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A uniqueness theorem for functions holomorphic in a half-plane

By

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1. Introduction

In [1, p. 75], R. P. Boas states and proves the following uniqueness theorem.

Theorem 1. *If $f(z)$ is an entire function of exponential type and if for some θ, $|f(re^{i\theta})| \leq e^{-A(r)r}$, where $\lim_{r\to\infty} A(r)=\infty$, then $f(z)=0$ identically.*

This theorem shows that if the function $f(z)$ tends to zero sufficiently rapidly along some ray $\arg z=\theta$, then it must be identically zero. Instead of a ray, in [4, Theorem 1], we have extended the same result to an arbitrary curve having a limit point at infinity. Moreover, the same type of uniqueness theorems can also be considered in a half-plane instead of the entire plane, see [3, 6]. In this connection, instead of a curve, we can also regard only a sequence of Jordan arcs tending to the point at infinity, see [5]. The shape of Jordan arcs can behave in different ways. The one considered in [5] looks like arc, and the other which we are going to discuss looks like radial. In this case, our results are somewhat equivalent to those of V. I. Gavrilov [2] and D. C. Rung [8] in the unit disk.

To introduce our results, let us begin with the following two definitions.

Definition 1. Let $H=\{z: \operatorname{Re} z>0\}$ be the right halfplane and let $\{\gamma_n\}$ be a sequence of disjoint Jordan arcs in H. Denote α_n the angle subtended by γ_n at the origin and set

$$l_n=\min_{z\in\gamma_n}|z|,\ L_n=\max_{z\in\gamma_n}|z|,\ \theta_n=\min_{z\in\gamma_n}\arg z.$$

We call $\{\gamma_n\}$ a radial-like sequence if it tends to infinity nontangentially and satisfies the following two conditions

(i) $$\lim_{n\to\infty} l_n=\lim_{n\to\infty} L_n=\infty, \quad \text{and}$$

(ii) $$0<\varliminf_{n\to\infty} l_n/L_n=\varlimsup_{n\to\infty} l_n/L_n<1.$$

Definition 2. Let $\{\gamma_n\}$ be a radial-like sequence and let α_0 be a fixed angle satisfying $0 \leq \alpha_n < \alpha_0 \leq \pi$. Then there is a δ with $0 < \delta < \alpha_0$ such that the angle $\beta_n = \theta_n - \delta$ satisfies

$$-\pi/2 \leq \beta_n < \theta_n < \theta_n + \alpha_n < \beta_n + \alpha_0 \leq \pi/2.$$

We define

$$F_n(\alpha_0) = \{z : 0 < |z| < L_n, \ \beta_n < \arg z < \beta_n + \alpha_0\}$$

and

$$M(f, F) = \max\,(\sup_{z \in F} \log |f(z)|, 1), \text{ where } F = F_n(\alpha_0).$$

With the help of the above two definitions, we are now able to state the following main result.

Theorem 2. *Let $f(z)$ be a function holomorphic in $H \cup \{0\}$ and let $\{\gamma_n\}$ be a radial-like sequence in H. If $\{A_n\}$ is a sequence of positive numbers tending to infinity and satisfying*

$$\overline{\lim_{n \to \infty}}\, M(f, F_n(\alpha_0))/A_n < \infty, \quad \text{and}$$

$$|f(z)| \leq e^{-A_n |z|^{\pi/\alpha_0}}, \qquad \text{for} \quad z \in \gamma_n, \ n = 1, 2, \ldots,$$

then $f(z) = 0$ identically.

2. Harmonic measure

To prove Theorem 2, we shall need the following estimate of harmonic measure, due to R. Nevanlinna [7, p. 103].

Lemma 1. *Let D be the unit disk and let Γ_0 be a simple arc starting from the origin and tending to the boundary of D. Let $\omega(z, \Gamma_0)$ denote the harmonic measure at the point z of the arc Γ_0 relative to the domain $D - \Gamma_0$, then the lower bound of $\omega(z, \Gamma_0)$ satisfies*

$$\omega(z, \Gamma_0) \geq \frac{2}{\pi} \sin^{-1} \frac{1 - |z|}{1 + |z|}$$

$$\geq \frac{1}{\pi}(1 - |z|), \textit{ for every } z \in D - \Gamma_0.$$

Now, if Γ does not start from the origin, then we can consider the following conformal mapping:

$$z(w) = \frac{w - ae^{i\alpha}}{1 - ae^{-i\alpha} w}, \text{ where } a = \min_{w \in \Gamma} |w|, \ ae^{i\alpha} \in \Gamma.$$

It is obvious that

$$\frac{1}{\pi}(1 - |z|) = \frac{|1 - ae^{-i\alpha} w|^2 - |w - ae^{i\alpha}|^2}{\pi |1 - ae^{-i\alpha} w| (|1 - ae^{-i\alpha} w| + |w - ae^{i\alpha}|)}$$

$$\geq \frac{1}{8\pi}(1-|w|^2)(1-a^2)\,.$$

By the conformal invariance of harmonic measure, we thus have

$$\omega(w,\,\Gamma)=\omega(z,\,\Gamma_0),\ \text{where}\ \Gamma_0=z(\Gamma),$$

$$\geq \frac{1}{8\pi}(1-|w|^2)(1-a^2)\,.$$

This yields the following result.

Lemma 2. [8, Rung]. *Let Γ be an arc in D tending to the boundary of D. Let $\omega(w,\,\Gamma)$ be the harmonic measure at w of Γ relative to $D-\Gamma$. Then for each $w\in D-\Gamma$, we have*

$$\omega(w,\,\Gamma)\geq \frac{1}{8\pi}(1-|w|^2)(1-a^2),\ \textit{where}\ a=\min_{w\in\Gamma}|w|\,.$$

With the help of Lemma 2, we are able to prove the following lemma which is the is the only one we shall need in the proof of our main theorem. We remark that in Definitions 1 and 2, by choosing a subarc of γ_n, we may assume, for the proof of Theorem 2, that the complement $F_n(\alpha_0)-\gamma_n$ is connected.

Lemma 3. *Let $F_n(\alpha_0)$, γ_n, l_n and L_n be defined by Definitions 1 and 2. Let $\omega(re^{i\phi},\,\gamma_n)$ be the harmonic measure at $re^{i\phi}$ of γ_n relative to $F_n(\alpha_0)-\gamma_n$. Then for each $re^{i\phi}\in F_n(\alpha_0)-\gamma_n$, we have*

$$\omega(re^{i\phi},\,\gamma_n)\geq \frac{1}{8\pi}\left(\frac{r}{L_n}\right)^{\alpha}\left(\frac{l_n}{L_n}\right)^{\alpha}\left[1-\left(\frac{r}{L_n}\right)^{2\alpha}\right]\cdot\left[1-\left(\frac{l_n}{L_n}\right)^{2\alpha}\right]$$
$$\cdot\sin\alpha(\phi-\beta_n)\cdot\sin\alpha\delta,$$

where $\alpha=\pi/\alpha_0$, $\beta_n=\theta_n-\delta$, $0<\alpha(\phi-\beta_n)<\pi$, and $0<\alpha\delta<\pi$.

Proof. For simplicity, we write $F(\alpha_0)$, γ, l, L and θ instead of $F_n(\alpha_0)$, γ_n, l_n, L_n and θ_n. We map $F(\alpha_0)$ conformally onto the unit disk D_w by the following mapping

$$w(z)=([\quad]^2+i)/([\quad]^2-i)\,, \tag{1}$$

where
$$[\quad]=\left(i+\left(e^{-i\beta}\frac{z}{L}\right)^{\alpha}\right)\Big/\left(i-\left(e^{-i\beta}\frac{z}{L}\right)^{\alpha}\right),$$

$\alpha=\pi/\alpha_0$ and $\beta=\beta_n+\alpha_0/2$.
Let $|w(le^{i\theta})|=a$ and let Γ be the subarc of the image $w(\gamma)$ which connects the circle $|w|=a$ to the unit circle $|w|=1$. Then by Carleman's principle, for each $w\in D_w-w(\gamma)$, we have

$$\omega(w,\,w(\gamma))\geq\omega(w,\,\Gamma)\,. \tag{2}$$

According to Lemma 2, we know that

$$(3) \qquad \omega(w, \Gamma) \geq \frac{1}{8\pi}(1-|w|^2)(1-a^2).$$

By virtue of the conformal invariance of harmonic measure, inequalities (1), (2) and (3) allow us to write for $z \in F(\alpha_0) - \gamma$,

$$(4) \qquad \begin{aligned} \omega(z, \gamma) &= \omega(w, w(\gamma)) \\ &\geq \omega(w, \Gamma) \\ &\geq \frac{1}{8\pi}(1-|w|^2)(1-a^2) \\ &= \frac{1}{8\pi}(1-|w(z)|^2)(1-|w(le^{i\theta})|^2). \end{aligned}$$

We set $z = re^{i\phi}$, and we want to estimate $1-|w(z)|^2$ and $1-|w(le^{i\theta})|^2$. By (1), we have

$$(5) \qquad \begin{aligned} &1-|w(z)|^2 = \\ &= 1 - \left\{\frac{[\quad]^2+i}{[\quad]^2-i}\right\}\overline{\left\{\frac{[\quad]^2+i}{[\quad]^2-i}\right\}} \\ &= \frac{|[\quad]|^4+1+i\{[\quad]^2-\overline{[\quad]^2}\}-\{|[\quad]|^4+1+i\{\overline{[\quad]^2}-[\quad]^2\}\}}{|[\quad]^2-i|^2} \\ &= \frac{2i\{[\quad]^2-\overline{[\quad]^2}\}}{|[\quad]^2-i|^2} \\ &= \frac{2i\left\{\left[i+\left(e^{-i\beta}\frac{z}{L}\right)^{\alpha}\right]^2\overline{\left[i-\left(e^{-i\beta}\frac{z}{L}\right)^{\alpha}\right]^2}-\left[i-\left(e^{-i\beta}\frac{z}{L}\right)^{\alpha}\right]^2\overline{\left[i+\left(e^{-i\beta}\frac{z}{L}\right)^{\alpha}\right]^2}\right\}}{\left|\left[i+\left(e^{-i\beta}\frac{z}{L}\right)^{\alpha}\right]^2-i\left[i-\left(e^{-i\beta}\frac{z}{L}\right)^{\alpha}\right]^2\right|^2} \end{aligned}$$

Let $\left(e^{-i\beta}\frac{z}{L}\right)^{\alpha} = \rho e^{i\psi}$, then $\rho = \left(\frac{r}{L}\right)^{\alpha} < 1$, $\psi = \alpha(\phi-\beta)$, and

$$(6) \qquad \begin{aligned} &|[i+\rho e^{i\psi}]^2 - i[i-\rho e^{i\psi}]^2|^2 \\ &= 2|1+2\rho e^{i\psi}-\rho^2 e^{i2\psi}|^2 \\ &= 2\{-4\rho^2\cos^2\psi + 4\rho(1-\rho^2)\cos\psi + (1+6\rho^2+\rho^4)\} \\ &= 4\left\{-2\rho^2\left(\cos\psi - \frac{1-\rho^2}{2\rho}\right)^2 + (1+\rho^2)^2\right\} \\ &\leq 4(1+\rho^2)^2 \leq 16. \end{aligned}$$

$$(7) \qquad \begin{aligned} &2i\{[i+\rho e^{i\psi}]^2\overline{[i-\rho e^{i\psi}]^2} - [i-\rho e^{i\psi}]^2\overline{[i+\rho e^{i\psi}]^2}\} \\ &= 2i\{[(-1+\rho^2 e^{i2\psi})+2i\rho e^{i\psi}][(-1+\rho^2 e^{-i2\psi})+2i\rho e^{-i\psi}] \\ &\quad -[(-1+\rho^2 e^{i2\psi})-2i\rho e^{i\psi}][(-1+\rho^2 e^{-i2\psi})-2i\rho e^{-i\psi}]\} \\ &= 8\rho(1-\rho^2)(e^{i\psi}+e^{-i\psi}) \end{aligned}$$

$$= 16\rho(1-\rho^2)\cos\psi$$

$$= 16\left(\frac{r}{L}\right)^{\alpha}\left[1-\left(\frac{r}{L}\right)^{2\alpha}\right]\cos\alpha(\phi-\beta)\,.$$

Equalities (5), (7) and inequality (6) give

$$1-|w(z)|^2 \geq \left(\frac{r}{L}\right)^{\alpha}\left[1-\left(\frac{r}{L}\right)^{2\alpha}\right]\cos\alpha(\phi-\beta)\,. \tag{8}$$

Replacing r and ϕ by l and θ respectively, we get

$$1-|w(le^{i\theta})|^2 \geq \left(\frac{l}{L}\right)^{\alpha}\left[1-\left(\frac{l}{L}\right)^{2\alpha}\right]\cos\alpha(\theta-\beta)\,. \tag{9}$$

Since the angles $\beta=\beta_n+\alpha_0/2$ and $\theta-\beta=\delta-\alpha_0/2$, hence the lemma follows from inequalities (4), (8), and (9).

3. Proof of Theorem 2

According to the hypothesis, there is a subsequence which for convenience will still be denoted by the same sequence $\{n\}$ such that

$$\lim_{n\to\infty} A_n=\infty \tag{10}$$

$$\lim_{n\to\infty}\frac{M(f,\,F_n(\alpha_0))}{A_n}=M<\infty \tag{11}$$

$$\lim_{n\to\infty}\frac{l_n}{L_n}=l,\ 0<l<1. \tag{12}$$

Let γ_n^* be a subarc of γ_n such that γ_n^* meets the circles $|z|=l_n$ and $|z|=L_n$ only at the points $l_ne^{i\theta_n'}$ and $L_ne^{i\theta_n''}$, respectively. Since γ_n^* has the same associated lengths l_n and L_n as those of γ_n, hence by Definition 1, $\{\gamma_n^*\}$ is also a radial-like sequence. Moreover, the function $f(z)$ satisfies the same inequality in the hypothesis for each $z\in\gamma_n^*$, because γ_n^* is only a subarc of γ_n. Clearly $\{\gamma_n^*\}$ contains a subsequence which will be denoted by $\{\gamma_m\}$ such that

$$\lim_{m\to\infty}\theta_m=\theta,\ -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\,. \tag{13}$$

Let $\omega(z,\gamma_m)$ be the harmonic measure at z of γ_m relative to $F_m(\alpha_0)-\gamma_m$. Then by the hypothesis and the two-constants Theorem, we know that for each $z\in F_m(\alpha_0)-\gamma_m$, we have

$$\begin{aligned}\log|f(z)| &\leqq \omega(z,\gamma_m)\,(-A_ml_m^{\alpha})+(1-\omega(z,\gamma_m))M(f,\,F_m(\alpha_0))\\ &\leq -A_m\left\{l_m^{\alpha}\omega(z,\gamma_m)-\frac{M(f,\,F_m(\alpha_0))}{A_m}\right\}\end{aligned} \tag{14}$$

By virtue of Lemma 3, for $z=re^{i\phi}$, inequality (14) becomes the following

$$\log|f(re^{i\phi})| \leq -A_m\left\{\frac{1}{8\pi}r^\alpha\left(\frac{l_m}{L_m}\right)^{2\alpha}\left[1-\left(\frac{r}{L_m}\right)^{2\alpha}\right]\left[1-\left(\frac{l_m}{L_m}\right)^{2\alpha}\right]\right. \tag{15}$$

$$\left.\cdot\sin\alpha(\phi-\beta_m)\sin\alpha\delta-\frac{M(f,F_m(\alpha_0))}{A_m}\right\}.$$

For a given ε witn $0<\varepsilon<l<1$ and $l+\varepsilon<1$, by (11), (12), and (13), there exists a positive integer N such that $m\geq N$ implies

$$M-\varepsilon<\frac{M(f,F_m(\alpha_0))}{A_m}<M+\varepsilon$$

$$0<l-\varepsilon<\frac{l_m}{L_m}<l+\varepsilon<1 \quad \text{and}$$

$$\varepsilon<\sin\alpha(\phi-\beta_m), \text{ for some } \phi\neq\theta-\delta.$$

For sufficiently large m, inequality (15) becomes

$$\log|f(re^{i\phi})| \leq -A_m\left\{k_\varepsilon\cdot r^\alpha\left[1-\left(\frac{r}{L_m}\right)^{2\alpha}\right]-(M+\varepsilon)\right\} \tag{16}$$

where $k_\varepsilon=\frac{1}{8\pi}(l-\varepsilon)^{2\alpha}[1-(l+\varepsilon)^{2\alpha}]\varepsilon\sin\alpha\delta>0$. Let $g_m(r)=k_\varepsilon r^\alpha\left[1-\left(\frac{r}{L_m}\right)^{2\alpha}\right]-(M+\varepsilon)$. Then $g'_m(r)=\alpha k_\varepsilon r^{\alpha-1}\left[1-3\left(\frac{r}{L_m}\right)^{2\alpha}\right]>0$ provided $0<r<L_m/\sqrt{3}$. This shows that the function $g_m(r)$ is strictly increasing for $0<r<L_m/\sqrt{3}$. Moreover, for any fixed r, $g_m(r)$ is also strictly increasing for m, because $1-(r/L_m)^{2\alpha}$ increases to 1. Clearly, we can choose a sufficiently large $K\geq N$ so that

$$g_m(r)\geq g_K(r)\geq g_K(L_K/3)>0, \tag{17}$$

provided $m\geq K$ and $r\in I_K=[L_K/3, L_K/\sqrt{3}]$. It follows from (16) and (17) that for any $m\geq K$ and $r\in I_K$

$$\log|f(re^{i\phi})|\leq -A_m g_m(r)\leq -A_m g_K(L_K/3), \tag{18}$$

where the angle ϕ is chosen to satisfy $\sin\alpha(\phi-\beta_m)>\varepsilon$. Allowing $m\to\infty$ and applying (10) and (18), we thus establish that $\log|f(re^{i\phi})|=-\infty$ or $f(re^{i\phi})=0$ for any point $re^{i\phi}$ with $r\in I_K$. This concludes that $f(z)=0$ identically due to the classical uniqueness theorem.

4. Bounded functions

In this section, we shall develop some uniqueness theorems for functions bounded and holomorphic in H.

Corollary 1. *Let $f(z)$ be a function bounded and holomorphic in H. If $\{\gamma_n\}$ is a radial-like sequence in H for which*

$$|f(z)|\leq e^{-|z|^{1+\varepsilon}} \quad \textit{for each } z\in\gamma_n,$$

$n=1, 2, \dots,$ *and some* $\varepsilon>0$, *then* $f(z)\equiv 0$.

Proof. Let $A_n=l_n^{\varepsilon}$, then the inequality becomes

$$|f(z)|\leq e^{-|z|^{1+\varepsilon}}\leq e^{-l_n^{\varepsilon}|z|}=e^{-A_n|z|}, \qquad \text{for each} \quad z\in\gamma_n,\ n=1, 2, \dots .$$

Clearly we have $\lim_{n\to\infty} A_n=\infty$. In order to use Theorem 2, we choose $\beta_n=-\frac{\pi}{2}$, for each n and $\alpha_0=\pi$. According to Definition 2, we know that

$$F_n(\alpha_0)=F_n(\pi)=\left\{z: 0<|z|<L_n,\ -\frac{\pi}{2}<\arg z<\frac{\pi}{2}\right\}.$$

Thus $M(f, F_n(\pi))\leq M<\infty$, and therefore the result follows from Theorem 2.

Instead of a sequence of arcs, we now consider a curve.

Corollary 2. *If* $f(z)$ *is bounded and holomorphic in* H *and if* $|f(z)|\leq e^{-|z|^{1+\varepsilon}}$ *for each* z *lying on a curve* $\Gamma\subset H$ *which extends to infinity nontangentially and* $\varepsilon>0$, *then* $f(z)\equiv 0$.

Proof. To prove this corollary, we need only construct a radial-like sequence from the given curve Γ. Let $\Gamma_n=\{z: 2^{n-1}\leq|z|\leq 2^n,\ z\in\Gamma\}$ and let γ_n be a connected subset of Γ_n such that $l_n=\min_{z\in\gamma_n}|z|=2^{n-1}$, $L_n=\max_{z\in\gamma_n}|z|=2^n$. Then we have

(i) $\lim_{n\to\infty} l_n=\lim_{n\to\infty} L_n=\infty$ and

(ii) $0<\lim_{n\to\infty}\frac{l_n}{L_n}=\frac{1}{2}<1$.

It follows from Definition 1 that this sequence $\{\gamma_n\}$ is a radial-like sequence. The assertion now follows from Corollary 1.

5. Functions having angular limits.

Finally, we shall prove a uniqueness theorem for functions having angular limits. Our result is equivalent to that of Gavrilov [2] and Rung [8] for functions normal in the unit disk.

Corollary 3. *Let* $f(z)$ *be a function holomorphic in* H *and having angular limits at* 0 *and* ∞. *If* Γ *is a curve in* H *tending to infinity non-tangentially and if for some* $\varepsilon>0$,

$$|f(z)|\leq e^{-|z|^{1+\varepsilon}}, \qquad \textit{for each} \quad z\in\Gamma,$$

then $f(z)\equiv 0$.

Proof. By the hypothesis of Γ, we can choose N such that

$$\Gamma\subset F(\alpha_0)=\{z: 0<|z|<\infty,\ |\arg z|<\alpha_0/2\},\ \alpha_0=\pi/(1+\varepsilon/N).$$

Since $f(z)$ has angular limits at 0 and ∞, we have $M(f, F(\alpha_0))<\infty$.

Now, by the same argument as Corollary 2, we can construct a radial-like sequence $\{\gamma_n\}$ for which

$$|f(z)| \leq e^{-A_n|z|^{\pi/\alpha_0}}, \qquad \text{for each} \quad z \in \gamma_n, \; n=1, 2, \dots,$$

where $A_n = l_n^{(1-1/N)\varepsilon}$. The result now follows from Theorem 2.

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