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On the quasiconformal deformation of open Riemann surfaces and variations of some conformal invariants

By

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Introduction.

The quasiconformal mapping is an important object of the modern function theory. Specifically, it is very useful not only for the Teichmüller space theory but also for the theory of open Riemann surfaces (cf. [1], [3], [9], [17]).

In this paper, we shall study the variations of fundamental quantities on an open Riemann surface as it varies quasiconformally. Especially, we shall consider the variation of the value at each point of the solution of Dirichlet problem (§ II). Further we shall give its variational formula under a certain condition. It should be remarked that these investigations are applicable to harmonic functions which have not necessarily finite Dirichlet integral.

In § III, we shall consider the squeezing deformation of bordered Riemann surfaces and the variations of harmonic functions.

Finally, in § IV we shall show the continuity of a certain pseudo-metric related to harmonic functions with finite Dirichlet integral (For the detailed discussion of this pseudo-metric, see [13]). Then we shall prove that this result implies the continuity of Dirichlet integrals of certain reproducing kernel functions under quasiconformal deformations.

As for the basic terminologies and notations (e. g. *Dirichlet potential, maximal dilatation*, and spaces Γ, Γ_h etc.), we follow Ahlfors-Sario [5], Constantinescu-Cornea [6], Lehto-Virtanen [10], and Sario-Nakai [17].

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§ I. Basic definitions and results.

1. Let R_0, R_1 be open Riemann surfaces and $f: R_0 \to R_1$ be a quasiconformal mapping. f induces an isomorphism $f^{\#}: \Gamma(R_1) \to \Gamma(R_0)$. That is, for $\omega = a(\zeta)d\zeta + b(\zeta)d\bar{\zeta} \in \Gamma(R_1)$ $f^{\#}(\omega)$ is defined by

$$f^{\#}(\omega) = [a(f)f_z + b(f)(\bar{f})_z]dz + [a(f)f_{\bar{z}} + b(f)(\bar{f})_{\bar{z}}]d\bar{z},$$

where the mapping f is written as $\zeta=f(z)$ in terms of local parameters ζ and z on R_1 and R_0 respectively, and f_z, $f_{\bar z}$, $(\bar f)_z$, $(\bar f)_{\bar z}$ are distributional derivatives of f and $\bar f$. We put $f_h^{\#}=\mathrm{P}_h\circ f^{\#}$, where P_h is the orthogonal projection from Γ onto Γ_h. Then we know the following:

Proposition 1. ([14]) (i) *For any* $\omega\in\Gamma(R_1)$,

$$K_f^{-1/2}\|\omega\|_{R_1}\leqq\|f^{\#}(\omega)\|_{R_0}\leqq K_f^{1/2}\|\omega\|_{R_1},$$

where K_f *is the maximal dilatation of* f *and* $\|\cdot\|_{R_j}(j=0, 1)$ *denote the norms on* R_j.

(ii) $f_h^{\#}$ *gives an isomorphism from* $\Gamma_h(R_1)$ *onto* $\Gamma_h(R_0)$, *and for any* $\omega_h\in\Gamma_h(R_1)$

$$K_f^{-1/2}\|\omega_h\|_{R_1}\leqq\|f_h^{\#}(\omega_h)\|_{R_0}\leqq K_f^{1/2}\|\omega_h\|_{R_1}.$$

2. For an arbitrary open Riemann surface R we denote by R^* the *Royden's compactification* of R and by $\Delta(R)$ the *harmonic boundary* of R^*. (For the Royden's compactification see [6] or [17].) Then

Proposition 2. ([17]) *Let* $f: R_0\to R_1$ *be a quasiconformal mapping, then there exists a homeomorphism* $f^*: R_0^*\to R_1^*$ *such that* $f^*=f$ *on* R_0 *and* $f^*(\Delta(R_0))=\Delta(R_1)$. *Especially, if* v_0 *is a Dirichlet potential on* R_1, *then* $v_0\circ f$ *is also a Dirichlet potential on* R_0.

3. Let R be a Riemann surface which does not belong to class O_{HD}. For $z_1, z_2\in R$ we set

$$\mathrm{d}_H^R(z_1, z_2)=\sup\left\{\frac{|u(z_1)-u(z_2)|}{\sqrt{D_R(u)}}\ ;\ u\in HD(R),\ D_R(u)>0\right\}.$$

If $R\in O_{HD}$, we set $\mathrm{d}_H^R(z_1, z_2)=0$. Then it is known (e.g. see [13]) that $0\leqq \mathrm{d}_H^R(z_1, z_2)<\infty$, d_H^R is a pseudometric on R and $\mathrm{d}_H^R(\cdot, \cdot)$ is a continuous function with respect to each variable. Furthermore,

Proposition 3. ([13] or [18])

$$\mathrm{d}_H^R(z_1, z_2)=(2\pi)^{-1}\sqrt{D_R(p_0-p_1)},$$

where p_0 *and* p_1 *are harmonic on* $R-\{z_1, z_2\}$, $p_0+(-1)^j\log|w_j|$ *and* $p_1+(-1)^j\log|w_j|$ *are harmonic at* z_j *for the local parameters* w_j *with* $w_j(z_j)=0$ $(j=1, 2)$, *and* p_0 *and* p_1 *have respectively* L_0 *behavior and* $I(L)_1$ *behavior near the ideal boundary of* R.

Next, we note the relation between d_H^R and *the reduced extremal distance* on R.

For two distinct points z_1, z_2 on R and for sufficiently small numbers $r_1, r_2>0$, we take the local disks $D_j(r_j)=\{w_j: |w_j|<r_j\}$ $(j=1, 2)$. We denote by $\mathfrak{F}_{r_1,r_2}$ a curve family consisting of all curves which connect $\partial D_1(r_1)$ and $\partial D_2(r_2)$ on $R-D_1(r_1)\cup D_2(r_2)$. Then the reduced extremal distance $\lambda(z_1, z_2)$ is defined by the following:

$$\lambda(z_1, z_2)=\lim_{(r_1, r_2)\to 0}\{\lambda(\mathfrak{F}_{r_1, r_2})+(2\pi)^{-1}\log(r_1 r_2)\},$$

where $\lambda(\mathfrak{F}_{r_1, r_2})$ means the extremal length of $\mathfrak{F}_{r_1, r_2}$.

We define another reduced extremal distance. We consider the family $\tilde{\mathfrak{F}}_{r_1, r_2}$ consisting of all curves which connect $\partial D_1(r_1)$ and $\partial D_2(r_2)$ on $\tilde{R}-D_2(r_1)\cup D_2(r_2)$ where $\tilde{R}$ is the Alexandroff compactification of R. Just as in the case of $\lambda(z_1, z_2)$ we define the reduced extremal distance $\tilde{\lambda}(z_2, z_2)$:

$$\tilde{\lambda}(z_1, z_2)=\lim_{(r_1, r_2)\to 0}\{\lambda(\tilde{\mathfrak{F}}_{r_1, r_2})+(2\pi)^{-1}\log(r_1 r_2)\}.$$

Proposition 3. ([13])

$$\mathrm{d}_H^R(z_1, z_2)^2=\lambda(z_1, z_2)-\tilde{\lambda}(z_1, z_2).$$

Finally, when $R=D=\{|z|<1\}$, we note the following:

Proposition 4. ([13])

$$\mathrm{d}^{\mathrm{D}}(z_1, z_2)\geqq\sqrt{\pi}\,\mathrm{d}_H^D(z_1, z_2),$$

where $\mathrm{d}^{\mathrm{D}}(\,,)$ *is the hyperbolic (Poincaré) distance on D, and equality holds if and only if* $z_1=z_2$.

§II. The variation of Dirichlet solutions.

4. Let $R_n (n=0, 1, 2, \cdots)$ be open Riemann surfaces and $f_n: R_0\to R_n$ be quasiconformal mappings with maximal dilatations $K_n=K_{f_n}\to 1$ as $n\to\infty$. For each $u\in HD(R_0)$, $u\circ f_n^{-1}$ is a Dirichlet function on R_n, therefore we have the Royden's decomposition on R_n:

$$u\circ f_n^{-1}=u_n+v_{0,n}, \tag{1}$$

where $u_n\in HD(R_n)$ and $v_{0,n}$ is a *Dirichlet potential* on R_n. Then we define a mapping $\mathrm{P}^{f_n}: HD(R_0)\to HD(R_n)$ as $\mathrm{P}^{f_n}(u)=u_n$.

Lemma 1. P^{f_n} *is linear and isomorphic. Further,*

$$d\mathrm{P}^{f_n}(u)=(f_n^{-1})_h^{\#}(du). \qquad (u\in HD(R_0)) \tag{2}$$

Proof. Since the linearity and (2) are seen immediately from the definitions. we shall show only that P^{f_n} is isomorphic.

For $\mathrm{P}^{f_n}(u)\circ f_n$, we consider the Royden's decomposition on R_0:

$$\mathrm{P}^{f_n}(u)\circ f_n=\mathrm{P}^{f_n^{-1}}\circ\mathrm{P}^{f_n}(u)+v_0,$$

where v_0 is a Dirichlet potential on R_0.

Hence by (1)

$$u=\mathrm{P}^{f_n^{-1}}\circ\mathrm{P}^{f_n}(u)+(v_0+v_{0,n}\circ f_n).$$

Since $v_0+v_{0,n}\circ f_n$ is a Dirichlet potential on R_0 from Proposition 2, we have

$P^{f_n^{-1}} \circ P^{f_n}(u)=u$ from the uniqueness of the Royden's decomposition, that is, P^{f_n} is isomorphic.

Theorem 1. *Let* $R_n (n=0, 1, 2, \cdots)$ *be open Riemann surfaces and* $f_n: R_0 \to R_n$ *be quasiconformal mappings with* $K_n=K_{f_n} \to 1$ *as* $n \to \infty$. *Then for any* $u \in HD(R_0)$ *and for any* $z \in R_0$,

$$\lim_{n\to\infty} (P^{f_n}(u)) \circ f_n(z)=u(z).$$

Proof. From Lemma 1 and Proposition 1-(ii)

$$\lim_{n\to\infty} D_{R_n}(P^{f_n}(u))=D_{R_0}(u).$$

On the other hand,

$$\lim_{n\to\infty} D_{R_n}(u \circ f_n^{-1})=\lim_{n\to\infty} \|(f_n^{-1})^{\#}(du)\|_{R_n}^2=\|du\|_{R_0}^2=D_{R_0}(u).$$

Therefore, in the decomposition (1)

$$\lim_{n\to\infty} D_{R_n}(v_{0,n})=\lim_{n\to\infty} \{D_{R_n}(u \circ f_n^{-1})-D_{R_n}(P^{f_n}(u))\}=0.$$

It follows that from Proposition 1-(i)

$$\lim_{n\to\infty} D_{R_0}(v_{0,n} \circ f_n)=\lim_{n\to\infty} \|f_n^{\#}(dv_{0,n})\|_{R_0}^2=0.$$

That is, $\{v_{0,n} \circ f_n\}_1^\infty$ is a sequence of Dirichlet potentials on R_0 with Dirichlet norms converging to zero. Hence from [6] Hilfssatz 7. 8, there exists a Borel set E_u such that E_u is polar on R_0 and $\lim_{n\to\infty} v_{0,n} \circ f_n(a)=0$ for each $a \in R_0-E_u$. Thus our conclusion is valid on R_0-E_u.

If $a \in E_u$, we can take a point $a_\varepsilon \in R_0-E_u$ for each $\varepsilon>0$ such that a_ε is in a local disk $D(a)$ about a, $d^{D(a)}(a, a_\varepsilon)<\varepsilon$ and $d^{f_n(D(a))}(f_n(a), f_n(a_\varepsilon))<\varepsilon$ for sufficiently large n (cf. [10] Chapter II).

Then from Proposition 4 and the definition of d_H^R,

$$|u(a)-u(a_\varepsilon)| \leqq \frac{\varepsilon}{\sqrt{\pi}} D_{R_0}(u)^{1/2}, \tag{3}$$

$$\begin{aligned} |(P^{f_n}(u)) \circ f_n(a)-(P^{f_n}(u)) \circ f_n(a_\varepsilon)| &\leqq \frac{\varepsilon}{\sqrt{\pi}} D_{R_n}(P^{f_n}(u))^{1/2} \\ &\leqq \frac{\varepsilon}{\sqrt{\pi}} K_n^{1/2} D_{R_0}(u)^{1/2}. \end{aligned} \tag{4}$$

We can take sufficiently large n such that

$$|u(a_\varepsilon)-(P^{f_n}(u)) \circ f_n(a_\varepsilon)|<\varepsilon \tag{5}$$

because $a_\varepsilon \in R_0-E_u$.

Hence from (3), (4), and (5), we have

$$\begin{aligned} |u(a)-(P^{f_n}(u)) \circ f_n(a)| \leqq& |u(a)-u(a_\varepsilon)|+|u(a_\varepsilon)-(P^{f_n}(u)) \circ f_n(a_\varepsilon)| \\ &+|(P^{f_n}(u)) \circ f_n(a_\varepsilon)-(P^{f_n}(u)) \circ f_n(a)| \end{aligned}$$

$$\leqq \varepsilon\left(\frac{1}{\sqrt{\pi}} D_{R_0}(u)^{1/2}+\frac{1}{\sqrt{\pi}} K_n^{1/2} D_{R_0}(u)^{1/2}+1\right),$$

and the proof is complete.

Remark. We can show that $P^{f_n}(u)$ is the Dirichlet solution on R_n^* with the boundary value $u\circ(f_n^*)^{-1}$ on $\Delta(R_n)$ (cf. [6] Satz 7. 6 and Hilfssatz 8. 2). Hence Theorem 1 implies the continuity of the Dirichlet solutions on the Royden's compactifications under the quasiconformal deformation.

Corollary 1. *Let* R_n, f_n $(n=0, 1, 2, \cdots)$ *be the same as ones in Theorem* 1. *Then for any bounded continuous function* g *on* $R_0^*-R_0$ *and for any* $a\in R_0$,

$$\lim_{n\to\infty} H^{R_n^*}_{g\circ(f_n^*)^{-1}}(f_n(a))=H^{R_0^*}_g(a),$$

where $H_f^{R^*}$ *is the one defined in* [6] p. 86.

Proof. For any $\varepsilon>0$ there exists a continuous function g_ε on $\Delta(R_0)$ such that $\max\{|g(z)-g_\varepsilon(z)| : z\in\Delta(R_0)\}<\varepsilon$ and $H^{R_0^*}_{g_\varepsilon}\in HBD(R_0)$. From *the maximal principle,*

$$\sup\{|H^{R_n^*}_{g\circ(f_n^*)^{-1}}(f_n(a))-H^{R_n^*}_{g_\varepsilon\circ(f_n^*)^{-1}}(f_n(a))| : a\in R_0\}<\varepsilon,$$

$$\sup\{|H^{R_0^*}_g(a)-H^{R_0^*}_{g_\varepsilon}(a)| : a\in R_0\}<\varepsilon.$$

Hence for each $a\in R_0$ and for each $\varepsilon>0$, we have

$$\begin{aligned}&\varlimsup_{n\to\infty} |H^{R_n^*}_{g\circ(f_n^*)^{-1}}(f_n(a))-H^{R_0^*}_g(a)|\\ &\leqq\varlimsup_{n\to\infty}\{|H^{R_n^*}_{g\circ(f_n^*)^{-1}}(f_n(a))-H^{R_n^*}_{g_\varepsilon\circ(f_n^*)^{-1}}(f_n(a))|\\ &\quad+|H^{R_n^*}_{g_\varepsilon\circ(f_n^*)^{-1}}(f_n(a))-H^{R_0^*}_{g_\varepsilon}(a)|+|H^{R_0^*}_{g_\varepsilon}(a)-H^{R_0^*}_g(a)|\}\\ &\leqq 2\varepsilon.\end{aligned}$$

q. e. d.

If $S_n(n=0, 1, \cdots)$ are *compact bordered Riemann surfaces,* then K_n-quasiconformal mapping $f_n: S_0\to S_n$ can be extended to a homeomorphism of $S_0\cup\partial S_0$ into $S_n\cup\partial S_n$. We denote it by f_n again.

Corollary 2. *Let* $S_n, f_n(n=0, 1, \cdots)$ *be ones as above with* $\lim_{n\to\infty} K_n=1$. *Then for any bounded continuous function* g *on* ∂S_0,

$$\lim_{n\to\infty} H^{S_n}_{g\circ f_n^{-1}}(f_n(a))=H^{S_0}_g(a), \qquad \textit{for any} \quad a\in S_0.$$

If $\hat{g}$ *is bounded and upper semi-continuous on* ∂S_0,

$$\varlimsup_{n\to\infty} H^{S_n}_{\hat{g}\circ f_n^{-1}}(f_n(a))\leqq H^{S_0}_{\hat{g}}(a).$$

Proof. If g is a boundary value of some HD-function on S_0, $H^{S_n}_{g\circ f_n^{-1}}=P^{f_n}(H^{S_0}_g)$

from Bemerkung of Satz 7. 6 and Hilfssatz 8. 2[6]. Therefore the first statement is shown by the same proof as that of Corollary 1.

As for $\hat{g}$, we can prove easily the statement by considering the decreasing sequence of continuous functions converging to $\hat{g}$. q. e. d.

From an example such as in [4], we know that the Borel set on ∂S_0 *with harmonic measure zero* is not always preserved by a quasiconformal mapping. Therefore, *a resolutive function* on ∂S_0 is not always preserved by a quasiconformal mapping. That is, there exist compact bordered Riemann surfaces S_0, S_1 and a quasiconformal mapping $f: S_0 \rightarrow S_1$, and a resolutive function g on ∂S_0 such that $g \circ f^{-1}$ is not a resolutive function on ∂S_1. So, in order that the resolutiveness is preserved we have to assume a certain condition about $\{f_n\}_1^\infty$.

To this end we shall recall here some results about *fuchian groups* and *Poincaré series.*

Let S_0 be a compact bordered Riemann surface and G be a fuchsian group acting on the upper half plane U such that $S_0 = U/G$. Then for a function F on U we consider *the Poincaré series* of F:

$$\Theta(F, G)(z) = \sum_{A \in G} F(Az) A'(z) \qquad (z \in U).$$

Proposition 4. ([8]) *Let F be a rational function with no poles on the set of limit points of G. Then the Poincaré series $\Theta(F, G)(z)$ converges uniformly on every compact subset on $(\mathfrak{S}_0 \cup \partial \mathfrak{S}_0)$ — equivalent points of poles of F, where $\mathfrak{S}_0$ is a (certain) fundamental region of G on U.*

Let $B_1(G)$ be the set of Beltrami coefficient μ compatible with G (cf, [2], [16]) and $\|\mu\|_\infty < 1$. For each $\mu \in B_1(G)$ there is a quasiconformal automorphism f^μ of U which fixes 0, 1, and ∞, and satisfies $(f^\mu)_{\bar{z}} = \mu(z)(f^\mu)_z$ a. e.. The group $G^\mu = f^\mu \circ G \circ (f^\mu)^{-1}$ is also a fuchsian group.

Let W be a relatively compact open subset in S_0. We assume that $\{f_n\}_1^\infty$ satisfy the following condition (A) for W.

(A) *There exist $\nu_1, \cdots, \nu_m$ in $B_1(G)$ such that*

(i) *ν_i is infinitely differentiable in the real sense for $i = 1, \cdots, m$.*

(ii) *$\pi(\{support\ of\ \nu_i\}) \subset W$ for $i = 1, \cdots, m$, where π is the natural projection from U onto $U/G = S_0$.*

(iii) *For μ_n, the complex dilatation of the lift of f_n, there are uniquely determined real numbers $\alpha_{1,n}, \cdots, \alpha_{m,n}$ such that $\mu_n = \sum_{j=1}^m \alpha_{j,n} \nu_j$, and $\|\alpha^{(n)}\| = \left(\sum_{j=1}^m \alpha_{j,n}^2\right)^{1/2} \rightarrow 0$ as $n \rightarrow \infty$.*

Let g be a real valued resolutive function on ∂S_0, and V be a neighbourhood of ∂S_0 in S_0 such that $\bar{V} \cap \bar{W} = \emptyset$. If the condition (A) for W is satisfied for $f_n: S_0 \rightarrow S_n$ $(n = 1, 2, \cdots)$, then it is easy to show that $g \circ f_n^{-1}$ is a resolutive function on ∂S_n. Set $u = H_g^{S_0}$, $u_n = H_{g \circ f_n^{-1}}^{S_n}$, and $E_n d\zeta = 1/2(du + i^*du) - 1/2(d(u_n \circ f_n) + i^*d(u_n \circ f_n))$, where the local parameter is obtained by projecting the coordinate function of U.

Lemma 2. *Suppose that* $\{S_n\}_0^\infty$ *and* $\{f_n\}_1^\infty$ *are satisfying the condition* (A) *for* $W(\subset S_0)$. *Then there exists a constant* $M>0$ *not depending on* g *and* n *such that*

(6) $$|E_n|\leqq M\|\alpha^{(n)}\|\int_{\partial\mathfrak{S}_0}|g||dz| \qquad on \quad \mathcal{V}=\pi^{-1}(V)\cap\mathfrak{S}_0.$$

Proof. (cf. [16]) From the theory of the Dirichlet problem we have

$$\frac{1}{2}(du+i^*du)=\frac{i}{\pi}\Big(\int_{\partial\mathfrak{S}_0}g(z)\frac{\partial^2}{\partial z\partial\zeta}G_0(z, \zeta)dz\Big)d\zeta,$$

$$\frac{1}{2}(du_n+i^*du_n)=\frac{i}{\pi}\Big(\int_{\partial\mathfrak{S}_n}g\circ f_n^{-1}(z_n)\frac{\partial^2}{\partial z_n\partial\zeta_n}G_n(z_n, \zeta_n)dz_n\Big)d\zeta_n,$$

where $G_0(\cdot, \zeta)$ and $G_n(\cdot, \zeta_n)$ are the Green's functions of S_0 and S_n with poles at ζ and ζ_n respectively.

On the other hand, by a simple calculation we have

$$\Theta(K, G)(z)dzd\zeta=\frac{\partial^2}{\partial z\partial\zeta}G_0(z, \zeta)dzd\zeta,$$

where $K_\zeta(z)=K(z, \zeta)=-\frac{1}{2}(z-\zeta)^{-2}$.

Therefore, for $\zeta\in V$,

$$\frac{1}{2}(d(u_n\circ f_n)+i^*d(u_n\circ f_n))$$

$$=\frac{i}{\pi}\Big(\int_{\partial\mathfrak{S}_0}g(z)\Theta(K_{f_n(\zeta)}, G^n)(f_n(z))df_n(z)\Big)df_n(\zeta),$$

where $G^n=G^{\mu_n}$.

A direct computation (cf. [16]-(9)) gives

$$\Theta(K_{f_n(\zeta)}, G^n)(f_n(z))df_n(z)=\Theta(\underline{K}_{\zeta,n}, G)(z)(dz+\mu_n dz),$$

where $\underline{K}_{\zeta,n}(z)=K(f_n(z))(f_n)_z(z)$. Since $\mu_n\equiv 0$ on $\mathcal{V}$,

$$E_n d\zeta=-\frac{i}{\pi}\Big(\int_{\partial\mathfrak{S}_0}g(z)\Theta(K_{\zeta,n}-K, G)(z)dz\Big)d\zeta,$$

where $K_{\zeta,n}(z)=\underline{K}_{\zeta,n}(z)(f_n)_\zeta(\zeta)$.

Generally, when we set $K_\zeta^{t\nu}(z)=K(f^{t\nu}(z), f^{t\nu}(\zeta))(f^{t\nu})_z(z)(f^{t\nu})_\zeta(\zeta)$ for $\nu\in B_1(G)$ and $\theta(t, z, \zeta)=\Theta(K_\zeta^{t\nu}, G)(z)$ $(-1\leqq t\leqq 1)$, it is known that $\theta(t, z, \zeta)$ is differentiable about t and

(7) $$\frac{\partial}{\partial t}\theta(t, z, \zeta)|_{t=0}=\frac{i}{\pi}\iint_{\mathfrak{S}_0}\Theta(K_z, G)(w)\Theta(K_\zeta, G)(w)\nu(w)dwd\bar{w}$$

(cf. [16] Proposition 7.).

Further, we may show that $\theta(t, z, \zeta)-\theta(0, z, \zeta)$ is analytic for $(z, \zeta)\in\mathcal{V}\times\mathcal{V}$. Hence when we set

$$\theta(t, z, \zeta)-\theta(0, z, \zeta)=t\cdot\frac{\partial}{\partial t}(t, z, \zeta)|_{t=1}+e(t, z, \zeta),$$

$|e(t, z, \zeta)|\leqq\tilde{M}|t|$ for a certain constant $\tilde{M}>0$ and for any $(t, z, \zeta)\in[-1, 1]\times$

$\mathfrak{V}\times\mathfrak{V}$. Hence we have

$$|E_n(\zeta)|\leqq\frac{1}{\pi}\int_{\partial\mathfrak{S}_0}|g(z)||\Theta(K_\zeta-K_{\zeta,n},\ G)(z)||dz|$$
$$\leqq\frac{\|\alpha^{(n)}\|}{\pi}\left(\tilde{M}+\sup\left|\frac{\partial}{\partial t}\theta(t,\ z,\ \zeta)|_{t=0}\right|\right)\int_{\partial\mathfrak{S}_0}|g(z)||dz|\,.$$

q. e. d.

Theorem 2. *Let $S_n(n=0,1,2,\cdots)$ be compact bordered Riemann surfaces. Suppose that K_n-quasiconformal mappings $f_n: S_0\to S_n$ satisfy the condition* (A) *for some $W(\subset S_0)$.*

Then, for any $a\in S_0$ and for any resolutive function g on ∂S_0,

$$\lim_{n\to\infty}H^{S_n}_{g\circ f_n^{-1}}(f_n(a))=H^{S_0}_g(a)\,.$$

Proof. We may assume that g is real valued. From condition (A) and Lemma 2, we have

$$\lim_{n\to\infty}D_V(u-u_n\circ f_n)=0\,,$$

where V is the same one as in Lemma 2, $u=H^{S_0}_g$ and $u_n=H^{S_n}_{g\circ f_n^{-1}}$

Since $u-u_n\circ f_n$ is harmonic on V and vanishes identically on $\partial V\cap\partial S_0$, it can be extended as a harmonic function to $\hat{V}$, the double of V with respect to $\partial V\cap\partial S_0$. Hence $(u-u_n\circ f_n)\to 0$ as $n\to\infty$ uniformly on every compact subset on $\hat{V}$.

To prove this theorem on S_0-V, we consider a (relatively compact) regular subregion W' on S_0 such that $\overline{W}\subset W'$ and $\partial W'\subset V$.

From the above argument, for any $\varepsilon>0$ and for sufficiently large number n,

$$|u\circ f_n^{-1}-u_n|<\varepsilon \qquad \text{on} \quad \partial(f_n(W'))\,.$$

Hence, $|H^{f_n(W')}_{u\circ f_n^{-1}}-H^{f_n(W')}_{u_n}|<\varepsilon$ on $f_n(W')$.

On the other hand, from Corollary 1 we have

$$|H^{W'}_u(a)-H^{f_n(W')}_{u\circ f_n^{-1}}(f_n(a))|<\varepsilon$$

for sufficiently large n and for any $a\in W'$.

Noting $H^{S_0}_g=H^{W'}_u$ on W' and $H^{S_n}_{g\circ f_n^{-1}}=H^{f_n(W')}_{u_n}$ on $f_n(W')$, we can prove our conclusion from the above inequalities.

Theorem 3. *Let ν be in $B_1(G)$ whose support is contained in $\pi^{-1}(W)$ where W is a relatively compact open subset on S_0 such that $S_0-\overline{W}$ is connected. For t $(-1\leqq t\leqq 1)$ we denote by $f_{t\nu}$ the quasiconformal mapping from S_0 onto $S_{t\nu}=U/f^{t\nu}\circ G\circ(f^{t\nu})^{-1}$ such that $\pi\circ f_{t\nu}=f^{t\nu}\circ\pi$.*

Then for any $a\in S_0-\overline{W}$ and any resolutive function g,

$$\frac{\partial}{\partial t}H^{S_{t\nu}}_{g\circ f_{t\nu}^{-1}}(f_{t\nu}(a))|_{t=0}=-\frac{2}{\pi}\,\mathrm{Re}\int_{\partial S_0}g(z)\left(\int_{a_0}^{a}F(\zeta,\ z)d\zeta\right)dz\,,$$

where $F(\zeta,z)=\iint_U K(w,\zeta)\Theta(K_z,G)(w)\nu(w)dwd\bar{w}$, $a_0\in\partial S_0$ and an integral path *from* a_0 *to* a *is taken in* $S_0-\overline{W}$.

Proof. Set $u=H_g^{S_0}$, and $u_{t_\nu}=H_{g\circ f_{t_\nu}^{-1}}^{S_{t_\nu}}$. Since u, u_{t_ν} are real valued and $\lim_{S_0\ni z\to a_0}(u(z)-u_{t_\nu}(f_{t_\nu}(z)))=0$,

$$u_{t_\nu}(f_{t_\nu}(a))-u(a)=\mathrm{Re}\int_{a_0}^{a}\{d(u_{t_\nu}\circ f_{t_\nu})+i^*d(u_{t_\nu}\circ f_{t_\nu})-(du+i^*du)\}\,,$$

Hence from (7) and the definition of E_n, we can easily show the statement.

§III. The squeezing deformation and the Dirichlet problem.

5. Let S be a bordered Riemann surface with *nodes*. We denote by $N(S)$ the set of nodes of S and $S-N(S)$ by S', and a component of S' is called a *part* of S. In this section we assume that S has at most finite number of parts and each part is a compact bordered or compact Riemann surface with finite number of punctures. The one is called a *bordered part* and the other is called a *non-bordered part*.

A *deformation* $\langle S_1, S_2, f\rangle$ of marked bordered Riemann surfaces S_1 and S_1 (cf. [1], [13]) is a continuous surjection f from $S_1\cup\partial S_1$ to $S_2\cup\partial S_2$, which preserves the marking, such that

(i) $f^{-1}|_{S_2'}$ is a homeomorphism into S_1, where $f^{-1}|_{S_2'}$ is the restriction of f^{-1} on S_2',

(ii) $f|_{\partial S_1}$ is a homeomorphism onto ∂S_2, and

(iii) for every $p\in N(S_2)$, the set of $f^{-1}(p)$ is either a node of S_1 or a simple closed curve on S_1.

A deformation $\langle S_1, S_2, f\rangle$ of marked bordered surfaces S_1 and S_2 with nodes is called *factored throngh* S if there exist deformations $\langle S_1, S, f_1\rangle$ and $\langle S, S_2, f_2\rangle$ such that $f=f_2\circ f_1$.

Let a bordered Riemann surface S_0 with nodes be given, and a neighbourhood K of the nodes of S_0 and a positive constant ε be arbitrarily fixed, then a K, ε-conformal fundamental neighbourhood $N_{K,\varepsilon}$ of S_0 is defined by the set of S, a bordered Riemann surface with nodes such that there exists a deformation $\langle S, S_0, f\rangle$ and $f^{-1}|_{(S_0-K)}$ is a $(1+\varepsilon)$-quasiconformal mapping into S. Taking $\{N_{K,\varepsilon}: K$ is a neighbourhood of $N(S_0)$ and $\varepsilon>0\}$ as a fundamental neighbourhood system at S_0, we can define the convergence of $\{S_n\}_1^\infty$, a sequence of bordered Riemann surfaces with nodes, to S_0, and then we call it the convergence in the conformal topology.

Let $S^{i0}(i=1,\cdots,k)$ be bordered parts of S_0, and g be a bounded continuous function on ∂S_0. Then we define $H_g^{S_0}$, a Dirichlet solution with boundary value g, as follows:

$$H_g^{S_0}=H_{g|_{\partial\bar{S}^{i0}}}^{\bar{S}^{i0}}\qquad\text{on}\quad S^{i0}(i=1,\cdots,k)$$

$$H_g^{S_0}=0\qquad\text{on}\quad S_0'-\bigcup_{i=1}^{k}S^{i0}\,,$$

where $\bar{S}^{i0}$ is a compact bordered Riemann surface filled in the punctures.

Theorem 4. *Let g be a bounded continuous function on ∂S_0. Suppose that $\{S_n\}_1^\infty$ converges to S_0 in the conformal topology, and $f_n: S_n \to S_0$ $(n=1, 2, \cdots)$ are mappings corresponding to the convergence as above. Then for any $a \in S^{i0}(i=1, \cdots, k)$*

$$\lim_{n\to\infty} H_{g\circ f_n}^{S_n}(f_n^{-1}(a)) = H_g^{S_0}(a). \tag{8}$$

Proof. Set $M=\max|g|$. For given $\eta>0$, we can take a neighbourhood K_η of $N(S_0)$ sufficiently small such that

$$0 < x_0(a) < \eta/M, \tag{9}$$

where x_0 is the harmonic measure of $S^{i0} \cap \partial K_\eta$ with respect to $S^{i0}-K_\eta$. Set $\tilde{S}^{in}=f_n^{-1}(S^{i0})$ and denote by x_n the harmonic measure of $S^{in} \cap f_n^{-1}(\partial K_\eta)$ with respect to $\tilde{S}^{in}-f_n^{-1}(K_\eta)$. Since f_n are $(1+\varepsilon_n)$-quasiconformal mappings on $\tilde{S}^{in}-f_n^{-1}(K_\eta)$ and $\lim_{n\to\infty} \varepsilon_n=0$, we have from Corollary 2

$$|x_0(a)-x_n(f_n^{-1}(a))| < \eta \quad \text{for sufficiently large } n. \tag{10}$$

Let v_0 be a Dirichlet solution on $S^{i0}-K_\eta$ whose boundary value is g on ∂S^{i0} and zero on $S^{i0} \cap \partial K_\eta$. and v_n is harmonic on $\tilde{S}^{in}-f_n^{-1}(K_\eta)$ whose boundary value is $g\circ f_n$ on $f_n^{-1}(\partial S^{i0})$ and zero on $f_n^{-1}(\partial K_\eta) \cap \tilde{S}^{in}$. Then from the maximum principle,

$$\begin{cases} |H_g^{S_0}(a)-v_0(a)| \leqq Mx_0(a), \\ |H_{g\circ f_n}^{S_n}(f_n^{-1}(a))-v_n(f_n^{-1}(a))| \leqq Mx_n(f_n^{-1}(a)). \end{cases} \tag{11}$$

By using Corollary 2 again, we have

$$|v_0(a)-v_n(f_n^{-1}(a))| < \eta \quad \text{for sufficiently large } n. \tag{12}$$

Thus from (9)-(12), we conclude

$$\begin{aligned} |H_g^{S_0}(a)-H_{g\circ f_n}^{S_n}(f_n^{-1}(a))| &\leqq |H_g^{S_0}(a)-v_0(a)| \\ &\quad + |v_0(a)-v_n(f_n^{-1}(a))| + |v_n(f_n^{-1}(a))-H_{g\circ f_n}^{S_n}(f_n^{-1}(a))| \\ &< 2\eta + Mx_n(f_n^{-1}(a)) \leqq 2\eta + M(\eta + x_0(a)) \\ &< (3+M)\eta, \end{aligned}$$

for sufficiently large n. This implies (8). q. e. d.

If $a \in S_0$ is in a *non-bordered part* of S_0, (8) is not true. Furthermore, we can give an example such that $a \in S_0$ is in non-bordered part of S_0 and $f_n^{-1}(a)$ is in bordered part of S_n for each $n(>0)$ but $\lim H_{g\circ f_n}^{S_n}(f_n^{-1}(a))$ does not exist.

In fact, let $S_0=\{w_1: 0<|w_1|<1\} \cup \{w_2: 0<|w_2|<\infty\} \cup \{w_3: 1<|w_3|<\infty\}$ be a bordered Riemann surface with nodes $\{w_1=0=w_2\}$ and $\{w_2=\infty=w_3\}$. We take $S_n=\{z: 1<|z|<8n^3\}$, and define $f_n: S_n \to S_0$ as follows;

$$f_n|_{1<|z|<1+2n}:\{1<|z|<1+2n\} \longrightarrow \{0<|w_1|<1\},$$

$$f_n|_{1+2n<|z|<n(5+7n)/2}:\{1+2n<|z|<n(5+7n)/2\} \longrightarrow \{0<|w_2|<\infty\},$$

$$f_n|_{n(5+7n)/2<|z|<8n^3}:\{n(5+7n)/2<|z|<8n^3\} \longrightarrow \{1<|n_3|<\infty\},$$

and

$$f_n(\{|z|=1+2n\})=\{w_1=0=w_2\},\ f_n(\{|z|=n(5+7n)/2\})=\{w_2=\infty=w_3\},$$

and the resticted maps are all surjective. Furthermore, $f_n|_{1<|z|<n}$ and $f_n|_{7n^2<|z|<8n^3}$ are $1/z$ onto $\{w_1: n^{-1}<|w_1|<1\}$ and $8n^3/z$ onto $\{w_3: 1<|w_3|<(8/7)n\}$ respectively. When $n=2m(m=1, 2, \cdots)$, $f_n|_{3n<|z|<2n^2}(z)=n^{-3/2}z$ onto $\{w_2: 3n^{-1/2}<|w_2|<2n^{1/2}\}$. When $n=2m+1(m=1, 2, \cdots)$, $f_n|_{3n<|z|<2n^2}(z)=n^{-4/3}z$ onto $\{w_2: 3n^{1/3}<|w_2|<2n^{2/3}\}$. Then we can easily verify that $\{S_n\}_1^\infty$ converges to S_0 in the conformal topology.

We take a continuous function g on ∂S_0 such that $g=0$ on $\{w_1: |w_1|=1\}$ and $=1$ on $\{w_3: |w_3|=1\}$ and a point a corresponding to $w_2=1$. Then

$$H^{S_n}_{g\circ f_n}(f_n^{-1}(a))=(\log u)(2\log 2n)^{-1};\ n=2m$$

$$H^{S_n}_{g\circ f_n}(f_n^{-1}(a))=(4\log n)(9\log 2n)^{-1};\ n=2m+1.$$

Thus, a desired example is obtained.

§IV. The continuity of d_H^R.

6. The aim of this section is to show the following theorem.

Theorem 5. *Let R_0 be an arbitrary open Riemann surface and $f_n: R_0\to R_n$ be quasiconformal mappings $(n=1, 2, \cdots)$ such that $\lim_{n\to\infty} K_n=1$. Then for any a_0, $b_0\in R_0$,*

$$\lim_{n\to\infty} \mathrm{d}_H^{R_n}(a_n, b_n)=\mathrm{d}_H^{R_0}(a_0, b_0), \tag{13}$$

where $a_n=f_n(a_0)$ and $b_n=f_n(b_0)$.

To prove this theorem we need some lemmas.

Lemma 3. *Let $S_n(n=0, 1, 2, \cdots)$ be compact bordered Riemann surfaces and W be a relatively compact open set on S_0 such that $S_0-\overline{W}$ is connected. Suppose that quasiconformal mappings $f_n: S_0\to S_n$ $(n=1, 2, \cdots)$ satisfy the condition (A) for W. Then for any a_0, $b_0\in S_0-\overline{W}$,* (13) *is valid.*

Proof. In general, we consider a Hilbert space

$$HD_a(R)=\{u\in HD(R): u(a)=0\}$$

for a fixed point $a\in R$. We denote by $u(R; a, b)$ *the reproducing kernel function* in $HD_a(R)$ such that for any $u\in HD_a(R)$

$$(du, du(R; a, b))_R=u(b).$$

Then we can easily show that $d_H^R(a, b)=\|du(R; a, b)\|_R$.

And we define a mapping $P_{f_n}: HD_{a_0}(S_0)\to HD_{a_n}(S_n)$ as $P_{f_n}(u)=P^{f_n}(u)-P_{f^n}(u)(a_n)$ for each n. Then as Lemma 1 it is shown that P_{f_n} is linear and isomrphic, furthermore, $dP_{f_n}(u)=(f_n^{-1})_h^\#(du)$.

From Lemma 2 and Schwarz's inequality, we have

$$\|du-d(P_{f_n}(u)\circ f_n)\|_V^2\leqq M_n\int_{\partial S_0}|u|^2|dz|\,,$$

where V is a connected neighbourhood of ∂S_0 in S_0-W, and $M_n>0$ $(n=1, 2, \cdots)$ are constants with $\lim_{n\to\infty} M_n=0$.

Since we know that there exists a constant $\tilde{C}>0$ such that $\int_{\partial S_0}|u|^2|dz|\leqq\tilde{C}\|du\|_{S_0}^2$ (cf. [11]), $\|du-d(P_{f_n}(u)\circ f_n)\|_V^2\leqq\tilde{C}M_n\|du\|_{S_0}^2$. Hence we have

$$(14)\qquad \begin{aligned}|u(b_0)-P_{f_n}(u)(b_n)|^2&=|u(b_0)-P_{f_n}(u)\circ f_n(b_0)|^2\\ &\leqq d_H^V(a_0, b_0)^2\|du-d(P_{f_n}(u)\circ f_n)\|_V^2\\ &\leqq\tilde{C}M_n d_H^V(a_0, b_0)^2\|du\|_{S_0}^2\,.\end{aligned}$$

On the other hand, for any $\omega_1, \omega_2\in\Gamma_h(S_n)$ from [12] Lemma 3,

$$({}^*(f_n)_h^\#(\omega_1), (f_n)_h^\#({}^*\omega_2))_{S_0}=(\omega_1, \omega_2)_{S_n}\,.$$

Thus we have

$$\begin{aligned}P_{f_n}(u)(b_n)&=(dP_{f_n}(u), du(S_n; a_n, b_n))_{S_n}\\ &=((f_n^{-1})_h^\#(du), du(S_n; a_n, b_n))_{S_n}\\ &=(du, -{}^*(f_n)_h^\#({}^*du(S_n; a_n, b_n)))_{S_0}\,.\end{aligned}$$

Hence if we denote by $dh(S_0; b_n)$ $(h(S_0; b_n)\in HD_{a_0}(S_0))$ the Γ_{he}-projection of $-{}^*(f_n)_h^\#({}^*du(S_0; a_n, b_n))$,

$$u(b_0)-P_{f_n}(u)(b_n)=(du, du(S_0; a_0, b_0)-dh(S_0; b_n))_{S_0}\,.$$

Hence from (14) we have

$$(15)\qquad \lim_{n\to\infty}\|du(S_0; a_0, b_0)-dh(S_0; b_n)\|_{S_0}=0\,.$$

On the other hand, since P_{f_n} is ismorphic, there is $v_n\in HD_{a_0}(S_0)$ such that $P_{f_n}(v_n)=u(S_n; a_n, b_n)$. Then

$$\begin{aligned}\|dh(S_0; b_n)\|_{S_0}&=\sup\{|P_{f_n}(u)(b_n)|/\|du\|_{S_0}: u\in HD_{a_0}(S_0)\}\\ &\geqq|P_{f_n}(v_n)(b_n)|/\|dv_n\|_{S_0}\\ &=|(dP_{f_n}(v_n), du(S_n; a_n, b_n))_{S_n}|/\|dv_n\|_{S_0}\\ &=\|du(S_n; a_n, b_n)\|_{S_n}\|dP_{f_n}(v_n)\|_{S_n}/\|dv_n\|_{S_0}\,.\end{aligned}$$

Since $\|dP_{f_n}(v_n)\|_{S_n}/\|dv_n\|_{S_0}\to1$ as $n\to\infty$, from (15) we have

$$(16)\qquad \|du(S_0; a_0, b_0)\|_{S_0}=\lim_{n\to\infty}\|dh(S_0; b_n)\|_{S_0}$$

$$\geqq \varlimsup_{n\to\infty} \|du(S_n\,;\,a_n,\,b_n)\|_{S_n}\,.$$

But from the definition of $h(S_0\,;\,b_n)$ and Proposition 1,

$$\lim_{n\to\infty} \|dh(S_0\,;\,b_n)\|_{S_0} \leqq \varliminf_{n\to\infty} \|(f_n)_h^{\#}(*du(S_n\,;\,a_n,\,b_n)\|_{S_0} \tag{17}$$

$$=\varliminf_{n\to\infty} \|du(S_n\,;\,a_n,\,b_n)\|_{S_n}\,.$$

Thus from (16) and (17) we conclude that

$$\lim_{n\to\infty} \mathrm{d}_H^{S_n}(a_n,\,b_n)=\lim_{n\to\infty} \|du(S_n\,;\,a_n,\,b_n)\|_{S_n}$$

$$=\|du(S_0\,;\,a_0,\,b_0)\|_{S_0}=\mathrm{d}_H^{S_n}(a_0,\,b_0)\,. \qquad \text{q. e. d.}$$

Lemma 4. *Let* $S_n(n=0,\,1,\,2,\,\cdots)$*be compact bordered Riemann surfaces and* $f_n\colon S_0\to S_n$ *be* K_n*-quasiconformal mappings with* $\lim_{n\to\infty} K_n=1$. *Then* (13) *is valid for any* $a_0,\,b_0\in S_0$.

Proof. At first, we assume that S_0 is conformal equivalent to neither an annulus or the unit disk. Then we can consider $T^{\#}(S_0)$, *the reduced Teichmüller space* of S_0 as follows.

Consider all pairs $(S,\,f)$ where S is a compact bordered Riemann surface and f is a quasiconformal mapping from S_0 onto S. We call $(S,\,f)$ and $(S',\,f')$ *equivalent* if $f'\circ f^{-1}$ is homotopic to a conformal mapping of S on S'. The reduced Teichmüller space $T^{\#}(S_0)$ is the set of eqivalent classes.

It is known (cf. [16] Proposition 6) that for sufficiently large n, there are compact bordered Riemann surfaces $\underline{S}_n$ and $\underline{K}_n$-quasiconformal mappings such that $\{(\underline{S}_n,\,\underline{f}_n)\}$ satisfies the condition (A) for a regular subregion W as Lemma 3 and $(\underline{S}_n,\,\underline{f}_n)$ is equivalent to $(S_n,\,f_n)$ in $T^{\#}(S_0)$.

Hence there exist conformal mappings $\phi_n\colon S_n\to\underline{S}_n$ such that ϕ_n is homotopic to $\underline{f}_n\circ f_n^{-1}$ for sufficiently large n. Thus $F_n=\underline{f}_n^{-1}\circ\phi_n\circ f_n\colon S_0\to S_0$ are quasiconformal mappings homotopic to the identity and $K_{F_n}\to 1$ as $n\to\infty$. Therefore $\{F_n\}$ converges to the identity uniformly on every compact subset on S_0.

When S_0 is an annulus or the unit disk, we can take a conformal mappings $\phi_n\colon S_n\to S_n$ and quasiconformal mappings $\underline{f}_n\colon S_0\to S_n$ $(n=1,\,2,\,\cdots)$ such that $\{\underline{f}_n\}_1^{\infty}$ satisfies the condition (A) for a regular subregion W with $\overline{W}\not\ni a_0,\,b_0$ and $F_n=\underline{f}_n^{-1}\circ\phi_n\circ f_n\colon S_0\to S_0$ converges to the identity uniformly on every compact subset on S_0 as $n\to\infty$.

Any way, we take quasiconformal mappings $\{F_n\}$ as above.

For any $\varepsilon>0$, by using Proposition 4 as the proof of Theorem 1, we have for sufficiently large n

$$|\mathrm{d}_H^{S_0}(a_0,\,b_0)-\mathrm{d}_H^{S_0}(F_n(a_0),\,F_n(b_0))|<\varepsilon\,,$$

$$|\mathrm{d}_H^{\underline{S}_n}(\underline{f}_n(a_0),\,\underline{f}_n(b_0))-\mathrm{d}_H^{\underline{S}_n}(\phi_n\circ f_n(a_0),\,\phi_n\circ f_n(a_0))|<\varepsilon\,,$$

where $\underline{S}_n=S_n$ if S_0 is an annulus or the unit disk.

From Lemma 3, we have

$$|\mathrm{d}_H^{S_0}(a_0, b_0)-\mathrm{d}_H^{S_n}(\underline{f}_n(a_0), \underline{f}_n(b_0))|<\varepsilon,$$

for sufficiently large n. Since ϕ_n is conformal,

$$\mathrm{d}_{\underline{H}}^{S_n}(\phi_n\circ f_n(a_0), \phi_n\circ f_n(b_0))=\mathrm{d}_H^{S_n}(a_n, b_n).$$

Therefore, we conclude

$$|\mathrm{d}_H^{S_0}(a_0, b_0)-\mathrm{d}_H^{S_n}(a_n, b_n)|<2\varepsilon,$$

for sufficiently large n. q.e.d.

Lemma 5. *Let R_0 be an orbitrary open Riemann surface and $f_n: R_0\to R_n$ be quasiconformal mappings $(n=1, 2, \cdots)$ with $\lim_{n\to\infty} K_{f_n}=1$. Then for any $a_0, b_0\in R_0$*

$$\varliminf_{n\to\infty} \mathrm{d}_H^{R_n}(a_n, b_n)\geqq \mathrm{d}_H^{R_0}(a_0, b_0),$$

where $a_n=f_n(a_0)$ and $b_n=f_n(b_0)$.

Proof. As in the proof of Lemma 3 we have

$$\mathrm{P}_{f_n}(u)(b_n)=(du, dh(R_0; b_n))_{R_0},$$

where $u\in HD_{a_0}(R_0)$ and $dh(R_0; b_n)$ $(h(R_0; b_n)\in HD_{a_0}(R_0))$ is the Γ_{he}-projection of $-*(f_n)_h^{\#}(*du(R_n; a_n, b_n))$. From Theorem 1 and the definition of P_{f_n},

$$\lim_{n\to\infty}(du, dh(R_0; b_n))_{R_0}=\lim_{n\to\infty}\mathrm{P}_{f_n}(u)(b_n)=u(b_0)$$
$$=(du, du(R_0; a_0, b_0))_{R_0}.$$

That is, $\{h(R_0; b_n)\}_1^\infty$ converges to $u(R_0; a_0, b_0)$ weakly in $HD_{a_0}(R_0)$. Therefore,

$$\varliminf_{n\to\infty}\mathrm{d}_H^{R_n}(a_n, b_n)=\varliminf_{n\to\infty}\|du(R_n; a_n, b_n)\|_{R_n}$$
$$=\varliminf_{n\to\infty}\|(f_n)_h^{\#}(*du(R_n; a_n, b_n))\|_{R_0}$$
$$\geqq\varliminf_{n\to\infty}\|dh(R_0; b_n)\|_{R_0}\geqq\|du(R_0; a_0, b_0)\|_{R_0}$$
$$=\mathrm{d}_H^{R_0}(a_0, b_0).$$

q.e.d.

7. *Proof of Theorem* 5. There exists a Borel set E on R_0 with mes $E=0$ such that $H_n(a)\leqq K_n$ for all f_n and for all $a\in R_0-E$, where $H_n(a)$ is *the circular dilatation* of f_n at a (cf. [8]).

At first, we assume that $a_0, b_0\in R_0-E$. Let z_1, z_2 be local parameters of a_0, b_0 respectively. For any sequences $\{\mathrm{r}_{j,m}\}_{m=1}^\infty$ $(j=1, 2)$ of positive numbers with $\lim_{m\to\infty}\mathrm{r}_{j,m}=0$, we set

$$\underline{\mathrm{d}}_{j,m}^n=\min\{|f_n(z_j)|:|z_j|=\mathrm{r}_{j,m}\},$$

and

$$\mathrm{d}_{j,m}^n=\max\{|f_n(z_j)|:|z_j|=\mathrm{r}_{j,m}\}\qquad(j=1, 2),$$

where local parameters of a_n, b_n are fixed for each n.

Considering the curve families as in § I-3, we set

$$\mathfrak{F}(m)=\mathfrak{F}_{r_{1,m}r_{2,m}}\,,\ \tilde{\mathfrak{F}}(m)=\tilde{\mathfrak{F}}_{r_{1,m}r_{2,m}} \qquad \text{on} \quad R_0$$

$$\mathfrak{F}(n,\underline{m})=\mathfrak{F}_{\underline{d}^n_{1,m}\underline{d}^n_{2,m}},\ \tilde{\mathfrak{F}}(n,\underline{m})=\tilde{\mathfrak{F}}_{\underline{d}^n_{1,m}\underline{d}^n_{2,m}}\,,$$

and

$$\mathfrak{F}(n,m)=\mathfrak{F}_{d^n_{1,m}d^n_{2,m}},\ \tilde{\mathfrak{F}}(n,m)=\tilde{\mathfrak{F}}_{d^n_{1,m}d^n_{2,m}} \qquad \text{on} \quad R_n\,.$$

Then obviously

$$\lambda(\mathfrak{F}(n,m))-\lambda(\tilde{\mathfrak{F}}(n,\underline{m}))\leqq\lambda(f_n(\mathfrak{F}(m)))-\lambda(f_n(\tilde{\mathfrak{F}}(m)))$$
$$\leqq\lambda(\mathfrak{F}(n,\underline{m}))-\lambda(\tilde{\mathfrak{F}}(n,m))\,.$$

From the quasiconformality of f_n, there exist constants A^m_n, $\tilde{A}^m_n$ such that $K_n^{-1}\leqq A^m_n$, $\tilde{A}^m_n\leqq K_n$ and $\lambda(f_n(\mathfrak{F}(m))=A^m_n\lambda(\mathfrak{F}(m))$, $\lambda(f_n(\tilde{\mathfrak{F}}(m)))=\tilde{A}^m_n\lambda(\tilde{\mathfrak{F}}(m))$ for $m, n=1, 2, \cdots$. Therefore

(18) $$\lambda(\mathfrak{F}(n,m))-\lambda(\tilde{\mathfrak{F}}(n,\underline{m}))$$
$$\leqq A^m_n\{\lambda(\mathfrak{F}(m))-\lambda(\tilde{\mathfrak{F}}(m))\}+(A^m_n-\tilde{A}^m_n)\lambda(\tilde{\mathfrak{F}}(m))$$
$$\leqq\lambda(\mathfrak{F}(n,\underline{m}))-\lambda(\tilde{\mathfrak{F}}(n,m))\,.$$

On the other hand, from Proposition 3 we have

$$\varliminf_{m\to\infty}\{\lambda(\mathfrak{F}(n,m))-\lambda(\tilde{\mathfrak{F}}(n,\underline{m}))\}$$
$$=\varliminf_{m\to\infty}\{\lambda(\mathfrak{F}(n,m))-\lambda(\tilde{\mathfrak{F}}(n,m))+\lambda(\tilde{\mathfrak{F}}(n,m))$$
$$+\frac{1}{2\pi}\log(d^n_{1,m}d^n_{2,m})-\lambda(\tilde{\mathfrak{F}}(n,\underline{m}))-\frac{1}{2\pi}\log(\underline{d}^n_{1,m}\underline{d}^n_{2,m})$$
$$-\frac{1}{2\pi}\log(d^n_{1,m}/\underline{d}^n_{1,m})-\frac{1}{2\pi}\log(d^n_{2,m}/\underline{d}^n_{2,m})\}$$
$$=d^{R_n}_H(a_n,b_n)^2-\frac{1}{2\pi}\log H_n(a_0)H_n(b_0)\,,$$

and

$$\varlimsup_{n\to\infty}\{\lambda(\mathfrak{F}(n,\underline{m}))-\lambda(\tilde{\mathfrak{F}}(n,m))\}$$
$$=d^{R_n}_H(a_n,b_n)^2+\frac{1}{2\pi}\log H_n(a_0)H_n(b_0)\,.$$

Hence from (18) we have

(19) $$d^{R_n}_H(a_n,b_n)^2-\frac{1}{2\pi}\log H_n(a_0)H_n(b_0)$$
$$\leqq\underline{A}_n d^{R_0}_H(a_0,b_0)^2+\underline{c}_n\leqq\bar{A}_n d^{R_0}_H(a_0,b_0)^2+\bar{c}_n$$
$$\leqq d^{R_n}_H(a_n,b_n)^2+\frac{1}{2\pi}\log H_n(a_0)H_n(b_0)\,,$$

where $\underline{A}_n=\varliminf_{m\to\infty}A^m_n$, $A_n=\varlimsup_{m\to\infty}A^m_n$, $\underline{c}_n=\varliminf_{m\to\infty}(A^m_n-\tilde{A}^m_n)\lambda(\tilde{\mathfrak{F}}(m))$, and $\bar{c}_n=\varliminf_{m\to\infty}(A^m_n-\tilde{A}^m_n)$

$\lambda(\mathfrak{F}(m))$. Since $\lim_{n\to\infty} H_n(a_0)=\lim_{n\to\infty} H_n(b_0)=\lim_{n\to\infty} \underline{A}_n=\lim_{n\to\infty} \bar{A}_n=1$, we have

$$\overline{\lim_{n\to\infty}}\, \mathrm{d}_H^{R_n}(a_n, b_n)^2 \leqq \mathrm{d}_H^{R_0}(a_0, b_0)^2+\overline{\lim_{n\to\infty}}\, \underline{c}_n \tag{20}$$

$$\leqq \mathrm{d}_H^{R_0}(a_0, b_0)^2+\overline{\lim_{n\to\infty}}\, \bar{c}_n \leqq \overline{\lim_{n\to\infty}}\, \mathrm{d}_H^{R_n}(a_n, b_n)^2 .$$

If $\overline{\lim}_{n\to\infty} \bar{c}_n>0$, then there is a certain constant $\eta>0$ and a subsequenci $\{\bar{c}_{n_p}\}$ such that $\lim_{n_p\to\infty} \bar{c}_{n_p}>2\eta$. Eor sufficiently large n_p, $c_{n_p}-(1/2\pi)\log H_{n_p}(a_0)H_{n_p}(b_0)>\eta$. It is known (cf. [18]) that $\|dp_0-dp_0^{\tilde{W}}\|_{\tilde{W}}\to 0$, $\|dp_1-dp_1^{\tilde{W}}\|_{\tilde{W}}\to 0$ as $\tilde{W}\nearrow R_0$, where $\tilde{W}$ is a regular subregion of R_0 and $p_0^{\tilde{W}}$, $p_1^{\tilde{W}}$ are principal functions on $\tilde{W}$ defined as in § I-3. Hence we conclude from Proposition 3 that $\mathrm{d}_H^{R_0}(a_0, b_0)=\lim_{\tilde{W}\nearrow R_0} \mathrm{d}_H^{\tilde{W}}(a_0, b_0)$. So, we may take a regular subregion W_η such that $W_\eta \ni a_0, b_0$, and

$$\mathrm{d}_H^{W_\eta}(a_0, b_0)^2-\eta/4 \leqq \mathrm{d}_H^{R_0}(a_0, b_0)^2 \leqq \mathrm{d}_H^{W_\eta}(a_0, b_0)^2 .$$

From the definition of d_H^R, $\mathrm{d}_H^{R_n}(a_n, b_n)\leqq \mathrm{d}_H^{f_n(W_\eta)}(a_n, b_n)$. Hence we have

$$\mathrm{d}_H^{R_{n_p}}(a_{n_p}, b_{n_p})^2-\bar{A}_{n_p}\mathrm{d}_H^{R_0}(a_0, b_0)^2 \tag{21}$$

$$\leqq \mathrm{d}_H^{f_{n_p}(W_\eta)}(a_{n_p}, b_{n_p})^2-\bar{A}_{n_p}\mathrm{d}_H^{W_\eta}(a_0, b_0)^2+\frac{\eta}{4}\bar{A}_{n_p} .$$

Since W_η is a compact bordered Riemann surface, we may assume from Lemma 4 for sufficiently large n_p

$$\mathrm{d}_H^{f_{n_p}(W_\eta)}(a_{n_p}, b_{n_p})^2-\bar{A}_{n_p}\mathrm{d}_H^{W_\eta}(a_0, b_0)^2<\eta/4 .$$

So, by (21)

$$\mathrm{d}_H^{R_{n_p}}(a_{n_p}, b_{n_p})^2-\bar{A}_{n_p}\mathrm{d}_H^{R_0}(a_0, b_0)^2<\frac{\eta}{4}(1+\bar{A}_{n_p}) \tag{22}$$

We can easily show that (19) and (22) contradict each other. Therefore, $\overline{\lim}_{n\to\infty} \bar{c}_n\leqq 0$. Then from (20)

$$\overline{\lim_{n\to\infty}}\, \mathrm{d}_H^{R_n}(a_n, b_n)\leqq \mathrm{d}_H^{R_0}(a_0, b_0) .$$

Hence it follows from Lemma 5 that if $a_0, b_0\in R_0-E$, (13) is valid.

If a_0 or $b_0\in E$, then by using the method of the proof of Theorem 1, we can conclude that (13) is valid. Hence the proof is complete.

Corollary 3. *Let R_0 and $\{R_n, f_n\}_1^\infty$ satisfy the same condition as Theorem* 5, *then*

$$\lim_{n\to\infty} \|(f_n)_h^\#(^*du_n)-{}^*du_0\|_{R_0}=0 ,$$

where u_0 and u_n are $u(R_0; a_0, b_0)$ and $u(R; a_n, b_n)$ respectively which are the same as ones defined in the proof of Lemma 3.

Proof. We can write

$$-{}^*(f_n)_h^\#(^*du_n)=dh(R_0; b_n)+\omega_n \tag{23}$$

($\omega_n \in {}^*\Gamma_{ho}(R_0)$). Then obviously,

$$\|-{}^*(f_n)_h^\#({}^*du_n)-du_0\|_{R_0}^2=\|dh(R_0\,;\,b_n)-du_0\|_{R_0}^2+\|\omega_n\|_{R_0}^2. \tag{24}$$

From Theorem 5 and its proof, we have

$$\|du_0\|_{R_0}=\lim_{n\to\infty}\|(f_n)_h^\#({}^*du_n)\|_{R_0}\geqq\varliminf_{n\to\infty}\|dh(R_0\,;\,b_n)\|_{R_0}\geqq\|du_0\|_{R_0}.$$

Hence from (23) we have $\|\omega_n\|_{R_0}\to 0$ as $n\to\infty$. Since $\{h(R_0\,;\,b_n)\}_1^\infty$ converges to u_0 weakly in $HD_{a_0}(R_0)$, $\|dh(R_0\,;\,b_n)-du_0\|_{R_0}\to 0$ as $n\to\infty$. Thus from (24) the statement follows.

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