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Simple transcendental extensions of valued fields

Dedicated to A. Seidenberg on his 65th birthday

By

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Let $K_0 \subset K = K_0(x)$ be fields with x transcendental over K_0; let v_0 be a valuation of K_0 and v be an extension of v_0 to K; and let $V_0 \subset V$, $k_0 \subset k$, and $G_0 \subset G$ be the respective valuation rings, residue fields, and value groups.

If k is not algebraic over k_0, then there exists $y \in V$ such that y specializes to a transcendental y^* over k_0 under the canonical homomorphism $V \to k$; if this y should happen to be a generator of K/K_0, then it is easily seen that $k = k_0(y^*)$ and $G = G_0$. Our main theorem asserts that, under the assumption that char $k_0 = 0$, if v_0 is henselian, then the converse holds: if k/k_0 is simple transcendental and $G = G_0$, then there exists a generator of K/K_0 which specializes to a transcendental over k_0. We also prove that "v_0 is henselian" can be replaced by "v_0 is rk 1" and that for arbitrary *finite* rk v_0 one must assume, in addition, that for every valuation ring $W \supset V$ of K, the residue field of W is simple transcendental over the residue field of $W \cap K_0$.

It requires no new considerations to prove this theorem under the a priori weaker hypothesis that k_0 is algebraically closed in k and $\neq k$ and K/K_0 is generically of index 1 (i.e. every *generator* of K/K_0 has value in G_0), and in this form the theorem yields as a corollary the char 0 case of the following conjecture of Nagata:

Ruled Residue Conjecture. k is either algebraic or ruled over k_0.

("Ruled" means that there should be a field k_1 with $k_0 \subset k_1 \subset k$ and k simple transcendental over k_1; in the present setting such a k_1 is necessarily finite algebraic over k_0.) Nagata [7] has proved, without assumption on the characteristic, that this conjecture holds for discrete v_0 and that k is always either algebraic over k_0 or *contained in* a finite algebraic extension of k_0 followed by a simple transcendental extension.

The paper divides into two parts. Part I, consisting of §§ 1–5, is devoted to proving the above theorem for henselian v_0 (3.7) and to deriving the above conjecture in char 0 from it (4.6). In Part II (§§ 6–8) the corresponding theorem for v_0 of finite rk is proved.

The main portion of this work was done (1978–79) while the author was on sabbatical leave from LSU, during which time he enjoyed the hospitality of the University of Wisconsin-Milwaukee.

Notation and terminology.

We fix fields $K_0 < K$ with K a simple extension of K_0, i.e. there exists $x \in K$, $\notin K_0$ such that $K = K_0(x)$. Usually x will be transcendental (abbreviated tr.) over K_0, but we do not a priori assume this. We also fix a valuation v of K and its restriction v_0 to K_0. Moreover, we shall *consistently use x to denote a generator of K/K_0 of value* 0; there always exists such a generator since one of x, $1+x$, or $1+(1/x)$ must have value 0.

The valuation ring, residue field, and value group of v will be denoted V, k, and G respectively; a subscript 0 will indicate the corresponding objects for v_0; and $K^\wedge$, $v^\wedge$ will denote the henselization (cf. [4] or [9]) of K, v. By the *index* of K/K_0 we shall mean $[G: G_0]$; and we shall say that K/K_0 (or v/v_0) is *generically of index* 1 if for every generator z of K/K_0, $v(z) \in G_0$. For example, K/K_0 is generically of index 1 if $[G: G_0] = 1$. This condition will be used in § 3 and will be discussed in § 4.

The notation $(\ )^*$ will be reserved for image under the canonical homomorphism $V \to V/m_V = k$; thus, if $a \in V$, a^* denotes the image of a under $V \to k$. To enlarge on this notation, $K \xrightarrow{v} k$ will signify in our diagrams that k is the residue field of v; and for $a \in K$, $a \xrightarrow{v} a^*$ (read "a specializes to a^* under v") will mean $a \in V$ and a^* is the image of a under $V \to k$. The reference to v will be omitted when the valuation involved is clear. Similarly, if $f(X) \in V[X]$, $f(X)^*$ will denote the image of $f(X)$ under the homomorphism $V[X] \to k[X]$ obtained by specializing coefficients.

In addition, we shall use Z to denote the integers, Q the rationals, C the complex numbers, and X an indeterminate.

Part I: The theorem for henselian v_0, and the Ruled Residue Conjecture.

1. Preliminaries.

As specified above, $K = K_0(x)$, $x \notin K_0$, with x either transcendental or algebraic over K_0 and $v(x) = 0$.

In a few special cases it is easy to describe a generating set for k/k_0. To begin with, note that we always have $k_0(x^*) \subset k$ since $k_0 \subset k$ and $x^* \in k$.

1.1. Inf extensions (See also 4.3).

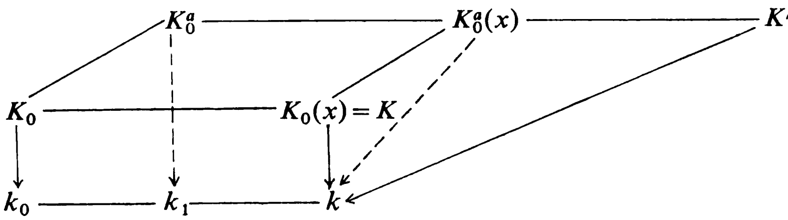
For any $z \in K$, v will be called the inf extension (to $K_0(z)$) of v_0 w.r.t. $v(z)$ if for every $\xi = a_0 + a_1 z + \cdots + a_n z^n$, $a_i \in K_0$, $v(\xi) = \inf\{v_0(a_i) + iv(z) \mid i = 0, \ldots, n\}$. If z is tr. over K_0, then it is easily verified that an extension of v_0 to $K_0(z)$ may be so defined (cf. [2, p. 160, Lemma 1]). We are mainly interested in the inf extension of v_0 w.r.t. $v(x) = 0$, for which the following simple fact is basic: *x^* is tr. over $k_0 \Leftrightarrow x$ is tr. over K_0 and v is the inf extension of v_0 w.r.t.* $v(x) = 0$; *and when this is the case,* *then* $k = k_0(x^*)$ *and* $G = G_0$ (cf. [2, p. 161, Prop. 2]).

1.2. Suppose x is algebraic over K_0. Then K is algebraic over K_0, and therefore also k is algebraic over k_0. Moreover, it is a classical result that $[K: K_0] \geq [k: k_0] \times [G: G_0]$ (cf. [2, p. 138, Lemma 2]). Therefore if $[K_0(x): K_0] = [k_0(x^*): k_0]$, then $k = k_0(x^*)$ and $G = G_0$; or if $[K_0(x): K_0] = [G: G_0]$, then $k = k_0$. (A strong form of the above inequality [2, p. 143, Theorem 1] shows that in these two cases v is the *only* extension of v_0, up to equivalence.) Note also that the inequality implies that x is tr. over K_0 whenever k is not finite algebraic over k_0.

1.3. Suppose k is not algebraic over k_0. Then there exists $\alpha \in k$ such that α is tr. over k_0. Let y be a preimage in V for α. By 1.1, y is tr. over K_0 and the restriction of v to $K_0(y)$ has residue field $k_0(\alpha)$ and value group G_0. Since x is algebraic over $K_0(y)$, by the inequality of 1.2 we have $[G: G_0] < \infty$ and $[k: k_0(\alpha)] < \infty$. In particular, k is then a finitely generated extension of k_0 of tr. degree 1; so if k/k_0 is an algebraic extension followed by a simple tr. extension, then it is automatically a *finite* algebraic extension followed by a simple tr. extension. (If k/k_0 is algebraic, then it can happen that $[k: k_0] = \infty$; see 5.1.)

1.4. The residue field and value group for the henselization $v^\wedge$, $K^\wedge$ of v, K are again k and G (cf. [4, p. 136]). If $\alpha \in k$ is separably algebraic of deg n over k_0 then by Hensel's lemma (cf. [4, p. 118, Cor. 16.6]) there exists a preimage $a \in K^\wedge$ for α such that a is separably algebraic over K_0 of deg n. It follows from 1.2 that G_0 is the value group and $k_0(\alpha)$ the residue field of $v^\wedge$ restricted to $K_0(a)$.

A consequence is that if K_0^g is the separable algebraic closure of K_0 in $K^\wedge$, then the restriction v_0^g of $v^\wedge$ to K_0^g has a residue field k_1 which contains the separable algebraic closure of k_0 in k, and hence which is itself separably algebraically closed in k, and a value group G_1 such that $G_0 \subset G_1 \subset G$, and the restriction of $v^\wedge$ to $K_0^g(x)$ has residue field k and value group G:



Moreover, v_0^g, K_0^g is henselian [4, p. 130, Theorem 17.9]. Thus, *in considering the Ruled Residue Conjecture, we may assume* k_0 *is separably algebraically closed in* k *and* K_0 *is henselian.*

A word of caution is in order, however. In passing from K_0 to K_0^g, the notion of "generator" changes; for if $r \in K_0^g \backslash K_0$, then $x - r$ is a generator of $K_0^g(x)$ over K_0^g but is not a generator of $K_0(x)$ over K_0, since it is not even in $K_0(x)$.

2. Generating pairs.

Throughout § 2 x, y *will be elements of* k *of value* 0, *with* x *tr. over* K_0.

2.1. Definition. x will be called a *generator* for y if $y \in K_0[x]$, or, equivalentl if $y=af(x)$ for some $a \neq 0 \in K_0$ and some primitive $f(X) \in V_0[X]$. ($f(X) \in V_0[X]$ is called *primitive* if some coefficient has value 0.) The pair $x, f(X)$ will be called a *generating pair* for y. The a and $f(X)$ are unique up to unit multiples from V_0; to be precise, if $y=a_1 f_1(x)$ for some $a_1 \neq 0 \in K_0$ and some primitive $f_1(X) \in V_0[X]$, then there exists a unit $u \in V_0$ such that $a=ua_1$ and $f(X)=(1/u)f_1(X)$. Note also that $v(y)=0$ implies $v(f(x))=-v(a) \geq 0$.

2.2. Multiplicity. The generator for y (or the generating pair x, $f(X)$) will be said to have *multiplicity* n (≥ 0) if x^* is a root of multiplicity n for $f(X)^*$, i.e. if $f(X)^*=(X-x^*)^n h(X)$, with $h(X) \in k_0[X]$ and $h(x^*) \neq 0$.

Suppose $r \in V$ is such that $r^*=x^*$. We may write $f(X)=a_0+a_1(X-r)+\cdots+a_n(X-r)^n+\cdots+a_m(X-r)^m$, where the a_i are uniquely determined elements of $V_0[r]$; in fact, $a_i=f^{(i)}(r)$, where $f^{(i)}(r)$ shall denote the i^{th} derivative of $f(X)$ *with the coefficients formally divided by* $i!$, evaluated at r. Then x^* is a root of multiplicity n for $f(X)^* \Leftrightarrow a_0^*=\cdots=a_{n-1}^*=0$ and $a_n^* \neq 0$. For further reference, note also that if $f(X)=b_0+b_1X+\cdots+b_nX^n+\cdots+b_mX^m$, then x^* is a root of multiplicity $\leq n$ for $f(X)^*$ if $v(b_n)=0$ and $v(b_j)>0$ for $j>n$; for then $f^{(n)}(x)=b_n+$(terms of value >0), so $f^{(n)}(x)^*=b_n^* \neq 0$.

2.3. Multiplicity 0. *x is a generator for y of multiplicity* $0 \Leftrightarrow y \in V_0[x]$. For, suppose $x, f(X)$ is a generating pair for y. Since $y=af(x)$, $a \in K_0$, and $v(y)=0$, $v(a)=0 \Leftrightarrow v(f(x))=0 \Leftrightarrow f(x^*)^* \neq 0 \Leftrightarrow x, f(X)$ has multiplicity 0. Thus, if $x, f(X)$ is a generating pair of multiplicity 0, then $a \in V_0$ and hence $y \in V_0[x]$. Conversely, if $y \in V_0[x]$, then there exists $a \in V_0$ and a primitive $f(X) \in V_0[X]$ such that $y=af(x)$. Since $v(a) \geq 0$, it follows from $v(y)=0$ that $v(a)=0$; so $x, f(X)$ has multiplicity 0.

2.4. Existence of generating pairs.

Proposition. *Assume $[G:G_0]=n<\infty$, let x be a tr. generator of K over K_0 of value 0, and let l be any field such that $k_0 \subset l \subset k$. If there exists $\alpha \in k$ such that $\alpha^n \notin l$, then there exists $y \in K_0[x]$ of value 0 such that $y^* \notin l$.*

Proof. Choose a preimage $a \in K$ for α. Since $K=K_0(x)$, $a=f_1(x)/f_2(x)$, $f_i(X) \in K_0[X]$. Let $b=a^n=f_1(x)^n/f_2(x)^n$. The hypothesis $[G:G_0]=n$ implies $v(f_i(x)^n) \in G_0$, $i=1, 2$. Therefore there exist $c_i \in K_0$ such that $v(c_i f_i(x)^n)=0$; and then $b=(c_2/c_1)(c_1f_1(x)^n/c_2f_2(x)^n)$, where b, c_2/c_1, $c_if_i(x)^n$, $i=1, 2$, all have value 0. But then $b^*=(c_2/c_1)^*[(c_1f_1(x)^n)^*/(c_2f_2(x)^n)^*]$ implies either $(c_1f_1(x)^n)^*$ or $(c_2f_2(x)^n)^*$ is not in l since $b^*=\alpha^n \notin l$. Thus, for $i=1$ or 2, $y=c_if_i(x)^n$ is the required element.

Corollary. *Let x be a generator of K over K_0 of value 0, and suppose k is not algebraic over k_0. Then there exists $y \in K_0[x]$ of value 0 such that y^* is tr. over k_0 If, moreover, x is a generator of multiplicity 0 for this y, then x^* is tr. over k_0 and $k=k_0(x^*)$.*

Proof. For the first assertion, note that $[G:G_0]<\infty$ by 1.3, and then apply the above proposition with $l=$algebraic closure of k_0 in k. For the second assertion, apply 2.3 to conclude $y \in V_0[x]$. It follows that $y^* \in k_0[x^*]$ and hence that x^* is tr. over k_0. Then by 1.1, $k = k_0(x^*)$.

2.5. Nagata's proof [7, p. 91, Thm. 5] that k/k_0 is either algebraic or k is contained in a finite algebraic extension of k_0 followed by a simple tr. extension:

Suppose K_0 is algebraically closed and k/k_0 is not algebraic. By 2.4 there exists $y \in K_0[x]$ such that y^* is tr. over k_0. Factor: $y = a(x-r_1)\cdot\cdots\cdot(x-r_m)$, a, $r_i \in K_0$. Since $[G: G_0] < \infty$ (1.3) and G_0 is now divisible, we have $G = G_0$. Therefore there exist $b_1,\ldots, b_m \in K_0$ such that $v(x-r_i) = b_i$. Then $y^* = (ab_1\cdot\cdots\cdot b_m)^*((x-r_1)/b_1)^*\cdot\cdots\cdot((x-r_m)/b_m)^*$, so y^* is tr. over k_0 implies $(x-r_i)/b_i$ is tr. over k_0 for some i. Thus, we have found a generator $x_1 = (x-r_i)/b_i$ of K/K_0 such that x_1^* is tr. over k_0. By 1.1, $k = k_0(x_1^*)$.

If K_0 is not algebraically closed, pass to the algebraic extension $K_0' = K_0(r_i, b_i)$. The residue field of $K_0'(x)$ is $k_0'(x_1^*)$, where k_0' is the residue field of K_0' and hence is finite algebraic over k_0. Thus, $k_0 \subset k \subset k_0'(x_1^*)$.

3. Proof of the theorem.

We remind the reader that x always denotes a generator of K over K_0 of value $0\,(x \notin K_0)$. *In addition, throughout* § 3 x *will be assumed tr. over* K_0 *and* y *will be an element of* K *of value* 0 *having a fixed generating pair* $x, f(x)$ *of multiplicity* $n > 0$.

3.1. Definition.. We shall call x *rational* if $x^* \in k_0$, or equivalently, if there exists $r \in K_0$ such that $v(x-r) > 0$. For such an r, $v(r) = 0$ and $r^* = x^*$.

Let $\mathfrak{J}(x) = \{x_1 \in K \mid \text{there exist } r,\ 0 \neq b \in K_0 \text{ such that } x_1 = (x-r)/b \text{ and } v(x-r) = v(b) > 0\}$. *Whenever we write* $x_1 = (x-r)/b \in \mathfrak{J}(x)$, *we shall be tacitly assuming that* $r, 0 \neq b \in K_0$ *and* $v(x-r) = v(b) > 0$. Note that $\mathfrak{J}(x) \neq \phi$ if x is rational and K is generically of index 1 over K_0. (Reminder: generically index 1 means every generator of K over K_0 has value in G_0.) If $x_1 \in \mathfrak{J}(x)$ and there exist r_1, $0 \neq b_1 \in K_0$ such that $v(x_1 - r_1) = v(b_1) > 0$, then $x_2 = (x_1 - r_1)/b_1 \in \mathfrak{J}(x)$ too. Thus, every $x_1 \in \mathfrak{J}(x)$ is a generator of K over K_0 of value 0, and $\mathfrak{J}(x_1) \subset \mathfrak{J}(x)$.

The next lemma is crucial to the proof of the main theorem.

3.2. Lemma. *Suppose there exists* $x_1 = (x-r)/b \in \mathfrak{J}(x)$ *such that* x_1 *is not a generator for* y *of multiplicity* $< n$, *and write* $f(X) = a_0 + a_1(X-r) + \cdots + a_n(X-r)^n + \cdots + a_m(X-r)^m$, $a_i \in V_0[r] (\subset V_0)$. *Then*

i) $v(a_i(x-r)^i) \geq v((x-r)^n))$ *for* $i = 0,\ldots, n-1$;

ii) x_1 *is a generator for* y *of multiplicity* n; *and*

iii) *if* char $k \nmid n$, *then* $v(a_{n-1}) = v(x-r)$.

Remark. Since we are assuming throughout § 3 that x is a generator for y of multiplicity $n > 0$, ii) may be rephrased: if x is a generator for y of multiplicity $n > 0$, then every element of $\mathfrak{J}(x)$ is a generator for y of multiplicity $\leq n$. Also, iii) implies $a_{n-1} \neq 0$ because $x \notin K_0$ implies $x - r \neq 0$.

Proof. Note to begin with that $v(a_n) = 0$ since r^* is a root of multiplicity n of $f(X)^*$.

i): Suppose there exists $i<n$ such that $v(a_i(x-r)^i)<v((x-r)^n)$. Choose q to be the largest integer in $\{0,\ldots,n-1\}$ such that $v(a_q(x-r)^q)=\min\{v(a_j(x-r)^j)\mid j=0,\ldots,n-1\}$, i.e. choose $q\in\{0,\ldots,n-1\}$ such that

$$(\sharp)\qquad \begin{cases} v(a_q(x-r)^q)<v(a_j(x-r)^j),\ j=q+1,\ldots,n, \\ \text{and}\quad v(a_q(x-r)^q)\le v(a_j(x-r)^j),\ j=0,\ldots,q. \end{cases}$$

It follows that $v(a_q(x-r)^q)<v(a_j(x-r)^j),\ j>n$, since $v(a_n(x-r)^n)=v((x-r)^n)<v(a_j(x-r)^j),\ j>n$.

Now consider $(1/a_qb^q)f(x)=b_0+b_1x_1+\cdots+b_nx_1^n+\cdots+b_mx_1^m$, where $b_j=a_j/a_qb^{q-j},\ j=0,\ldots,m$. By $(\sharp)$,

$$\begin{cases} v(b_j)\ge 0, & j=0,\ldots,q, \\ b_q=1, & \\ v(b_j)>0, & j=q+1,\ldots,m. \end{cases}$$

Let $f_1(X)=b_0+b_1X+\cdots+b_mX^m$. Then $y=af(x)=aa_qb^qf_1(x_1)$, so $x_1,f_1(X)$ is a generating pair for y. Moreover, by 2.2 the multiplicity of $x_1,f_1(X)$ is $\le q<n$. Thus, we have a contradiction to the hypothesis that x_1 is not a generator for y of multiplicity $<n$.

ii): Consider $(1/a_nb^n)f(x)=b_0+b_1x_1+\cdots+b_mx_1^m$, where now $b_j=a_j/a_nb^{n-j}$, $j=0,\ldots,m$; and again let $f_1(X)=b_0+b_1X+\cdots+b_mX^m$. By i), $v(b_j)\ge 0$ for $j=0,\ldots,n$; and also $v(b_j)>0$ for $j=n+1,\ldots,m$ since $v(b)>0$. By 2.2 we again see that $x_1,f_1(X)$ is a generating pair for y of multiplicity $\le n$; and the hypothesis that x_1 is not a generator for y of multiplicity $<n$ yields the equality.

iii): Let $f_1(X)$ be as in ii). Then $f_1^{(n-1)}(x_1)=b_{n-1}+nx_1+c_2b_{n+1}x_1^2+\cdots+c_{m-n+1}b_mx_1^{m-n+1}$, where the c_i are natural numbers. Therefore $f_1^{(n-1)}(x_1)^*=b_{n-1}^*+nx_1^*$ since $v(b_j)>0,\ j=n+1,\ldots,m$. But $nx_1^*\ne 0$ because $\operatorname{char}k\nmid n$; so we must have $b_{n-1}^*\ne 0$ too, for otherwise x_1 would be a generator for y of multiplicity $<n$, contrary to hypothesis. But $b_{n-1}^*\ne 0$ implies $v(b_{n-1})=0$, so $v(a_{n-1})=v(a_nb)=v(b)=v(x-r)$.

3.3 Corollary. *Suppose* $\operatorname{char}k\nmid n$ *and* K/K_0 *is generically of index* 1. *If* x *is rational and* $\deg f(X)=n$, *then there exists* $x_1\in\mathfrak{J}(x)$ *which is a generator for* y *of multiplicity* $<n$.

Proof. Since x is rational, there exists $r\in K_0$ such that $v(x-r)>0$. Then $r\in V_0$, and $f(X)=a_0+a_1(X-r)+\cdots+a_n(X-r)^n$, $a_i\in V_0[r]=V_0$. By our initial assumption, x, $f(X)$ is a generating pair for y of multiplicity n, so $a_n^*\ne 0$ and $a_{n-1}^*=0$. Let $t=-a_{n-1}/na_n$. Then $t\in K_0$ and $v(t)>0$. Now let $r_1=r+t$, and rewrite $f(X)=b_0+b_1(X-r_1)+\cdots+b_n(X-r_1)^n$, where $b_n=a_n$, $b_{n-1}-nb_nt=a_{n-1},\ldots$. Since K/K_0 is generically of index 1, there exists $b\ne 0\in K_0$ such that $v(x-r_1)=v(b)>0$, and hence $x_1=(x-r_1)/b\in\mathfrak{J}(x)$. But t was chosen so that $b_{n-1}=0$. Thus, the failure of 3.2–iii) yields the conclusion that x_1 must be a generator for y of multiplicity $<n$.

3.4. Lemma. *Suppose x is rational and K_0 is henselian. Then there exists $s \in V_0[x]$ of value 0 such that y/s has a generating pair x, $g(X)$ of multiplicity n and with $g(X)$ monic of deg n.*

Proof. Since x^* is a root of multiplicity $n>0$ of $f(X)^*$, $f(X)^*=(X-x^*)^n h_1(X)$, $h_1(X) \in k_0[X]$ and $h_1(x^*) \neq 0$. By Hensel's lemma, [6, p. 189, Thm. 44.4] or [9, p. 185, Thm. 4], there exist $g(X)$, $h(X) \in V_0[X]$ such that $g(X)$ is monic of deg n, $f(X)=g(X)h(X)$, and $g(X)^*=(X-x^*)^n$, $h(X)^*=h_1(X)$. Let $s=h(x) \in V_0[x]$. Since $y=af(x)$ for some $a \in K_0$, $y=ag(x)h(x)$ and $y/s=ag(x)$; so x, $g(X)$ is a generating pair for y/s of the required type. Q.E.D.

Note that for the s of 3.4, $s \in V_0[x]$ and $v(s)=0$ imply $0 \neq s^* \in k_0[x^*]=k_0$.

3.5. Proposition. *Suppose K_0 is henselian, K/K_0 is generically of index* 1, *and* char $k \nmid n$. *If x is rational, then there exists $x_1 \in \mathfrak{J}(x)$ such that x_1 is a generator for y of multiplicity $<n$.*

Proof. By 3.4 there exists $s \in V_0[x]$ of value 0 and a generating pair x, $g(X)$ for y/s of multiplicity n, with $g(X)$ monic of deg n. By 3.3 there exists $x_1 \in \mathfrak{J}(x)$ which is a generator for y/s of multiplicity $<n$. This means there exists $a \in K_0$ and a primitive $f_1(X) \in V_0[X]$ such that $y/s=af_1(x_1)$ and x_1^* is a root of multiplicity $<n$ for $f_1(X)^*$. If we write $s=s(x) \in V_0[x]$, and if $x_1=(x-r)/b$, then $s(x)=s(x_1 b+r)$ $=s_1(x_1) \in V_0[x_1]$. Moreover, $s_1(x_1^*)^*=s^* \neq 0$, so x_1^* is a root of multiplicity 0 of $s_1(X)^*$. Thus, $y=as_1(x_1)f_1(x_1)$, and it follows that x_1, $s_1(X)f_1(X)$ is a generating pair for y of multiplicity $<n$.

3.6 Corollary. *Suppose K_0 is henselian, K/K_0 is generically of index* 1, *and* char $k=0$. *If every element of $\mathfrak{J}(x) \cup \{x\}$ is rational, then there exists $x_1 \in \mathfrak{J}(x)$ such that x_1 is a generator for y of multiplicity* 0.

Proof. Since x is rational and K/K_0 is generically index 1, $\mathfrak{J}(x) \neq \emptyset$. Moreover, by 3.2 every element of $\mathfrak{J}(x)$ is a generator for y of multiplicity $\leq n$. Choose $x_1 \in \mathfrak{J}(x)$ of multiplicity μ and such that no element of $\mathfrak{J}(x)$ has multiplicity $<\mu$. If $\mu=0$, we are done; if not, by 3.5 there exists $x_2 \in \mathfrak{J}(x_1) \subset \mathfrak{J}(x)$ such that x_2 is a generator for y of multiplicity $<\mu$, a contradiction to the choice of x_1.

3.7 Theorem. *Assume $K=K_0(x)$, where x is tr. over K_0 and $v(x)=0$;* char $k=0$; *and K_0 is henselian. If K/K_0 is generically of index* 1 *and k_0 is algebraically closed in k and $\neq k$, then there exists $x_1 \in \mathfrak{J}(x) \cup \{x\}$ such that x_1^* is tr. over k_0.*

Proof. If there exists $x_1 \in \mathfrak{J}(x) \cup \{x\}$ such that $x_1^* \notin k_0$, then by hypothesis x_1^* is tr. over k_0 and we are done. Thus we may assume every element of $\mathfrak{J}(x) \cup \{x\}$ is rational.

By 2.4-Corollary, there exists $y_1 \in K$ of value 0 such that x is a generator for y_1 and y_1^* is tr.$/k_0$; and also by 2.4-Corollary, we may further assume that x is a generator for y_1 of multiplicity $n>0$. But then by 3.6 there exists $x_1 \in \mathfrak{J}(x)$ such that x_1 is a generator for y_1 of multiplicity 0, which means $y_1 \in V_0[x_1]$. Therefore $y_1^* \in k_0[x_1^*]$, and hence x_1^* is tr.$/k_0$. Q.E.D.

In view of the reduction of 1.4 whereby k_0 may be assumed separably algebraically closed in k and K_0 henselian, 3.7 yields the Ruled Residue Conjecture (char 0) in the case that $[G: G_0]=1$. For by 1.1 if a *generator* of K/K_0 specializes to a tr., then k/k_0 is simple transcendental.

4. Extensions generically of index 1.

We assume throughout § 4 *that* $K=K_0(x)$, x *tr. over* K_0 *and* $v(x)=0$.

Before proceeding to the final ingredient in the proof of the Ruled Residue Conjecture (char 0), we shall make a couple of comments on the notion of "generically index 1". Recall that K/K_0 is of index 1 means $v(\xi)\in G_0$ for every $\xi\in K$ and that K/K_0 is generically of index 1 was defined to mean $v(\xi)\in G_0$ for every *generator* ξ of K/K_0.

4.1 Proposition. *The following are equivalent:*

i) K/K_0 *is generically of index* 1.

ii) *If* $r\in K_0$ *and* $v(x-r)>0$, *then* $v(x-r)\in G_0$.

iii) *Either* $\{v(x-r)\mid r\in K_0 \text{ and } v(x-r)>0\}$ *has no maximal element, or its maximal element is in* G_0.

Proof. Since $x-r$ is a generator of K/K_0 for all $r\in K_0$, the implications i)$\Rightarrow$ ii)$\Rightarrow$iii) are immediate. ii)$\Rightarrow$i): Every generator of $K_0(x)/K_0$ is of the form $\xi=(ax+b)/(cx+d)$; $a, b, c, d\in K_0$, $ad-bc\neq 0$ (cf. [10, p. 198]). Therefore it suffices to show $v(ax+b)\in G_0$ whenever $a\neq 0$, $b\in K_0$, or equivalently, to show $v(x+(b/a))\in G_0$. Since $v(x)=0$, either $v(b/a)<0$ and $v(x+(b/a))=v(b/a)\in G_0$, or $v(b/a)\geq 0$, in which case $v(x+(b/a))\geq 0$ and ii) applies. iii)$\Rightarrow$ii): If there exist $r, r'\in K_0$ such that $0<v(x-r)<v(x-r')$, then $v(x-r)=v((x-r)-(x-r'))=v(r'-r)\in G_0$. Thus, if $v(x-r)$ is not a maximal element of the set, then it is automatically in G_0.

Q. E. D.

4.2 Example of K/K_0 which is generically of index 1 but not of index 1 and which has x rational, i.e. $x^*\in k_0$.

Let v be the X-adic valuation of $Q(\sqrt{2}, \pi)(X)$, i.e. v is the inf extension of the 0-valuation of $Q(\sqrt{2}, \pi)$ *w.r.t.* $v(X)=1$; let $K_0=Q(X^2)$; and let $K=K_0(x)$, where $x=1+\sqrt{2}X^2+\pi X^3$. In view of 4.1 to prove K/K_0 is generically of index 1 it suffices to show $v(x-r)>0$, $r\in K_0$, implies $v(x-r)=2$. Note first that $v(x-r)>0$ implies $1=x^*=r^*$, so $r=1-a$, $a\in K_0$ and $v(a)>0$. Therefore $x-r=a+\sqrt{2}X^2+\pi X^3$ and $(x-r)/X^2=(a/X^2)+\sqrt{2}+\pi X$; so it remains to show $v((a/X^2)+\sqrt{2})=0$. But $a\in K_0$ and $v(a)>0$ implies $v(a)\geq 2$. Then $(a/X^2)+\sqrt{2}\to(a/X^2)^*+\sqrt{2}$; and since $a/X^2\in K_0$, $(a/X^2)^*\in k_0=Q$. Since $\sqrt{2}\notin Q$, it follows that $(a/X^2)^*+\sqrt{2}\neq 0$. Hence $v((a/X^2)+\sqrt{2})=0$.

Finally, to see that K/K_0 is not of index 1, note that $[(x-1)/X^2]^2-2=2\sqrt{2}\pi X+\pi^2X^2$ has value $1\notin G_0$. Thus, $G_0=2Z$ and $G=Z$. Q. E. D.

Exactly when generically index 1 does imply index 1 for fields K/K_0 is not clear. For example, a consequence of 6.2 is that this implication holds if $rk\ v=1$,

k_0 is algebraically closed in k and $\neq k$, and either char $k=0$ or v is discrete.

The following proposition relates arbitrary inf extensions to those defined with respect to value 0.

4.3 Proposition (*continuation of* 1.1). *Let z be a (tr.) generator of K/K_0, let $v(z)=g$, and suppose $g+G_0$ is of finite order $n\geq 1$ in G/G_0. Let $v_1=v\mid K_1$, where $K_1=K_0(z^n)$, and let k_1 be the residue field of v_1. Then the following are equivalent:*

i) *v is the inf extension of v_0 w.r.t. $v(z)=g$.*

ii) *v_1 is the inf extension of v_0 w.r.t. $v_1(z^n)=ng$.*

iii) *There exists $b\neq 0\in K_0$ such that v_1 is the inf extension of v_0 w.r.t. $v_1(z^n/b)=0$.*

iv) *There exists $b\neq 0\in K_0$ such that $z^n/b \xrightarrow{v_1} \alpha$ tr. over k_0.*

Moreover, when these hold, then $k=k_1=k_0(\alpha)$ and G/G_0 is cyclic, generated by $g+G_0$.

Proof. i)$\Rightarrow$ii)$\Rightarrow$iii) are immediate from the definitions, and iii)$\Leftrightarrow$iv) by 1.1. It remains to show iii)$\Rightarrow$i). The value group of v_1 is G_0 and the residue field is $k_0(\alpha)$ by 1.1. Since $[K\colon K_1]=n$ and $[G\colon G_0]\geq n$, it follows from 1.2 that $[G\colon G_0]=n$, $[k\colon k_1]=1$, and v_1 extends uniquely, up to equivalence, to K. In particular, then $G=G_0+Zg$ and $k=k_1$. But the inf extension w of v_0 w.r.t. $w(z)=g$ is an extension of v_1 to K (cf 1.1), so w is equivalent to v. Since $G=G_0+Zg$ and $w(z)=g=v(z)$, we must actually have $w=v$. Q.E.D.

We are now ready for the technical device (4.4 and 4.5) needed to complete the proof of the Ruled Residue Conjecture (char 0).

4.4 Lemma. *Let $\xi\in K$, $\notin K_0$ and $v(\xi)=g$, where $g+G_0$ is of finite order $n\geq 1$ in G/G_0; let t be tr. over K, and let v_t denote the inf extension of v (to $K(t)$) w.r.t. $v_t(t)=g$; and let $v_t^\wedge$, $K(t)^\wedge$ be the henselization of v_t, $K(t)$.*

If char $k\nmid n$, *k_0 is algebraically closed in k, and v is not the inf extension of v_0 (to $K_0(\xi)$) w.r.t. $v(\xi)=g$, then there exists $b\in K(t)^\wedge$ algebraic over $K_0(t)$ with the following properties:*

i) *$b\to b^*$ tr. over k.*

ii) *The residue fields of $K'=K(t, b)$ and $K_0'=K_0(t, b)$ are $k(b^*)$ and $k_0(b^*)$, respectively.*

iii) *The value groups of K' and K_0' are G and G_0+Zg, respectively.*

Proof. Since $v_t(t^n)=ng\in G_0$, there exists $d\in K_0$ such that $v_t(t^n)=v_t(d)$; and by 4.3, $t^n/d\to\alpha$ tr. over k_0 and the residue field of $K_0(t)$ is $k_0(\alpha)$. Also, by 1.1, $t/\xi\to\beta$ tr. over k and the residue field of $K(t)$ is $k(\beta)$. But $v(d)=v(\xi^n)$ implies there exists $u\in K$ of value 0 such that $\xi^n=ud$; and therefore $(t/\xi)^n=(1/u)(t^n/d)$, and consequently $\beta^n=(1/u^*)\alpha$.

Claim: $u^*\in k_0$. For otherwise u^* is tr. over k_0 by hypothesis. But then $u=\xi^n/d\to u^*$ tr. over k_0 implies by 4.3 that v is the inf extension of v_0 w.r.t. $v(\xi)=g$, a contradiction to our hypotheses.

Thus, β is separably algebraic of deg n over $k_0(\alpha)$; so by Hensel's lemma [4, p.

118, Cor. (16.6)] there exists $b \in K(t)^\wedge$ algebraic of deg n over $K_0(t)$ such that $b \to \beta$. Then the residue field and value group for $K(t, b)$ are $k(\beta)$ and G since $K(t) \subset K(t, b) \subset K(t)^\wedge$. By 1.2 and 4.3 the residue field and value group for $K_0(t, b)$ are $k_0(\alpha, \beta) = k_0(\beta)$ and $G_0 + Zg =$ value group of $K_0(t)$. Q.E.D.

Note that if ξ is a *generator* of K/K_0, then by 4.3 k/k_0 is not simple transcendental implies v is not the inf extension of v_0 w.r.t. $v(\xi)$. This is how we shall fulfill the above hypothesis in the following corollary.

4.5 Corollary. *If there exist (valued) fields $K \supset K_0$ such that*

i) *K/K_0 is simple tr. and* char $k=0$,

ii) *K_0 is henselian,*

iii) *k_0 is algebraically closed in k and $k \neq k_0$.*

iv) *k/k_0 is not simple tr.,*

then there exist such fields with the additional property that K/K_0 is generically of index 1.

Proof. Suppose there exists a generator z of K/K_0 such that $v(z) = g \notin G_0$. By 4.4 there exist fields $K_0' \subset K' = K_0'(z)$ having residue fields $k_0' = k_0(\beta)$, $k' = k(\beta)$, respectively, β tr. over k, and value groups G_0', G, respectively, with $[G: G_0'] < [G: G_0]$. It follows from [11, p. 167, Lem. 2] that k'/k_0' satisfies iii) and from the generalized Lüroth theorem [8, p. 137, Thm. 4.12.2] that k'/k_0' satisfies iv). Now replace K_0' by its henselization $(K_0')^\wedge$ (inside $(K')^\wedge$) and K' by $(K_0')^\wedge(z)$; this does not alter the residue fields or value groups (cf. [4, p. 136, Thm. 17.19] or [8, p. 193, Thm. 5.11.11]). Thus, under the assumption that K/K_0 is not generically index 1 we have found fields $(K_0')^\wedge \subset (K_0')^\wedge(z)$ satisfying i)–iv) and the additional condition that $[G: G_0'] < [G: G_0]$. The corollary now follows by induction on $[G: G_0]$.

4.6 Ruled Residue Theorem (char 0). *Let K_0 and $K = K_0(x)$ be fields with x tr. over K_0, let v be a valuation of K with residue field k, and let k_0 be the residue field of $v \mid K_0$. Suppose* char $k=0$ *and k is not algebraic over k_0. Then there exists a finite algebraic extension k_1 of k_0 and an α tr. over k_1 such that $k = k_1(\alpha)$.*

Proof. By 1.3 it suffices to show k is of the form $k_1(\alpha)$, k_1 algebraic over k_0 and α tr. over k_1. By 1.4 we may assume K_0 is henselian and k_0 is algebraically closed in k, and by 4.5 we may additionally assume K/K_0 is generically of index 1. The theorem now follows from 3.7. Q.E.D.

4.7 Remarks.

1. It is only in the reduction step of 4.5 that field extensions of K lying outside $v^\wedge$, $K^\wedge$ are used. If one wants to think in terms of working inside a fixed valued field, he can proceed as follows: If order of $G/G_0 = s$, choose preimages $g_1, \dots, g_s \in G$ for the elements of G/G_0. Then let $t_1, \dots, t_s$ be indeterminates, and extend v to $K(t_1, \dots, t_s)$ by infs w.r.t. $v(t_i) = g_i$. Now the construction of 4.5 can be carried out inside the henselization $K(t_1, \dots, t_s)^\wedge$.

2. On the char $k=0$ assumption: It is not at all clear how to adapt our

methods to the non-zero characteristic case. As noted in the introduction, Nagata has proved without restriction on the characteristic that the statement of 4.6 remains valid a) if v is discrete, rk n, i.e. if G is a lexicographic direct sum of n copies of Z, or b) if the conclusion is weakened to $k \subset k_1(\alpha)$ (cf. [7, Thms. 1 and 5], [8, p. 198, Thm. 5.12.1]). When $K_0 = Q$, it seems that the discrete, rk 1 case of a) (from which a) follows by induction) is implicit in the early paper [5] of Mac Lane, although the terminology of that paper obscures this conclusion (See [5, Thms. 8.1, 12.1, and 14.1]). As for further progress in removing the characteristic 0 assumption from 4.6, in generalizing from Nagata's result a) above there are two extreme cases to take into account: one is the case of discrete, infinite rk v, i.e. G is the lexicographic direct sum of infinitely many copies of Z; and the other (probably the more difficult) is the case of non-discrete, rk 1 v, e.g. $G = Q$.

3. Addendum (Oct., 1980). W. Heinzer, after reading a preprint of this paper, has pointed out that the Ruled Residue Conjecture for k_0 perfect can be proved as follows: Let $D = K_0[x] \cap V$; and note that $V = D_S$, where $S = \{\text{units of } V\} \cap D$. For, if $\xi \in V$, write $\xi = f_1/f_2$, $f_i \in K_0[x]$; since $[G\colon G_0] < \infty$, there exist $a \in K_0$ and an integer $n > 0$ such that $v(f_2^n) = v(a)$; and therefore $(f_2^n/a)\xi \in D$ and $\xi \in D_S$. It follows that k is the quotient field of D^*, where $D \to D^*$. Next, Nagata's argument (cf. 2.5) shows there exists a finite algebraic extension K_0' of K_0 and an $x_1 = (x - r)/b \in K_0'[x] = K_0'[x_1]$ such that x_1^* is tr. over k_0. By 1.1, then $K_0'(x_1)$, v' is the inf extension of K_0', v_0' w.r.t. $v'(x_1) = 0$, from which it follows that $D' \to k_0'[x_1^*]$, where $D' = K_0'[x_1] \cap V'$. Thus, we have $k_0 \subset D^* \subset k_0'[x_1^*]$; so by [1, p. 322, (2.9)] the integral closure of D^* is of the form $k_0''[z]$, k_0'' algebraic over k_0 and z tr. over k_0''. But then $k = k_0''(z)$.
Q.E.D.

The theorem of [1] on which Heinzer's proof rests requires two non-elementary facts about 1-dim function fields: i) genus does not decrease under a finite separable extension of the base field and ii) genus 0 plus the existence of a rational place implies simple tr. Thus, while his proof yields the more general case of a perfect k_0, it is not nearly as simple-minded as our proof of 4.6. In any case, both approaches should be of interest in further efforts to remove the restrictive hypothesis involving the characteristic.

5. Complements.

We begin with a class of examples to illustrate that all of the possibilities for k/k_0 suggested by theorem 4.6 can occur.

5.1. Let k_0 be a subfield of C = complex numbers, let $C((t))$ be the field of formal Laurent series in the indeterminate t with coefficients in C, and let v be the t-adic valuation of $C((t))$. Let $x = a_0 + a_1 t + a_2 t^2 + \cdots \in C[[t]]$, and consider the residue fields given by

$$\begin{array}{ccccc} K_0 = k_0(t) & \text{——} & K = k_0(t)\,(x) & \text{——} & C((t)) \\ \downarrow & & \downarrow & & \downarrow \\ k_0 & \text{————} & k & \text{————} & C \end{array}$$

What is a generating set for k over k_0?

Lemma. *If* $a_0, a_1, \dots, a_i$ $(i \geq 0)$ *are algebraic over* k_0, *then* $a_0, a_1, \dots, a_i, a_{i+1} \in k$.

Proof. Note that $x \to a_0$ implies $a_0 \in k$. Let $f(X) \in k_0[X]$ be the irreducible polynomial for a_0 over k_0, and let $y_1 = f(x)/t = f'(a_0)((x-a_0)/t) + (tf''(a_0)/2)((x-a_0)/t)^2 + \cdots$. Since $(x-a_0)/t = a_1 + a_2 t + \cdots$, we can write $y_1 = f'(a_0)a_1 + (f'(a_0)a_2 + b_2^{(1)})t + (f'(a_0)a_3 + b_3^{(1)})t^2 + \cdots$, where $b_j^{(1)} \in k_0(a_0, \dots, a_{j-1})$. But $y_1 \to y_1^* = f'(a_0)a_1$ and $f'(a_0) \neq 0$, so $a_1 \in k$ since y_1^* and $f'(a_0)$ are in k.

Now let $f_1(X) \in k_0[X]$ be the irreducible polynomial for y_1^* over k_0, and let $y_2 = f_1(y_1)/t = f_1'(y_1^*)((y_1 - y_1^*)/t) + (tf_1''(y_1^*)/2)((y_1 - y_1^*)/t)^2 + \cdots$. Since $(y_1 - y_1^*)/t = (c^{(1)}a_2 + b_2^{(1)}) + (c^{(1)}a_3 + b_3^{(1)})t + \cdots$, where $c^{(1)} = f'(a_0) \neq 0 \in k_0(a_0)$ and $b_j^{(1)} \in k_0(a_0, \dots, a_{j-1})$, we can write $y_2 = (c^{(2)}a_2 + b_2^{(2)}) + (c^{(2)}a_3 + b_3^{(2)})t + \cdots$, with $c^{(2)} \neq 0 \in k_0(a_0, a_1)$ and $b_j^{(2)} \in k_0(a_0, \dots, a_{j-1})$. Then $y_2 \to y_2^* = c^{(2)}a_2 + b_2^{(2)}$ implies $a_2 \in k_0(a_0, a_1, y_2^*) \subset k$.

We have thus demonstrated the lemma for $i = 0, 1$; the general case is by induction on i and is identical to the $i = 1$ case.

Corollary. *If* $a_0, \dots, a_{n-1}$ $(n \geq 1)$ *are algebraic over* k_0 *and* a_n *is tr. over* k_0, *then* $k = k_0(a_0, \dots, a_{n-1}, a_n)$. *If* $a_0, a_1, \dots$ *are all algebraic over* k_0, *then* $k = k_0(a_0, a_1, \dots)$.

Proof. The inclusion $\supset$ is by the lemma. Suppose a_n is tr. over k_0, and consider the finite algebraic extension of $K_0 = k_0(t)$, $L = k_0(t, a_0, \dots, a_{n-1})$. Then $L(x) = L(x_n)$, where $x_n = a_n + a_{n+1}t + \cdots$. The residue field of L is $k_0(a_0, \dots, a_{n-1})$. Moreover, since $x_n \to a_n$ tr. over $k_0(a_0, \dots, a_{n-1})$, by 1.1 the residue field of $L(x)$ must be $k_0(a_0, \dots, a_{n-1})(a_n)$. But $K \subset L(x)$ implies k is $\subset$ the residue field $k_0(a_0, \dots, a_{n-1}, a_n)$ of L. Thus, we have proved the first assertion of the corollary. For the second, observe that $K \subset k_0(a_0, a_1, \dots)((t))$ implies $k \subset k_0(a_0, a_1, \dots)$. Q.E.D.

Note that x is necessarily tr. over $k_0(t)$ whenever k/k_0 is not finite algebraic, by 1.2. In conclusion, the corollary shows that it is possible to get the residue field k to be an arbitrary finite algebraic extension of k_0 followed by a simple tr. extension (actually, it is only necessary to take $n = 1$ in the corollary since any finite algebraic extension of k_0 can be realized as a simple extension), or to be an arbitrary countably generated algebraic extension of k_0. See also [2, p. 173, Exercise 1] and [12, p. 104, Example 4] for examples of this latter type. (Incidentally, the Remark on p. 162 of [2] seems to ignore examples of the former type.)

It is interesting to pursue this example a bit further and inquire about the completion v^c, K^c of v, K in $C((t))$ when, say, a_0 is algebraic over k_0 and a_1 tr. over k_0. First observe that $V = k_0(y_1)[x]_{(f(x))}$, where $f(X)$ is the irreducible polynomial for a_0 over k_0. For, we have seen that y_1 specializes to a tr. over k_0, which implies $k_0(y_1) \subset V$; and since $f(X)$ is irreducible over k_0 and therefore also over $k_0(y_1)$, $k_0(y_1)[x]_{(f(x))}$ is a DVR contained in V and having the same quotient field $k_0(t, x)$ as V, and hence must be V. We have also seen that the residue field k of V is $k_0(a_0, a_1)$, so by Hensel's lemma (cf. [4, p. 120, 16.7]) there exists a preimage for a_0

in V^c which is algebraic over k_0. But the only such preimage in $C[[t]]$ is a_0 itself, so $a_0 \in V^c$. Thus, $k_0(y_1)[a_0]=k_0(y_1, a_0)\subset V^c$ is a coefficient field for V^c, and V^c is the t-adic topological closure of $k_0(a_0, y_1)[t]_{(t)}$ in $C[[t]]$; so V^c may be thought of as being the subset of $C[[t]]$ obtained by taking power series in t with coefficients in $k_0(a_0, y_1)$ and rewriting them as power series with coefficients in C.

5.2. As mentioned in the introduction, Nagata [7, p. 91, Thm. 5] has proved that if k/k_0 is not algebraic, then k is contained in a (finite) algebraic extension of k_0 followed by a simple tr. extension. Does this result in itself imply 4.6? That is, given fields $k_0\subset k\subset k_1(t)$ with k_1 finite algebraic over k_0, t tr. over k_1, and k/k_0 not algebraic, is k necessarily a finite algebraic extension of k_0 followed by a simple tr. extension? The following example (cf. [3, p. 23] and [8, p. 144, 2]) shows that the answer is "no".

Let $k_0=$reals; $k=k_0(x, y)$, where $x^2+y^2+1=0$; and $k_1=C=$complexes. Then $k_0\subset k\subset C(x+iy)$. For $x-iy=-1/(x+iy)$ implies $x-iy, x+iy\in C(x+iy)$, and hence $x, y\in C(x+iy)$.

Next observe that k_0 is algebraically closed in k, which amounts to verifying $i\notin k$. For, if $i\in k$, then $k_0(x, y)=k_0(x, y, i)$; and hence $[k_0(x, y, i): k_0(x)]=2$. But $[k_0(x, i): k_0(x)]=2$, and it follows from Gauss's lemma that Y^2+x^2+1 is irreducible over $k_0(x, i)=C(x)$; so $[k_0(x, y, i): k_0(x)]=4$.

Now suppose k/k_0 is simple tr.. Then there exists a valuation v of k/k_0 having residue field k_0. If $v(x)\geq 0$, then $y^2+x^2+1=0$ implies $v(y)\geq 0$ too; and therefore in the residue field k_0, $y^{*2}+x^{*2}+1=0$, which is impossible because $k_0=$reals. If $v(x)<0$, then the same argument applied to $(y/x)^2+(1/x)^2+1=0$ works. Thus, k is not a simple tr. extension of k_0.

The function field k/k_0 is known to have genus 0, but the additional fact needed to be able to conclude that k is a simple tr. extension of k_0 is the existence of a k_0-rational place. See [3, p. 23].

5.3. An application of the Ruled Residue Theorem (inspired by the applications of Nagata in [7]. See also [8, p. 199, Thm. 5.12.2]).

Let $k_0<k$ be fields of char. 0 and G be any torsion-free abelian group (written additively). Let $k[G]$ be the group ring of G with coefficients in k, i.e. $k[G]=\oplus\{kX^g \mid g\in G\}$, with multiplication defined linearly by $X^gX^h=X^{g+h}$. Let $k(G)$ denote the quotient field of $k[G]$. Then $k_0(G)\subset k(G)$.

Cancellation theorem. *If $k(G)$ is a simple tr. extension of $k_0(G)$, then k is a simple tr. extension of k_0.*

Proof. Since G is torsion-free, G can be totally ordered. Then any $\xi\in k[G]$ may be written $\xi=a_1X^{g_1}+\cdots+a_tX^{g_t}$, $a_i\neq 0\in k$, $g_1<\cdots<g_t\in G$. Define v: $k[G]\to G$ by $v(\xi)=\inf\{g_i \mid i=1,\dots, t\}$; and extend to a valuation v of $k(G)$ having value group G and residue field k. The restriction v_0 of v to $k_0(G)$ is similarly a valuation with residue field k_0.

Claim: k_0 is algebraically closed in $k(G)$, and hence a fortiori in k. Since $k_0(G)$ is algebraically closed in $k(G)$ by hypothesis, it suffices to show k_0 is algebraically closed in $k_0(G)$. If $\alpha \in k_0(G)$ is algebraic over k_0, then $k_0[\alpha] = k_0(\alpha) \subset V_0$, and hence $k_0(\alpha)$ would map isomorphically under the residue map $V_0 \to k_0$, thereby yielding $\alpha \in k_0$.

Thus, by theorem 4.6 and the fact that k_0 is algebraically closed in k and $\neq k$, we conclude that k is a simple tr. extension of k_0. Q. E. D.

In the statement of the cancellation theorem, we can replace the hypothesis that $k(G)$ is a simple tr. extension of $k_0(G)$ by the weaker hypothesis that $k(G)$ is $\subset$ a simple tr. extension of $k_0(G)$, for by Lüroth's theorem the former hypothesis is a consequence of the latter. Finally, the cancellation theorem may be rephrased in terms of quotient fields of group rings as follows: If G is identified with $0 \oplus G$ in $Z \oplus G$, then $k_0(Z \oplus G) = k(G)$ implies $k \cong k_0(Z)$.

5.4. The set $\mathfrak{J}(x) \cup \{x\}$. The statements of 3.6 and 3.7 concerning elements of $\mathfrak{J}(x) \cup \{x\}$ imply comparable statements for arbitrary generators of value 0, as we shall now show. Assume $K = K_0(x)$, where x is tr. over K_0 of value 0.

Proposition. *Suppose K/K_0 is generically of index 1, and let l be a field such that $k_0 \subset l \subset k$. If there exists a generator y of K/K_0 of value 0 such that $y^* \notin l$, then there exists $x_1 \in \mathfrak{J}(x) \cup \{x\}$ such that $x_1^* \notin l$.*

Proof. By [10, p. 198], $y = (ax+b)/(cx+d)$, $a, b, c, d \in K_0$, $ad - bc \neq 0$. Since K/K_0 is generically of index 1, there exists $e \neq 0 \in K_0$ such that $v(ax+b) = v(cx+d) = v(e)$. Then $y = ((ax+b)/e)/((cx+d)/e)$ implies one of $((ax+b)/e)^*$ or $((cx+d)/e)^* \notin l$. Therefore we may assume $y = (ax+b)/e$. Dividing a, b, e by the element of least value from among a, b, e, we may further assume a, b, e have value ≥ 0 and one of them has value 0. If $v(e) = 0$, then $y^* = (a^*/e^*)x^* + (b^*/e^*)$ implies $x^* \notin l$, so $x_1 = x$ works; if $v(e) > 0$ but $v(a) = 0$, then $x_1 = y = (x + (b/a))/(e/a) \in \mathfrak{J}(x)$; and if $v(e) > 0$ and $v(b) = 0$, then $v(ax+b) = v(e) > 0$ implies $v(a) = 0$ and we are in the previous case. Q. E. D.

By taking $l = k_0$ (resp., l = algebraic closure of k_0 in k), we have

Corollary. *Suppose K/K_0 is generically of index 1. If there exists a generator y of K/K_0 such that $y^* \notin k_0$ (resp., y^* is tr. over k_0), then there exists $x_1 \in \mathfrak{J}(x) \cup \{x\}$ such that $x_1^* \notin k_0$ (resp., x_1^* is tr. over k_0).*

To carry this a bit further, let us define K to be *generically rational* over K_0 if for every gnerator y of K/K_0 of value 0, $y^* \in k_0$. Then under the assumption that K/K_0 is generically of index 1, the condition of 3.6 "every element of $\mathfrak{J}(x) \cup \{x\}$ is rational" is equivalent to "K is generically rational over K_0".

Part II: The theorem for v_0 of finite rk.

We retain the notation established in the introduction; in particular, $K = K_0(x)$, where $v(x) = 0$. In addition, *we assume throughout II that x is tr. over K_0.*

6. Theorem 3.7 revisited.

Theorem 3.7 is false without the assumption that K_0 is henselian if rk $v>1$, as example 7.2 will show; indeed, the henselian hypothesis was employed precisely to deal with valuations of infinite rk, and if we restrict attention to valuations of finite rk, a sharper result, which in the rk 1 case amounts to deleting the henselian hypothesis and in the discrete, rk 1 case amounts to deleting both the henselian and char 0 hypotheses, can be obtained. Since we are ignorant of the status of this result in the cases of infinite rk or of non-zero characteristic and arbitrary value group, we shall first phrase it as a conjecture.

6.1 Conjecture. For every valuation overring W of V ($W \subset K$), the residue field l_0 of $W \cap K_0$ is algebraically closed in the residue field l of W, $k_0 \neq k$, and K/K_0 is generically of index $1 \Rightarrow$ there exists a generator x of K/K_0 such that v is the inf extension of v_0 w.r.t. $v(x)=0$; or, equivalently, there exists a generator x of K/K_0 which specializes to a tr. over k_0.

What we know about this conjecture, aside from the henselian case of 3.7, is summed up in the following theorem.[1]

6.2 Theorem *The implication* $\Rightarrow$ *of* 6.1 *is true if either a)* rk v *is finite and* char $k=0$, *or b)* v *is discrete.*

The converse implication $\Leftarrow$ to 6.1 is always true. For, if $x \xrightarrow{v} x^*$ tr. over k_0 and w is the valuation of K whose ring is W, then there exists a valuation u of the residue field l of w such that $x \xrightarrow{w} x' \xrightarrow{u} x^*$. (See § 7). But x^* is tr. over k_0, so x' is tr. over l_0, and therefore 1.1 yields l/l_0 is simple tr., and hence l_0 is algebraically closed in l.

In b) rk v is necessarily finite, since by definition of discrete, G is a lexicographic direct sum of finitely many copies of Z; but char k may be arbitrary. In both a) and b) the crux of the proof lies in the rk 1 case, from which the finite rk case follows by induction.

The remainder of § 6 will be devoted to establishing a) and b) for rk 1 v. Just as theorem 3.7 follows from 3.6, this will follow from

6.3 Proposition. *Suppose* v *is* rk 1 *and either a)* char $k=0$ *or b)* v *is discrete, and suppose* K/K_0 *is generically of index* 1 *and every element of* $\mathfrak{Z}(x) \cup \{x\}$ *is rational. If* y *is an element of* K *of value* 0 *and* x *is a generator for* y *of multiplicity* >0, *then there exists* $x_1 \in \mathfrak{Z}(x)$ *such that* x_1 *is a generator for* y *of multiplicity* 0.

Proof. We first need a lemma.

Lemma. *Suppose* $y \in K$ *has a generating pair* $x, f(X)$ *of multiplicity* $n>0$, *where* char $k \nmid n$. *If* $x_1=(x-r)/b \in \mathfrak{Z}(x)$, *then either* x_1 *is a generator for* y *of multiplicity* $<n$ *or there exists a generating pair* $x_1, f_1(X)$ *for* y *of multiplicity* n

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1) Added August, 1981: I now have an example (to appear in a sequel) in char. p for which $G_0=G=Q$, k/k_0 is simple tr., and yet no generator of K/K_0 specializes to a tr. over k_0. Thus, the remaining undecided case of 6.1 is char $k=0$ and rk v infinite.

and an $r_1 \in K_0$ *such that* $f(x)=b^n f_1(x_1)$, $v(x_1-r_1)>0$, *and* $v(f_1^{(n-1)}(r_1)) \geq 2v(f^{(n-1)}(r))=2v(b)$.

Proof of lemma. Suppose x_1 is not a generator for y of multiplicity $<n$. We may write $f(X)=a_0+a_1(X-r)+\cdots+a_n(X-r)^n+\cdots+a_m(X-r)^m$, where the a_i are in V_0, $a_0^*=\cdots=a_{n-1}^*=0$, $a_n^*\neq 0$, and $a_{n-1}=f^{(n-1)}(r)$ (cf. 2.2). By 3.2, $v(a_i(x-r)^i) \geq v((x-r)^n)$ for $i=0,\dots,n-1$, and $v(a_{n-1})=v(x-r)=v(b)$. Therefore if we write $b^{-n}f(x)=b_0+b_1((x-r)/b)+\cdots$, where $b_i=a_i/b^{n-i}$, then $v(b_i)\geq 0$, $i=0,\dots,n-1$, and $v(b_{n-1})=0$; moreover, the b_i for $i\geq n$ are of the form $b_n=a_n$, $b_{n+1}=a_{n+1}b,\dots$, and hence are also in V_0. Let $f_1(X)=b_0+b_1X+\cdots+b_mX^m$. Then $x_1, f_1(X)$ is a generating pair for y, and $f(x)=b^n f_1(x_1)$. Moreover, computing $f_1^{(n)}(X)=a_n+b(\cdots)$, we see that $f_1^{(n)}(x_1^*)^*=a_n^*\neq 0$. Therefore $x_1, f_1(X)$ is a generating pair for y of multiplicity $\leq n$, and hence by our initial assumption of multiplicity n.

It remains to show there exists $r_1\in K_0$ with the specified properties. We have $f_1^{(n-1)}(x_1)=(a_{n-1}/b)+na_nx_1+b(\cdots)$, and $f_1^{(n-1)}(x_1)^*=0$ since $x_1, f_1(X)$ has multiplicity n; so $0=(a_{n-1}/b)^*+na_n^*x_1^*$ and $x_1^*=-(a_{n-1}/b)^*/na_n^*$. Let $\alpha=-(a_{n-1}/b)/na_n$. Now, as far as the requirement $v(x_1-r_1)>0$ is concerned, we are free to choose r_1 to be any element of the form $r_1=\alpha+t$, $t\in K_0$ and $v(t)>0$. For any such r_1, $f_1^{(n-1)}(r_1)=(a_{n-1}/b)+na_nr_1+((n+1)n/2)a_{n+1}br_1^2+b^2(\cdots)=na_nt+((n+1)n/2)a_{n+1}b\alpha^2+$(terms involving bt, t^2, and b^2). Therefore if we choose $t=-((n+1)/2a_n)\times(a_{n+1}b\alpha^2)$ (Note: If char $K=2$, our hypotheses imply $n+1$ is even.), then $f_1^{(n-1)}(r_1)=$(terms involving bt, t^2, and b^2). It follows that $v(t)\geq v(b)>0$ and $v(f_1^{(n-1)}(r_1))\geq 2v(b)=2v(a_{n-1})$. Q. E. D.

We shall only use the inequality of the lemma in the weak form $v(f_1^{(n-1)}(r_1))\geq v(f^{(n-1)}(r))$. We now continue the proof of 6.3.

Choose $x_1\in\mathfrak{J}(x)\cup\{x\}$ such that x_1 is a generator for y of multiplicity n and no element of $\mathfrak{J}(x)\cup\{x\}$ is a generator for y of multiplicity $<n$. If $n=0$, we are done, so assume $n>0$. Every element of $\mathfrak{J}(x_1)\subset\mathfrak{J}(x)$ is rational by hypothesis, and by 3.2 every element of $\mathfrak{J}(x_1)$ is a generator for y of multiplicity n. Thus, by replacing x by x_1 in the formulation of proposition 6.3, we may additionally assume that every element of $\mathfrak{J}(x)$ is a generator for y of multiplicity $n>0$.

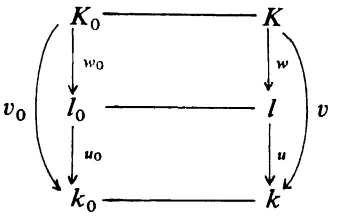
Proof of 6.3–*a*): Assume char $k=0$. Suppose we have a generating pair $x_i, f_i(X)$ of multiplicity n for y, $x_i\in\mathfrak{J}(x)$, and an $r_i\in K_0$ such that $v(x_i-r_i)>0$. Since K/K_0 is generically of index 1, there exists $b_i\in K_0$ such that $(x_i-r_i)/b_i=x_{i+1}\in\mathfrak{J}(x_i)\subset\mathfrak{J}(x)$. By the above lemma, there exists a generating pair $x_{i+1}, f_{i+1}(X)$ for y of multiplicity n and an $r_{i+1}\in K_0$ such that $f_i(x_i)=b_i^n f_{i+1}(x_{i+1})$, $v(x_{i+1}-r_{i+1})>0$, and $v(f_{i+1}^{(n-1)}(r_{i+1}))\geq v(f_i^{(n-1)}(r_i))=v(b_i)$. We thus define inductively a sequence x_i, $f_i(X)$, $i=1,2,\dots$, of generating pairs for y and elements $b_i\in K_0$ such that $f_i(x_i)=b_i^n f_{i+1}(x_{i+1})$ and $v(b_{i+1})\geq v(b_i)$. Then $y=af_1(x_1)=ab_1^n f_2(x_2)=ab_1^nb_2^nf_3(x_3)=\cdots$, where $0<v(b_1)\leq v(b_2)\leq\cdots$. Since v is rk 1, for sufficiently large t $v(ab_1^n\cdots b_t^n)>0$. But then $v(y)>0$, a contradiction.

Proof of 6.3–*b*): Assume v is discrete. To every generating pair $x_1, f_1(X)$ for y

with $x_1 \in \mathfrak{Z}(x)$ there is associated a coefficient $a = y/f_1(x_1) \in K_0$. Since $v(f_1(x_1))>0$ because x_1 is assumed to be a generator for y of multiplicity >0, we have $v(a)<0$. Choose a generating pair $x_1, f_1(X)$ of this type (i.e. for y with $x_1 \in \mathfrak{Z}(x)$) for which $-v(a)$ is minimal. (This uses v is discrete, rk 1.) Since x_1 is rational and K/K_0 is generically of index 1, there exists $x_2=(x_1-r_1)/b_1 \in \mathfrak{Z}(x_1) \subset \mathfrak{Z}(x)$. Expand: $f_1(X)= a_0+a_1(X-r_1)+\cdots+a_n(X-r_1)^n+\cdots+a_m(X-r_1)^m$, $a_i \in V_0$. Then $f_1(x_1)=b_1^n[c_0+c_1((x_1-r_1)/b_1)+\cdots+c_m((x_1-r_1)/b_1)^m]$, where $c_i=a_ib_1^{i-n}$. By 3.2–*i*), $c_0,\dots,c_{n-1} \in V_0$; and $c_n=a_n$, $c_{n+1}=a_{n+1}b_1,\dots, c_m=a_mb_1^{m-n}$ are also in V_0. Therefore if $f_2(X)= c_0+c_1X+\cdots+c_mX^m$ and $x_2=(x_1-r_1)/b_1$, it follows that $x_2, f_2(X)$ is a generating pair for y. But $y=af_1(x_1)=ab_1^nf_2(x_2)$, and $v(b_1)>0$ (since $v(b_1)=v(x_1-r_1)>0$); so $-v(ab_1^n)<-v(a)$, a contradiction to our choice of $x_1, f_1(X)$.

7. Composite valuations and the induction step for 6.2.

Recall (cf. [12, pp. 43,53]) that a valuation v of K is called *composite* with valuations w of K and u of l if $V \subset W$, l is the residue field of w, and the image V' of V under $W \to W/m_w = l$ is the valuation ring U of u. The canonical homomorphism $V \to V/m_v = k$ may then be factored: $V \to V' = U \to k$. In terms of specialization maps (or "places"; cf. [12, p. 3]), one should keep in mind the following diagram:



7.1. We shall now finish the proof of 6.2 by induction on rk v, the rk 1 case having been established in § 6. If rk $v>1$ (and finite), then v is composite with valuations w and u of strictly smaller rk.

First observe that w/w_0 is generically of index 1. For, v/v_0 is generically of index 1 implies for any generator z of K/K_0 there exists $a \in K_0$ such that z/a is a unit of V. But $V \subset W$, so z/a is also a unit of W, and therefore $w(z)=w(a)$ and w/w_0 is generically of index 1.

By induction hypothesis applied to w, there exists a generator z of K/K_0 such that $z \xrightarrow{w} z'$ tr. over l_0. Replacing z by either $1+z$ or $1+(1/z)$ if necessary, we may further assume $v(z)=0$ and hence also $u(z')=0$. Now let $l_1=l_0(z') \subset l$, and let $u_1=u \mid l_1$. We want to check next that the hypotheses of 6.1 hold for u_1/u_0.

Claim: u_1/u_0 is generically of index 1. First observe that for any element $\beta \neq 0$ of l which has a w-preimage $b \in K$ which is a *generator* of K/K_0, $u(\beta) \in u(l_0)$. For v/v_0 is generically of index 1 implies there exists $a \neq 0 \in K_0$ such that b/a is a unit of $V \subset W$. Then $w(a)=w(b)=0$, $a \xrightarrow{w} \alpha \neq 0 \in l_0$, and $b/a \xrightarrow{w} \beta/\alpha$. But b/a is a unit of V implies β/α is a unit of $V'=U$, so $u(\beta)=u(\alpha) \in u(l_0)$. Next observe that to check u_1/u_0 is generically of index 1, it suffices by 4.1 to show that for any $r' \in l_0$ such that

$u(z'-r')>0$, $u(z'-r')\in u(l_0)$. But $z'-r'$ has a w-preimage $z-r$, $r\in K_0$, in K which is a generator of K/K_0; so the previous observation applies.

Claim: Given any valuation overring R_1 of U_1 in l_1, the residue field of $R_1\cap l_0=R_0$ is algebraically closed in the residue field of R_1. To see this, first note that there exists a valuation overring R of U in l such that $R\cap l_1=R_1$ (cf. [12, p. 53, Lemma 4]). The inverse image of R under $W\to l$ is a valuation ring T lying between V and W; so by the hypothesis on V, the residue field $\mathcal{O}_0$ of $T\cap K_0$ is algebraically closed in the residue field $\mathcal{O}$ of T. But $\mathcal{O}$, $\mathcal{O}_0$ are also the residue fields of R, R_0, respectively, and the residue field of R_1 lies between $\mathcal{O}_0$ and $\mathcal{O}$, thereby establishing our assertion.

Claim: The residue field of u_0, $l_0(=k_0)\neq$ residue field of u_1, l_1. For, l is algebraic over l_1 implies k is algebraic over the residue field of u_1. Since k/k_0 is not algebraic by hypothesis, $k_0\neq$ residue field of u_1.

Thus, we may apply the induction hypothesis to u_1/u_0 to conclude there exists a generator of l_1/l_0 which specializes under u to a tr. over k_0. By 5.4-Corollary this generator may be assumed to be of the form $(z'-r')/s'$, for some r', $0\neq s'\in l_0$. But then if r, s are w-preimages in K_0 for r', s', $(z-r)/s\xrightarrow{w}(z'-r')/s'$; and therefore $(z-r)/s$ is the desired generator of K/K_0 which specializes under v to a tr. over k_0.

Q. E. D.

7.2. We give next an example to show "K_0 is henselian" cannot be omitted from 3.7 and the condition on the residue fields in 6.1 cannot be weakened to "k_0 is algebraically closed in k". The example will have the following properties: v, v_0 are discrete, rk 2; index of $v/v_0=1$; k/k_0 is simple tr.; $k_0=Q$. The idea is to construct discrete, rk 1 valuations w, u such that v is composite with w and u and such that (in the initial notation of § 7) l/l_0 is not simple tr. Then no generator of K/K_0 can specialize under v to a tr. over k_0; for if it did, it would also specialize under w to a tr. over l_0, and by 1.1 this would imply l/l_0 is simple tr.

Let s, z be complex numbers algebraically independent over Q, and let t be an indeterminate over C. Let $K_0=Q(s,t)$ and $K=K_0(x)$, where $x=(1+s)^{1/2}+zt$, and let w be the restriction of the t-adic valuation of $C(t)$ to K. Then $l_0=Q(s)$ and $l=l_0((1+s)^{1/2}, z)$, as we have seen in 5.1. Now let u_0 be the s-adic valuation of l_0; extend first to a valuation u_1 of $l_0((1+s)^{1/2})$ and then to a valuation u of l by infs w.r.t. $u(z)=0$.

The residue field k_0 of u_0 is Q; and the residue field k_1 of u_1 remains Q, since u_0 extends in two ways to $l_0((1+s)^{1/2})$ (because if $\xi=(1+s)^{1/2}$, then $s=\xi^2-1=(\xi-1)\cdot(\xi+1)$ implies u_0 extends to $l_0(\xi)=Q(\xi)$ either by $u_1(\xi-1)=1$, $u_1(\xi+1)=0$, or the reverse). Therefore by 1.1 the residue field k of u is $Q(z^*)$, where $z\xrightarrow{u}z^*$.

Finally, v/v_0 is of index 1 because w/w_0 and u/u_0 are of index 1. (To see this, let $a\neq 0\in K$. Then w/w_0 is of index 1 implies there exists $a_0\neq 0\in K_0$ such that $a/a_0\xrightarrow{w}\beta\neq 0$. Similarly, u/u_0 is of index 1 implies there exists $\beta_0\neq 0\in l_0$ such that $\beta/\beta_0\xrightarrow{u}\gamma\neq 0$. Let b_0 be a w-preimage for β_0 in K_0. Then $a/a_0b_0\xrightarrow{w}\beta/\beta_0\xrightarrow{u}\gamma\neq 0$, so $v(a)=v(a_0b_0)\in v(K_0)$.)

7.3. We conclude § 7 with a proposition on composite valuations needed in § 8.

Proposition. *Let z be a generator of K/K_0, and suppose $[G: G_0]<\infty$ and v is composite with a valuation w of K. If v is the inf extension of v_0 w.r.t. $v(z)$, then w is the inf extension of w_0 w.r.t. $w(z)$ (and w/w_0 is of finite index).*

Proof. Let H be the value group of w. If the coset $v(z)+G_0$ has order n in G/G_0, then $w(z)+H_0$ has order n_1 dividing n in H/H_0. For, if there exists $b\neq 0\in K_0$ such that $v(z^n/b)=0$, then z^n/b is a unit of V and a fortiori a unit of W; and therefore $w(z^n)=w(b)\in H_0$. Thus, $n=n_1m$ for some integer $m\geq 1$.

By 4.3, there exists $b\neq 0\in K_0$ such that $z^n/b\xrightarrow{v}\eta$ tr. over k_0, which implies $z^n/b\xrightarrow{w}\eta'$ tr. over l_0. Also, there exists $c\neq 0\in K_0$ such that $w(z^{n_1})=w(c)$. Then $(z^{n_1}/c)^m=z^n/c^m=z^n/db$, d a unit of W_0. Hence $(z^{n_1}/c)^m\xrightarrow{w}\eta'/d'$, $d'\in l_0$. But η' is tr. over l_0, so we must have z^{n_1}/c also specializes under w to a tr. over l_0. Therefore by 4.3 w is the inf extension of w_0 w.r.t. w(z).

8. Conjecture 6.1 for arbitrary inf extensions.

What is the appropriate generalization of conjecture 6.1 to arbitrary inf extensions? It is a somewhat surprising fact that the obvious reformulation is not quite correct; one needs an extra condition, "every generator of $K_0(z^n)/K_0$ has value in G_0" below, as we shall show in example 8.2.

8.1. Conjecture.

For every valuation overring $W\subset K$ of V the residue field l_0 of $W\cap K_0$ is algebraically closed in the residue field l of W; $k_0\neq k$; and there exists a generator z of K/K_0 with $v(z)+G_0$ of order $n\geq 1$ in G/G_0 such that every generator of K/K_0 has value in $\{iv(z)+G_0 \mid i=0,\dots, n-1\}$ and every generator of $K_0(z^n)/K_0$ has value in $G_0(\Longleftrightarrow v$ is the inf extension of v_0 w.r.t. $v(z_1)$ for some generator z_1 of K/K_0 such that $v(z_1)+G_0$ has order n in G/G_0.

Note that the converse ($\Leftarrow$) to the conjecture is true: if v is the inf extension of v_0 w.r.t. $v(z)$, then the value group of $K_0(z^n)/K_0$ is G_0 by 4.3; the group G/G_0 is cyclic generated by $v(z)+G_0$ by the definition of inf extension w.r.t. $v(z)$; and l/l_0 is simple tr., by 7.3 and 4.3, and a fortiori satisfies the hypothesis of the conjecture.

8.2. Examples. If Γ is any totally ordered abelian group and L a field, then the group ring $L[\Gamma]=\oplus\{LX^\gamma \mid \gamma\in\Gamma\}$, with multiplication defined by $X^\gamma X^\delta=X^{\gamma+\delta}$, may be given a valuation w by defining $w(a_0X^{\gamma_0}+\cdots+a_tX^{\gamma_t})=\inf\{\gamma_i \mid i=0,\dots, t\}$; and, as usual, this valuation extends to the quotient field $L(\Gamma)$ of $L[\Gamma]$. Moreover, the value group of w is Γ, and one verifies easily that the residue field is L.

Let $Q(t)$ be a simple tr. extension of Q, let Γ be the additive subgroup of the reals consisting of $\{\alpha+\beta\pi \mid \alpha, \beta\in Z\}$, and let w be the (rk 1) valuation of $Q(t)(\Gamma)$ described above. Let $z=X^1+tX^\pi$, let $K=K_0(z)$, where $K_0\subset Q(t)(\Gamma)$ will be described presently, and let v_0, v be the restrictions of w to K_0, K respectively. a) Example where k is simple tr. over k_0 but G/G_0 is not cyclic (and hence v cannot be an inf extension of v_0 w.r.t. any choice of generator of K/K_0). Take $K_0=Q(G_0)$, where

G_0 is the subgroup of Γ consisting of $\{\alpha+\beta\pi \mid \alpha, \beta \in 2Z\}$. Then the value group of v_0 is G_0 and the residue field k_0 is Q. Since $z^2=X^2+2tX^{1+\pi}+t^2X^{2\pi}$ and $X^2=r\in K_0$, $z^2-r\in K$, and therefore $v(z^2-r)=1+\pi$ is in the value group G of v. Since $v(z)=1$, it follows that $1, \pi\in G$; so $G=\Gamma$. Then $G/G_0\cong(Z/2Z)\oplus(Z/2Z)$.

Now let us compute k. (Incidentally, we know $Q=k_0\subset k\subset Q(t)=$ residue field of w, so without further ado we already know by Lüroth's theorem that k/k_0 is simple tr.) We have $(z^2-r)^2=4t^2X^{2+2\pi}+4t^3X^{1+3\pi}+t^4X^{4\pi}$. Let $s=4X^{2+2\pi}\in K_0$. Then $\xi=(z^2-r)^2/s\xrightarrow{v}t^2$. Since t^2 is tr. over k_0, it follows that the residue field of $K_0(\xi)$ is $Q(t^2)$ (cf. 1.1); and the value group of $K_0(\xi)$ is G_0. But then $[G: G_0]=4$ and $[K: K_0(\xi)]\leq 4$ imply (by 1.2) that the resdidue field k of K is also $Q(t^2)$.

Remark. In light of this example, it would be interesting to know just what finite groups G/G_0 can occur when k is simple tr. over k_0 (and, of course, also K is simple tr. over K_0).[2] If $k=k_0$, results of this type, due to Mac Lane-Schilling, are discussed in [12, p. 102].

b) Example to show that the hypothesis "every generator of $K_0(z^n)/K_0$ has value in G_0" is needed in 8.1. Take $K_0=Q(G_0)$, where G_0 is the subgroup of Γ consisting of $\{\alpha+\beta\pi \mid \alpha\in 2Z, \beta\in Z\}$. Then $v(z)=1$ implies the value group G of K is Γ. Therefore $G/G_0\cong Z/2Z$, and $v(z)+G_0$ generates G/G_0.

Let $K_1=K_0(z^2)$. We have seen in a) that $v(z^2-r)=1+\pi$, so the value group G_1 of K_1 is $\Gamma=G$. Therefore $[G_1: G_0]=2$. Since $G_1\neq G_0$, v_1 is not the inf extension of v_0 w.r.t. $v_1(z^2)=2(\in G_0)$, and hence by 4.3 v cannot be the inf extension of v_0 w.r.t. $v(z)=1$.

Claim: v cannot be the inf extension of v_0 w.r.t. *any* generator of K/K_0. Note first that for any $s\in K_0$, $v(z)=1\neq v(s)$. If $v(s)<v(z)$, then $v(z-s)=v(s)$ and $(z-s)^2/s^2\to -1$. If, on the other hand, $v(z)<v(s)$, then $v(z-s)=v(z)=1$ and $(z-s)^2/X^2\to 1$. The claim now follows from the Proposition below, which asserts that if v is the inf extension of v_0 w.r.t. some generator of K/K_0, then there exists $s\in K_0$ such that for any $d\neq 0\in K_0$ with $v(d)=v((z-s)^2)$, $(z-s)^2/d$ specializes to a tr. over k_0.

Lemma. *Let $\xi\in K$. If $\xi/b\to$ tr. over k_0 for some $b\neq 0\in K_0$, then $\xi/b'\to$ tr. over k_0 for every $b'\in K_0$ such that $v(b')=v(\xi)$.*

Proof. $v(b')=v(\xi)=v(b)$ implies there exists a unit u of V_0 such that $b'=ub$. Therefore $\xi/b'=(1/u)(\xi/b)\to(1/u^*)(\xi/b)^*$. But $1/u^*\in k_0$.

Proposition (4.3 *continued*). *Suppose z_1 is a (tr.) generator of K/K_0 such that $v(z_1)+G_0$ has finite order $n\geq 1$ in G/G_0. If v is the inf extension of v_0 w.r.t. $v(z_1)$, then for any generator z of K/K_0, there exists $s\in K_0$ such that for any $d\in K_0$ with $v(d)=nv(z-s)$, $(z-s)^n/d\to$ tr. over k_0.*

Proof. By 4.3, there exists $b\neq 0\in K_0$ such that $z_1^n/b\to$tr. over k_0. We may write $z_1=(a_1z-c_1)/(a_2z-c_2)$, a_i, $c_i\in K_0$, $a_1c_2-a_2c_1\neq 0$. Since $[G: G_0]=n$, there

2) Added August, 1981: W. Heinzer has now proved that G/G_0 may be any finite abelian group.

exist $d_i \in K_0$ such that $nv(a_iz-c_i)=v(d_i)$, $i=1, 2$. Therefore $z_1^n/(d_1/d_2)=N_1/N_2$, where $N_i=(a_iz-c_i)^n/d_i$ has value 0. By the lemma, $z_1^n/(d_1/d_2)\to$tr. over k_0, so either N_1 or N_2 specializes to a tr. over k_0 also; say N_1 does. Then $a_1 \neq 0$ and $N_1=(z-(c_1/a_1))^n/(d_1/a_1^n)$. In view of the above lemma, we are done. Q. E. D.

In order to apply this example to 8.1, it remains to verify k_0 is algebraically closed in k. (As in a) we know a priori by Lüroth's theorem that k/k_0 is simple tr., but it is also easy to compute k directly.) We have seen in a) that the residue field of $K_0(\xi)$ is $Q(t^2)$ and the value group is G_0. Since $[K_1: K_0(\xi)]\leq 2$ and $[G_1: G_0]=2$, it follows that the residue field of K_1 must remain $Q(t^2)$. But $[K: K_1]\leq 2$ and $(z^2-r)/2X^\pi z \xrightarrow{v} t \in k$, so we must have $k=Q(t)$.

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