivial. So we assume $0<m<1$. Let P^λ be the probability distribution on $C([0,\infty),\mathcal{S}'(R^r))$ induced by N_t^λ. By Lemma 5.3, Lemma 5.5 and Lemma 5.7 the condition of Lemma 5.1 are fulfilled. Accordingly the family $\{P^\lambda\}_{\lambda\geq 1}$ is tight.

Let $\{\lambda_n\}$ be any sequence, tending to $+\infty$, so that $\{P^{\lambda_n}\}$ converges to some limit P^∞. We claim that for any $f\in C_0^2(R^1)$

$$f(\eta_t(\phi))-\int_0^t \eta_s(L\phi)f'(\eta_s(\phi))ds-\rho\|\phi\|^2\int_0^t f''(\eta_s(\phi))ds \tag{5.55}$$

is a P^∞-martingale, and moreover

$$P^\infty[\eta_0=0]=1 . \tag{5.56}$$

By Lemma 5.5 (i)

$$E^{P^\infty}[(\eta_0(\phi))^2]=\lim_{\lambda\to\infty}E[(N_0^\lambda(\phi))^2]=0 . \tag{5.57}$$

Thus we get (5.56). Next, we notice by Lemma 5.2 that for any $f\in C_0^2(R^1)$

$$\begin{aligned} &f(N_t^\lambda(\phi))-\int_0^t \lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(\lambda^2 s)f'(N_s^\lambda(\phi))ds \\ &\quad-\int_0^t \lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 s)(1-x_i(\lambda^2 s))f''(N_s^\lambda(\phi))ds \end{aligned} \tag{5.58}$$

is a martingale. Hence (5.55) follows easily from (5.58), Lemma 5.3, Lemma 5.5 and Lemma 5.6.

On the other hand it is known that the uniqueness holds for the martingale problem (5.55) with (5.56), (cf. [4] Theorem 1.4). Therefore P^∞ is uniquely determined and this implies that P^λ converges to P^∞ as $\lambda\to\infty$.

Also, denoting $W_t(\phi)=\dfrac{1}{\sqrt{2\rho m(1-m)}}(\eta_t(\phi)-\int_0^t \eta_s(L\phi)ds)$, it is easy to see that W_t is a $\mathcal{S}'(R^r)$-valued standard Wiener process. Thus, we complete the proof of Theorem 5.1′.

Next, we proceed to the proof of Theorem 5.2′. If $m=0$ or 1, it is trivial. Let $\mu_0=\nu_m$ with $0<m<1$. Then $\{\boldsymbol{x}(t)\}$ is a $[0, 1]^{Z^r}$-valued stationary Markov process. Accordingly, $\{N_t^\lambda\}$ also is $\mathcal{S}'(R^r)$-valued stationary process for each $\lambda\geqq 1$.

For any $\phi\in\mathcal{S}(R^r)$, set $\ll\phi\gg=\int_{R^r}\int_{R^r}\dfrac{|\phi(x)||\phi(y)|}{|x-y|^{r-2}}dxdy$. Then we have

Lemma 5.8. *Let $p>r$ be fixed. For any $T>0$ there are some constants $C_1>0$, $C_2>0$ and $C_T>0$ such that*

(i) $E[(N_0^\lambda(\phi))^2]\leqq C_1\|\phi\|_{(p+r)}^2$,

(ii) $\overline{\lim}_{\lambda\to\infty} E[(N_0^\lambda(\phi))^2]\leqq C_2\ll\phi\gg$,

and

(iii) $\overline{\lim}_{\lambda\to\infty} E[(N_t^\lambda(\phi)-N_s^\lambda(\phi))^2]\leqq C_T(\|\phi\|^2+\ll L\phi\gg)|t-s|$

for any $\phi\in\mathcal{S}(R^r)$, $0\leqq t-s\leqq T$ and $\lambda\geqq 1$.

Proof. By (5.31) and Theorem 3.1

(5.59) $$E[f_\alpha(x(0))]=\langle\nu_m, f\rangle=\lim_{t\to\infty} T_t f_\alpha(m)=\lim_{t\to\infty}\boldsymbol{E}_\alpha[m^{|\alpha_t|}],$$

and

(5.60) $$\begin{aligned}E[(N_0^\lambda(\phi))^2]&=\lambda^{-r-2}\sum_{i\in Z^r}\sum_{j\in Z^r}\phi_\lambda(i)\phi_\lambda(j)E[(x_i(0)-m)(x_j(0)-m)]\\&=\lambda^{-r-2}\sum_{i\in Z^r}\sum_{j\in Z^r}\phi_\lambda(i)\phi_\lambda(j)(\lim_{t\to\infty}\boldsymbol{E}_{\varepsilon i+\varepsilon j}[m^{|\alpha_t|}]-m^2)\\&\leqq\lambda^{-r-2}\sum_{i\in Z^r}\sum_{j\in Z^r}|\phi_\lambda(i)||\phi_\lambda(j)|\boldsymbol{P}_{\varepsilon i+\varepsilon j}[\zeta<+\infty]m(1-m)\end{aligned}$$

Further we notice by (5.7)

(5.61) $$\begin{aligned}\boldsymbol{P}_{\varepsilon i+\varepsilon j}[\zeta<+\infty]&\leqq\boldsymbol{P}_{\varepsilon i+\varepsilon j}[\sigma_1<+\infty]=\hat{P}_{i-j}[\sigma_{\{0\}}<+\infty]\\&\leqq\frac{C}{(1+|i-j|)^{r-2}}.\end{aligned}$$

Also, it is not hard to check that

(5.62) $$\phi^2(x)(1+|x|^{2p})\leqq C'\sum_{\substack{|\alpha|\leqq p\\|\beta|\leqq r}}\|x^\alpha D^\beta\phi\|^2\leqq C''\|\phi\|_{(p+r)}^2.$$

Here the second inequality is found in appendix of [4]. By making use of the above estimates we obtain (i). (ii) is immediate from (5.60) and (5.61). (iii); For any $f\in C_0^2(R^1)$

$$f(M_t^\lambda(\phi)-M_s^\lambda(\phi))-\int_s^t \lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(\lambda^2 u)(1-x_i(\lambda^2 u))f''(M_u^\lambda(\phi)-M_s^\lambda(\phi))du$$

is a martingale for $t\geqq s$. So it follows

$$E[(M_t^\lambda(\phi)-M_s^\lambda(\phi))^2]\leqq\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2|t-s|. \tag{5.63}$$

Noting (5.60), (5.61) and the stationarity, we see

$$\begin{aligned}(5.64)\quad & E\Big[\Big(\int_s^t \lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(\lambda^2 u)du\Big)^2\Big] \\ &\leqq(t-s)^2\lambda^{-r-2}E([\sum_{i\in Z^r}Q_\lambda\phi(i)(x_i(0)-m))^2] \\ &\leqq(t-s)^2\lambda^{-r-2}\sum_{i\in Z^r}\sum_{j\in Z^r}|Q_\lambda\phi(i)||Q_\lambda\phi(j)|\frac{C}{(1+|i-j|)^{r-2}}.\end{aligned}$$

Since $\{q_i\}$ is finitely supported, we have some constant $c>0$ satisfying

$$q_i=0 \qquad \text{for any } i\in Z^r \text{ with } |i|>c. \tag{5.65}$$

By making use of the Taylor expansion it holds

$$\begin{aligned}(5.66)\quad & |Q_\lambda\phi(i)-(L\phi)_\lambda(i)| \\ &\leqq\sum_{j\in Z^r}q_j\sum_{u=1}^r\sum_{v=1}^r|j_u j_v|\sup_{|x|\leqq c}\Big|D_uD_v\phi\Big(\frac{i+x}{\lambda}\Big)-D_uD_v\phi\Big(\frac{i}{\lambda}\Big)\Big|.\end{aligned}$$

Accordingly, (ii) follows easily from (5.63), (5.64) and (5.66).

Lemma 5.9.

(i) $\lim_{\lambda\to\infty}E[(\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(0)-N_0^\lambda(L\phi))^2]=0.$

(ii) $\lim_{\lambda\to\infty}E[(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2x_i(0)(1-x_i(0))-\rho\|\phi\|^2)^2]=0.$

Proof. (i); We note by (5.59) and (5.61) that

$$E[(x_i(0)-m)(x_j(0)-m)]\leqq\frac{C}{(1+|i-j|)^{r-2}}. \tag{5.67}$$

So, by (5.66)

$$\begin{aligned}&\lim_{\lambda\to\infty}E[(\lambda^{-(r+2)/2}\sum_{i\in Z^r}Q_\lambda\phi(i)x_i(0)-N_0^\lambda(L\phi))^2] \\ &\leqq\lim_{\lambda\to\infty}\lambda^{-r-2}\sum_{i\in Z^r}\sum_{j\in Z^r}|Q_\lambda\phi(i)-(L\phi)_\lambda(i)||Q_\lambda\phi(j)-(L\phi)_\lambda(j)|\frac{C}{(1+|i-j|)^{r-2}} \\ &=0.\end{aligned}$$

(ii);

$$E[(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2x_i(0)^2)^2]=\lambda^{-2r}\sum_{i\in Z^r}\sum_{j\in Z^r}\phi_\lambda(i)^2\phi_\lambda(j)^2\boldsymbol{E}_{2\varepsilon_i+\varepsilon_j}[m^{|\alpha_\infty|}].$$

Since

$$|\boldsymbol{E}_{2\varepsilon i+2\varepsilon j}[m^{|\alpha_\infty|}]-\boldsymbol{E}_{2\varepsilon i}[m^{|\alpha_\infty|}]\boldsymbol{E}_{2\varepsilon j}[m^{|\alpha_\infty|}]|$$
$$\leqq 4\hat{P}_{i-j}[\sigma_{\{0\}}<+\infty] \longrightarrow 0 \quad \text{as } |i-j|\to+\infty,$$

we get

$$\lim_{\lambda\to\infty} E[(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(0)^2)^2]=(\|\phi\|^2\boldsymbol{E}_{2\varepsilon 0}[m^{|\alpha_\infty|}])^2. \tag{5.68}$$

In the same way we have

$$\lim_{\lambda\to\infty} E[(\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2 x_i(0))^2]=\|\phi\|^4 m^2. \tag{5.69}$$

Hence (ii) follows from (5.68) and (5.69).

Proof of Theorem 5.2′.

In order to show the convergence of finite dimensional distributions we shall adopt Tanaka's method in [19]. Let $m\geqq 1$ and $0\leqq t_1<t_2<\cdots<t_m$ be fixed. We will claim that the distribution of $(N_{t_1}^\lambda, N_{t_2}^\lambda, \cdots, N_{t_m}^\lambda)$ converges as $\lambda\to+\infty$ to the corresponding joint distribution of the Ornstein-Uhlenbeck process defined by the stochastic integral equation (5.15)′. We assume that t_i is of the form $t_k^L=k2^{-L}$ ($k=0, 1, 2, \cdots$, $L=1, 2, \cdots$). Once this case is proved the general case can be easily driven by noticing Lemma 5.8 (iii).

Define a $\mathcal{S}'(R^r)$-valued continuous process $N_t^{\lambda, L}$ by

$$N_t^{\lambda, L}=2^L(t_{k+1}^L-t)N_{t_k^L}^\lambda+2^L(t-t_k^L)N_{t_{k+1}^L}^\lambda \quad \text{for } t_k^L\leqq t\leqq t_{k+1}^L. \tag{5.70}$$

Then it follows from Lemma 5.8 (i) that

$$E[\sup_{0\leq t\leq T}(N_t^{\lambda, L}(\phi))^2]\leqq C_T^L\|\phi\|_{(p+r)}^2, \tag{5.71}$$

and

$$E[(N_t^{\lambda, L}(\phi)-N_s^{\lambda, L}(\phi))^2]\leqq C_T^L\|\phi\|_{(p+r)}^2|t-s|^2 \quad \text{for } 0<t-s<T, \tag{5.72}$$

where $C_T^L>0$ is a constant depending on L and T. So, denoting by P_L^λ the probability distribution on $C([0, \infty), \mathcal{S}'(R^r))$ induced by $N_t^{\lambda, L}$, it follows from Lemma 5.1 that the family $\{P_L^\lambda\}_{\lambda\geq 1}$ is tight for each L.

Now, let $\{\lambda_n\}$ be any sequence, tending to $+\infty$, along which the distribution of $(N_{t_1}^{\lambda_n}, N_{t_2}^{\lambda_n}, \cdots, N_{t_m}^{\lambda_n})$ converges to some distribution μ on $\overset{m}{\otimes}\mathcal{S}'(R^r)$. We will prove that μ agrees with the corresponding distribution of N_t defined by (5.15)′. By the diagonal method we can choose a subsequence $\{\lambda_n'\}$ so that for each L $P_L^{\lambda_n'}$ converges to some probability distribution P_L^∞ on $C([0, \infty), S'(R^r))$. It follows from Lemma 5.8 (iii) that if $0<t-s<T$

$$E^{P_L^\infty}[(\eta_t(\phi)-\eta_r(\phi))^2]\leqq C_T(\|\phi\|^2+\ll L\phi\gg)|t-s|. \tag{5.73}$$

Denote by $\mathcal{W}$ the space of all $\mathcal{S}'(R^r)$-valued functions defined on $[0, \infty)$ and denote by $\mathcal{B}(\mathcal{W})$ the usual σ-field on $\mathcal{W}$. Let us define the projection $\Pi_L: \mathcal{W}\to C([0, \infty), \mathcal{S}'(R^r))$ by

$$(\Pi_L\xi)_t=2^L(t_{k+1}^L-t)\xi_{t_k^L}+2^L(t-t\ )\xi_{t_{k+1}^L} \quad \text{for } t_k^L\leqq t\leqq t_{k+1}^L. \tag{5.74}$$

Then, making use of (5.73) and the Kolmogorov extension theorem, we can easily see that the family $\{P_L\}_{L=1,2,\dots}$ determines a unique probability measure P^∞ on $(\mathcal{W}, \mathcal{B}(\mathcal{W}))$ such that

(5.75) $$\Pi_L P^\infty = P_L^\infty \qquad \text{for each } L \geqq 1,$$

and

(5.76) $$E^{P^\infty}[(\xi_t(\phi)-\xi_r(\phi))^2] \leqq C_T(\|\phi\|^2 + \ll L\phi \gg)|t-s| \qquad (0<t-s<T).$$

Furthermore, by Lemma 5.8 (ii) we have

(5.77) $$E^{P^\infty}[(\xi_0(\phi))^2] \leqq C_2 \ll \phi \gg.$$

Also, by reducing to a finite dimensional case, we can show that the process (ξ_t, P^∞) has a progressively measurable and separable modification, which we denote by $\{\bar{\xi}_t\}$. Here notice that $(\bar{\xi}_t, P^\infty)$ is a $\mathcal{S}'(R^r)$-valued stationary process.

Next, we claim that for any $\phi \in \mathcal{S}(R^r)$ and any $f \in C_0^2(R^1)$

(5.78) $$f(\bar{\xi}_t(\phi)) - \int_0^t f'(\bar{\xi}_s(\phi))\bar{\xi}_s(L\phi)ds - \rho\|\phi\|^2 \int_0^t f''(\bar{\xi}_s(\phi))ds$$

is a P^∞-martingale. Denote

$$R_t^{\lambda,L} = f(N_t^{\lambda,L}(\phi)) - \int_0^t N_s^{\lambda,L}(L\phi) f'(N_s^{\lambda,L}(\phi))ds - \rho\|\phi\|^2 \int_0^t f''(N_s^{\lambda,L}(\phi))ds$$

$$- f(N_t^\lambda(\phi)) - \int_0^t N_s^\lambda(L\phi) f'(N_s^\lambda(\phi))ds - \rho\|\phi\|^2 \int_0^t f''(N_s^\lambda(\phi))ds.$$

It follows from Lemma 5.8 and Lemma 5.9 that

(5.79) $$\lim_{L\to\infty} \lim_{\lambda\to\infty} \sup_{0\leq t\leq T} E[|R_t^{\lambda,L}|] = 0 \qquad \text{for any } T>0.$$

For any $\phi, \phi_1, \cdots, \phi_k \in \mathcal{S}(R^r)$, any $g \in C_0(R^k)$ and $0 \leqq s_1 < \cdots < s_k \leqq s < t$

$$E^{P^\infty}\Big[(f(\bar{\xi}_t(\phi)) - f(\bar{\xi}_s(\phi)) - \int_s^t f'(\bar{\xi}_u(\phi))\bar{\xi}_u(L\phi)du$$

$$- \rho\|\phi\|^2 \int_0^t f''(\bar{\xi}_u(\phi))du) g(\bar{\xi}_{s_1}(\phi_1), \cdots, \bar{\xi}_{s_k}(\phi_k))\Big]$$

$$= \lim_{L\to\infty} \lim_{n'\to\infty} E\Big[(f(N_t^{\lambda_{n'},L}(\phi)) - f(N_s^{\lambda_{n'},L}(\phi)) - \int_s^t f'(N_u^{\lambda_{n'},L}(\phi)) N_u^{\lambda_{n'},L}(L\phi)du$$

$$- \rho\|\phi\|^2 \int_s^t f''(N_u^{\lambda_{n'},L}(\phi))du) g(N_{s_1}^{\lambda_{n'},L}(\phi_1), \cdots, N_{s_k}^{\lambda_{n'},L}(\phi_k))\Big]$$

$$= \lim_{L\to\infty} \lim_{n'\to\infty} E\Big[(f(N_t^{\lambda_{n'}}(\phi)) - f(N_s^{\lambda_{n'}}(\phi)) - \int_s^t f'(N_u^{\lambda_{n'}}(\phi)) N_u^{\lambda_{n'}}(L\phi)du$$

$$- \int_s^t \lambda^{-r} \sum_{i\in Z^r} \phi_\lambda(i)^2 x_i(\lambda^2 u)(1-x_i(\lambda^2 u)) f''(N_u^{\lambda_{n'}}(\phi))du$$

$$+ R_t^{\lambda_{n'},L} - R_s^{\lambda_{n'},L}) g(N_{s_1}^{\lambda_{n'}}(\phi_1), \cdots, N_{s_k}^{\lambda_{n'}}(\phi_k))\Big] = 0 \quad \text{(by (5.79) and Lemma 5.9).}$$

Thus, we obtain (5.78). It is known that the martingale problem (5.78) is uniquely solvable if the distribution of the initial condition $\bar{\xi}_0$ is uniquely determined, (cf. [4] Theorem 1.4).

L generates a unique strongly continuous contraction semi-group on $C_\infty(R^r)$, which we denote by $\{S_t\}_{t\geq 0}$. Also, S_t is self-adjoint and contractive on $L^2(R^r)$, and satisfies

(5.80) $$\int_0^\infty S_t\phi(x)dt=G\phi(x)=\int_{R^r} G(y)\phi(x-y)dy$$

where $G(x)$ is defined by (5.17). It is not hard to see that $S_t\phi\in\mathcal{S}(R^r)$ holds for any $\phi\in\mathcal{S}(R^r)$, and denoting $\phi_t=S_t\phi$, for any $f\in C_0^2(R^1)$

(5.81) $$f(\bar{\xi}_t(\phi_{T-t}))-\int_0^t\|\phi_{T-u}\|^2 f''(\bar{\xi}_u(\phi_{T-u}))du$$

is a P^∞-martingale for $0\leqq t\leqq T$. Taking $f(x)=e^{ix}$, we have

$$E^{P^\infty}[\exp(i\bar{\xi}_t(\phi_T))]=E^{P^\infty}[\exp(i\bar{\xi}_0(\phi_T))]-\rho\int_0^t\|\phi_{T-u}\|^2E^{P^\infty}[\exp(i\bar{\xi}_u(\phi_{T-u}))]du \quad (0\leqq t\leqq T),$$

and this implies

(5.82) $$E^{P^\infty}[\exp(i\bar{\xi}_t(\phi_{T-t})]=E^{P^\infty}[\exp(i\bar{\xi}_0(\phi_T)]\exp\left(-\rho\int_0^t\|\phi_{T-u}\|^2du\right).$$

Setting $T=t$, and taking account of the stationarity of $\bar{\xi}_t$, we see

(5.83) $$E^{P^\infty}[\exp(i\bar{\xi}_0(\phi))]=E^{P^\infty}[\exp(i\bar{\xi}_0(\phi_t))]\exp\left(-\rho\int_0^t\|\phi_u\|^2du\right)$$

for any $t\geqq 0$.

On the other hand since A is positive definite it follows from (5.77) and (5.17)

(5.84) $$E^{P^\infty}[(\bar{\xi}_0(\phi))^2]\leqq C_2\ll\phi\gg\leqq C_3(G|\phi|,\ |\phi|)_{L^2(R^r)}.$$

Accordingly, we obthin

$$\lim_{t\to\infty}E^{P^\infty}[(\bar{\xi}_0(\phi_t))^2]\leqq C_3\lim_{t\to\infty}(GS_t|\phi|,\ S_t|\phi|)_{L^2(R^r)}$$

$$=C_3\lim_{t\to\infty}\left(\int_{2t}^\infty S_u|\phi|du,\ |\phi|\right)=0.$$

Hence, letting $t\to+\infty$ in (5.83), we have

(5.85) $$E^{P^\infty}[\exp(i\bar{\xi}_0(\phi))]=\exp\left(-\rho\int_0^\infty\|S_u\phi\|^2du\right)=\exp\left(-\frac{\rho}{2}(G\phi,\ \phi)_{L^2(R^r)}\right).$$

Thus the distribution of $\bar{\xi}_0$ is uniquely determined. Therefore the distribution of $(\bar{\xi}_t, P^\infty)$ coincides with that of the stationary Ornstein-Uhlenbeck process defined by (5.15)′. This completes the proof of Theorem 5.2′.

8) $C_\infty(R^r)$ denotes the Banach space of all continuous functions defined on R^r vanishing at ∞ with the uniform norm.

§ 6. Scaling limit (II)

In the preceding section we studied the limiting process by scaling both in time and space. In this section we shall discuss another type of scaling limit, that is, only in space.

Let $X=[0, 1]^{Z^r}$. Let us consider the following stochastic differential equation,

$$dx_i(t)=a(x_i(t))dB_i(t)+\sum_{i\in Z^r}q_{ji}x_j(t)dt \qquad (i\in Z^r) \tag{6.1}$$

$$\boldsymbol{x}(0)=\{x_i(0)\}\in X,$$

where $\{B_i(t)\}_{i\in Z^r}$ is an independent system of one-dimensional standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ and $\boldsymbol{x}(0)$ is $\mathcal{F}_0$-measutable.

Furthermore we assume

(6.2) *$a(x)$ is a 1/2 Hölder continuous function defined on $[0, 1]$ and satisfies $a(0)=a(1)=0$,*

and

(6.2) $q_{ji}=q_{j-i,0}\ (=q_{j-i})$ *for any j and $i\in Z^r$, $q_i\geqq 0$ $(i\neq 0)$ and* $\sum_{i\in Z^r}q_i=0$.

Then it is known that (6.1) has a unique X-valued strong solution and $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}\,;\boldsymbol{x}(t))$ is a diffusion process on X, (cf. [16]). Denoting $M_t^\lambda(\phi)=\lambda^{-r/2}\sum_{i\in Z^r}\phi_\lambda(i)(x_i(t)-E[x_i(t)])$ for each $\phi\in\mathcal{S}(R^r)$, M_t is a $\mathcal{S}'(R^r)$-valued continuous process. Then we obtain

Theorem 6.1. *Let $r\geqq 1$. Suppose that $\{x_i(0)\}_{i\in Z^r}$ are independent and identically distributed. Then M_t^λ converges as $\lambda\to+\infty$ to a $\mathcal{S}'(R^r)$-valued Gaussian process M_t, which is defined below, in the sense of probability distributions on $C([0, \infty), \mathcal{S}'(R^r))$.*

$$M_t(\phi)=M_0(\phi)+\int_0^t\sqrt{g(s)}dW_s(\phi) \qquad \text{for any } \phi\in\mathcal{S}(R^r), \tag{6.4}$$

where W_t is a $\mathcal{S}'(R^r)$-valued standard Wiener process, M_0 is a $\mathcal{S}'(R^r)$-valued Gaussian random variable independent of W_t satisfying

$$E[e^{iM_0(\phi)}]=\exp\left(-\frac{v}{2}\|\phi\|^2\right) \qquad \text{where } m=E[x_0(0)] \tag{6.5}$$

and $v=E[(x_0(0)-m)^2]$, and

$$g(t)=E[a(x_0(t))^2]. \tag{6.6}$$

Outline of the proof.

1°. Denoting by P^λ the probability distribution on $C([0, \infty), \mathcal{S}'(R^r))$ induced by M_t^λ, we can show by the same argument as Theorem 5.1′ that the family $\{P^\lambda\}_{\lambda\geqq 1}$ is tight.

2°. It is easy to see $\lim_{\lambda\to\infty}E[e^iM_0^\lambda(\phi)]=\exp(-v\|\phi\|^2/2)$.

3°. For any $t>0$ the distribution of $\{x_i(t)\}_{i\in Z^r}$ is Z^r-shift invariant and mixing with respect to Z^r-shift.
In fact, for any finite subset V of Z^r, denote by $\{x_i^V(t)\}$ the solution of the following stochastic differential equation.

(6.7) $$x_i(t)=0 \quad \text{for } i\notin V\,,$$
$$x_i(t)=x_i(0)+\int_0^t a(x_i(s))dB_i(s)+\int_0^t \sum_{j\in Z^r} q_{ji}x_j(s)ds \qquad \text{for } i\in V\,.$$

Then it is known that

(6.8) $$\lim_{V\nearrow Z^r} E[|x_i^V(t)-x_i(t)|]=0 \quad \text{for any } i\in Z^r \text{ and } t>0 \text{ cf. [16])}.$$

Notice that $\{x_i(t)-x_i^V(t)\}_{i\in V}$ and $\{x_j(t)-x_j^{V+k}(t)\}_{j\in V+k}$ have the same distribution and that $\{x_i^V(t)\}_{i\in V}$ and $\{x_j^{V+k}(t)\}_{j\in V+k}$ are independent if $V\cap V+k=\phi$. So, using (6.8) we can show the mixing property at any fixed $t>0$.

4°. For any $\phi\in\mathcal{S}(R^r)$ and $f\in C_0^2(R^1)$

$$f(M_t^\lambda(\phi))-\int_0^t f'(M_s^\lambda(\phi))\sum_{i\in Z^r}\lambda^{-r/2}\sum_{j\in Z^r}q_{ij}\phi_\lambda(j)x_i(s)ds$$
$$-\frac{1}{2}\int_0^t f''(M_s^\lambda(\phi))\sum_{i\in Z^r}\lambda^{-r}\phi_\lambda(i)^2a(x_i(s))^2ds$$

is a martingale.

5°. It follows from 3° that

(6.9) $$\lim_{\lambda\to\infty} E[(\sum_{i\in Z^r}\lambda^{-r}\phi_\lambda(i)^2a(x_i(t))^2-g(t)\|\phi\|^2)^2]=0$$

for any $t>0$.

6°. By making use of the Poisson formula on Fourier transform, it is easy to see

(6.10) $$\lim_{\lambda\to\infty}\lambda^{-r}\sum_{i\in Z^r}(q*\phi_\lambda(i))^2=0\,.$$

Notice that Lemma 5.4 also is true in the present case. So, we see

(6.11) $$\lim_{\lambda\to\infty} E\left[\int_0^t f'(M_s^\lambda(\phi))\sum_{i\in Z^r}(\lambda^{-r/2}\sum_{j\in Z^r}q_{ij}\phi_\lambda(j))x_i(s)ds)^2\right]$$
$$\leqq \text{const.}\lim_{\lambda\to\infty} t\lambda^{-r}\int_0^t E[(\sum_{i\in Z^r}q*\phi_\lambda(i)(x_i(s)-m))^2]ds$$
$$\leqq \text{const.}\lim_{\lambda\to\infty}\lambda^{-r}\sum_{i\in Z^r}(q*\phi_\lambda(i))^2=0\,.$$

7°. For any limiting point P^∞ of $\{P^\lambda\}$ as $\lambda\to+\infty$,

(6.12) $$f(\eta_t(\phi))-\frac{1}{2}\|\phi\|^2\int_0^t g(s)f''(\eta_s(\phi))ds$$

7) $q*\phi(i)=\sum_{j\in Z^r}q_{j-i}\phi(j)$.

is a P^{∞}-martingale for any $\phi\in\mathcal{S}(R^r)$ and $f\in C_0^2(R^1)$. Also, it follows from 2° that (η_0, P^{∞}) has the same distribution as M_0 of (6.5). Therefore we obtain the conclusion since the martingale problem (6.12) has a unique solution.

Corollary 6.1. *Let $V_n=[-n, n]\times\cdots\times[-n, n]\subset Z^r$ and define $z_n(t)=(2n+1)^{-r/2}\sum_{i\in V_n}(x_i(t)-E[x_i(t)])$. Then under the same assumption of Theorem 6.1, $z_n(t)$ converges as $n\to+\infty$ to a real Gaussian process $z(t)$, which is defined below, in the sense of probability distributions on $C([0, \infty), R^1)$.*

$$z(t)=z(0)+\int_0^t\sqrt{g(s)}\,dB(s)\,, \tag{6.13}$$

where $B(t)$ is a one-demensional standard Brownian motion, and $z(0)$ is a Gaussian random variable independent of $B(t)$ with the mean 0 and the variance v.

Theorem 6.1 asserts that the scaling limit process of the above type does not involve the migration rate $\{q_{ji}\}$ explicitly. But it should be noted that $g(t)$ depends on $\{q_{ji}\}$.

On the other hand when $\{q_{ji}\}$ is replaced by $\{\lambda^2 q_{ji}\}$ the limiting process also is an $\mathcal{S}'(R^r)$-valued Ornstein-Uhlenbeck process, of which drift term is determined by $\{q_{ji}\}$.

Denote by $(\Omega, \mathcal{F}, P; \boldsymbol{x}^{\lambda}(t)=\{x_i^{\lambda}(t)\})$ the diffusion process defined by the stochastic differential equation,

$$\begin{aligned} dx_i^{\lambda}(t)&=a(x_i^{\lambda}(t))dB_i(t)+\lambda^2\sum_{j\in Z^r}q_{ji}x_j^{\lambda}(t)dt \\ \boldsymbol{x}^{\lambda}(0)&=\{x_i(0)\}_{i\in Z^r}\in X \end{aligned} \tag{6.13}$$

where $\{B_i(t)\}_{i\in Z^r}$ is an independent system of one-dimensional standard Brownian motions on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ and $x(0)$ is $\mathcal{F}_0$-measurable.

We assume the condition [C] of §5 and (6.2). Denoting

$$K_t^{\lambda}(\phi)=\lambda^{-r/2}\sum_{i\in Z^r}\phi_{\lambda}(i)(x_i^{\lambda}(t)-E[x_i^{\lambda}(t)]) \qquad \text{for each } \phi\in\mathcal{S}(R^r)\,,$$

K_t^{λ} is a $\mathcal{S}'(R^r)$-valued continuous process for each $\lambda>0$. Then we obtain the following.

Theorem 6.2. *Let $r\geqq 1$. Suppose that $\{x_i(0)\}_{i\in Z^r}$ is independent and identically distributed. Then K_t^{λ} converges as $\lambda\to+\infty$ to a $\mathcal{S}'(R^r)$-valued Ornstein-Uhlenbeck process K_t, which is defined below, in the sense of probability distributions on $C([0, \infty), \mathcal{S}'(R^r))$.*

$$K_t(\phi)=K_0(\phi)+a(m)W_t(\phi)+\int_0^t K_s(L\phi)ds \qquad \textit{for any } \phi\in\mathcal{S}(R^r)\,, \tag{6.14}$$

where W_t is a $\mathcal{S}'(R^r)$-valued standard Wiener process, and K_0 is a $\mathcal{S}'(R^r)$-valued Gaussian random variable independent of W_t satisfying

$$E[e^{iK_0(\phi)}]=\exp\left(-\frac{v}{2}\|\phi\|^2\right) \qquad \textit{with } v=E[(x_0(0)-m)^2] \tag{6.15}$$

and $m=E[x_0(0)]$, *and* L *is of* (5.13).

Outline of the proof.

1°. Denoting by P^λ the probability distribution on $C([0,\infty), \mathcal{S}'(R^r))$ induced by K_t^λ, we can show by the same argument as the proof of Theorem 5.1′ that the family $\{P^\lambda\}_{\lambda\geq 1}$ is tight.

2°. $$\lim_{\lambda\to\infty} E[(x_i^\lambda(t)-m)^2]=0 \qquad \text{for any } i\in Z^r \text{ and } t>0\,.$$

In fact, denoting $h^\lambda(t;i,j)=E[(x_i^\lambda(t)-m)(x_j^\lambda(t)-m)]$, it follows by using, Ito's formula that

$$\frac{d}{dt}h^\lambda(t;i,j)=\lambda^2\sum_{m\in Z^r}q_{mi}h^\lambda(t;m,j)+\lambda^2\sum_{m\in Z^r}q_{mj}h^\lambda(t;i,m)+\delta_{ij}E[a(x_0(t))^2]\,.$$

Noting the independence of $\{x_i(0)\}_{i\in Z^r}$, we have

$$h^\lambda(t;i,i)=\sum_{j\in Z^r}P_{\lambda^2 t}(i,j)^2E[(x_0(0)-m)^2] \tag{6.16}$$

$$+\int_0^t\sum_{j\in Z^r}P_{\lambda^2 t}(i,j)^2E[a(x_0^\lambda(t-s))^2]ds\,.$$

Here notice that $\lim_{t\to\infty}\sum_{j\in Z^r}P_t(i,j)^2=0$ for all $i\in Z^r$, because $P_tP_t^*$ is a spatially homogeneuos transition probability on Z^r. Hence we obtain $\lim_{\lambda\to\infty}h^\lambda(t;i,i)=\lim_{\lambda\to\infty}E[(x_i^\lambda(t)-m)^2]=0$ for any $t>0$.

3°. For any $f\in C_0^2(R^1)$

$$f(K_t^\lambda(\phi))-\int_0^t\lambda^{-r/2}\sum_{i\in Z^r}Q_\lambda\phi(i)(x_i^\lambda(s)-m)f'(K_s^\lambda(\phi))ds$$

$$-\frac{1}{2}\int_0^t\lambda^{-r}\sum_{i\in Z^r}\phi_\lambda(i)^2a(x_i^\lambda(s))^2f''(K_s^\lambda(\phi))ds$$

is a martihgale.

4°. It follows from 2° that

$$\lim_{\lambda\to\infty}E\Big[\Big(\int_0^t\sum_{i\in Z^r}\lambda^{-r}\phi_\lambda(i)^2a(x_i^\lambda(s))^2f''(K_s^\lambda(\phi))ds \tag{6.17}$$

$$-a(m)^2\|\phi\|^2\int_0^t f''(K_s^\lambda(\phi))ds\Big)^2\Big]=0\,.$$

5°. $$E\Big[\Big(\int_0^t\lambda^{-r/2}\sum_{i\in Z^r}Q_\lambda\phi(i)(x_i^\lambda(s)-m)f'(K_s^\lambda(\phi))ds-\int_0^t K_s^\lambda(L\phi)f'(K_s^\lambda(\phi))ds\Big)^2\Big]$$

$$\leqq t\|f'\|_\infty^2\int_0^t\lambda^{-r}\sum_{i\in Z^r}\sum_{j\in Z^r}(Q_\lambda\phi(i)-(L\phi)_\lambda(i))(Q_\lambda\phi(j)-(L\phi)_\lambda(j))h^\lambda(s;i,j)ds$$

$$\leqq t\|f'\|_\infty^2\lambda^{-r}\sum_{i\in Z^r}(Q_\lambda\phi(i)-(L\phi)_\lambda(i))^2\int_0^t\sup_{i\in Z^r}\sum_{j\in Z^r}|h^\lambda(s;i,j)|ds$$

$$\longrightarrow 0 \qquad \text{as } \lambda\to+\infty \text{ by Lemma 5.3.}$$

6°. For any limiting point of P^{∞} of $\{P^{\lambda}\}$ as $\lambda\to+\infty$,

$$f(\eta_t(\phi))-\int_0^t \eta_s(L\phi)f'(\eta_s(\phi))ds-\frac{1}{2}a(m)^2\|\phi\|^2\int_0^t f''(\eta_s(\phi))ds$$

is a P^{∞}-martingale for any $\phi\in\mathcal{S}(R^r)$ and $f\in C_0^2(R^1)$. Also, it follows that (η_0, P^{∞}) has the same distribution as M_0 of (6.5). Hence we complete the proof of Theorem 6.2 since the martingale problem of 6° has a unique solution.

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