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Notes on liaison and duality

By

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1. Introduction.

In classifying algebraic curves in projective three space Noether [8] introduced the notion of the residual intersection (liaison). Two curves $C, C' \subset \boldsymbol{P}_k^3$, k an algebraically closed field, are linked geometrically, if they have no components in common and $C \cup C'$ is a complete intersection. In his paper [10] Rao studied the following graded S-module: $M(C) = \bigoplus_{s \in \boldsymbol{Z}} H^1(\boldsymbol{P}, \mathcal{I}_C(s))$, where $S = k[X_0, \cdots, X_3]$ and $\mathcal{I}_C$ denotes the ideal sheaf of C. If C, C' are linked, then $M(C)$ and $M(C')$ are isomorphic up to duality and shifts in gradings. One of the main results of our paper is to extend Rao's result to an arbitrary Cohen-Macaulay variety $X \subset \boldsymbol{P}_k^n$. That is, we define a formal vector $\mathfrak{m}(X)$ which is shown to be an invariant up to duality and shifts in gradings under the liaison, compare 5.

Because most of our results are valid for Gorenstein ideals instead of complete intersection ideals, we consider Gorenstein liaison, compare 2.1 resp. 5.1 for the exact definition. In the context of local algebra two ideals $\mathfrak{a}$, $\mathfrak{b}$ of a local Gorenstein ring R are linked geometrically if $\mathfrak{a}$, $\mathfrak{b}$ are of pure height, have no primary components in common, and $\mathfrak{a} \cap \mathfrak{b}$ is a Gorenstein ideal. Now the question is, what kind of properties of $R/\mathfrak{a}$ can be transformed into properties of $R/\mathfrak{b}$. Even in the case of liaison with respect to complete intersection ideals most of our results are new. As a main point we define in 3. a certain complex $J_{\mathfrak{a}}^{\cdot}$, the truncated dualizing complex of $R/\mathfrak{a}$, which is shown to be an invariant (up to a shift and duality) under liaison, i. e. $J_{\mathfrak{b}}^{\cdot}$ is up to a shift and duality isomorphic to $J_{\mathfrak{a}}^{\cdot}$. As an application it follows that $R/\mathfrak{a}$ is a Cohen-Macaulay ring (locally a Cohen-Macaulay ring, resp. a Buchsbaum ring) if and only if the corresponding property holds for $R/\mathfrak{b}$. As another application of this type we show in 4.1 that the Serre condition S_r for $R/\mathfrak{a}$ is equivalent to the vanishing of the local cohomology groups $H_{\mathfrak{m}}^i(R/\mathfrak{b}) = 0$ for $\dim R/\mathfrak{b} - r < i < \dim R/\mathfrak{b}$. In the case $\mathfrak{a}$ is linked to itself that leads to a Cohen-Macaulay criterion. For this and related results compare 4. We conclude our considerations with some geometric applications, compare 5. In particular, in 5.4 we extend Rao's invariant to an arbitrary Cohen-Macaulay variety.

As a main technical tool to establish the relation between $R/\mathfrak{a}$ and $R/\mathfrak{b}$ we use the theory of dualizing complexes. Compare Hartshorne [5], chapter V or Sharp [15] for the definitions and basic results in this theory. Once more it shows the usefulness of dualizing complexes in attacking questions in local algebra and algebraic geometry. For non-explained terminology see any textbook about commutative algebra.

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2. Liaison with respect to Gorenstein ideals.

First of all we recall the definition of the liaison studied in the papers [1], [9], [10] of Artin, Nagata, Peskine, Szpiro, and Rao. For our purposes here we extend the notion a little. We consider liaison with respect to Gorenstein ideals instead of complete intersection ideals. Let R denote a local d-dimensional Gorenstein ring with its unique maximal ideal $\mathfrak{m}$ and its residue field k. An ideal $\mathfrak{c}\subset R$ is called a Gorenstein ideal, if $R/\mathfrak{c}$ is a Gorenstein ring.

Definition 2.1. Two ideals $\mathfrak{a}$, $\mathfrak{b}\subset R$ are linked (algebraically) by a Gorenstein ideal $\mathfrak{c}$ of height g, if

a) $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of pure height g with $\mathfrak{c}\subset\mathfrak{a}\cap\mathfrak{b}$, and
b) $\mathfrak{b}/\mathfrak{c}\cong\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{c})$ and $\mathfrak{a}/\mathfrak{c}\cong\mathrm{Hom}_R(R/\mathfrak{b}, R/\mathfrak{c})$.

Furthermore, $\mathfrak{a}$ and $\mathfrak{b}$ are linked geometrically, if $\mathfrak{a}$, $\mathfrak{b}$ are ideals of pure height, have no primary components in common, and $\mathfrak{a}\cap\mathfrak{b}$ is a Gorenstein ideal.

If $\mathfrak{c}$ is a complete intersection ideal, the Definition 2.1 coincides with the usual definition, compare [9], § 1 and § 2. Assume two ideals $\mathfrak{a}$, $\mathfrak{b}\subset R$ are linked geometrically, then they are linked. The converse is true under the additional assumption, that $\mathfrak{a}$ and $\mathfrak{b}$ have no primary components in common, see Proposition 2.3. The equivalence relation generated by the linkage is called "*liaison*". Serre [16] showed that in a regular local ring a Gorenstein ideal of height two is a complete intersection. Under this additional assumption our liaison coincides with the usual one.

By identifying $\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{c})=\mathfrak{c}:\mathfrak{a}/\mathfrak{c}$ we can write b) in 2.1 as

$$\mathfrak{b}=\mathfrak{c}:\mathfrak{a} \quad\text{and}\quad \mathfrak{a}=\mathfrak{c}:\mathfrak{b}.$$

For ideals $\mathfrak{a}$, $\mathfrak{b}\subset R$ of pure height g in a Gorenstein ring R one of these relations is equivalent to the other.

Proposition 2.2. *Let $\mathfrak{a}\subset R$ denote an ideal of pure height g. Let $\mathfrak{c}\subset\mathfrak{a}$ be a Gorenstein ideal of height g. Then $\mathfrak{a}$ and $\mathfrak{c}:\mathfrak{a}$ are linked by $\mathfrak{c}$.*

Proof. By passing to the Gorenstein ring $R/\mathfrak{c}$ we reduce the problem to the following: Let $\mathfrak{a}$ be an ideal of height zero in a Gorenstein ring R. Then $\mathfrak{a}$ and $0:\mathfrak{a}$ are linked. To this end it is enough to show that $\mathfrak{a}$ is reflexive, i.e. $0:(0:\mathfrak{a})=\mathfrak{a}$. That is true for $\mathfrak{a}R_{\mathfrak{p}}$ in the zero-dimensional Gorenstein ring $R_{\mathfrak{p}}$, $\mathfrak{p}\in \mathrm{Ass}_R R/\mathfrak{a}$, because every ideal in $R_{\mathfrak{p}}$ is reflexive. Now we have $\mathfrak{a}\subset 0:(0:\mathfrak{a})$ and the inclusion

$$0:(0:\mathfrak{a})/\mathfrak{a} \hookrightarrow R/\mathfrak{a}.$$

Since $\mathfrak{a}$ is an ideal of pure height, we get

$$0_{R_{\mathfrak{p}}}:(0_{R_{\mathfrak{p}}}:\mathfrak{a}R_{\mathfrak{p}})=\mathfrak{a}R_{\mathfrak{p}}$$

for all $\mathfrak{p}\in \mathrm{Ass}_R(0:(0:\mathfrak{a})/\mathfrak{a})$. Hence $0:(0:\mathfrak{a})=\mathfrak{a}$, which proves our statement. □

Let $\mathfrak{a}, \mathfrak{b}\subset R$ denote two ideals linked by a Gorenstein ideal $\mathfrak{c}$ of height g. Then we have a canonical isomorphism

$$\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{c})\cong \mathrm{Hom}_{R/\mathfrak{c}}(R/\mathfrak{a}, R/\mathfrak{c}) \quad \text{resp.}$$

$$\mathrm{Hom}_R(R/\mathfrak{b}, R/\mathfrak{c})\cong \mathrm{Hom}_{R/\mathfrak{c}}(R/\mathfrak{b}, R/\mathfrak{c}).$$

Because $\dim R/\mathfrak{a}=\dim R/\mathfrak{b}=\dim R/\mathfrak{c}$ and $R/\mathfrak{c}$ is a Gorenstein ring it follows that

$$\mathrm{Hom}_{R/\mathfrak{c}}(R/\mathfrak{a}, R/\mathfrak{c}) \quad \text{resp.}$$

$$\mathrm{Hom}_{R/\mathfrak{c}}(R/\mathfrak{b}, R/\mathfrak{c})$$

is the canonical (or dualizing) module of $R/\mathfrak{a}$ resp. $R/\mathfrak{b}$. Compare Herzog and Kunz [7]. We abbreviate it by $K_{\mathfrak{a}}$ resp. $K_{\mathfrak{b}}$. Furthermore there is a canonical isomorphism

$$K_{\mathfrak{a}}\cong \mathrm{Ext}_R^g(R/\mathfrak{a}, R) \quad \text{resp.}$$

$$K_{\mathfrak{b}}\cong \mathrm{Ext}_R^g(R/\mathfrak{b}, R).$$

So the liaison of two ideals is closely related to properties of their canonical modules.

At this moment we want to add a proposition concerning the liaison of two ideals with no primary components in common.

Proposition 2.3. *Let* $\mathfrak{a}, \mathfrak{b}$ *denote two ideals of* R *linked by a Gorenstein ideal* $\mathfrak{c}$. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *have no primary components in common, then*

$$\mathfrak{c}=\mathfrak{a}\cap\mathfrak{b},$$

i.e. $\mathfrak{a}$ *and* $\mathfrak{b}$ *are linked geometrically.*

Proof. By passing to the Gorenstein ring $R/\mathfrak{c}$ it is enough to show $(0)=\mathfrak{a}\cap\mathfrak{b}$. In view of the definition of the liaison we get

$$0:\mathfrak{a}=\mathfrak{b} \quad \text{and} \quad 0:\mathfrak{b}=\mathfrak{a}.$$

It follows immediately

$$\mathfrak{a}\cap\mathfrak{b}=0:(\mathfrak{a}, \mathfrak{b}).$$

Therefore, $\mathfrak{a}\cap\mathfrak{b}=(0)$ if and only if $(\mathfrak{a}, \mathfrak{b})$ contains a non-zero divisor. This is true, because $\mathrm{height}(\mathfrak{a}, \mathfrak{b})\geqq 1$ by our additional assumption on the primary components. □

Example. Let $\mathfrak{c}\subset R:=k[|X_1, \cdots, X_6|]$ denote the ideal generated by the Pfaffians of the following skew symmetric matrix

$$\begin{pmatrix} 0 & X_1 & 0 & 0 & X_2 \\ -X_1 & 0 & X_3 & X_4 & 0 \\ 0 & -X_3 & 0 & X_5 & 0 \\ 0 & -X_4 & -X_5 & 0 & X_6 \\ -X_2 & 0 & 0 & -X_6 & 0 \end{pmatrix}.$$

It follows $\mathfrak{c}=(X_3X_6, X_2X_5, X_1X_6+X_2X_4, X_2X_3, X_1X_5)$, and $\mathfrak{c}$ is a Gorenstein ideal of height 3 in R, see [3], Theorem 2.1. We have the following primary decomposition

$$\mathfrak{c}=(X_1, X_2, X_3)\cap(X_1, X_2, X_6)\cap(X_2, X_5, X_6)\cap(X_3, X_5, X_1X_6+X_2X_4).$$

For instance,

$$\mathfrak{a}=(X_1, X_2, X_6) \quad\text{and}$$

$$\mathfrak{b}=(X_1, X_2, X_3)\cap(X_2, X_5, X_6)\cap(X_3, X_5, X_1X_6+X_2X_4)$$

are linked by $\mathfrak{c}$.

It seems to be a natural question to ask whether it is possible to define a liaison with respect to a Cohen-Macaulay ideal. Compare Artin and Nagata [1] for some results in this direction. But it is not clear which kind of properties are preserved under this "liaison". The Cohen-Macaulayness is no longer an invariant. Put $R:=k[|X_1, \cdots, X_4|]$. Then the ideals $\mathfrak{a}=(X_1, X_2)\cap(X_3, X_4)$ and $\mathfrak{b}=(X_2, X_3)$ are linked to the Cohen-Macaulay ideal $\mathfrak{a}\cap\mathfrak{b}$. But $R/\mathfrak{b}$ is a Cohen-Macaulay ring, while $R/\mathfrak{a}$ is not a Cohen-Macaulay ring.

3. On an invariant of the liaison.

In the sequel let R denote a local d-dimensional Gorenstein ring. Let $\mathfrak{a}\subset R$ be an ideal of pure height g. Let $E_R^{\cdot}$ be the minimal injective resolution of R over itself, i.e.

$$E_E^i\cong \bigoplus_{P\in\operatorname{Spec} R,\, \dim R/P=d-i} E_R(R/P),$$

where $E_R(R/P)$ denotes the injective hull of R/P as R-module. We define a complex

$$I_{\mathfrak{a}}^{\cdot}=\operatorname{Hom}_R(R/\mathfrak{a}, E_R^{\cdot}).$$

It follows by well-known arguments that

$$I_{\mathfrak{a}}^i\cong \bigoplus_{\mathfrak{p}\in\operatorname{Spec} R/\mathfrak{a},\, \dim R/\mathfrak{p}=d-i} E_{R/\mathfrak{a}}(R/\mathfrak{p}).$$

Therefore, we get $I_{\mathfrak{a}}^i=0$ for $i<g$ and $i>d$. In fact, the complex $I_{\mathfrak{a}}^{\cdot}$ is the dualizing complex of $R/\mathfrak{a}$, compare Hartshorne [5], Chapter V, or Sharp [15]. Note that in the derived category $I_{\mathfrak{a}}^{\cdot}$ is isomorphic to $\underline{R}\operatorname{Hom}_R(R/\mathfrak{a}, R)$. The first non-vanishing cohomology module of $I_{\mathfrak{a}}^{\cdot}$ is

$$H^g(I_{\mathfrak{a}}^{\cdot})=\mathrm{Ext}_R^g(R/\mathfrak{a},\ R)\cong K_{\mathfrak{a}}\,,$$

the canonical module of $R/\mathfrak{a}$. Factoring out it in $I_{\mathfrak{a}}^{\cdot}$ we get a short exact sequence of complexes

$$0 \longrightarrow K_{\mathfrak{a}}[-g] \longrightarrow I_{\mathfrak{a}}^{\cdot} \longrightarrow J_{\mathfrak{a}}^{\cdot} \longrightarrow 0$$

where $K_{\mathfrak{a}}[-g]$ denotes $K_{\mathfrak{a}}$ viewed as a complex and shifted g places to the right. The complex $J_{\mathfrak{a}}^{\cdot}$ is the truncated dualizing complex. We have

$$H^i(J_{\mathfrak{a}}^{\cdot})=\begin{cases} H^i(I_{\mathfrak{a}}^{\cdot}) & \text{for } i\neq g \\ 0 & \text{for } i=g. \end{cases}$$

By the Local Duality Theorem, [5], V, Theorem 6.2, we get the following canonical isomorphisms

$$\mathrm{Hom}_R(H^i(J_{\mathfrak{a}}^{\cdot}),\ E)\cong H_{\mathfrak{m}}^{d-i}(R/\mathfrak{a})\,, \qquad i\neq g,$$

where E denotes the injective hull of the residue field k. Furthermore, we call a homomorphism $\varphi^{\cdot}: M^{\cdot}\to N^{\cdot}$ of two complexes of R-modules a quasi-isomorphism, if $\varphi^{\cdot}$ induces isomorphisms on the cohomology modules. In the next we make use of the theory of derived functors and derived categories in the sense of [5], Chapter I.

Theorem 3.1. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *be two ideals of* R *linked by a Gorenstein ideal* $\mathfrak{c}$ *of height* g. *Then there exists a canonical isomorphism*

$$J_{\mathfrak{b}}^{\cdot}[g]\cong \underline{R}\ \mathrm{Hom}_R(J_{\mathfrak{a}}^{\cdot},\ R)$$

in the derived category of R, *i.e. up to a shift and dauality the truncated dualizing complex is an invariant under the liaison.*

Proof. By virtue of 2. we have the following isomorphisms

$$K_{\mathfrak{a}}\cong \mathrm{Hom}_R(R/\mathfrak{a},\ R/\mathfrak{c})\cong \mathfrak{b}/\mathfrak{c}\,.$$

Therefore, we get the canonical short exact sequence

$$0 \longrightarrow K_{\mathfrak{a}} \longrightarrow R/\mathfrak{c} \longrightarrow R/\mathfrak{b} \longrightarrow 0\,.$$

Note that $R/\mathfrak{c}$ is a Gorenstein ring. Thus we have the canonical isomorphism

$$R/\mathfrak{c}\cong \underline{R}\ \mathrm{Hom}_R(R/\mathfrak{c},\ R)[g]\,,$$

because the dualizing complex of $R/\mathfrak{c}$ is up to a shift isomorphic to the ring itself. Using this and the canonical map

$$\underline{R}\ \mathrm{Hom}_R(R/\mathfrak{a},\ R) \longrightarrow \underline{R}\ \mathrm{Hom}_R(R/\mathfrak{c},\ R)$$

induced by the epimorphism $R/\mathfrak{c}\to R/\mathfrak{a}$ we get the following commutative diagram of complexes with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{\mathfrak{a}}[-g] & \longrightarrow & \underline{R}\operatorname{Hom}_R(R/\mathfrak{c}, R) & \longrightarrow & R/\mathfrak{b}[-g] & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow \varphi^{\cdot} & & \\ 0 & \longrightarrow & K_{\mathfrak{a}}[-g] & \longrightarrow & \underline{R}\operatorname{Hom}_R(R/\mathfrak{a}, R) & \longrightarrow & J^{\cdot}_{\mathfrak{a}} & \longrightarrow & 0 \end{array}$$

where $\varphi^{\cdot}$ is defined in an obvious manner. By applying the dualizing functor $\underline{R}\operatorname{Hom}_R(\cdot, R)$ we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{R}\operatorname{Hom}_R(R/\mathfrak{b}, R)[g] & \longrightarrow & R/\mathfrak{c} & \longrightarrow & \underline{R}\operatorname{Hom}_R(K_{\mathfrak{a}}, R)[g] & \longrightarrow & 0 \\ & & \downarrow \psi^{\cdot} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \underline{R}\operatorname{Hom}_R(J_{\mathfrak{a}}, R) & \longrightarrow & R/\mathfrak{a} & \longrightarrow & \underline{R}\operatorname{Hom}_R(K_{\mathfrak{a}}, R)[g] & \longrightarrow & 0, \end{array}$$

where $\psi^{\cdot}=\underline{R}\operatorname{Hom}_R(\varphi^{\cdot}, R)$. Now we shall show that $\psi^{\cdot}$ induces a quasi-isomorphism between $J^{\cdot}_{\mathfrak{b}}[g]$ and $\underline{R}\operatorname{Hom}_R(J^{\cdot}_{\mathfrak{a}}, R)$. To this end we consider the induced homomorphisms on the cohomology modules. We get the commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \operatorname{Ext}^g_R(R/\mathfrak{b}, R) & \longrightarrow & R/\mathfrak{c} & \longrightarrow & \operatorname{Ext}^g_R(K_{\mathfrak{a}}, R) & \longrightarrow & \operatorname{Ext}^{g+1}_R(R/\mathfrak{b}, R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ext}^0_R(J^{\cdot}_{\mathfrak{a}}, R) & \longrightarrow & R/\mathfrak{a} & \longrightarrow & \operatorname{Ext}^g_R(K_{\mathfrak{a}}, R) & \longrightarrow & \operatorname{Ext}^1_R(J^{\cdot}_{\mathfrak{a}}, R) & \longrightarrow & 0 \end{array}$$

and isomorphisms for $i\geqq 1$

$$\begin{array}{ccc} \operatorname{Ext}^{g+i}_R(K_{\mathfrak{a}}, R) & \xrightarrow{\sim} & \operatorname{Ext}^{g+i+1}_R(R/\mathfrak{b}, R) \\ \| & & \downarrow \\ \operatorname{Ext}^{g+i}_R(K_{\mathfrak{a}}, R) & \xrightarrow{\sim} & \operatorname{Ext}^{i+1}_R(J^{\cdot}_{\mathfrak{a}}, R). \end{array}$$

Using $\operatorname{Ext}^g_R(R/\mathfrak{b}, R)\cong K_{\mathfrak{b}}\cong\mathfrak{a}/\mathfrak{c}$ we see that

$$\operatorname{Ext}^1_R(J^{\cdot}_{\mathfrak{a}}, R)\cong\operatorname{Ext}^{g+1}_R(R/\mathfrak{b}, R) \quad \text{and}$$

$$\operatorname{Ext}^0_R(J^{\cdot}_{\mathfrak{a}}, R)=0.$$

Therefore, $\psi^{\cdot}$ induces the desired quasi-isomorphism. □

In the case of the liaison with respect to a complete intersection ideal Theorem 3.1 is contained implicitely in [13], compare also [2], Theorem 1.

Remark 3.2. Let $C\subset \boldsymbol{P}^3_k$ be a curve, i.e. an equidimensional subvariety of $\boldsymbol{P}^3_k$ of codimension two which is a generic complete intersection. In his paper [10] Rao studied the following graded $k[X_0, \cdots, X_3]$-module

$$M(C)=\bigoplus_{s\in \boldsymbol{Z}} H^1(\boldsymbol{P}^3_k, \mathcal{I}_C(s)),$$

where $\mathcal{I}_C$ denotes the ideal sheaf of C. Let C, C' be two curves linked by a complete intersection, then Rao [10], 2.2 showed that $M(C')$ is up to duality and a shift in grading isomorphic to $M(C)$. In fact, our Theorem 3.1 shows Rao's result. Furthermore, in 5. we shall define a corresponding invariant for an arbitrary Cohen-Macaulay variety $X\subset\boldsymbol{P}^3_k$.

Using 3.1 we shall prove that the liaison preserves certain properties of linked ideals. Let $\mathfrak{a}\subset R$ denote an ideal of pure height. Then $R/\mathfrak{a}$ is called locally a Cohen-Macaulay ring (outside $\mathfrak{m}$), if $(R/\mathfrak{a})_{\mathfrak{p}}$ is a Cohen-Macaulay ring for all prime ideals $\mathfrak{p}\in\operatorname{Spec} R/\mathfrak{a}\setminus\{\mathfrak{m}\}$. For example, by [11], Satz 4, this is equivalent to the finite length of the local cohomology modules $H_{\mathfrak{m}}^{i}(R/\mathfrak{a})$, $0\leqq i<\dim R/\mathfrak{a}$. Buchsbaum rings form a particular class of locally Cohen-Macaulay rings. The concept of Buchsbaum rings was studied by Stückrad and Vogel in [17] and the literature quoted there. Denote by $e_0(x\,;A)$ the multiplicity of a local noetherian ring A with respect to a system of parameters $x=\{x_1,\cdots,x_n\}$, $n=\dim A$, Then A is said to be a Buchsbaum ring, if the difference

$$L_A(A/xA)-e_0(x\,;A)$$

is an invariant of A not depending on the system of parameters. In particular, a Cohen-Macaulay ring is a Buchsbaum ring. Compare the Example for non-Cohen-Macaulay Buchsbaum rings.

Corollary 3.3. *Let $\mathfrak{a}$, $\mathfrak{b}$ denote two ideals of R linked by a Gorenstein ideal $\mathfrak{c}$ of height g.*

a) *If $R/\mathfrak{a}$ is locally a Cohen-Macaulay ring, then $R/\mathfrak{b}$ is locally a Cohen-Macaulay ring.*

b) *If $R/\mathfrak{a}$ is a Buchsbaum ring, then $R/\mathfrak{b}$ is a Buchsbaum ring. In both cases there are canonical isomorphisms*

$$H_{\mathfrak{m}}^{i}(R/\mathfrak{b})\cong\operatorname{Hom}_R(H_{\mathfrak{m}}^{n-i}(R/\mathfrak{a}),E),$$

for $i=1,\cdots,n-1$, where $n=\dim R/\mathfrak{a}=\dim R/\mathfrak{b}$.

Proof. First of all we shall show a). By the above remark it is enough to prove the statement on the canonical isomorphisms. Because by Matlis duality the finite length of $H_{\mathfrak{m}}^{i}(R/\mathfrak{b})$, $i=1,\cdots,n-1$, follows by the finite length of $H_{\mathfrak{m}}^{n-i}(R/\mathfrak{a})$, $i=1,\cdots,n-1$. By 3.1 we have

$$\operatorname{Hom}_R(J_{\mathfrak{b}}^{\cdot},E)\cong\operatorname{Hom}_R(\underline{R}\operatorname{Hom}(J_{\mathfrak{a}}^{\cdot},R),E)[g].$$

In the derived category R is up to a shift of d places to the left isomorphic to the (normalized) dualizing complex of R. Hence, the Local Duality Theorem yields

$$\operatorname{Hom}_R(J_{\mathfrak{b}}^{\cdot},E)\cong\underline{R}\Gamma_{\mathfrak{m}}(J_{\mathfrak{a}}^{\cdot})[d+g].$$

By our assumption $J_{\mathfrak{a}}^{\cdot}$ is a complex whose cohomology modules are of finite length. Therefore, $\underline{R}\Gamma_{\mathfrak{m}}(J_{\mathfrak{a}}^{\cdot})\cong J_{\mathfrak{a}}^{\cdot}$ in the derived category. By using the isomorphisms

$$\operatorname{Hom}_R(H^i(J_{\mathfrak{b}}^{\cdot}),E)\cong H_{\mathfrak{m}}^{d-i}(R/\mathfrak{b}),\qquad i\neq g,$$

the assertion follows.

To prove b) we recall that $R/\mathfrak{a}$ is a Buchsbaum ring if and only if $J_{\mathfrak{a}}^{\cdot}$ is quasi-isomorphic to a complex of k-vector spaces, compare [13] or [14], Theorem 2.3. Because this property of a complex is preserved under the dualizing functor, b) follows by view of 3.1. □

In particular, we recover the result of Peskine and Szpiro [9], Proposition 1.3, that the Gorenstein liaison preserves the Cohen-Macaulayness of linked ideals. In the case of liaison with respect to complete intersection ideals (3.3), b) was proved in [13], compare also [2], Theorem 1. Furthermore, the paper [2] contains results concerning the classification of curves $C\subset \boldsymbol{P}_k^3$ for which Rao's invariant $M(C)$ is a finite dimensional k-vector space, i. e. those curves which are projectively Buchsbaum.

Example. Let C denote the following rational curve

$$C=\operatorname{Proj}(k[s^{4n}, s^{2n+1}t^{2n-1}, s^{2n-1}t^{2n+1}, t^{4n}]),\ n\geqq 1, \quad \text{in } \boldsymbol{P}_k^3 .$$

Let $V\subset \boldsymbol{P}_k^3$ be the complete intersection defined by the equations

$$X_0X_3-X_1X_2=0 \quad \text{and} \quad X_0X_2^{2n}-X_1^{2n}X_3=0 .$$

Let D denote the union of the two skew lines

$$X_0=X_1=0 \quad \text{and} \quad X_2=X_3=0$$

Then it follows $V=C\cup D$, i. e. C and D are linked by the complete intersection V. Now it is easy to see that D is a projectively Buchsbaum curve. So, it follows that the vertex of the affine cone over C is a non-Cohen-Macaulay Buchsbaum ring.

4. On the vanishing of local cohomology.

As it was shown in 3.3 Gorenstein liaison preserves the Cohen-Macaulayness. In the sequel we shall generalize it to the vanishing of certain local cohomology modules. To this we recall the condition S_r, $r\geqq 1$ an integer. Let $\mathfrak{a}\subset R$ denote an ideal in the d-dimensional Gorenstein ring R. The local ring $R/\mathfrak{a}$ satisfies S_r, if

$$\operatorname{depth}(R/\mathfrak{a})_{\mathfrak{p}}\geqq \min(r, \dim(R/\mathfrak{a})_{\mathfrak{p}})$$

for all prime ideals $\mathfrak{p}$ of R. It is easily seen that $R/\mathfrak{a}$ satisfies S_1 if and only if the ideal $\mathfrak{a}$ is of pure height.

Theorem 4.1. *Let $\mathfrak{a}$, $\mathfrak{b}$ denote ideals in the Gorenstein ring R linked by a Gorenstein ideal $\mathfrak{c}\subset R$ of height g. For an integer $r\geqq 1$ the following conditions are equivalent:*

(i) *$R/\mathfrak{a}$ satisfies S_r, and*
(ii) *$H_{\mathfrak{m}}^i(R/\mathfrak{b})=0$ for $\dim R/\mathfrak{b}-r<i<\dim R/\mathfrak{b}$.*

Proof. In [12] we gave a characterization of the condition S_r for the ring $R/\mathfrak{a}$ ($\mathfrak{a}$ is of pure height) in terms of the vanishing of the local cohomology groups $H_{\mathfrak{m}}^i(K_{\mathfrak{a}})$ of the canonical module $K_{\mathfrak{a}}$. In fact, by [12], (2.2), the condition (i) is equivalent to

(iii) *The canonical map*

$$H^n_{\mathfrak{m}}(K_{\mathfrak{a}}) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, E)$$

is bijective (resp. surjective for $r=1$) *and*

$$H^i_{\mathfrak{m}}(K_{\mathfrak{a}})=0 \quad \textit{for} \quad n-r+1<i<n\,,$$

where $n=\dim R/\mathfrak{a}=\dim R/\mathfrak{b}$.

To prove our assertion it is enough to establish a relation between the local cohomology groups of $R/\mathfrak{b}$ and $K_{\mathfrak{a}}$. This is done in 4.2, and our Theorem follows. □

Lemma 4.2. *Let* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\subset R$ *denote ideals as before. Then there exist a canonical exact sequence*

$$0 \longrightarrow H^{n-1}_{\mathfrak{m}}(R/\mathfrak{b}) \longrightarrow H^n_{\mathfrak{m}}(K_{\mathfrak{a}}) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, E) \longrightarrow 0$$

and canonical isomorphisms

$$H^{i-1}_{\mathfrak{m}}(R/\mathfrak{b})\cong H^i_{\mathfrak{m}}(K_{\mathfrak{a}}) \qquad \textit{for} \quad i<n\,,$$

where $n=\dim R/\mathfrak{a}=\dim R/\mathfrak{b}$.

Proof. As we have seen in the proof of 3.1 there exist a canonical exact sequence

$$0 \longrightarrow R/\mathfrak{a} \longrightarrow \mathrm{Ext}^g_R(K_{\mathfrak{a}}, R) \longrightarrow \mathrm{Ext}^{g+1}_R(R/\mathfrak{b}, R) \longrightarrow 0$$

and canonical isomorphisms

$$\mathrm{Ext}^{g+i}_R(K_{\mathfrak{a}}, R)\cong \mathrm{Ext}^{g+i+1}_R(R/\mathfrak{b}, R)$$

for all $i\geqq 1$. The statement now follows from the Local Duality Theorem for the Gorenstein ring R. □

By using 3.1 we can prove 4.1 more directly with a spectral sequence argument as it was done in a more general setting in [12]. So, 4.1 can be viewed as a consequence of our liaison invariant. Furthermore, Theorem 4.1 yields another proof of the fact, that the liaison preserves the Cohen-Macaulayness. For an ideal $\mathfrak{a}\subset R$ of pure height g which is linked to itself by a Gorenstein ideal $\mathfrak{c}$, i.e. $\mathfrak{a}=\mathfrak{c}:\mathfrak{a}$, we get a Cohen-Macaulay criterion. To this end we recall the condition C on $R/\mathfrak{a}$ introduced by Hartshorne and Ogus [6], Definition 1.7. We say that $R/\mathfrak{a}$ satisfies C, if

$$\mathrm{depth}(R/\mathfrak{a})_{\mathfrak{p}}\geqq \min(\dim(R/\mathfrak{a})_{\mathfrak{p}}, \frac{1}{2}(\dim(R/\mathfrak{a})_{\mathfrak{p}}+1))$$

for all prime ideals $\mathfrak{p}$ of R. This is equivalent to saying that $R/\mathfrak{a}$ satisfies S_2, and that whenever $\dim(R/\mathfrak{a})_{\mathfrak{p}}\geqq 3$, then

$$\mathrm{depth}(R/\mathfrak{a})_{\mathfrak{p}}\geqq \frac{1}{2}(\dim(R/\mathfrak{a})_{\mathfrak{p}}+1)\,.$$

Thus, a Cohen-Macaulay ring satisfies the condition C. It implies S_2 but not S_3.

Proposition 4.3. *Let $\mathfrak{a}\subset R$ be an ideal which is linked to itself. Assume $R/\mathfrak{a}$ satisfies condition C. Then $R/\mathfrak{a}$ is a Cohen-Macaulay ring.*

Proof. We make an induction on the dimension of $R/\mathfrak{a}$. If $\dim R/\mathfrak{a}\leqq 1$, $R/\mathfrak{a}$ is a Cohen-Macaulay ring by C. So suppose $\dim R/\mathfrak{a}\geqq 2$. Because C is a local property, the local rings $(R/\mathfrak{a})_{\mathfrak{p}}$ for $\mathfrak{p}\neq\mathfrak{m}$ are Cohen-Macaulay rings by the induction hypothesis, i.e. $R/\mathfrak{a}$ is locally a Cohen-Macaulay ring. By virtue of 3.3 we get

$$H^i_{\mathfrak{m}}(R/\mathfrak{a})=\mathrm{Hom}_R(H^{n-i}_{\mathfrak{m}}(R/\mathfrak{a}),\ E)\,,\quad n=\dim R/\mathfrak{a}\,,$$

for $i=1, \cdots, n-1$. Now depth $R/\mathfrak{a}\geqq\min\left(n,\ \frac{1}{2}(n+1)\right)$, i.e. $H^i_{\mathfrak{m}}(R/\mathfrak{a})=0$ for $i\neq n$. Thus, $R/\mathfrak{a}$ is a Cohen-Macaulay ring. □

Next we want to show an application of the liaison concerning the openess of the S_r locus in the Zariski topology. For a local noetherian ring $R/\mathfrak{a}$ we put

$$S_r(R/\mathfrak{a})=\{\mathfrak{p}\in\mathrm{Spec}\ R\,|\,(R/\mathfrak{a})_{\mathfrak{p}} \text{ satisfies } S_r\}\,,$$

$r\geqq 1$ an integer.

Proposition 4.4. *Let $\mathfrak{a}\subset R$ denote an ideal of pure height g. For an integer $r\geqq 2$ we have*

$$S_r(R/\mathfrak{a})=\mathrm{Supp}\ R/\mathfrak{a}\setminus\bigcup_{i=1}^{r-1}\mathrm{Supp}\ \mathrm{Ext}^{g+i}_R(R/\mathfrak{b},\ R)\,,$$

where $\mathfrak{b}\subset R$ is an ideal linked to $\mathfrak{a}$ by a Gorenstein ideal of height g.

Proof. First we consider the case

$$\bigcup_{i=1}^{r-1}\mathrm{Supp}\ \mathrm{Ext}^{g+i}_R(R/\mathfrak{b},\ R)=\emptyset\,.$$

That means $\mathrm{Ext}^{g+i}_R(R/\mathfrak{b},\ R)=0$, $i=1, \cdots, r-1$. By virtue of the Local Duality Theorem for R and by virtue of 4.1 this is equivalent to $S_r(R/\mathfrak{a})=\mathrm{Supp}\ R/\mathfrak{a}$. The remaining case follows the same way by a localization argument. □

5. A geometric application.

Next we shall consider liaison among projective varieties in $\boldsymbol{P}^n_k$, $n\geqq 3$, k an algebraically closed field. We call a projective variety $Z\subset\boldsymbol{P}^n_k$ projectively Cohen-Macaulay (resp. Gorenstein), if the homogeneous coordinate ring $R(Z)=S/I_Z$ is a Cohen-Macaulay (resp. Gorenstein) ring, where $S=k[X_0, \cdots, X_n]$ and I_Z denotes the homogeneous defining ideal of Z in S. By $\mathcal{I}_Z$ we denote the ideal sheaf of Z.

Definition 5.1. Two varieties $X, Y \subset \boldsymbol{P}_k^n$ are linked (algebraically) by a projectively Gorenstein variety $Z \subset \boldsymbol{P}_k^n$ of codimension g, if

a) X, Y are of pure codimension g with $Z \supset X \cup Y$, and
b) $\mathcal{I}_Y/\mathcal{I}_Z \cong \mathrm{Hom}_{\mathcal{O}_P}(\mathcal{O}_X, \mathcal{O}_Z)$ and $\mathcal{I}_X/\mathcal{I}_Z \cong \mathrm{Hom}_{\mathcal{O}_P}(\mathcal{O}_Y, \mathcal{O}_Z)$

Furthermore, X, Y are linked geometrically, if X, Y are of pure codimension g, have no components in common, and $X \cup Y = Z$ scheme theoretically, i.e. $\mathcal{I}_X \cap \mathcal{I}_Y = \mathcal{I}_Z$.

Now we shall prove a global analogue of Theorem 4.1. Moreover, it is possible to translate other results of 4. into a geometric context. But we shall restrict ourselves to 5.2 which seems to be the most important and most typical result.

Proposition 5.2. *Let $X, Y \subset \boldsymbol{P}_k^n$ be projective varieties linked by a projectively Gorenstein variety $Z \subset \boldsymbol{P}_k^n$ of codimension g. For an integer $r \geqq 1$ the following statements are equivalent:*

(i) *X satisfies the condition S_r, i.e.* $\operatorname{depth} \mathcal{O}_{X,x} \geqq \min(r, \dim \mathcal{O}_{X,x})$ *for $x \in X$, and*
(ii) *there is an integer t such that*

$$H^i(Y, \mathcal{O}_Y(s)) = 0 \quad \textit{for all} \quad |s| > t \quad \textit{and} \quad \max(0, \dim Y - r) < i < \dim Y .$$

Proof. First we want to show that $\mathrm{Ext}_{\mathcal{O}_P}^{g+i}(\mathcal{O}_Y, \mathcal{O}_P) = 0$, $0 < i < r$, is equivalent to condition (i). Let $x \in \boldsymbol{P}_k^n$, then

$$\mathcal{E}xt_{\mathcal{O}_P}^{g+i}(\mathcal{O}_Y, \mathcal{O}_P)_x \cong \mathrm{Ext}_{\mathcal{O}_{P,x}}^{g+1}(\mathcal{O}_{Y,x}, \mathcal{O}_{P,x}) = 0 , \quad 0 < i < r ,$$

is equivalent to S_r for $\mathcal{O}_{X,x}$ by virtue of the Local Duality Theorem and 4.1. But this holds for all $x \in \boldsymbol{P}$, so it proves our statement. Let $\mathfrak{a}, \mathfrak{b} \subset S$ denote the defining ideals of X, Y resp. Then we have

$$\mathcal{E}xt_{\mathcal{O}_P}^{g+i}(\mathcal{O}_Y, \mathcal{O}_P) \cong \mathrm{Ext}_S^{g+i}(S/\mathfrak{b}, S)^{\sim}.$$

That is, (i) is equivalent to the finite length of $\mathrm{Ext}_S^{g+i}(S/\mathfrak{a}, S)$, $0 < i < r$, as a graded S-module. By the Local Duality Theorem it is equivalent to the finite length of $H_{\mathfrak{m}}^i(S/\mathfrak{b})$, $\dim S/\mathfrak{b} - r < i < \dim S/\mathfrak{b}$, as a graded S-module. There exist a canonical exact sequence of graded S-modules

$$0 \longrightarrow S/\mathfrak{b} \longrightarrow \bigoplus_{s \in \boldsymbol{Z}} H^0(Y, \mathcal{O}_Y(s)) \longrightarrow H_{\mathfrak{m}}^1(S/\mathfrak{b}) \longrightarrow 0$$

and canonical isomorphisms of graded S-modules

$$\bigoplus_{s \in \boldsymbol{Z}} H^i(Y, \mathcal{O}_Y(s)) \cong H_{\mathfrak{m}}^{i+1}(S/\mathfrak{b}) , \quad i \geqq 1 .$$

Because $\mathfrak{b}$ is an ideal of pure height, $H_{\mathfrak{m}}^0(S/\mathfrak{b}) = 0$ and $H_{\mathfrak{m}}^1(S/\mathfrak{b})$ is a module of finite length. This proves the equivalence of (i) and (ii). □

As a Corollary it follows that the Gorenstein liaison preserves the Cohen-Macaulayness of linked varieties.

Corollary 5.3. *Let $X, Y, Z\subset \boldsymbol{P}_k^n$ as before. Suppose X is a Cohen-Macaulay variety. Then we get the following conditions:*

a) *Y is a Cohen-Macaulay variety, and*
b) *there are canonical isomorphisms*

$$H^i(\boldsymbol{P}, \mathcal{I}_Y(s))\cong H^{r-i}(\boldsymbol{P}, \mathcal{I}_X(e-s))^{\vee}, \qquad s\in \boldsymbol{Z},$$

for $0<i<r$, $r=\dim X=\dim Y$, where $\vee$ denotes the k-vector space dual. Furthermore, e is the integer defined by $\omega_Z=\mathcal{O}_Z(e)$ for the projectively Gorenstein variety Z.

Proof. The assertion a) follows immediately from 5.2. For the second part we remark that

$$[H_{\mathfrak{m}}^i(S/\mathfrak{a})]_s\cong H^i(\boldsymbol{P}, \mathcal{I}_X(s)), \qquad s\in \boldsymbol{Z},$$

for $0<i<r$. Here $[M]_s$ denotes the s-th graded piece of a graded S-module M. By the proof of 5.2 we see that X is a Cohen-Macaulay variety if and only if $S/\mathfrak{a}$ is locally a Cohen-Macaulay ring. So the isomorphisms follow by a graded version of 3.3. □

In particular, if Z is a complete intersection defined by forms $F_1=0, \cdots, F_g=0$, we get $e=\sum_{i=1}^{g} d_i-n-1$, where $d_i=\deg F_i$, $i=1, \cdots, g$.

Let $X\subset \boldsymbol{P}_k^n$ denote a Cohen-Macaulay variety. It follows that all the graded S-modules

$$H_*^i(\mathcal{I}_X):=\bigoplus_{s\in \boldsymbol{Z}} H^i(\boldsymbol{P}, \mathcal{I}_X(s)), \quad 0<i<\dim X,$$

are modules of finite length.

Definition 5.4. For a Cohen-Macaulay variety $X\subset \boldsymbol{P}_k^n$ we define the formal vector

$$\mathfrak{M}(X)=(H_*^1(\mathcal{I}_X), \cdots, H_*^{r-1}(\mathcal{I}_X))$$

$r=\dim X$. Furthermore, $\mathfrak{M}^{\vee}(X)$ denotes the following formal vector

$$\mathfrak{M}^{\vee}(X)=(H_*^{r-1}(\mathcal{I}_X)^{\vee}, \cdots, H_*^1(\mathcal{I}_X)^{\vee}),$$

where $^{\vee}$ is the k-vector space dual. For two pure r-dimensional Cohen-Macaulay varieties $X, Y\subset \boldsymbol{P}_k^n$ we say that $\mathfrak{M}(X)$ and $\mathfrak{M}(Y)$ are isomorphic up to a shift in degree e if

$$H_*^i(\mathcal{I}_X)\cong H_*^i(\mathcal{I}_Y)(e), \quad 1<i<r,$$

as graded S-modules, i.e. $\mathfrak{M}(X)\cong\mathfrak{M}(Y)(e)$.

In the case of a curve $C\subset \boldsymbol{P}_k^3$ the formal vector $\mathfrak{M}(C)$ coincides with the graded S-module $M(C)$ of Rao [10], 2.2. Next we want to extend Rao's result to an arbitrary Cohen-Macaulay variety.

Proposition 5.5. *Let $X, Y\subset \boldsymbol{P}_k^n$ be projective Cohen-Macaulay varieties linked* *by a projectively Gorenstein variety $Z\subset \boldsymbol{P}_k^n$. Then we have*

$$\mathfrak{M}(Y)\cong\mathfrak{M}^{\vee}(X)(-e).$$

for a certain integer e. Furthermore, for any formal vector $\mathfrak{M}=(H^1, \cdots, H^{r-1})$ of graded S-modules H^i, $0<i<r$, of finite length there exists an irreducible integral Cohen-Macaulay variety $X\subset \boldsymbol{P}_k^{r+1}$, $r=\dim X$, such that $\mathfrak{M}(X)\cong\mathfrak{M}$ up to a shift in gradings.

Proof. The first part of the Proposition is proved in 5.3. By Theorem A of Evans and Griffith [4] there exists a homogeneous prime ideal $\mathfrak{p}\subset S=k[X_0, \cdots, X_{r+1}]$ such that

$$H_{\mathfrak{m}}^i(S/\mathfrak{p})\cong H^i, \quad 0<i<r.$$

Now we define $X=V(\mathfrak{p})\subset \boldsymbol{P}_k^{r+1}$. Then it follows that $\mathfrak{M}(X)\cong\mathfrak{M}$ up to a shift in gradings. □

Does $\mathfrak{M}$ define a unique liaison class in $\boldsymbol{P}_k^{r+1}$? More generally, for an arbitrary projective variety $X\subset \boldsymbol{P}_k^n$ the truncated dualizing complex $J_{\mathcal{O}_X}^{\bullet}$ is up to duality an invariant under liaison.

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