J. Math. Kyoto Univ. (JMKYAZ)
22-2 (1982) 257-259

The nilpotency of elements of the equivariant stable homotopy groups of spheres

By

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(Communicated by Prof. H. Toda, Jan. 29, 1981)

Let G be a finite group. Put

$$\omega_G^\alpha = \{\Sigma^0, \Sigma^0\}_G^\alpha$$

for $\alpha \in RO(G)$ and

$$\omega_G^* = \sum_{\alpha \in RO(G)} \omega_G^\alpha,$$

which is a graded ring with unity. In this article we will prove

Theorem. *Every torsion element of ω_G^* is nilpotent.*

Notation and elementary results of [4] are used freely.

The author is grateful to Dr. A. Kono for his valuable suggestions.

1. Fixed-point exact sequence

Let G be a finite group, X and Y pointed G-spaces and α an element of the real representation ring $RO(G)$ of G. By $\{X, Y\}_G^\alpha$ we denote the abelian group of the stable G-homotopy classes of pointed G-maps of degree α from X to Y.

Put

$$\tilde{\omega}_G^\alpha(X) = \{X, \Sigma^0\}_G^\alpha$$

for a pointed G-space X and $\alpha \in RO(G)$. Let V and W be effective G-modules, that is $V^G = W^G = \{0\}$. Then in a similar way to [1, 2] the G-cofibration

$$S(W)_+ \xrightarrow{\eta_{W,W+V}} S(W \oplus V)_+ \xrightarrow{\xi_{W+V,V}} S(W \oplus V)/S(W) \approx \Sigma^W S(V)_+$$

induces the following exact sequence

$$\begin{aligned} &\cdots \to \tilde{\omega}_G^{\alpha+V-1}(S(V)_+ \wedge X) \xrightarrow{\xi^*_{W+V,V}} \tilde{\omega}_G^{\alpha+W+V-1}(S(W \oplus V)_+ \wedge X) \\ (1.1)\quad &\xrightarrow{\eta^*_{W,W+V}} \tilde{\omega}_G^{\alpha+W+V-1}(S(W)_+ \wedge X) \xrightarrow{\delta_{V,W}} \tilde{\omega}_G^{\alpha+V}(S(V)_+ \wedge X) \to \cdots. \end{aligned}$$

Let U be another effective G-module, then it is easy to check that

$$\xi^*_{W+V+U,U} = \xi^*_{W+V+U,V+U} \circ \xi^*_{V+U,U} \tag{1.2}$$

(cf. [2], Proposition 2.1, iii)). Thus for fixed $\alpha \in RO(G)$, $\{\tilde{\omega}_G^{\alpha+V-1}(S(V)_+ \wedge X), \xi^*_{W+V,V}\}$ forms a direct system of abelian groups. Then put

$$\tilde{\lambda}_G^\alpha(X) = \operatorname{Colim} \tilde{\omega}_G^{\alpha+V-1}(S(V)_+ \wedge X). \tag{1.3}$$

Let X be a finite dimensional pointed G-complex and consider the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots \to \tilde{\omega}_G^{\alpha-1}(X) & \xrightarrow{\chi_V} & \tilde{\omega}_G^{\alpha+V-1}(X) & \xrightarrow{\beta_V} & \tilde{\omega}_G^{\alpha+V-1}(S(V)_+ \wedge X) & \xrightarrow{\delta_V} & \cdots \\
\| & & \downarrow \chi_W & & \downarrow \xi^*_{W+V,V} & & \\
\cdots \to \tilde{\omega}_G^{\alpha-1}(X) & \xrightarrow{\chi_{W+V}} & \tilde{\omega}_G^{\alpha+W+V-1}(X) & \xrightarrow{\beta_{W+V}} & \tilde{\omega}_G^{\alpha+W+V-1}(S(W\oplus V)_+ \wedge X) & \xrightarrow{\delta_{W+V}} & \cdots.
\end{array}$$

Since the fixed-point homomorphism $\phi_G\colon \tilde{\omega}_G^{\alpha+V-1} \to \tilde{\omega}_G^{\alpha^G-1}(X^G)$ is isomorphic for $|\alpha^H| \geq \dim X^H + 2 - \dim V^H$ for all proper subgroups H of G by [4], Theorem 2.6, passing to the colimit of the above diagram, we get the following exact sequence:

$$\cdots \longrightarrow \tilde{\omega}_G^{\alpha^G-1}(X^G) \longrightarrow \tilde{\lambda}_G^\alpha(X) \xrightarrow{\delta_G} \tilde{\omega}_G^\alpha(X) \xrightarrow{\phi_G} \tilde{\omega}_G^{\alpha^G}(X^G) \longrightarrow \cdots. \tag{1.4}$$

We call this exact sequence the G-fixed-point exact sequence of ω_G^*. This exact sequence is a special case of the exact sequence of T. tom Dieck [3].

2. Proof of Theorem

Put

$$\omega_G^{2*} = \sum_{\alpha \in RO(G)} \omega_G^{2\alpha},$$

which is a commutative ring with unit. For an element ξ of ω_G^{2*} put

$$S(\xi) = \{1, \xi, \xi^2, \ldots, \xi^n, \ldots\}.$$

$S(\xi)$ is the multiplicative subset of ω_G^{2*}. It is sufficient to prove Theorem for an element of ω_G^{2*}. Thus Theorem is equivalent to the following theorem.

Theorem 2.1. *For every torsion element ξ of ω_G^{2*} we have $S(\xi)^{-1}\omega_G^* = 0$.*

We will prove Theorem 2.1 by induction on the order of G. If $G = \{e\}$, G. Nishida [5] proved the theorem. Hence, fix a finite group G and assume that Theorem 2.1 is valid for all proper subgroups of G.

For a pointed G-space X we put

$$\tilde{\omega}_G^*(X) = \sum_{\alpha \in RO(G)} \tilde{\omega}_G^\alpha(X),$$

which is an ω_G^{2*}-module. $f^*\colon \omega_G^*(Y) \to \omega_G^*(X)$, which is induced by a pointed G-map, is an ω_G^{2*}-module homomorphism and so is the suspension isomorphism $\sigma^V\colon \tilde{\omega}_G^*(X)$

$\rightarrow\tilde{\omega}_G^*(X)$. Moreover for a subgroup H of G the forgetful homomorphism ψ_H: $\omega_G^{2*}\rightarrow\omega_H^{2*}$ is a ring homomorphism.

Noting these fact we obtain the following lemma by a parallel argument to [4], Lemma 2.3.

Lemma 2.2. *Let V be an effective* G-*module, then*

$$S(\xi)^{-1}\tilde{\omega}_G^*(S(V)_+)=0$$

for any torsion element ξ of ω_G^{2*}.

Proof of Theorem 2.1. If we put

$$\lambda_G^*=\sum_{\alpha\in RO(G)}\tilde{\lambda}_G(\Sigma^0),$$

then it is an ω_G^{2*}-module by definition. Then by Lemma 2.2 $S(\xi)^{-1}\lambda_G^*=0$ since the localization commutes with the colimit.

Since there is the G-fixed-point exact sequence (1.4), there is the exact sequence

$$\lambda_G^* \xrightarrow{\delta_G} \omega_G^* \xrightarrow{\phi_G} \omega^*.$$

Obviously δ_G is an ω_G^{2*}-module homomorphism. Since ϕ_G: $\omega_G^{2*}\rightarrow\omega^{2*}$ is a ring homomorphism, we may regard ω^* as an ω_G^{2*}-module. Then ϕ_G is an ω_G^{2*}-module homomorphism. Since $S(\xi)^{-1}\lambda_G^*=0$, $S(\xi)^{-1}\omega^*=0$ and the localization preserves exact sequences, $S(\xi)^{-1}\omega_G^*=0$, which completes the proof.

Remark. An element ξ of ω_G^* is a torsion element if and only if $\phi_H(\xi)\in\omega^*$ is a torsion element for all subgroups H of G. This fact is easily showed by a parallel argument to [4].

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