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On the poles of the scattering matrix for two strictly convex obstacles

Dedicated to the memory of Prof. Hitoshi Kumano-go

By

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§ 1. Introduction

Let $\mathcal{O}$ be a bounded open set in $\boldsymbol{R}^3$ with sufficiently smooth boundary Γ. We set $\Omega=\boldsymbol{R}^3-\bar{\mathcal{O}}$. Suppose that Ω is connected. Consider the following acoustic problem

$$(1.1)\qquad \begin{cases} \Box u(x, t)=\dfrac{\partial^2 u}{\partial t^2}-\triangle u=0 & \text{in} \quad \Omega\times(-\infty, \infty) \\ u(x, t)=0 & \text{on} \quad \Gamma\times(-\infty, \infty) \end{cases}$$

where $\triangle=\sum_{j=1}^{3}\dfrac{\partial^2}{\partial x_j^2}$. Denote by $\mathscr{S}(\sigma)$ the scattering matrix for this problem. Concerning the definition of the scattering matrix see, for example, Lax and Phillips [8, page 9]. It is known that $\mathscr{S}(\sigma)$ is a unitary operator from $L^2(S^2)$ onto itself for all $\sigma\in\boldsymbol{R}$ and

Theorem 5.1 of Chapter V of [8]. *$\mathscr{S}(\sigma)$ extends to an operator valued function $\mathscr{S}(z)$ analytic in* $\operatorname{Im} z\leq 0$ *and meromorphic in the whole plane.*

Concerning how the scattering matrix $\mathscr{S}(\sigma)$ is related to the geometric properties of obstacles

Theorem 5.6 of Chapter V of [8]. *The scattering matrix determines the scattering.*

About a question as to a concrete correspondance of geometric properties of $\mathcal{O}$ to analytic properties of $\mathscr{S}(\sigma)$, Majda and Ralston [10], Petkov [14] and Petkov and Popov [15] made clear relationships between $\mathcal{O}$ and the asymptotic behavior of the scattering phase of $\mathscr{S}(\sigma)$ for $\sigma\to\pm\infty$ when $\mathcal{O}$ is non-trapping. But concerning relationships between $\mathcal{O}$ and the poles of $\mathscr{S}(z)$ we know a few facts. The results we want to show in this paper are

Theorem 1. *Let $\mathcal{O}=\mathcal{O}_1\cup\mathcal{O}_2$, $\bar{\mathcal{O}}_1\cap\bar{\mathcal{O}}_2=\emptyset$. Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ are strictly convex, that is, the Gaussian curvatures of the boundary Γ_j of $\mathcal{O}_j$, $j=1, 2$ never vanish. Then there exist positive constants c_0 and c_1 such that*

(i) *for any* $\varepsilon>0$ *a region*

$$\{z;\ \mathrm{Im}\, z \leq c_0+c_1-\varepsilon\} - \bigcup_{j=-\infty}^{\infty} \{z;\ |z-z_j| \leq C(|j|+1)^{-1/2}\}$$

contains only a finite number of poles of $\mathscr{S}(z)$, *where*

$$z_j = ic_0 + \frac{\pi}{d} j, \quad d = distance\ (\mathcal{O}_1, \mathcal{O}_2),$$

and C *is a constant independent of* ε,

(ii) *there exist infinitely many poles of* $\mathscr{S}(z)$ *in*

$$\bigcup_{j=-\infty}^{\infty} \{z;\ |z-z_j| \leq C(|j|+1)^{-1/2}\}.$$

Remark on constants c_0 and c_1. Let a_j, $j=1, 2$ be the points such that $a_j \in \Gamma_j$ and $|a_1-a_2|=d=$distance $(\mathcal{O}_1, \mathcal{O}_2)$. The constant c_0 is determined by d and the principal curvatures and the principal directions of Γ_j at a_j, $j=1, 2$. An explicit formula for c_0 will be given in §6, and c_1 is also estimated by using d and the principal curvatures and directions of Γ_j at a_j.

Concerning the location of the poles of $\mathscr{S}(z)$, Lax and Phillips [7], with the results on the uniform decay of local energy by Morawetz, Ralston and Strauss [13] and Melrose [11], shows that "if $\mathcal{O}$ is non-trapping there exist a, $b>0$ such that a region

$$\{z;\ \mathrm{Im}\, z \leq a \log(1+|z|)+b\}$$

contains no poles". On the other hand, Bardos, Guillot and Ralston [1] shows, under the same assumption on $\mathcal{O}$ as our Theorem 1, the existence of an infinite number of poles of $\mathscr{S}(z)$ in $\{z;\ \mathrm{Im}\, z \leq \varepsilon \log|z|\}$ for any $\varepsilon>0$. Note that $\mathcal{O}$ is always trapping if $\mathcal{O}$ consists of two disjoint objects. Then their result shows a difference in locations of poles of the scattering matrices between cases of trapping obstacles and of non-trapping obstracles.

Our Theorem 1 gives a very precise information on the position of poles of $\mathscr{S}(z)$, and represents clearly a reflection of some geometric properties of $\mathcal{O}$ in the distribution of poles. At the same time it shows that a conjecture of Lax and Phillips [8, page 158] on poles of the scattering matrix for trapping obstacles is not correct in general. Namely even in a case of a trapping obstacle, when it consists of two strictly convex objects, all the poles of $\mathscr{S}(z)$ have the imaginary parts $\geq a>0$.[1)]

If we take account of another part of Theorem 5.1 of Chapter V of [8] the first part of Theorem 1 is derived immediately from

Theorem 2. *Suppose that* $\mathcal{O}$ *satisfies the assumption in Theorem* 1. *Denote by* $U(\mu)g$ *a solution in* $\bigcap_{m>0} H^m(\Omega)$ *of a problem*

1) This fact is already shown in Theorem 2.1 of [5].

$$
(1.2) \qquad \begin{cases} (\mu^2-\triangle)u=0 & \text{in} \quad \Omega \\ u=g & \text{on} \quad \Gamma \end{cases}
$$

for $\mathrm{Re}\,\mu>0$ *and* $g\in C^\infty(\Gamma)$. *Then* $U(\mu)$ *is analytic in* $\mathrm{Re}\,\mu>0$ *as* $\mathscr{L}(C^\infty(\Gamma), C^\infty(\bar{\Omega}))$ *valued function and prolonged analytically into a region*

$$
\mathscr{D}_\varepsilon=\{\mu;\ \mathrm{Re}\,\mu\ge -c_0-c_1+\varepsilon\}-\bigcup_{j=-\infty}^{\infty}\{\mu;\ |\mu-\mu_j|\le C(|j|+1)^{-1/2}\}-\{\mu;\ |\mu|\ge C_\varepsilon\}
$$

for any $\varepsilon>0$, *where* $\mathscr{L}(C^\infty(\Gamma), C^\infty(\bar{\Omega}))$ *denotes a set of all linear continuous mappings from* $C^\infty(\Gamma)$ *into* $C^\infty(\bar{\Omega})$,

$$
\mu_i=-c_0+i\frac{\pi}{d}j=iz_j,
$$

and C *is a constant independent of* ε. *Moreover an estimate*

$$
\sum_{|\beta|\le m}\sup_{x\in\Omega_R}|D_x^\beta(U(\mu)g)(x)|\le C_{R,m,\varepsilon}\sum_{j=0}^{m+7}|\mu|^j\|g\|_{H^{m+7-j}(\Gamma)}
$$

holds for all $\mu\in\mathscr{D}_\varepsilon$, *where* $\Omega_R=\Omega\cap\{x;\ |x|<R\}$.

The second part of Theorem 1 follows from

Theorem 3. *$U(\mu)$ has an infinite number of poles in*

$$
\bigcup_{j=-\infty}^{\infty}\{\mu;\ |\mu-\mu_j|\le C(|j|+1)^{-1/2}\}.
$$

In order to prove Theorem 2 we shall construct a parametrix of a mixed problem

$$
(P) \qquad \begin{cases} \Box u=0 & \text{in} \quad \Omega\times\boldsymbol{R} \\ u=f & \text{on} \quad \Gamma\times\boldsymbol{R} \\ \mathrm{supp}\,u\subset\bar{\Omega}\times[0,\infty) \end{cases}
$$

for $f\in C_0^\infty(\Gamma\times(0,\infty))$. Our method of construction of a parametrix is the same one used in the previous paper [5]. But we examine carefully properties of asymptotic solutions constructed there, and pick up some typical behavior of solutions caused by the existence of a ray which plys a_1 and a_2.

§ 2. Properties of phase functions

Without loss of generality we may suppose

$$
a_1=(0,0,0),\ a_2=(0,0,d) \qquad (d>0).
$$

Let

$$
\Gamma_{10}=\{y(\sigma);\ \sigma\in(-\sigma_{10},\sigma_{10})\times(-\sigma_{20},\sigma_{20})=I_1\}\,(\sigma_{10},\sigma_{20}>0)
$$

and

$$\Gamma_{20}=\{z(\eta);\ \eta\in(-\eta_{10},\ \eta_{10})\times(-\eta_{20},\ \eta_{20})=I_2\}\ (\eta_{10},\ \eta_{20}>0)$$

be representations of Γ_1 near a_1 and Γ_2 near a_2 respectively, and suppose that they satisfy

$$y(0)=a_1,\ z(0)=a_2,$$

(2.1)
$$\frac{\partial y}{\partial\sigma_j}(0)=\frac{\partial z}{\partial\eta_j}(0)=Y_j,\quad j=1,\ 2,$$

where $Y_1=(1,\ 0,\ 0)$, $Y_2=(0,\ 1,\ 0)$. Set $Y_3=(0,\ 0,\ 1)$.

Let $\varphi(x)$ be a real valued C^∞ function defined near Γ_{10} which satisfies

(2.2)
$$|\nabla\varphi(x)|=\left(\sum_{j=1}^{3}\left|\frac{\partial\varphi}{\partial x_j}(x)\right|^2\right)^{1/2}=1.$$

Set $\nabla\varphi(y(\sigma))=i(\sigma)=(i_1(\sigma),\ i_2(\sigma),\ i_3(\sigma))$ and

(2.3)
$$\begin{cases}\dfrac{\partial i}{\partial\sigma_j}(\sigma)=(\kappa_{j1}(\sigma),\ \kappa_{j2}(\sigma),\ \kappa_{j3}(\sigma)),\ j=1,\ 2,\\[2ex] \mathscr{K}(\sigma)=\begin{bmatrix}\kappa_{11}(\sigma) & \kappa_{12}(\sigma)\\ \kappa_{21}(\sigma) & \kappa_{22}(\sigma)\end{bmatrix}.\end{cases}$$

We suppose that

(2.4)
$$|i(\sigma)-Y_3|\leq\delta,\quad \mathscr{K}(\sigma)>0\qquad\text{for all}\quad\sigma\in I_1$$

where δ is a small positive constant. Remark that from $i_3(\sigma)\geq 1-\delta$, $|i_l(\sigma)|\leq\delta$, $l=1,\ 2$ and

(2.5)
$$\frac{\partial i_3}{\partial\sigma_j}i_3=-\sum_{l=1}^{2}\frac{\partial i_l}{\partial\sigma_j}i_l$$

it follows that

(2.6)
$$\left|\frac{\partial i_3}{\partial\sigma_j}(\sigma)\right|\leq\delta/(1-\delta)\|\mathscr{K}(\sigma)\|,$$

where $\|\mathscr{K}(\sigma)\|$ denotes the operator norm of $\mathscr{K}(\sigma)$. Define a mapping Φ from $\Gamma_{10}\times[0,\ \infty)$ into $\boldsymbol{R}^3$ by

$$\Phi(y(\sigma),\ l)=y(\sigma)+li(\sigma).$$

Since

(2.7)
$$\sum_{j=1}^{2}\left|\frac{\partial y}{\partial\sigma_j}(\sigma)-Y_j\right|\leq C\delta_0\qquad\text{for all}\quad y(\sigma)\in S_1(\delta_0)^{2)}$$

the Jacobian determinant of $(\sigma,\ l)\to y(\sigma)+li(\sigma)$ satisfies

$$\left|\frac{D(\Phi)}{D(\sigma_1,\ \sigma_2,\ l)}\right|\geq\det\,[I+l\mathscr{K}(\sigma)]-C(\delta_0+\delta)$$

2) $S_j(\delta_0)$ is the connected component containing a_j of $\{x=(x_1,\ x_2,\ x_3);\ x\in\Gamma_j,\ x_1^2+x_2^2\leq\delta_0^2\}$, and $S(\delta_0)=S_1(\delta_0)\cup S_2(\delta_0)$.

where I denotes the unit matrix in $\boldsymbol{R}^2$. Then when δ_0 and δ are small Φ is a one to one mapping. Then for each $y(\sigma)\in S_1(\delta_0)$ there exists $l(\sigma)\in \boldsymbol{R}$ such that

$$y(\sigma)+l(\sigma)i(\sigma)\in\Gamma_{20}.$$

Set

$$z(\eta)=y(\sigma)+l(\sigma)i(\sigma). \tag{2.8}$$

Lemma 2.1. *For $y(\sigma)\in S_1(\delta_0)$ and $z(\eta)\in S_2(\delta_0)$ linked by relation* (2.8) *we have*

$$\left\|\left[\frac{\partial\sigma_p}{\partial\eta_q}\right]_{\substack{p\to 1,2\\ q\downarrow 1,2}}-[I+d\mathscr{K}(\sigma)]^{-1}\right\|\le C(\delta_0+\delta) \tag{2.9}$$

$$\left|\frac{\partial l}{\partial\sigma_p}\right|\le C(\delta_0+\delta),\ p=1,2, \tag{2.10}$$

where C is a constant independent of δ_0 and $\mathscr{K}$.

Proof. A differentiation of (2.8) by η_q gives

$$\frac{\partial z}{\partial\eta_q}=\sum_{p=1}^{2}\left(\frac{\partial y}{\partial\sigma_p}+l\frac{\partial i}{\partial\sigma_p}+\frac{\partial l}{\partial\sigma_p}i\right)\frac{\partial\sigma_p}{\partial\eta_q}, \tag{2.11}$$

from which it follows that

$$\frac{\partial z}{\partial\eta_q}-\left(\frac{\partial z}{\partial\eta_q}\cdot i\right)i=\sum_{p=1}^{2}\left[\frac{\partial y}{\partial\sigma_p}-\left(\frac{\partial y}{\partial\sigma_p}\cdot i\right)i+l\frac{\partial i}{\partial\sigma_p}\right]\frac{\partial\sigma_p}{\partial\eta_q}. \tag{2.12}$$

Taking account of (2.6), (2.7) and

$$\sum_{q=1}^{2}\left|\frac{\partial z}{\partial\eta_q}-Y_q\right|\le C\delta_0,$$

$$\left|\frac{\partial z}{\partial\eta_q}\cdot i\right|+\left|\frac{\partial y}{\partial\sigma_p}\cdot i\right|\le C(\delta_0+\delta)\qquad\text{for all}\quad z(\eta)\in S_2(\delta_0),$$

we have by the projection to (x_1, x_2)-plane of (2.12)

$$\left\|I-(I+l(\sigma)\mathscr{K}(\sigma))\left[\frac{\partial\sigma_p}{\partial\eta_q}\right]_{\substack{p\to 1,2\\ q\downarrow 1,2}}\right\|\le C(\delta_0+\delta).$$

Thus we have (2.9). The scalar product of (2.12) with i gives

$$\frac{\partial z}{\partial\eta_q}\cdot i=\sum_{q=1}^{2}\left(\frac{\partial y}{\partial\sigma_p}\cdot i+\frac{\partial l}{\partial\sigma_p}\right)\left(\frac{\partial\sigma_p}{\partial\eta_q}\right),\quad q=1,2.$$

Since

$$\left|\frac{\partial z}{\partial\eta_q}\cdot i\right|,\ \left|\frac{\partial y}{\partial\eta_p}\cdot i\right|\le C(\delta_0+\delta)$$

we have (2.10) with the aid of (2.9). Q.E.D.

Hereafter we will denote often $\sigma\in S_1(\delta_0)$, $\eta\in S_2(\delta_0)$ instead of $y(\sigma)\in S_1(\delta_0)$, $z(\eta)\in S_2(\delta_0)$ for brevity. Denote a mapping from $S_1(\delta_0)$ into Γ_{20} defined by (2.8) as

$$z(\eta)=\Theta(y(\sigma)).$$

For $x=(x_1, x_2, x_3)$ denote by x' a point $(x_1, x_2, 0)$. Suppose that

$$y(\sigma)\cdot i(\sigma)'=y(\sigma)'\cdot i(\sigma)'\geq 0 \quad \text{for all} \quad y(\sigma)\in\partial S_1(\delta_0) \tag{2.13}$$

holds. Then we have for $y(\sigma)\in\partial S_1(\delta_0)$

$$\begin{aligned}|\Phi(y(\sigma), l)'|^2&=|y(\sigma)'|^2+2ly(\sigma)'\cdot i(\sigma)'+l^2|i(\sigma)'|^2\\&\geq|y(\sigma)'|^2.\end{aligned}$$

Therefore

$$\Theta(S_1(\delta_0))\supset S_2(\delta_0).$$

Note that

$$|i(\sigma)-Y_3|\leq 2\delta_0/d \quad \text{if} \quad \Theta(y(\sigma))\in S_2(\delta_0),\ \sigma\in S_1(\delta_0). \tag{2.14}$$

Then for each $\eta\in S_2(\delta_0)$ there exists uniquely $\sigma\in S_1(\delta_0)$ such that $\Theta(y(\sigma))=z(\eta)$. We denote this correspondance η to σ by

$$\sigma=\Psi(\eta). \tag{2.15}$$

Let $r(\eta)$ be a $\boldsymbol{R}^3$-valued C^∞ function defined by

$$r(\eta)=i(\sigma)-2(i(\sigma)\cdot m(\eta))m(\eta) \tag{2.16}$$

where $m(\eta)$ is the unit outer normal of Γ_2 at $z(\eta)$, and σ and η are linked by (2.15).

Lemma 2.2. *Suppose that* (2.13) *holds. Then* $r(\eta)$ *defined by* (2.16) *satisfies*

$$r(\eta)\cdot z(\eta)'>0 \quad \textit{for all} \quad z(\eta)\in\partial S_2(\delta_0). \tag{2.17}$$

Proof. From (2.13) it follows that $\Theta^{-1}(z(\eta))=y(\sigma)\in S_1(\delta_0)$ for $z(\eta)\in\partial S_2(\delta_0)$, from which

$$|(y(\sigma)+li(\sigma))'|^2\leq|(y(\sigma)+l(\sigma)i(\sigma))'|^2 \quad \text{for all} \quad 0\leq l\leq l(\sigma)$$

follows. Then we have $\left[\frac{d}{dl}|(y(\sigma)+li(\sigma))'|^2\right]_{l=l(\sigma)}\geq 0$, which is equivalent to

$$2z(\eta)\cdot i(\eta)'\geq 0.$$

On the other hand the strict convexity of Γ_2 implies

$$m(\eta)\cdot z(\eta)'\geq c|z(\eta)'|^2 \qquad (c>0),$$

and (2.14) and $m(0)=-Y_3$ imply $-i(\sigma)\cdot m(\eta)\geq 1-C\delta_0$. Then we have for all $z(\eta)\in\partial S_2(\delta_0)$

$$r(\eta)\cdot z(\eta)' = i(\sigma)'\cdot z(\eta)' - 2(i(\sigma)\cdot m(\eta))m(\eta)\cdot z(\eta)'$$
$$\geq 2(1-C\delta_0)c|z(\eta)'|^2 > 0. \qquad \text{Q. E. D.}$$

Let us set

$$\begin{cases} \dfrac{\partial m(\eta)}{\partial \eta_p} = \sum_{h=1}^{3} k_{hp}^{(2)}(\eta) Y_h,\ p=1,2, \\ K_2 = [k_{hp}^{(2)}(0)]_{\substack{h\to 1,2,\\ p\downarrow 1,2}} \end{cases}$$

and

$$\mathscr{G}_c = \{\mathscr{K};\ \mathscr{K} \text{ is a } 2\times 2 \text{ real matrix such that } \mathscr{K}\geq cI\}.$$

Define a mapping F_2 from $\mathscr{G}_0$ into istelf by

$$F_2(\mathscr{K}) = \mathscr{K}(I+d\mathscr{K})^{-1} + 2K_2.$$

For $r(\eta)$ defined by (2.16) set

$$\text{(2.18)} \qquad \begin{cases} \dfrac{\partial r}{\partial \eta_p}(\eta) = \sum_{h=1}^{3} \tilde{\kappa}_{hp}(\eta) Y_h,\ p=1,2, \\ \tilde{\mathscr{K}}(\eta) = [\tilde{\kappa}_{hp}(\eta)]_{\substack{h\to 1,2.\\ p\downarrow 1,2}} \end{cases}$$

Lemma 2.3. *Let $r(\eta)$ be defined by* (2.16) *for $i(\sigma)$ which satisfies* (2.13) *and* (2.14). *Then it holds that*

$$\text{(2.19)} \qquad \|\tilde{\mathscr{K}}(\eta) - F_2(\mathscr{K}(\sigma))\| \leq C\delta_0 \|\mathscr{K}(\sigma)\| \qquad \textit{for all} \quad \eta\in S_2(\delta_0)$$

$$\text{(2.20)} \qquad |\tilde{\kappa}_{31}(\eta)| + |\tilde{\kappa}_{32}(\eta)| \leq C\delta_0\|\mathscr{K}(\sigma)\| \qquad \textit{for all} \quad \eta\in S_2(\delta_0)$$

where C is a positive constant depending only on Γ.

Proof. A differentiation of (2.16) by η_p gives

$$\text{(2.21)} \qquad \begin{aligned} \frac{\partial r}{\partial \eta_p} &= \sum_{h=1}^{2} \frac{\partial i}{\partial \sigma_h}\frac{\partial \sigma_h}{\partial \eta_p} - 2\sum_{h=1}^{2}\Big(\frac{\partial i}{\partial \sigma_h}\cdot m\Big)\frac{\partial \sigma_h}{\partial \eta_p} m(\eta) \\ &\quad - 2\Big(i\cdot \frac{\partial m}{\partial \eta_p}\Big)m - 2(i\cdot m)\frac{\partial m}{\partial \eta_p}. \end{aligned}$$

From $\dfrac{\partial i}{\partial \sigma_h}\cdot i = 0$, $m\cdot \dfrac{\partial m}{\partial \sigma_h} = 0$ and (2.14) we have

$$\Big|\frac{\partial i}{\partial \sigma_h}\cdot m\Big| \leq C\delta_0\Big|\frac{\partial i}{\partial \sigma_h}\Big|, \qquad \Big|i\cdot \frac{\partial m}{\partial \eta_p}\Big| \leq C\delta_0.$$

Therefore by comparing the x_1 and x_2 components of the both sides of (2.21), with the aid of Lemma 2.1, we have (2.19). And x_3 component of the right hand side of (2.21) is estimated by $C\delta_0 \sum_{h=1}^{2}\Big|\dfrac{\partial i}{\partial \sigma_h}\Big|$, from which (2.20) follows. Q. E. D.

Corollary. *Suppose that $\boldsymbol{R}^3$-valued C^∞ function $r(\eta)$ defined in $S_2(\delta_0)$ satisfies*

(2.17), $|r(\eta)|=1$ *and*

$$|r(\eta)-(-Y_3)|\leq C\delta_0,\quad \tilde{\mathscr{K}}(\eta)\geq 0. \tag{2.22}$$

Correspond $y(\sigma)\in\Gamma_{10}$ *to* $z(\eta)$ *by a relation*

$$y(\sigma)=z(\eta)+h(\eta)r(\eta)\qquad (h(\eta)>0), \tag{2.23}$$

and define $i(\sigma)$ *near* $S_1(\delta_0)$ *by*

$$i(\sigma)=r(\eta)-2(r(\eta)\cdot n(\sigma))n(\sigma), \tag{2.24}$$

where $n(\sigma)$ *denotes the unit outer normal of* Γ_1 *at* $y(\sigma)$. *Then it holds that*

$$\|\mathscr{K}(\sigma)-F_1(\tilde{\mathscr{K}}(\eta))\|\leq C\delta_0\|\tilde{\mathscr{K}}(\eta)\| \tag{2.25}$$

$$|\kappa_{31}(\sigma)|+|\kappa_{32}(\sigma)|\leq C\delta_0\|\tilde{\mathscr{K}}(\eta)\|, \tag{2.26}$$

where

$$F_1(\mathscr{K})=\mathscr{K}(I+d\mathscr{K})^{-1}+2K_1\qquad \textit{for}\quad \mathscr{K}\in\mathscr{G}_0,$$

$$K_1=[k_{pq}^{(1)}(0)]_{\substack{p\to 1,2\\ q\downarrow 1,2}},\qquad \frac{\partial n(\sigma)}{\partial\sigma_p}=\sum_{h=1}^{3}k_{hp}^{(1)}(\sigma)Y_h.$$

Next we consider estimates of higher order derivatives of $r(\eta)$. For $f(x)$ defined on Γ_{10} we set

$$|f|_m(y(\sigma))=\max_{\substack{|b^{(l)}|\leq 1,1\leq l\leq j\\ 0\leq j\leq m}}|X_{b^{(1)}}X_{b^{(2)}}\cdots\cdots X_{b^{(j)}}f(y(\sigma))|,$$

$$|f|_m(S_1(\delta_0))=\max_{y(\sigma)\in S_1(\delta_0)}|f|_m(y(\sigma)),$$

where

$$X_{b^{(j)}}=b_1^{(j)}\frac{\partial}{\partial\sigma_1}+b_2^{(j)}\frac{\partial}{\partial\sigma_2},\ b_l^{(j)},\ l=1,\ 2 \text{ are constants},$$

and for $\tilde{f}(x)$ defined on Γ_{20} we set

$$|\tilde{f}|_m(z(\eta))=\max_{\substack{|b^{(l)}|\leq 1,\\ 1\leq j\leq m}}|\tilde{X}_{b^{(1)}}\cdots\tilde{X}_{b^{(j)}}\tilde{f}(z(\eta))|$$

$$|\tilde{f}|_m(S_2(\delta_0))=\max_{z(\eta)\in S_2(\delta_0)}|\tilde{f}|_m(z(\eta)),$$

$$\tilde{X}_{b^{(j)}}=b_1^{(j)}\frac{\partial}{\partial\eta_1}+b_2^{(j)}\frac{\partial}{\partial\eta_2}.$$

Lemma 2.4. *Let* $r(\eta)$ *be defined by* (2.16) *for* $i(\sigma)$ *satisfying* (2.13) *and* (2.14). *Suppose that* $\mathscr{K}(\sigma)\geq c>0$ *for all* $\sigma\in S_1(\delta_0)$. *Then we have for* $m=2, 3,\ldots$.

$$|r|_m(S_2(\delta_0))\leq((1+cd)^{-1}+C\delta_0)^{m+1}|i|_m(S_1(\delta_0))+C_m|i|_{m-1}(S_1(\delta_0)), \tag{2.27}$$

where C_m is a constant independent of i.

Proof. A differentiation of (2.11) by η_p gives

$$(2.28)\quad \frac{\partial^2 z}{\partial\eta_p\partial\eta_q}=\sum_{h,s=1}^{2}\left(\frac{\partial^2 y}{\partial\sigma_h\partial\sigma_s}+\frac{\partial^2 l}{\partial\sigma_h\partial\sigma_s}i+l\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}\right)\frac{\partial\sigma_h}{\partial\eta_p}\frac{\partial\sigma_s}{\partial\eta_q}$$
$$+\sum_{h=1}^{2}\left(\frac{\partial y}{\partial\sigma_h}+\frac{\partial l}{\partial\sigma_h}i+l\frac{\partial i}{\partial\sigma_h}\right)\frac{\partial^2\sigma_h}{\partial\eta_p\partial\eta_q}+2\sum_{h,s=1}^{2}\frac{\partial l}{\partial\sigma_h}\frac{\partial i}{\partial\sigma_s}\frac{\partial\sigma_h}{\partial\eta_p}\frac{\partial\sigma_s}{\partial\eta_q}.$$

Set

$$H=\left[{}^t\!\left(P\left(\frac{\partial y}{\partial\sigma_1}-\left(\frac{\partial y}{\partial\sigma_1}\cdot i\right)i+l\frac{\partial i}{\partial\sigma_1}\right)\right),\ {}^t\!\left(P\left(\frac{\partial y}{\partial\sigma_2}-\left(\frac{\partial y}{\partial\sigma_2}\cdot i\right)i+l\frac{\partial i}{\partial\sigma_2}\right)\right)\right],$$

where P denotes an orthogonal projection from $\boldsymbol{R}^3$ onto $\boldsymbol{R}^2$, that is,

$$P(x_1, x_2, x_3)=(x_1, x_2).$$

Consider components orthogonal to i in (2.28) and we have

$$\left|H\left[\frac{\partial^2\sigma_h}{\partial\eta_p\partial\eta_q}\right]_{h\downarrow 1,2}+lP\sum_{h,s=1}^{2}\left(\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}-\left(\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}\cdot i\right)i\right)\frac{\partial\sigma_h}{\partial\eta_p}\frac{\partial\sigma_s}{\partial\eta_q}\right|$$
$$\leq C\{|i|_1(S_1(\delta_0))+|z|_2(S_2(\delta_0))+|y|_2(S_1(\delta_0))\}$$

for all $p, q=1, 2$. Since $\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}\cdot i=-\frac{\partial i}{\partial\sigma_h}\cdot\frac{\partial i}{\partial\sigma_s}$ follows from $\frac{\partial i}{\partial\sigma_h}\cdot i=0$, we have

$$(2.29)\quad \left[\frac{\partial^2\sigma_h}{\partial\eta_p\partial\eta_q}\right]_{h\downarrow 1,2}\equiv -H^{-1}l\sum_{h,s=1}^{2}P\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}\frac{\partial\sigma_h}{\partial\eta_p}\frac{\partial\sigma_s}{\partial\eta_q},$$

where $A\equiv B$ means $|A-B|\leq C\{|i|_1(S_1(\delta_0))+|y|_2(S_1(\delta_0))+|z|_2(S_2(\delta_0))\}$ for some constant C independent of i and r. Since

$$\frac{\partial^2 i}{\partial\eta_p\partial\eta_q}=\sum_{h,s=1}^{2}\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}\frac{\partial\sigma_h}{\partial\eta_p}\frac{\partial\sigma_s}{\partial\eta_q}+\sum_{h=1}^{2}\frac{\partial i}{\partial\sigma_h}\frac{\partial^2\sigma_h}{\partial\eta_p\partial\eta_q}$$

we have by using (2.29)

$$(2.30)\quad \begin{aligned}P\frac{\partial^2 i}{\partial\eta_p\partial\eta_q}&\equiv\sum_{h,s=1}^{2}\left(I-\left[{}^t\!\left(P\frac{\partial i}{\partial\sigma_1}\right),\ {}^t\!\left(P\frac{\partial i}{\partial\sigma_2}\right)\right]lH^{-1}\right)P\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}\frac{\partial\sigma_h}{\partial\eta_p}\frac{\partial\sigma_s}{\partial\eta_q}\\ &=\sum_{h,s=1}^{2}\left[{}^t\!\left(P\left(\frac{\partial y}{\partial\sigma_1}-\left(\frac{\partial y}{\partial\sigma_1}\cdot i\right)i\right)\right),\ {}^t\!\left(P\left(\frac{\partial y}{\partial\sigma_2}-\left(\frac{\partial y}{\partial\sigma_2}\cdot i\right)i\right)\right)\right]H^{-1}P\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}\frac{\partial\sigma_h}{\partial\eta_p}\frac{\partial\sigma_s}{\partial\eta_q}\end{aligned}$$

Then

$$\begin{aligned}\tilde{X}_{b(1)}\tilde{X}_{b(2)}\,Pi(\sigma)&=\sum_{p,q=1}^{2}b_p^{(1)}b_q^{(2)}P\frac{\partial^2 i}{\partial\eta_p\partial\eta_q}\\ &\equiv\sum_{h,s=1}^{2}\left(\sum_{p=1}^{2}\frac{\partial\sigma_h}{\partial\eta_p}b_p^{(1)}\right)\left(\sum_{q=1}^{2}\frac{\partial\sigma_s}{\partial\eta_q}b_q^{(2)}\right)\tilde{Y}H^{-1}P\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}\end{aligned}$$

where $\tilde{Y}=\left[{}^t\left(P\left(\frac{\partial y}{\partial\sigma_1}-\left(\frac{\partial y}{\partial\sigma_1}\cdot i\right)i\right)\right),\ {}^t\left(P\left(\frac{\partial y}{\partial\sigma_2}-\left(\frac{\partial y}{\partial\sigma_2}\cdot i\right)i\right)\right)\right]$. Note that

$$\left|\left(\sum_{p=1}^{2}\frac{\partial\sigma_h}{\partial\eta_p}b_p^{(l)}\right)_{h\downarrow 1,2}\right|\leq((1+cd)^{-1}+C\delta_0)\|b^{(l)}\|$$

holds for $l=1,2$ from (2.9). Then by using $\|\tilde{Y}H^{-1}\|\leq(1+cd)^{-1}+C\delta_0$ we have

$$|Pi|_2(z(\eta))\leq((1+cd)^{-1}+C\delta_0)^{-3}|Pi|_2(y(\sigma))$$
$$+C|i|_1(y(\sigma))(1+|z|_2(S_2(\delta_0))+|y|_2(S_1(\delta_0)))\,.$$

By using (2.5) we have

$$|i|_2(z(\eta))\leq((1+cd)^{-1}+C_0)^{-3}|i|_2(y(\sigma))$$
$$+C|i|_1(y(\sigma))(1+|z|_2(S_2(\delta_0))+|y|_2(S_1(\delta_0)))\,,$$

from which (2.27) for $m=2$ follows immediately. For $m>2$ we may obtain the desired estimate by the same reasoning.

Corollary. *Suppose that* $r(\eta)$ *satisfies* $|r(\eta)|=1$, (2.17) *and* (2.22). *Then for* $i(\sigma)$ *defined by* (2.23) *and* (2.24), *if*

$$\tilde{\mathscr{K}}(\eta)\geq c\qquad\text{for all}\quad \eta\in S_2(\delta_0)$$

holds, we have

$$|i|_m(S_1(\delta_0))\leq((1+cd)^{-1}+\delta_0)^{-m-1}|r|_m(S_2(\delta_0))+C_m|r|_{m-1}(S_2(\delta_0))\,.$$

Let $\varphi(x)$ be a real valued C^∞ function defined near $S_1(\delta_0)$ satisfying $|\nabla\varphi(x)|=1$. Set $\nabla\varphi(y(\sigma))=i(\sigma)$ and suppose that (2.13) and (2.14) hold. Then, by extending φ by $\varphi(y+l\nabla\varphi(y))=\varphi(y)+l$, $\varphi(x)$ may be considered a function in $\Phi(S_1(\delta_0)\times[0,\infty))$ verifying $|\nabla\varphi(x)|=1$. Note that we have $\Phi(S_1(\delta_0)\times[0,\infty))\subset S_2(\delta_0)$ from (2.13). Denote this φ by φ_0 and define $\varphi_1(x)$ by

$$\text{(2.31)}\qquad\begin{cases}|\nabla\varphi_1(x)|=1 & \\ \varphi_1(x)=\varphi_0(x) & \text{on}\quad S_2(\delta_0)\\ \dfrac{\partial\varphi_1}{\partial\nu}(x)=-\dfrac{\partial\varphi_0}{\partial\nu}(x) & \text{on}\quad S_2(\delta_0)\,,\end{cases}$$

where ν denotes the unit outer normal of Γ at x. From (2.31) it follows that

$$\text{(2.32)}\qquad \nabla\varphi_1(x)=\nabla\varphi(x)-2(\nabla\varphi(x)\cdot\nu(x))\nu(x)\qquad\text{for all}\quad x\in S_2(\delta_0)\,,$$

that is, by setting $r(\eta)=\nabla\varphi_1(z(\eta))$

$$r(\eta)=i(\sigma)-2(i(\sigma)\cdot n(\eta))m(\eta)\,.$$

Then Lemma 2.2 assures that

$$r(\eta)\cdot z(\eta)'\geq 0\qquad\text{for all}\quad \eta\in\partial S_2(\delta_0)\,.$$

And from Lemma 2.3

$$\tilde{\mathscr{K}}(\eta) \geq F_2(\mathscr{K}(\sigma)) - C\delta_0\|\mathscr{K}(\sigma)\| \geq 2K_2 + \|\mathscr{K}(\sigma)\|((1+cd)^{-1} - C\delta_0).$$

As remarked on $\varphi(x)$, $\varphi_1(x)$ can be defined in $\{x(\eta)+hr(\eta);\ \eta \in S_2(\delta_0),\ h \geq 0\}(\subset S_1(\delta_0))$ verifying $|\nabla\varphi_1(x)|=1$. Then we can define $\varphi_2(x)$ by

$$\text{(2.33)} \qquad \begin{cases} |\nabla\varphi_2(x)| = 1 & \\ \varphi_2(x) = \varphi_1(x) & \text{on} \quad S_1(\delta_0) \\ \dfrac{\partial\varphi_2}{\partial\nu}(x) = -\dfrac{\partial\varphi_1}{\partial\nu}(x) & \text{on} \quad S_1(\delta_0), \end{cases}$$

and define successively $\varphi_3, \varphi_4, \ldots, \varphi_q, \varphi_{q+1} \cdots$ by

$$\text{(2.34)} \qquad \begin{cases} |\nabla\varphi_q| = 1 & \\ \varphi_q(x) = \varphi_{q-1}(x) & \text{on} \quad S_{\epsilon(q)}(\delta_0) \\ \dfrac{\partial\varphi_q}{\partial\nu}(x) = -\dfrac{\partial\varphi_{q-1}}{\partial\nu}(x) & \text{on} \quad S_{\epsilon(q)}(\delta_0), \end{cases}$$

where

$$\epsilon(q) = \begin{cases} 1 & \text{for } q \text{ even} \\ 2 & \text{for } q \text{ odd.} \end{cases}$$

Set

$$i_q(\sigma) = \nabla\varphi_{2q}(y(\sigma)),\quad r_q(\eta) = \nabla\varphi_{2q+1}(z(\eta)),$$

$$\frac{\partial i_q(\sigma)}{\partial\sigma_h} = \sum_{s=1}^{3} \kappa_{sh}^{(q)}(\sigma)Y_s,\quad \frac{\partial r_q(\eta)}{\partial\eta_h} = \sum_{s=1}^{3} \tilde{\kappa}_{sh}^{(q)}(\eta)Y_h$$

$$\mathscr{K}_q(\sigma) = [\kappa_{sh}^{(q)}(\sigma)]_{\substack{s\to 1,2\\ h\downarrow 1,2}},\quad \tilde{\mathscr{K}}_q(\eta) = [\tilde{\kappa}_{sh}^{(q)}(\eta)]_{\substack{s\to 1,2\\ h\downarrow 1,2}}.$$

Note that from (2.33) or (2.34)

$$\text{(2.35)} \qquad r_q(\eta) = i_q(\sigma) - 2(i_q(\sigma)\cdot m(\eta))m(\eta)$$

$$\text{(2.36)} \qquad i_{q+1}(\sigma) = r_q(\eta) - 2(r_q(\eta)\cdot n(\sigma))n(\sigma)$$

hold for $q=0, 1, 2, \ldots,$.

By using Lemmas 2.1 ~ 2.3 at each step we have

$$\text{(2.37)} \qquad \|\tilde{\mathscr{K}}_q(\eta) - F_2(\mathscr{K}_q(\sigma))\| \leq C\delta_0\|\mathscr{K}_q(\sigma)\|,$$

$$\text{(2.38)} \qquad \|\mathscr{K}_q(\sigma) - F_1(\tilde{\mathscr{K}}_{q-1}(\eta))\| \leq C\delta_0\|\tilde{\mathscr{K}}_{q-1}(\eta)\|.$$

Let

$$K_1,\ K_2 \geq C_0.$$

Then from (2.37) and (2.38)

$$\mathscr{K}_q(\sigma),\ \tilde{\mathscr{K}}_q(\eta) \geq 2C_0 - C\delta_0 \qquad \text{for all} \quad q \geq 1. \tag{2.39}$$

Remark that, since

$$F_l(\mathscr{K}) - F_l(\mathscr{K}') = (I + d\mathscr{K})^{-1}(\mathscr{K} - \mathscr{K}')(I + d\mathscr{K}')^{-1}$$

we have for $\mathscr{K},\ \mathscr{K}' \in \mathscr{G}_c$

$$\|F_l(\mathscr{K}) - F_l(\mathscr{K}')\| \leq (1 + cd)^{-2}\|\mathscr{K} - \mathscr{K}'\|, \quad l = 1, 2.$$

Set

$$\mathscr{F}_1(\mathscr{K}) = F_1(F_2(\mathscr{K})), \quad \mathscr{F}_2(\mathscr{K}) = F_2(F_1(\mathscr{K})).$$

Then for any $\mathscr{K},\ \mathscr{K}' \in \mathscr{G}_c$

$$\|\mathscr{F}_l(\mathscr{K}) - \mathscr{F}_l(\mathscr{K}')\| \leq (1 + cd)^{-4}\|\mathscr{K} - \mathscr{K}'\|, \quad l = 1, 2. \tag{2.40}$$

Proposition 2.5. *$\{\nabla\varphi_q;\ q = 0, 1,\ldots\}$ is a bounded set in $C^\infty(\bar{\omega})$, where ω is a domain surrounded by $S_l(\delta_0)$, $l = 1, 2$ and $|x'| = \delta_0$.*

Proof. Since for all $\mathscr{K} \geq 0$

$$\|\mathscr{K}(I + d\mathscr{K})^{-1}\| \leq 1/d,$$

we have

$$\|F_l(\mathscr{K})\| \leq 1/d + 2K_l,\ l = 1, 2.$$

Then for all $q \geq 1$

$$\|\mathscr{K}_q(\sigma)\|,\ \|\tilde{\mathscr{K}}_q(\eta)\| \leq 1/d + 2\max(\|K_1\|, \|K_2\|) + C\delta_0.$$

From Lemma 2.3 and its corollary we have for all $q \geq 1$

$$|\nabla\varphi_q|_1(S_{\epsilon(q)}(\delta_0)) \leq 1/d + 2\max(\|K_1\|, \|K_2\|) + C\,\delta_0.$$

Next suppose that for $m \geq 1$

$$|\nabla\varphi_q|_m(S_{\epsilon(q)}(\delta_0)) \leq C_m. \tag{2.41}$$

Applying Lemma 2.4 and its corollary we have

$$|i_q|_{m+1}(S_1(\delta_0)) \leq ((1 + cd)^{-1} + C\delta_0)^{-m-1}|r_{q-1}|_{m+1}(S_2(\delta_0)) + C'_m$$

$$|r_q|_{m+1}(S_2(\delta_0)) \leq ((1 + cd)^{-1} + C\delta_0)^{-m-1}|i_q|_{m+1}(S_1(\delta_0)) + C'_m$$

for some C'_m. Therefore we have

$$|i_q|_{m+1}(S_1(\delta_0)) \leq \sum_{j=0}^{2q} C'_m((1 + cd)^{-1} + C\delta_0)^{-(m+1)j}$$
$$+ ((1 + cd)^{-1} + C\delta_0)^{-(m+1)2q}|i_0|_{m+1}(S_1(\delta_0)).$$

Similarly

$$|r_q|_{m+1}(S_2(\delta_0)) \leq \sum_{j=0}^{2q} C'_m((1+cd)^{-1}+C\delta_0)^{-(m+1)j}$$
$$+((1+cd)^{-1}+C\delta_0)^{-(m+1)2q}|i_0|_{m+1}(S_1(\delta_0)).$$

Thus we have

$$|\nabla\varphi_q|_{m+1}(S_{\epsilon(q)}(\delta_0)) \leq C_{m+1} \qquad \text{for all} \quad q.$$

By induction (2.41) holds for all m. Since we have

$$\sup_{x\in\omega} \sum_{|\beta|\leq m} |D_x^\beta(\nabla\varphi_q(x))| \leq C_m|\nabla\varphi_q|_m(S_{\epsilon(q)}(\delta_0))$$

Proposition follows from (2.41).

§3. Convergence of phase functions

Let $i(\sigma)$, $j(\sigma)$ be $\boldsymbol{R}^3$-valued C^∞ functions satisfying $|i(\sigma)|=|j(\sigma)|=1$ and (2.13), (2.14). We denote 2×2 matrices defined by (2.3) for $i(\sigma)$ and $j(\sigma)$ by $\mathscr{K}(\sigma)$ and $\mathscr{H}(\sigma)$ respectively. Suppose that

(3.1) $$\mathscr{K}(\sigma),\ \mathscr{H}(\sigma) \geq 2C_0 \qquad \text{for all} \quad \sigma\in S_1(\delta_0).$$

Lemma 3.1. *Suppose that $y(\sigma)$, $y(\tilde\sigma)\in S_1(\delta_0)$ and*

(3.2) $$z(\eta)=y(\sigma)+l(\sigma)i(\sigma)=y(\tilde\sigma)+h(\tilde\sigma)j(\tilde\sigma)\in S_2(\delta_0).$$

Then it holds that

(3.3) $$|l(\sigma)-h(\tilde\sigma)| \leq C\delta_0|\sigma-\tilde\sigma|.$$

Proof.

$$y(\sigma)-y(\tilde\sigma)=\sum_{h=1}^{2}(\sigma_h-\tilde\sigma_h)\int_0^1\left(\frac{\partial y}{\partial\sigma_h}\right)(\sigma+t(\tilde\sigma-\sigma))dt$$
$$=\sum_{h=1}^{2}(\sigma_h-\tilde\sigma_h)Y_h(\sigma,\tilde\sigma).$$

From (2.7) it holds that for all σ, $\tilde\sigma\in S_1(\delta_0)$

(3.4) $$|Y_h-Y_h(\sigma,\tilde\sigma)| \leq C\delta_0,$$

from whict it follows that

$$\big||y(\sigma)-y(\tilde\sigma)|^2-|\sigma-\tilde\sigma|^2\big| \leq C\delta_0|\sigma-\tilde\sigma|^2.$$

Since $l(\sigma)i(\sigma)=y(\tilde\sigma)-y(\sigma)+h(\tilde\sigma)j(\tilde\sigma)$ we have

$$l(\sigma)^2=|y(\tilde\sigma)-y(\sigma)|^2+h(\tilde\sigma)^2+2h(\tilde\sigma)\sum_{h=1}^{2}(\tilde\sigma_h-\sigma_h)(Y_h(\tilde\sigma,\sigma)\cdot j(\sigma)).$$

(3.4) and (2.14) imply

$$|Y_h(\tilde\sigma,\sigma)\cdot j(\tilde\sigma)| \leq C\delta_0.$$

Then we have

$$|l(\sigma)^2-h(\tilde{\sigma})^2|\leq(1+C\delta_0)|\sigma-\tilde{\sigma}|^2+C\delta_0|\sigma-\tilde{\sigma}|.$$

Thus we have by using $l(\sigma)+h(\tilde{\sigma})\geq 2d$

$$|l(\sigma)-h(\tilde{\sigma})|\leq\frac{1}{2d}((1+C\delta_0)|\sigma-\tilde{\sigma}|+C\delta_0)|\sigma-\tilde{\sigma}|$$

from which (3.3) follows because of $|\sigma-\tilde{\sigma}|\leq C\delta_0$. Q. E. D.

Denote by $\sigma(\eta)$ and $\tilde{\sigma}(\eta)$ mappings from $S_2(\delta_0)$ to $S_1(\delta_0)$ defined by

$$z(\eta)=y(\sigma)+l(\sigma)i(\sigma),$$

$$z(\eta)=y(\tilde{\sigma})+h(\tilde{\sigma})j(\tilde{\sigma}),$$

respectively.

Lemma 3.2. *It holds that*

(3.5) $$\max_{\eta\in S_2(\delta_0)}|j(\tilde{\sigma}(\eta))-i(\sigma(\eta))|\leq((1+C_0d)^{-1}+C\delta_0)\max_{\sigma\in S_1(\delta_0)}|i(\sigma)-j(\sigma)|.$$

Proof. Set

$$\mathscr{X}(\sigma,\tilde{\sigma})=\left[{}^t\!\left(P\int_0^1\frac{\partial i}{\partial\sigma_1}(\sigma+t(\tilde{\sigma}-\sigma))dt\right),\quad {}^t\!\left(P\int_0^1\frac{\partial i}{\partial\sigma_2}(\sigma+t(\tilde{\sigma}-\sigma))dt\right)\right],$$

$$Y(\sigma,\tilde{\sigma})=\left[{}^t\!\left(P\int_0^1\frac{\partial y}{\partial\sigma_1}(\sigma+t(\tilde{\sigma}-\sigma))dt\right),\quad {}^t\!\left(P\int_0^1\frac{\partial y}{\partial\sigma_2}(\sigma+t(\tilde{\sigma}-\sigma))dt\right)\right],$$

$$A=\max_{\sigma\in S_1(\delta_0)}|i(\sigma)-j(\sigma)|.$$

From (3.2) we have

$$\begin{aligned}&y(\tilde{\sigma})-y(\sigma)+l(\sigma)(i(\tilde{\sigma})-i(\sigma))\\&\quad=(l(\sigma)-h(\tilde{\sigma}))j(\tilde{\sigma})+l(\sigma)(i(\tilde{\sigma})-j(\tilde{\sigma})).\end{aligned}$$

Then we have

(3.6) $$\begin{aligned}&[Y(\sigma,\tilde{\sigma})+l(\sigma)\mathscr{X}(\sigma,\tilde{\sigma})]^t(\tilde{\sigma}-\sigma)\\&\quad=(l(\sigma)-l(\tilde{\sigma}))Pj(\tilde{\sigma})+l(\sigma)P(i(\tilde{\sigma})-j(\tilde{\sigma})).\end{aligned}$$

Since $Y(\sigma,\tilde{\sigma})+l(\sigma)\mathscr{X}(\sigma,\tilde{\sigma})\geq 1+C_02d-C\delta_0$ we have

$$|\tilde{\sigma}-\sigma|\leq(1+C_0d)^{-1}\{|l(\sigma)-h(\tilde{\sigma})|+l(\sigma)A\}$$

by using (3.3)

$$\leq(1+C_0d)^{-1}C\delta_0|\tilde{\sigma}-\sigma|+(1+C_0d)^{-1}(d+\delta_0)A.$$

Then

(3.7) $$|\tilde{\sigma}-\sigma|\leq(1-C\delta_0)^{-1}(1+C_0d)^{-1}(d+\delta_0)A.$$

Substituting this estimate into (3.3) we have

$$|l(\sigma)-h(\tilde{\sigma})|\leq C\delta_0 A. \tag{3.8}$$

Note that

$$P(i(\tilde{\sigma})-i(\sigma))=\mathscr{K}(\sigma,\tilde{\sigma})^t(\tilde{\sigma}-\sigma)$$

by using (3.6)

$$=\mathscr{K}(\sigma,\tilde{\sigma})[Y(\sigma,\tilde{\sigma})+l(\sigma)\mathscr{K}(\sigma,\tilde{\sigma})]^{-1}\{l(\sigma)P(i(\tilde{\sigma})-j(\tilde{\sigma}))$$
$$+(l(\sigma)-h(\tilde{\sigma}))Pj(\tilde{\sigma})\}.$$

Then

$$P(j(\tilde{\sigma})-i(\sigma))=P(i(\tilde{\sigma})-i(\sigma))+P(j(\tilde{\sigma})-i(\tilde{\sigma}))$$
$$=\{l(\sigma)\mathscr{K}(\sigma,\tilde{\sigma})[Y(\sigma,\tilde{\sigma})+l(\sigma)\mathscr{K}(\sigma,\tilde{\sigma})]^{-1}-I\}P(i(\tilde{\sigma})-j(\tilde{\sigma}))$$
$$+\mathscr{K}(\sigma,\tilde{\sigma})[Y(\sigma,\tilde{\sigma})+l(\sigma)\mathscr{K}(\sigma,\tilde{\sigma})]^{-1}(l(\sigma)-h(\tilde{\sigma}))Pj(\tilde{\sigma}).$$

By using (3.8) and a relation

$$l(\tilde{\sigma})\mathscr{K}(\sigma,\tilde{\sigma})[Y(\sigma,\tilde{\sigma})+l(\sigma)\mathscr{K}(\sigma,\tilde{\sigma})]^{-1}-I$$
$$=-Y(\sigma,\tilde{\sigma})[Y(\sigma,\tilde{\sigma})+l(\sigma)\mathscr{K}(\sigma,\tilde{\sigma})]^{-1}$$

we have

$$|P(j(\tilde{\sigma})-i(\sigma))|\leq((1+C_0 d)^{-1}+C\delta_0)A+C\delta_0 A.$$

Since

$$|j_3(\tilde{\sigma})-i_3(\sigma)|=\sqrt{1-|Pj(\tilde{\sigma})|^2}-\sqrt{1-|Pi(\sigma)|^2}\leq C\delta_0|Pj(\tilde{\sigma})-Pi(\sigma)|$$

we have (3.5) from above estimates. Q.E.D.

Lemma 3.3. *Let $i(\sigma)$ and $j(\sigma)$ be $\boldsymbol{R}^3$-valued functions defined on $S_1(\delta_0)$ verifying $|i(\sigma)|=|j(\sigma)|=1$, (2.13), (2.14) and (3.1). For $i(\sigma)$ and $j(\sigma)$ define $r(\eta)$ and $s(\eta)$ by*

$$r(\eta)=i(\sigma)-2(i(\sigma)\cdot m(\eta))m(\eta)$$
$$s(\eta)=j(\tilde{\sigma})-2(j(\tilde{\sigma})\cdot m(\eta))m(\eta).$$

Then we have

$$|r-s|_0(S_2(\delta_0))\leq((1+C_0 d)^{-1}+C\delta_0)|i-j|_0(S_1(\delta_0)) \tag{3.9$_0$}$$

and for $m\geq 1$

$$|r-s|_m(S_2(\delta_0))\leq((1+C_0 d)^{-1}+C\delta_0)^{m+1}|i-j|_m(S_1(\delta_0))$$
$$+C_m\{|i|_{m+1}(S_1(\delta_0))+|j|_{m+1}(S_1(\delta_0))\}|i-j|_{m-1}(S_1(\delta_0)). \tag{3.9$_m$}$$

Proof. Set

$$Y(i;\sigma)=\left[{}^tP\Big(\frac{\partial y(\sigma)}{\partial\sigma_1}-\Big(\frac{\partial y(\sigma)}{\partial\sigma_1}\cdot i(\sigma)\Big)i(\sigma)\Big),\quad {}^tP\Big(\frac{\partial y(\sigma)}{\partial\sigma_2}-\Big(\frac{\partial y(\sigma)}{\partial\sigma_2}\cdot i(\sigma)\Big)i(\sigma)\Big)\right]$$

$$Z(i;\eta)=\left[{}^tP\Big(\frac{\partial z(\eta)}{\partial\eta_1}-\Big(\frac{\partial z(\eta)}{\partial\eta_1}\cdot i(\sigma)\Big)i(\sigma)\Big),\quad {}^tP\Big(\frac{\partial z(\eta)}{\partial\eta_2}-\Big(\frac{\partial z(\eta)}{\partial\eta_2}\cdot i(\sigma)\Big)i(\sigma)\Big)\right].$$

We define $Y(j;\tilde\sigma)$, $Z(j;\eta)$ by the same way. Then (2.12) may be written as

$$Z(i;\eta)=[Y(i;\sigma)+l(\sigma)\mathscr{K}(\sigma)]\left[\frac{\partial\sigma}{\partial\eta}\right],$$

where

$$\left[\frac{\partial\sigma}{\partial\eta}\right]=\left[\frac{\partial\sigma_p}{\partial\eta_q}\right]_{\substack{p\to 1,2\\ q\downarrow 1,2}}.$$

Similarly we have

$$Z(j;\eta)=[Y(j;\tilde\sigma)+h(\tilde\sigma)\mathscr{H}(\tilde\sigma)]\left[\frac{\partial\tilde\sigma}{\partial\eta}\right].$$

On the other hand

$$\begin{aligned}\mathscr{M}(\eta)&=\left[{}^t\Big(P\frac{\partial i(\sigma(\eta))}{\partial\eta_1}\Big),\quad {}^t\Big(P\frac{\partial i(\sigma(\eta))}{\partial\eta_2}\Big)\right]\\ &=\left[{}^t\Big(P\frac{\partial i}{\partial\sigma_1}\Big),\quad {}^t\Big(P\frac{\partial i}{\partial\sigma_2}\Big)\right]\left[\frac{\partial\sigma}{\partial\eta}\right]\\ &=\mathscr{K}(\sigma)[Y(i;\sigma)+l(\sigma)\mathscr{K}(\sigma)]^{-1}Z(i;\eta)\\ &=\mathscr{K}(\sigma)Y(i;\sigma)^{-1}[I+l(\sigma)\mathscr{K}(\sigma)Y(i;\sigma)^{-1}]^{-1}Z(i;\eta)\end{aligned}$$

holds. Similarly we have

$$\begin{aligned}\mathscr{N}(\eta)&=\left[{}^t\Big(P\frac{\partial j(\tilde\sigma(\eta))}{\partial\eta_1}\Big),\quad {}^t\Big(P\frac{\partial j(\tilde\sigma(\eta))}{\partial\eta_2}\Big)\right]\\ &=\mathscr{H}(\tilde\sigma)Y(j;\tilde\sigma)^{-1}[I+h(\tilde\sigma)\mathscr{H}(\tilde\sigma)Y(j;\tilde\sigma)^{-1}]^{-1}Z(j;\eta).\end{aligned}$$

Set

$$E=\mathscr{K}(\sigma)Y(i;\sigma)^{-1}-\mathscr{H}(\tilde\sigma)Y(j;\tilde\sigma)^{-1}.$$

Then

$$\begin{aligned}E&=[\mathscr{K}(\sigma)-\mathscr{K}(\tilde\sigma)]Y(i;\sigma)^{-1}+\mathscr{K}(\tilde\sigma)(Y(i;\sigma)^{-1}-Y(i;\tilde\sigma)^{-1})\\ &\quad+\mathscr{K}(\tilde\sigma)(Y(i;\tilde\sigma)^{-1}-Y(j;\tilde\sigma)^{-1})+(\mathscr{K}(\tilde\sigma)-\mathscr{H}(\tilde\sigma))Y(j;\tilde\sigma)^{-1}\\ &=E_1+E_2+E_3+E_4.\end{aligned}$$

We have

$$E_1\le|\sigma-\tilde\sigma|\left|\int_0^1\frac{\partial\mathscr{K}}{\partial\sigma}(\sigma+t(\tilde\sigma-\sigma))dt\right|$$

by using (3.7)

$$\leq C|i|_2(S_1(\delta_0))\,|i-j|_0(S_1(\delta_0)).$$

Similarly we have

$$\|E_2\| \leq \|\mathscr{K}(\sigma)\|\,|y|_2(S_1(\delta_0))\,|i-j|_0(S_1(\delta_0)),$$

$$\|E_3\| \leq \|\mathscr{K}(\tilde{\sigma})\|\,|y|_1(S_1(\delta_0))\,|i-j|_0(S_1(\delta_0)),$$

$$\|E_4\| \leq (1+C\delta_0)\max_{\sigma\in S_1(\delta_0)}\|\mathscr{K}(\sigma)-\mathscr{H}(\sigma)\|.$$

Then

$$\begin{aligned}
&\mathscr{M}(\eta)-\mathscr{N}(\eta)\\
&=\mathscr{K}(\sigma)Y(i;\sigma)^{-1}[I+l(\sigma)\mathscr{K}(\sigma)Y(i;\sigma)^{-1}]^{-1}(Z(i;\eta)-Z(j;\eta))\\
&\quad+[I+l(\sigma)\mathscr{K}(\sigma)Y(i;\sigma)^{-1}]^{-1}E[I+l(\sigma)\mathscr{H}(\tilde{\sigma})Y(j;\tilde{\sigma})^{-1}]^{-1}Z(j;\eta)\\
&\quad+\mathscr{H}(\tilde{\sigma})Y(j;\tilde{\sigma})^{-1}([I+l(\sigma)\mathscr{H}(\tilde{\sigma})Y(j;\tilde{\sigma})^{-1}]^{-1}-[I+h(\tilde{\sigma})\mathscr{H}(\tilde{\sigma})Y(j;\tilde{\sigma})^{-1}]^{-1})\\
&\qquad\cdot Z(j;\eta)=M_1+M_2+M_3.\\
&\|M_1\|\leq C\|\mathscr{K}(\sigma)\|\,|i-j|_0(S_1(\delta_0)),\\
&\|M_2\|\leq((1+C_0d)^{-1}+C\delta_0)^2\{(1+C\delta_0)\max_{\sigma\in S_1(\delta_0)}\|\mathscr{K}(\sigma)-\mathscr{H}(\sigma)\|\\
&\qquad\qquad+C(|i|_2(S_1(\delta_0))+|y|_2(S_1(\delta_0)))\,|i-j|_0(S_1(\delta_0))\},\\
&\|M_3\|\leq C(|j|_1(S_1(\delta_0))+C\delta_0)\,|i-j|_0(S_1(\delta_0)).
\end{aligned}$$

Set

$$\tilde{\mathscr{K}}(\eta)=\left[{}^t\!\left(P\frac{\partial r(\eta)}{\partial\eta_1}\right),\ {}^t\!\left(P\frac{\partial r(\eta)}{\partial\eta_2}\right)\right]$$

$$\tilde{\mathscr{H}}(\eta)=\left[{}^t\!\left(P\frac{\partial s(\eta)}{\partial\eta_1}\right),\ {}^t\!\left(P\frac{\partial s(\eta)}{\partial\eta_2}\right)\right]$$

and we have from the definitions of $r(\eta)$ and $s(\eta)$

$$\begin{aligned}
\|\tilde{\mathscr{K}}(\eta)-\tilde{\mathscr{H}}(\eta)\| &\leq \|\mathscr{M}(\eta)-\mathscr{N}(\eta)\|+C|i-j|_0(S_2(\delta_0))\\
&\leq((1+C_0d)^{-1}+C\delta_0)^{-2}(1+C\delta_0)\max_{\sigma\in S_1(\delta_0)}\|\mathscr{K}(\sigma)-\mathscr{H}(\sigma)\|\\
&\quad+C(|i|_2+|j|_2+|y|_2+|z|_2)\,|i-j|_0(S_1(\delta_0)).
\end{aligned}$$

This shows $(3.9)_1$.

Next consider the case of $m=2$. (2.30) may be written as

$$P\frac{\partial^2 i(\sigma(\eta))}{\partial\eta_p\partial\eta_q}=[I+l(\sigma)\mathscr{K}(\sigma)Y(i;\sigma)^{-1}]^{-1}\sum_{h,s=1}^{2}P\frac{\partial^2 i}{\partial\sigma_h\partial\sigma_s}\frac{\partial\sigma_h}{\partial\eta_p}\frac{\partial\sigma_s}{\partial\eta_q}+R(\eta)$$

$R(\eta)$ can be written by derivatives of $z(\eta)$ and $y(\sigma)$ of order ≤ 2 and derivatives of $i(\sigma)$ of order ≤ 1.
Similarly

$$P\frac{\partial^2 j(\tilde{\sigma}(\eta))}{\partial\eta_p\partial\eta_q}=[I+h(\tilde{\sigma})\mathscr{H}(\tilde{\sigma})Y(j;\tilde{\sigma})^{-1}]^{-1}\sum_{h,s=1}^{2}\frac{\partial^2 j}{\partial\sigma_h\partial\sigma_s}\frac{\partial\tilde{\sigma}_h}{\partial\eta_p}\frac{\partial\tilde{\sigma}_s}{\partial\eta_q}+\tilde{R}(\eta).$$

Since we have from (2.12)

$$\left[\frac{\partial\sigma_p}{\partial\eta_q}\right]_{\substack{p\to 1,2\\ q\downarrow 1,2}}=Z(i;\eta)Y(i;\sigma(\eta))^{-1}[I+l(\sigma)\mathscr{H}(\sigma)Y(i;\sigma)^{-1}]^{-1}$$

$$\left[\frac{\partial\tilde{\sigma}_p}{\partial\eta_q}\right]_{\substack{p\to 1,2\\ q\downarrow 1,2}}=Z(j;\eta)Y(j;\tilde{\sigma}(\eta))^{-1}[I+h(\tilde{\sigma})\mathscr{H}(\tilde{\sigma})Y(j;\tilde{\sigma})^{-1}]^{-1}.$$

Then from these relations we obtain

$$\begin{aligned}|\tilde{X}_a\tilde{X}_b(i(\sigma(\eta))-j(\tilde{\sigma}(\eta)))|&\leq((1+C_0d)^{-1}+C\delta_0)\,|X_{\tilde{a}}X_{\tilde{b}}(i(\sigma)-j(\sigma))|\\&\quad+C|i-j|_1(S_1(\delta_0))\{|y|_3(S_1(\delta_0))+|z|_3(S_2(\delta_0))\\&\quad+|j|_3(S_1(\delta_0))+|i|_3(S_1(\delta_0))\}\end{aligned}$$

where $\tilde{a}=\left[\frac{\partial\sigma}{\partial\eta}\right]a$ and $\tilde{b}=\left[\frac{\partial\sigma}{\partial\eta}\right]b$.

This shows that $(3.9)_2$ holds. And for $m\geq 3$ we can prove $(3.9)_m$ by the same way. Q. E. D.

Corollary. *Let $r(\eta)$ and $s(\eta)$ be $\boldsymbol{R}^3$-valued C^∞ function defined on $S_2(\delta_0)$ verifying $|r(\eta)|=|s(\eta)|=1$ and (2.17), (2.22). Then $i(\sigma)$ and $j(\sigma)$ defined by*

$$i(\sigma)=r(\eta(\sigma))-2(r(\eta(\sigma))\cdot n(\sigma))n(\sigma)$$

$$j(\sigma)=s(\tilde{\eta}(\sigma))-2(s(\tilde{\eta}(\sigma))\cdot n(\sigma))n(\sigma)$$

satisfy

$$(3.10)_0\qquad |i-j|_0(S_1(\delta_0))\leq((1+C_0d)^{-1}+C\delta_0)\,|r-s|_0(S_2(\delta_0))$$

and for $m\geq 1$

$$\begin{aligned}(3.10)\quad |i-j|_m(S_1(\delta_0))&\leq((1+C_0d)^{-1}+C\delta_0)^{m+1}|r-s|_m(S_2(\delta_0))\\&\quad+C|r-s|_{m-1}(S_2(\delta_0))\{|r|_{m+1}(S_2(\delta_0))+|s|_{m+1}(S_2(\delta_0))\\&\quad+|y|_{m+1}(S_1(\delta_0))+|z|_{m+1}(S_2(\delta_0))\},\end{aligned}$$

where $\eta(\sigma)$ and $\tilde{\eta}(\sigma)$ are defined by

$$y(\sigma)=z(\eta)+\tilde{l}(\eta)r(\eta)=z(\tilde{\eta})+\tilde{h}(\tilde{\eta})s(\tilde{\eta}).$$

Now we consider a convergence of a sequence of phase functions $\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_{q-1}, \varphi_q, \ldots$ constructed in the previous section. Fix $\delta_0>0$ so small that

$$\alpha=(1+C_0d)^{-1}+C\delta_0<1.$$

Note that from Proposition 2.5 $\{i_q\}_{q=0}^{\infty}$ and $\{r_q\}_{q=0}^{\infty}$ are bounded set of $\mathscr{B}^{\infty}(S_1(\delta_0))$ and $\mathscr{B}^{\infty}(S_2(\delta_0))$ respectively. Set

$$|i_1-i_0|_m(S_1(\delta_0))=A_m, \quad m=0, 1,\dots.$$

Taking account of (2.36) and (2.38) we have from Corollary

$$(3.10)_0 \qquad |i_{q+1}-i_q|_0(S_1(\delta_0))\le\alpha|r_q-r_{q-1}|_0(S_2(\delta_0)),$$

$$(3.11)_m \qquad |i_{q+1}-i_q|_m(S_1(\delta_0))$$
$$\le\alpha^{m+1}|r_q-r_{q-1}|_m(S_2(\delta_0))+C_m|r_q-r_{q-1}|_{m-1}(S_2(\delta_0)),$$

and Lemma 3.3 shows

$$(3.12)_0 \qquad |r_q-r_{q-1}|_0(S_2(\delta_0))\le\alpha|i_q-i_{q-1}|_0(S_1(\delta_0))$$

$$(3.12)_m \qquad |r_q-r_{q-1}|_m(S_2(\delta_0))$$
$$\le\alpha^{m+1}|i_q-i_{q-1}|_m(S_1(\delta_0))+C_m|i_q-i_{q-1}|_{m-1}(S_1(\delta_0)).$$

From $(3.11)_0$ and $(3.12)_0$ we have

$$|i_{q+1}-i_q|_0(S_1(\delta_0))\le\alpha^{2q}A_0,$$
$$|r_{q+1}-r_q|_0(S_2(\delta_0))\le\alpha^{2q+1}A_0.$$

Then there exists R^3-valued function $i_\infty(\sigma)$ on $S_1(\delta_0)$ and $r_\infty(\eta)$ on $S_2(\delta_0)$ such that

$$(3.13)_0 \qquad |i_q-i_\infty|_0(S_1(\delta_0))\le A_0\frac{\alpha^{2(q-1)}}{1-\alpha^2}$$

$$(3.14)_0 \qquad |r_q-r_\infty|_0(S_2(\delta_0))\le A_0\frac{\alpha^{2q-1}}{1-\alpha^2}.$$

Then by using $(3.11)_m$ and $(3.12)_m$ we have inductively for all $m\ge1$

$$(3.13)_m \qquad |i_q-i_\infty|_m(S_1(\delta_0))\le A_m\frac{\alpha^{2(q-1)}}{1-\alpha^2}+C_mA_{m-1}\alpha^{2(q-1)}$$

$$(3.14)_m \qquad |r_q-r_\infty|_m(S_2(\delta_0))\le A_m\frac{\alpha^{2q-1}}{1-\alpha^2}+C_mA_{m-1}\alpha^{2q-1}.$$

Thus we have

Proposition 3.4. *For a sequence of phase functions* $\{\varphi_q\}_{q=0}^{\infty}$ *there exist* R^3-*valued* C^∞ *function* $i_\infty(\sigma)$ *on* $S_1(\delta_0)$ *and* $r_\infty(\eta)$ *on* $S_2(\delta_0)$ *such that* $(3.13)_m$ *and* $(3.14)_m$ *hold for all* $m\ge0$.

Remark 1. i_∞ and r_∞ satisfy

$$(3.15) \qquad i_\infty(0)=Y_3, \quad r_\infty(0)=-Y_3.$$

Indeed, take a function $\psi(x)$ satisfying $|\nabla\psi|=1$, (2.4), (2.13) and $\nabla\psi(a_1)=Y_3$. Construct $\psi_0, \psi_1, \psi_2,\dots$ according to the process in §2 for ψ. Then it is evident that

$$\nabla\psi_{2q}(a_1)=Y_3, \quad \nabla\psi_{2q+1}(a_2)=-Y_3. \tag{3.16}$$

On the other hand, by using $(3.9)_0$ and $(3.10)_0$ successively we obtain

$$|\nabla\varphi_{2q}-\nabla\psi_{2q}|_0(S_1(\delta_0))\leq\alpha^{2(q-1)}|\nabla\varphi-\nabla\psi|_0(S_1(\delta_0)), \tag{3.17}$$

$$|\nabla\varphi_{2q+1}-\nabla\psi_{2q+1}|_0(S_2(\delta_0))\leq\alpha^{2q-1}|\nabla\varphi-\nabla\psi|_0(S_1(\delta_0)). \tag{3.18}$$

From $(3.13)_0$, (3.16) and (3.17) we have $i_\infty(0)=Y_3$ and from $(3.14)_0$, (3.16) and (3.18) we have $r_\infty(0)=-Y_3$.

Remark 2. Note that

$$r_\infty(\eta)=i_\infty(\sigma)-2(i_\infty(\sigma)\cdot m(\eta))m(\eta), \tag{3.19}$$

where σ and η are linked by $z(\eta)=y(\sigma)+l(\sigma)i_\infty(\sigma)$. And also it holds that

$$i_\infty(\sigma)=r_\infty(\eta)-2(r_\infty(\eta)\cdot n(\sigma))n(\sigma), \tag{3.20}$$

where $y(\sigma)=z(\eta)+h(\eta)r_\infty(\eta)$. Let $\mathscr{K}_\infty(\sigma)$ and $\tilde{\mathscr{K}}_\infty(\eta)$ be matrices defined by (2.3) for $i_\infty(\sigma)$ and $r_\infty(\eta)$. Since $\sigma=0$ corresponds to $\eta=0$ (2.38) for $\sigma=0$

$$\|\tilde{\mathscr{K}}_\infty(0)-F_2(\mathscr{K}_\infty(0))\|\leq C\delta_0\|\mathscr{K}_\infty(0)\|$$

holds for any $\delta_0>0$. This implies

$$\tilde{\mathscr{K}}_\infty(0)=F_2(\mathscr{K}_\infty(0)). \tag{3.21}$$

Similarly we have

$$\mathscr{K}_\infty(0)=F_1(\tilde{\mathscr{K}}_\infty(0)). \tag{3.22}$$

Then $\mathscr{K}_\infty(0)$ is the fixed point of $\mathscr{F}_1$ and $\tilde{\mathscr{K}}_\infty(0)$ is the fixed point of $\mathscr{F}_2$.

Remark 3. In the course of proof of Proposition 3.4 a constant $\alpha=(1+C_0d)^{-1}+C\delta_0$ in $(3.13)_m$ and $(3.14)_m$ is used as

$$\|(Y(i_q;\,\sigma)+l(\sigma)\mathscr{K}_q(\sigma))^{-1}\|\leq\alpha$$

$$\|(Z(r_q;\,\eta)+h(\eta)\tilde{\mathscr{K}}_q(\eta))^{-1}\|\leq\alpha.$$

But Proposition 3.4 assures that

$$\|[Y(i_q;\,\sigma)+l(\sigma)\mathscr{K}_q(\sigma)]^{-1}\|\leq\|(I+d\mathscr{K}_\infty(0))^{-1}\|+C\delta_0$$

$$\|[Z(r_q;\,\eta)+h(\eta)\tilde{\mathscr{K}}_q(\eta)]^{-1}\|\leq\|(I+d\tilde{\mathscr{K}}_\infty(0))^{-1}\|+C\delta_0$$

holds for large q. Therefore by setting

$$\alpha_0=\max\left(\|(I+d\mathscr{K}_\infty(0))^{-1}\|,\ \|(I+d\tilde{\mathscr{K}}_\infty(0))^{-1}\|\right)$$

we have

(3.13)$'_m$ $$|i_q-i_\infty|_m(S_1(\delta_0))\leq(\alpha_0+C\delta_0)^{2(q-1)}A'_m$$

(3.14)$'_m$ $$|r_q-r_\infty|_m(S_2(\delta_0))\leq(\alpha_0+C\delta_0)^{2q-1}A'_m$$

where A'_m is a constant determined by A_m and δ_0.

§4. Convergence of broken rays

We will use freely the notations concerning broken rays in §3 of [5]. Let $\varphi(x)$ be C^∞ function satisfying $|\nabla\varphi|=1$ and (2.3), (2.13). Let $\varphi_0, \varphi_1, \varphi_2,\dots$ be a sequence of phase functions constructed in §2. Denote by Φ_q the mapping Φ in §2 for $\nabla\varphi_q$, namely

$$\Phi_q\colon S_{\epsilon(q)}(\delta_0)\times[0, \infty)\longrightarrow \boldsymbol{R}^3$$

defined by

$$\Phi_q(x, l)=x+l\nabla\varphi_q(x).$$

And we denote by Θ_q a mapping Θ for $\nabla\varphi_q$, namely

$$\Theta_q\colon S_{\epsilon(q)}(\delta_0)\longrightarrow\Gamma_{\epsilon(q+1),0}.$$

As remarked in §2 we have from the assumption (2.3)

(4.1) $$\Theta_q(S_{\epsilon(q)}(\delta_0))\supset S_{\epsilon(q+1)}(\delta_0)\qquad\text{for all}\quad q.$$

By using this notation we have

(4.2) $$X_q(y, \nabla\varphi)=\Theta_q\circ\Theta_{q-1}\circ\cdots\circ\Theta_1\circ\Theta_0(y)\qquad\text{for}\quad y\in S_1(\delta_0).$$

Let Ψ_{2q} be a function defined by

$$\Theta_{2q}(y(\Psi_{2q}(\eta)))=z(\eta)\qquad\text{for}\quad \eta\in S_2(\delta_0)$$

and let Ψ_{2q+1} be a function defined by

$$\Theta_{2q+1}(z(\Psi_{2q+1}(\sigma)))=y(\sigma)\qquad\text{for}\quad \sigma\in S_1(\delta_0).$$

In other words

$$\Theta_{2q}^{-1}(z(\eta))=y(\Psi_{2q}(\eta)),\quad \Theta_{2q+1}^{-1}(y(\sigma))=z(\Psi_{2q+1}(\sigma)).$$

From Lemma 2.1 we have

(4.3) $$\begin{cases}\left\|\dfrac{\partial\psi_{2q}(\eta)}{\partial\eta}-[I+d\mathscr{K}_q(\psi_{2q}(\eta))]^{-1}\right\|\leq C\delta_0\\[2ex] \left\|\dfrac{\partial\psi_{2q+1}(\sigma)}{\partial\sigma}-[I+d\tilde{\mathscr{K}}_q(\psi_{2q+1}(\sigma))]^{-1}\right\|\leq C\delta_0.\end{cases}$$

Define $X_{-j}(x, \nabla\varphi_q)$ for $x\in S_{\epsilon(q)}(\delta_0)$ and $0\leq j\leq q$ by

$$X_{-j}(\cdot, \nabla\varphi_q)=\Theta_{q-j+1}^{-1}\circ\Theta_{q-j+2}^{-1}\circ\cdots\circ\Theta_q^{-1}(x).$$

And set

$$\Psi_{q,j}=\Psi_{q-j}\circ\Psi_{q-j+1}\circ\cdots\circ\Psi_{q-1}\circ\Psi_{q}.$$

Let $\varphi_\infty(x)$ and $\tilde{\varphi}_\infty(x)$ be real valued C^∞ functions such that

$$|\nabla\varphi_\infty|=1, \qquad |\nabla\tilde{\varphi}_\infty|=1,$$

$$\nabla\varphi_\infty(y(\sigma))=i_\infty(\sigma), \quad \varphi_\infty(a_1)=0,$$

$$\nabla\tilde{\varphi}_\infty(z(\eta))=r_\infty(\eta), \quad \tilde{\varphi}_\infty(a_2)=0.$$

Let us denote Ψ defined for i_∞ and r_∞ by Ψ_∞ and $\tilde{\Psi}_\infty$ respectively. And we denote by Θ_∞ and $\tilde{\Theta}_\infty$ mappings Θ defined for i_∞ and r_∞ respecitvely. Similarly we can define $X_{\pm j}(x, \nabla\varphi_\infty)$ or $X_{\pm j}(x, \nabla\tilde{\varphi}_\infty)$ for a sequence of phase functions $\nabla\varphi_\infty$, $\nabla\tilde{\varphi}_\infty$, $\nabla\varphi_\infty,\dots$. Set

$$X^\infty_{\pm j}(x)=\begin{cases} X_{\pm j}(x, \nabla\varphi_\infty) & \text{for} \quad x\in S_1(\delta_0)\\ X_{\pm j}(x, \nabla\tilde{\varphi}_\infty) & \text{for} \quad x\in S_2(\delta_0).\end{cases}$$

Define $\Psi_{\infty,j}(\sigma)$ and $\tilde{\Psi}_{\infty,j}(\eta)$ by

$$X^\infty_{-j}(y(\sigma))=\begin{cases} y(\Psi_{\infty,j}(\sigma)) & \text{for} \quad j \text{ even}\\ z(\Psi_{\infty,j}(\sigma)) & \text{for} \quad j \text{ odd},\end{cases}$$

$$X^\infty_{-j}(z(\eta))=\begin{cases} z(\tilde{\Psi}_{\infty,j}(\eta)) & \text{for} \quad j \text{ even}\\ y(\tilde{\Psi}_{\infty,j}(\eta)) & \text{for} \quad j \text{ odd}.\end{cases}$$

Hereafter we denote $\alpha_0+C\delta_0$ by α.

Lemma 4.1. *For $1\le j\le q$ we have*

$$\sum_{1\le\beta\le m}|\partial_\sigma^\beta\Psi_{q,j}(\sigma)|\le C_m\alpha^j, \tag{4.4}$$

where C_m is a constant independent of q and j.

Proof. From the chain rule of derivatives of composed functions we have

$$\left[\frac{\partial\Psi_{q,j}(\sigma)}{\partial\sigma}\right]=\left[\frac{\partial\psi_{q-j+1}}{\partial\sigma}(\Psi_{q,j}(\sigma))\right]\left[\frac{\partial\psi_{q-j+2}}{\partial\sigma}(\Psi_{q,j-1}(\sigma))\right]\cdots \cdots\left[\frac{\partial\psi_{q+1}}{\partial\sigma}(\Psi_{q,1}(\sigma))\right]\left[\frac{\partial\psi_q}{\partial\sigma}(\sigma)\right]. \tag{4.5}$$

Remark 3 of §3 says that

$$\left\|\left[\frac{\partial\psi_q}{\partial\sigma}\right]\right\|\le\alpha$$

except a finite number of q. Then we have

$$\left\|\left[\frac{\partial\Psi_{q,j}}{\partial\sigma}\right]\right\|\le C\alpha^j. \tag{4.6}$$

For derivatives of higher order diffderentiate the both sides of (4.5) and use the boundedness of $\{\partial_\sigma^\beta \psi_q\}_{q=0}^\infty$ for any β we have (4.4) for all m. Q. E. D.

By the same reasoning we have

Lemma 4.2. *For $j \geq 1$ we have*

$$\sum_{1 \leq |\beta| \leq m} |\partial_\sigma^\beta \Psi_{\infty,j}(\sigma)| \leq C_m \alpha^j, \tag{4.7}$$

$$\sum_{1 \leq \beta \leq m} |\partial_\eta^\beta \tilde{\Psi}_{\infty,j}(\eta)| \leq C_m \alpha^j. \tag{4.8}$$

Remark 1. Set

$$y(\tilde{\sigma}) = X_{-2j}(y(\sigma), \nabla\varphi_{2q}).$$

Then $\tilde{\sigma} = \Psi_{2q,2j}(\sigma)$. Therefore

$$|y(\tilde{\sigma}) - y(\tilde{\sigma}')| \leq \left\| \frac{\partial \Psi_{2q,2j}}{\partial \sigma} \right\| |\sigma - \sigma'| \leq C\alpha^{2j} |y(\sigma) - y(\sigma')|.$$

Namely it holds that for all $x,\ y \in S_1(\delta_0)$

$$|X_{-2j}(x, \nabla\varphi_{2q}) - X_{-2j}(y, \nabla\varphi_{2q})| \leq C\alpha^{2j}|x - y|.$$

Evidently an estimate of the above type holds for $x,\ y \in S_2(\delta_0)$. Then for all $0 \leq j \leq q$ and $x,\ y \in S_1(\delta_0)$ $(x,\ y \in S_2(\delta_0))$ we have

$$|X_{-j}(x, \nabla\varphi_q) - X_{-j}(y, \nabla\varphi_q)| \leq C\alpha^j |x - y|. \tag{4.9}$$

Lemma 4.3. *It holds that*

$$|X_{-2j}^\infty(x) - a_1| + |X_{-2j-1}^\infty(x) - a_2| \leq C\alpha^{2j}|x'| \tag{4.10}$$

for $x \in S_1(\delta_0)$ and

$$|X_{-2j}^\infty(x) - a_2| + |X_{-2j-1}^\infty(x) - a_1| \leq C\alpha^{2j}|x'| \tag{4.11}$$

for $x \in S_2(\delta_0)$.

Proof. Let $y(\sigma) \in S_1(\delta_0)$. Set $X_{-2j}^\infty(y(\sigma)) = y(\tilde{\sigma})$. Then $\tilde{\sigma} = \Psi_{\infty,2j}(\sigma)$. From (4.7)

$$\left\| \frac{\partial \tilde{\sigma}}{\partial \sigma} \right\| \leq C\alpha^{2j}.$$

By using $\Psi_{\infty,2j}(0) = 0$ we have $|\tilde{\sigma}| \leq C\alpha^{2j}|\sigma|$, which implies

$$|X_{-2j}^\infty(x) - a_1| = |y(\tilde{\sigma}) - y(0)| \leq C\alpha^{2j}|\sigma|.$$

Similarly we have

$$|X_{-2j-1}^\infty(x) - a_2| \leq C\alpha^{2j}|\sigma|.$$

Thus (4.10) is proved. (4.11) is also proved by the same reasoning.

Lemma 4.4. *For $1 \leq j \leq q$ it holds that*

(4.12) $$|X_{-j}(\cdot, \nabla\varphi_q)-X^{\infty}_{-j}(\cdot)|_m(S_{\epsilon(q)}(\delta_0)) \leq C_m\alpha^{q-j},$$

for $m=0, 1,\dots,$ where C_m is independent of q and j.

Proof. Let q is even. Since $X^{\infty}_{-1}(x)=X_{-1}(x, \nabla\varphi_\infty)$ we have with the aid of $(3.13)_m$

$$\sum_{|\beta|\leq m} |\partial^\beta_\sigma(X_{-1}(y(\sigma), \nabla\varphi_q)-X^{\infty}_{-1}(y(\sigma)))| \leq C_m\alpha^q.$$

Suppose that

$$\sum_{|\beta|\leq m} |\partial^\beta_\sigma(X_{-s}(y(\sigma), \nabla\varphi_q)-X^{\infty}_{-s}(y(\sigma)))| \leq C_m\alpha^{q-s+1}(1+\alpha^2+\cdots+\alpha^{2(s-1)}).$$

Since $X_{-s-1}(y(\sigma), \nabla\varphi_q)=X_{-1}(X_{-s}(y(\sigma), \nabla\varphi_q), \nabla\varphi_{q-s})$ and $X^{\infty}_{-s-1}(y(\sigma)) =X_{-1}(X^{\infty}_{-s}(y(\sigma), \nabla\varphi_\infty))$ we have

$$\begin{aligned} M= &\sum_{|\beta|\leq m} |\partial^\beta_\sigma(X_{-s-1}(y(\sigma), \nabla\varphi_q)-X^{\infty}_{-s-1}(y(\sigma)))| \\ \leq &\sum_{|\beta|\leq m} |\partial^\beta_\sigma(X_{-1}(X_{-s}(y(\sigma), \nabla\varphi_q), \nabla\varphi_{q-s})-X_{-1}(X_{-s}(y(\sigma), \nabla\varphi_q), \nabla\varphi_\infty))| \\ &+ \sum_{|\beta|\leq m} |\partial^\beta_\sigma(X_{-1}(X_{-s}(y(\sigma), \nabla\varphi_q), \nabla\varphi_\infty)-X_{-1}(X^{\infty}_{-s}(y(\sigma), \nabla\varphi_\infty)))| \\ = &M_1+M_2. \end{aligned}$$

Then from the above remark we have

$$M_1 \leq C_m\alpha^{q-s}.$$

And by using the assumption we have

$$M_2 \leq C_m\alpha\cdot\alpha^{q-s+1}(1+\alpha^2+\cdots+\alpha^{2s-2}).$$

Therefore

$$M \leq C_m\alpha^{q-s}(1+\alpha^2+\cdots+\alpha^{2s}).$$

Thus (4.12) is proved. Q.E.D.

Lemma 4.5. *There exists a point $A \in S_1(\delta_0)$ such that*

(4.13) $$|X_{2q}(A, \nabla\varphi_0)-a_1| \leq C\alpha^{2q}$$

(4.14) $$|X_{2q+1}(A, \nabla\varphi_0)-a_2| \leq C\alpha^{2q}.$$

Proof. Let $m>q\geqq 0$. Set

$$A_{m,q}=X_{-2m+q}(a_1, \nabla\varphi_{2m})$$

$$B_{m,q}=X_{-(2m+1)+q}(a_2, \nabla\varphi_{2m+1}).$$

Suppose that q is even. Then $A_{m,q}\in S_1(\delta_0)$. Let $n>m$.

$$(4.15)\qquad |X_{-2(n-m)}(a_1, \nabla\varphi_{2n})-a_1| = |X_{-2(n-m)}(a_1, \nabla\varphi_{2n})-X^{\infty}_{-2(n-m)}(a_1)|$$

from Lemma 4.4

$$\leq C\alpha^{2n-(2n-2m)}=C\alpha^{2m}.$$

Since

$$A_{n,q}-A_{m,q}=X_{-2m+q}(X_{-2(n-m)}(x, \nabla\varphi_{2n}), \nabla\varphi_{2m})-X_{-2m+q}(a_1, \nabla\varphi_{2m})$$

we have from Remark of Lemma 4.1 and the above estimate

$$|A_{n,q}-A_{m,q}|\leq C\alpha^{2m}\alpha^{2m-q}=C\alpha^{4m-q}.$$

Then for each q, $\{A_{m,q}\}_{m=1}^{\infty}$ is a Cauchy sequence. Therefore there exists $A_{\infty,q}$ such that

$$A_{m,q}\longrightarrow A_{\infty,q} \quad \text{as} \quad m\longrightarrow\infty.$$

Evidently it holds that

$$|A_{m,q}-A_{\infty,q}|\leq C\alpha^{4m-q}.$$

From the definition we have for $2m\geq p>q$

$$X_{p-q}(A_{m,q}, \nabla\varphi_q)=A_{m,p}.$$

Then

$$X_{p-q}(A_{\infty,q}, \nabla\varphi_q)=A_{\infty,p} \qquad \text{for all} \quad p>q.$$

This implies that

$$(4.16)\qquad X_q(A_{\infty,0}, \nabla\varphi_0)=A_{\infty,q} \qquad \text{for all} \quad q.$$

In (4.15) setting $m=q$ we have

$$|A_{n,2q}-a_1|\leq C\alpha^{2q}$$

and letting $n\to\infty$

$$|A_{\infty,2q}-a_1|\leq C\alpha^{2q}.$$

Taking account of (4.16) the above estimate shows (4.13).

By the same method we have

$$X_q(B_{\infty,0}, \nabla\varphi_0)=B_{\infty,q}$$

and

$$|B_{\infty,2q+1}-a_2|\leq C\alpha^{2q}.$$

On the other hand

$$|a_1-X_{-1}(a_2, \nabla\varphi_{2m+1})|\leq C\alpha^{2m}$$

and

$$|A_{m,0}-B_{m,0}|=|X_{-2m}(a_1, \nabla\varphi_{2m})-X_{-2m}(X_{-1}(a_2, \nabla\varphi_{2m+1}), \nabla\varphi_{2m})|\leq C\alpha^{4m}.$$

Then we have $A_{\infty,0}=B_{\infty,0}$. This completes the proof.

Proposition 4.6. *It holds that for* $0<j\leq q$

$$(4.17)\qquad \begin{cases} \text{(i)} & |X_{-j}(\cdot, \nabla\varphi_{2q})-X_{-j}^{\infty}(\cdot)|_m(S_1(\delta_0))\leq C_m\alpha^q \\ \text{(ii)} & |X_{-2q+j}(\cdot, \nabla\varphi_{2q})-X_j(A, \nabla\varphi_0)|_m(S_1(\delta_0))\leq C_m\alpha^q \end{cases}$$

and

$$(4.18)\qquad \begin{cases} \text{(i)} & |X_{-j}(\cdot, \nabla\varphi_{2q+1})-X_{-j}^{\infty}(\cdot)|_m(S_2(\delta_0))\leq C_m\alpha^q \\ \text{(ii)} & |X_{-2q-1+j}(\cdot, \nabla\varphi_{2q+1})-X_j(A, \nabla\varphi_0)|_m(S_2(\delta_0))\leq C_m\alpha^q. \end{cases}$$

Proof. (i) of (4.17) and (4.18) are nothing but Lemma 4.4. Let q is even. From Lemmas 4.3 and 4.4 we have

$$(4.19)\qquad |X_{-q}(x, \nabla\varphi_{2q})-a_1|\leq C\alpha^q.$$

Note that

$$X_{-2q+j}(x, \nabla\varphi_{2q})-X_j(A, \nabla\varphi_0)$$
$$=X_{-(q-j)}(X_{-q}(x, \nabla\varphi_{2q}), \nabla\varphi_q)-X_{-(q-j)}(X_q(A, \nabla\varphi_0), \nabla\varphi_q).$$

And (4.19) and (4.13) imply

$$|X_{-q}(x, \nabla\varphi_{2q})-X_q(A, \nabla\varphi_0)|\leq C\alpha^q.$$

Then Lemma 4.1 shows

$$|X_{-(q-j)}(X_{-q}(x, \nabla\varphi_{2q}), \nabla\varphi_q)-X_{-(q-j)}(X_q(A, \nabla\varphi_0), \nabla\varphi_q)|\leq C\alpha^{q+q-j}.$$

For $m\geq 1$, since

$$X_{-2q+j}(y(\sigma), \nabla\varphi_{2q})=y(\Psi_{2q,2q-j}(\sigma))$$

Lemma 4.1 shows that

$$|X_{-2q+j}(\cdot, \nabla\varphi_{2q})|_m(S_1(\delta_0))\leq C\alpha^{2q-j}.$$

Then (ii) of (4.17) is proved. We can show (ii) of (4.18) by the same method.

§5. Transport equations (1)

Let $\varphi(x)$ be a real valued C^∞ function verifying (2.2), (2.4) and (2.13) and let $\{\varphi_q\}_{q=q}^{\infty}$ be a sequence of phase functions constructed for φ following the procedure in §2. Set

$$T_q=2\frac{\partial}{\partial t}+2\nabla\varphi_q\cdot\nabla+\Delta\varphi_q.$$

Following §3 of [5] we choose $0<\delta_2<\delta_3<\delta_0$ so that Lemma 3.3 and its corollary of [5] hold.

Let $\upsilon_{jl}(x)$, j, $l=1, 2$ be functions defined on Γ_j satisfying

$$\upsilon_{j1}(x)=\begin{cases} 1 & x\in S_j(\delta_2) \\ 0 & x\notin S_j(\delta_3) \end{cases}$$

and $\upsilon_{j1}(x)+\upsilon_{j2}(x)=1$ on Γ_j. Set

$$\omega_q=\{\Phi_q(x, l);\ x\in S_{\epsilon(q)}(\delta_3),\ 0<l<|\Theta_q(x)-x|\}.$$

Note that, if $|i(0)-Y_3|\leq\delta_3$,

$$\omega_q\subset\omega \quad \text{for all} \quad q \tag{5.1}$$

where ω is the one defined in Proposition 2.5.

Definition 5.1. Let $\boldsymbol{f}=\{f_q\}_{q=0}^{\infty}$ be a sequence such that $f_q\in C_0^\infty(S_{\epsilon(q)}(\delta_0)\times(0, \infty))$ and $\boldsymbol{g}=\{g_q\}_{q=0}^{\infty}$ be a sequence such that $g_q\in C_0^\infty(\bar{\omega}\times(0, \infty))$. We say that a sequence $\boldsymbol{v}=\{v_q\}_{q=0}^{\infty}$ such that $v_q\in C_0^\infty(\bar{\omega}\times(0, \infty))$ is a solution of

$$\begin{cases} T\boldsymbol{v}=\boldsymbol{g} & \text{in} \quad \omega\times\boldsymbol{R} \\ \boldsymbol{v}=\boldsymbol{f} & \text{on} \quad S(\delta_2)\times\boldsymbol{R} \end{cases}$$

when

$$\begin{cases} T_q v_q=g_q & \text{in} \quad \bar{\omega}\times\boldsymbol{R} \\ v_q=\upsilon_{\epsilon(q),1}v_{q-1}+f_q & \text{on} \quad S_{\epsilon(q)}(\delta_0)\times\boldsymbol{R} \end{cases}$$

holds for all $q=0, 1,\dots$, where we set $v_{-1}=0$.

Let $\psi(x)$ be a real valued function defined in an open set $\mathscr{U}\subset\boldsymbol{R}^3$ satisfying $|\nabla\psi|=1$. Then any solution of an equation

$$2\frac{\partial v(x, t)}{\partial t}+2\nabla\psi(x)\cdot\nabla v(x, t)+\Delta\psi(x)v(x, t)=0 \qquad \text{in} \quad \mathscr{U}\times\boldsymbol{R}$$

satisfies

$$v(x+l\nabla\psi(x), t+l)=\left[\frac{G_\psi(x+l\nabla\psi(x))}{G_\psi(x)}\right]^{1/2} v(x, t) \tag{5.2}$$

for all x, $x+l\nabla\psi(x)\in\mathscr{U}$, where $G_\psi(x)$ denotes the Gaussian curvature of a surface $\mathscr{C}_\psi(x)=\{y;\ \psi(y)=\psi(x)\}$ at x (see, Keller, Lewis and Seckler [6] and Ikawa [3]).

Set for $x\in S_{\epsilon(q+1)}(\delta_0)$

$$\Lambda_q(x)=[G_{\varphi_q}(x)/G_{\varphi_q}(\Theta_q^{-1}(x))]^{1/2}.$$

Then for v_q satisfying $T_q v_q=0$ in $\bar{\omega}_q\times\boldsymbol{R}$ we have from (5.2)

$$v_q(x, t)=\Lambda_q(x)v_q(\Theta_q^{-1}(x), t-h_q(x)), \quad h_q(x)=|x-\Theta_q^{-1}(x)| \tag{5.3}$$

for all $x \in S_{\epsilon(q+1)}(\delta_0)$.

Let $f(x, t) \in C_0^\infty(S_1(\delta_2) \times \boldsymbol{R})$ and let j is a non negative integer. Set

(5.4) $$\boldsymbol{f} = \{f_q\}_{q=0}^\infty \text{ where } f_{2j} = f \text{ and } f_q = 0 \text{ for } q \neq 2j.$$

Let $\boldsymbol{v} = \{v_q\}_{q=0}^\infty$ be a solution of

(5.5) $$\begin{cases} T\boldsymbol{v} = 0 & \text{in} \quad \omega \times \boldsymbol{R} \\ \boldsymbol{v} = \boldsymbol{f} & \text{on} \quad S(\delta_0) \times \boldsymbol{R}. \end{cases}$$

The definition means that v_q, $q = 0, 1, \dots$ satisfy

(5.6) $$T_q v_q = 0 \quad \text{in} \quad \omega \times \boldsymbol{R} \qquad \text{for all} \quad q$$

and

(5.7) $$\begin{aligned} v_q &= 0 \qquad \text{for} \quad q < 2j \\ v_{2j}(x, t) &= f(x, t) \qquad \text{on} \quad S_1(\delta_0) \times \boldsymbol{R}, \end{aligned}$$

and for $q > 2j$

(5.8) $$v_q(x, t) = v_{\epsilon(q),1}(x) v_{q-1}(x, t) \qquad \text{on} \quad S_{\epsilon(q)}(\delta_0) \times \boldsymbol{R}.$$

Note that for all $x \in S_{\epsilon(q+1)}(\delta_3)$ we have $\Theta_q^{-1}(x) \in S_{\epsilon(q)}(\delta_2)$. Since

$$v_q(x, t) = v_{q-1}(x, t) \qquad \text{on} \quad S_{\epsilon(q)}(\delta_2)$$

follows from the definition of $v_{j,1}(x)$ and (5.8), we have the following lemma by applying (5.3) successively.

Lemma 5.1. *For any $q \geq 2j$ and $x \in S_{\epsilon(q+1)}(\delta_3)$*

(5.9) $$\begin{aligned} v_q(x, t) = A_q(x) \cdot A_{q-1}(X_{-1}(x, \nabla\varphi_q)) \cdots A_{2j}(X_{-(q-2j)}(x, \nabla\varphi_q)) \cdot \\ f(X_{-(q-2j)-1}(x, \nabla\varphi_q), t - h_{q,2j}(x)) \end{aligned}$$

holds where

$$h_{q,2j}(x) = \sum_{l=0}^{q-2j} h_{q-l}(X_{-l}(x, \nabla\varphi_q)).$$

Set for $x \in S_2(\delta_0)$

$$A_\infty(x) = [G_{\varphi_\infty}(x) / G_{\varphi_\infty}(\Theta_\infty^{-1}(x))]^{1/2}$$

$$\lambda = A_\infty(a_2), \ h_\infty(x) = |x - \Theta_\infty^{-1}(x)|$$

and for $x \in S_1(\delta_0)$

$$\tilde{A}_\infty(x) = [G_{\tilde{\varphi}_\infty}(x) / G_{\tilde{\varphi}_\infty}(\tilde{\Theta}_\infty^{-1}(x))]^{1/2}$$

$$\tilde{\lambda} = \tilde{A}_\infty(a_1), \ \tilde{h}_\infty(x) = |x - \tilde{\Theta}_\infty^{-1}(x)|.$$

Define $a_j(x)$ on $S_1(\delta_0)$ and $\tilde{a}_j(x)$ on $S_2(\delta_0)$ by

$$a_j(x)=\frac{\tilde{\Lambda}_\infty(x)}{\tilde{\lambda}}\;\frac{\Lambda_\infty(X^\infty_{-1}(x))}{\lambda}\cdots\frac{\tilde{\Lambda}_\infty(X^\infty_{-2j+2}(x))}{\tilde{\lambda}}\;\frac{\Lambda_\infty(X^\infty_{-2j+1}(x))}{\lambda}$$

$$\tilde{a}_j(x)=\frac{\Lambda_\infty(x)}{\lambda}\;\frac{\tilde{\Lambda}_\infty(X^\infty_{-1}(x))}{\tilde{\lambda}}\cdots\frac{\tilde{\Lambda}_\infty(X^\infty_{-2j+1}(x))}{\tilde{\lambda}}\;\frac{\Lambda_\infty(X^\infty_{-2j}(x))}{\lambda}\,.$$

Remark 1. Because of (2.1) and (3.15) the principal curvatures of $\mathscr{C}_{\varphi_\infty}(a_1)$ at a_1 are the eigenvalues of $\mathscr{K}_\infty(0)$ and those of $\mathscr{C}_{\varphi_\infty}(a_2)$ at a_2 are the eigenvalues of $\mathscr{K}_\infty(0)\,(I+d\mathscr{K}_\infty(0))^{-1}$. Therefore

$$\lambda=[\det(I+d\mathscr{K}_\infty(0))]^{-1/2}\,.$$

Similarly we have

$$\tilde{\lambda}=[\det(I+d\tilde{\mathscr{K}}_\infty(0))]^{-1/2}\,.$$

Lemma 5.2. *For $x\in S_1(\delta_0)$*

$$a(x)=\lim_{j\to\infty}a_j(x)$$

exists and

$$|a-a_j|_m(S_1(\delta_0))\le C_m\alpha^{2j} \tag{5.10}$$

holds. Similarly for $x\in S_2(\delta_0)$

$$\tilde{a}(x)=\lim_{j\to\infty}\tilde{a}_j(x)$$

exists and

$$|\tilde{a}-\tilde{a}_j|_m(S_2(\delta_0))\le C_m\alpha^{2j} \tag{5.11}$$

holds.

Proof. Since φ_∞ and $\tilde{\varphi}_\infty$ are C^∞ functions we have

$$\tilde{\Lambda}_\infty(x)\in C^\infty(S_1(\delta_0)),\quad \Lambda_\infty(x)\in C^\infty(S_2(\delta_0))\,.$$

Then (4.10) implies

$$|\Lambda_\infty(X^\infty_{-2p+1}(x))-\Lambda_\infty(a_2)|\le C\alpha^{2p}\qquad\text{for all}\quad x\in S_2(\delta_0)\,. \tag{5.12}$$

Note that

$$\Lambda_\infty(X^\infty_{-2p+1}(y(\sigma)))=\Lambda_\infty(y(\Psi_{\infty,2p-1}(\sigma)))\,.$$

Then Lemma 4.2 shows that

$$\sum_{1\le|\beta|\le m}|\partial^\beta_\sigma\Lambda_\infty(X^\infty_{-2p+1}(y(\sigma)))|\le C_m\alpha^{2p}\,.$$

Set

$$\Lambda_\infty(X^\infty_{-2p+1}(x))/\lambda=1+\gamma_{2p-1}(x)$$

and we have

$$|\gamma_{2p-1}|_m(S_1(\delta_0))\leq C_m\alpha^{2p}.$$

By the same way, if we set

$$\tilde{\Lambda}_\infty(X^\infty_{-2p}(x))/\lambda=1+\gamma_{2p}(x) \qquad \text{for} \quad x\in S_1(\delta_0)$$

we have

$$|\gamma_{2p}|_m(S_1(\delta_0))\leq C_m\alpha^{2p}.$$

Therefore we see that

$$a_j(x)=\prod_{p=0}^{2j}(1+\gamma_p(x))$$

converges to some function $a(x)$ and (5.10) holds. And (5.11) may be proved by the same way.

Remark 2. Since

$$a_{j+1}(x)=\frac{\tilde{\Lambda}_\infty(x)}{\tilde{\lambda}}\frac{\Lambda_\infty(X^\infty_{-1}(x))}{\lambda}\cdots\cdots\frac{\Lambda_\infty(X^\infty_{-2j}(X^\infty_{-1}(x)))}{\lambda}$$
$$=\frac{\tilde{\Lambda}_\infty(x)}{\tilde{\lambda}}\tilde{a}_j(X^\infty_{-1}(x)).$$

Letting $j\to\infty$ we have

$$a(x)=\frac{\tilde{\Lambda}_\infty(x)}{\tilde{\lambda}}\tilde{a}(X^\infty_{-1}(x)) \qquad \text{for all} \quad x\in S_1(\delta_0).$$

Similarly we have

$$\tilde{a}(x)=\frac{\Lambda_\infty(x)}{\lambda}a(X^\infty_{-1}(x)) \qquad \text{for all} \quad x\in S_2(\delta_0).$$

Lemma 5.3. *Set*

$$b_{2j,2h}=\frac{\Lambda_{2j}(X_{2j}(A,\nabla\varphi_0))}{\lambda}\frac{\Lambda_{2j+1}(X_{2j+1}(A,\nabla\varphi_0))}{\tilde{\lambda}}\cdots$$
$$\cdots\frac{\Lambda_{2(j+h)}(X_{2(j+h)}(A,\nabla\varphi_0))}{\lambda}.$$

Then $b_{2j}=\lim_{h\to\infty}b_{2j,2h}$ *exists and*

$$|b_{2j}-b_{2j,2h}|\leq C\alpha^{2(j+h)},\ |b_{2j}-1|\leq C\alpha^{2j} \tag{5.13}$$

holds.

Proof. From (3.13) we have

$$|\Lambda_{2p}(\cdot)-\Lambda_\infty(\cdot)|_m\leq C_m\alpha^{2p}.$$

Then (4.14) implies

$$|\Lambda_{2p}(X_{2p+1}(A, \nabla\varphi_0)) - \Lambda_\infty(a_2)| \le C\alpha^{2p}.$$

Similarly we have

$$|\Lambda_{2p+1}(X_{2p+2}(A, \nabla\varphi_0)) - \tilde{\Lambda}_\infty(a_1)| \le C\alpha^{2p}$$

Therefore from these estimates (5.13) follows by the same reasoning as in Lemma 4.2. Q. E. D.

For $x \in S(\delta_0)$

$$j_{\infty,q}(x) = h_\infty(X_{-q}^\infty(x)) - d.$$

Lemma 5.4. *For $x \in S(\delta_0)$ there exists $j_\infty(x) \in C^\infty(S(\delta_0))$ such that*

$$(5.14) \qquad |j_\infty(\cdot) - \sum_{p=1}^{q} j_{\infty,p}(\cdot)|_m(S(\delta_0)) \le C_m \alpha^q.$$

Proof. Let $x \in S_1(\delta_0)$. From the definition

$$\begin{aligned} h_\infty(X_{-2p}^\infty(x)) &= |X_{-2p}^\infty(x) - X_{-2p-1}^\infty(x)| \\ &\le |X_{-2p}^\infty(x) - a_1| + |X_{-2p-1}^\infty(x) - a_2| + |a_2 - a_1| \end{aligned}$$

by (4.10)

$$\le C\alpha^{2p} + d.$$

Taking account of $h_\infty(x) \ge d$ for all $x \in S(\delta_0)$

$$0 \le j_{\infty,2p} = h_\infty(X_{-2p}^\infty(x)) - d \le C\alpha^{2p} \qquad \text{for all} \quad x \in S_1(\delta_0).$$

By the same way we have $0 \le j_{\infty,2p+1} \le C\alpha^{2p+1}$. On the other hand for $|\beta| \ge 1$

$$\begin{aligned} |\partial_\sigma^\beta h_\infty(X_{-2p}^\infty(y(\sigma))| &= |\partial_\sigma^\beta(y(\Psi_{\infty,2p}(\sigma)) - z(\Psi_{\infty,2p+1}(\sigma)))| \\ &\le |\partial_\sigma^\beta y(\Psi_{\infty,2p}(\sigma))| + |\partial_\sigma^\beta z(\Psi_{\infty,2p+1}(\sigma))| \end{aligned}$$

by (4.7) and (4.8)

$$\le C_{|\beta|}\alpha^{2p}.$$

Thus we have

$$\sum_{1 \le |\beta| \le m} |\partial_\sigma^\beta j_{\infty,p}(y(\sigma))| \le C_m \alpha^{2p},$$

from which

$$|j_\infty(\cdot) - \sum_{p=1}^{\infty} j_{\infty,p}(\cdot)|_m(S_1(\delta_0)) \le C_m \alpha^q$$

follows. For $x \in S_2(\delta_0)$ we have the same estimates. Therefore (5.14) is proved. Q. E. D.

Remark 3. As Remark 2 we have

$$j_\infty(x) = j_{\infty,1}(x) + j_\infty(X_{-1}^\infty(x)) = h_\infty(x) + j_\infty(X_{-1}^\infty(x)) - d.$$

Lemma 5.5. *Set*

$$d_p = l_p(A, \nabla\varphi_0) - d.$$

Then

$$\lim_{h\to\infty} \sum_{p=1}^{j+h} d_p = d_{\infty,j}$$

exists and

$$|d_{\infty,j} - \sum_{p=j}^{j+h} d_p| \le C\alpha^{(j+h)}, \quad |d_{\infty,j}| \le C\alpha^j \tag{5.15}$$

holds.

Proof. $0 \le |X_{2p}(A, \nabla\varphi_0) - X_{2p+1}(A, \nabla\varphi_0)| - d$

$\le |X_{2p}(A, \nabla\varphi_0) - a_1| + |X_{2p+1}(A, \nabla\varphi_0) - a_2| + |a_1 - a_2| - d$

from Lemma 4.5

$\le C\alpha^{2p}$.

Then we have

$$0 \le d_{2p} \le C\alpha^{2p}.$$

Similarly we have

$$0 \le d_{2p+1} \le C\alpha^{2p+1}.$$

From these estimates (5.15) follows immediately.

Proposition 5.6. *Let $v = \{v_q\}_{q=0}^{\infty}$ be a solution of (5.5) for f of (5.4). Then v_q, $q \ge 2j$ are decomposed as*

$$v_q = w_q + z_q$$

where

$$w_q(x, t) = \begin{cases} \lambda^{p+1-j}\tilde{\lambda}^{p-j} a(x) b_{2j} f(A_{2j}, t - j_\infty(x) - d_{\infty,2j} - (2p+1-2j)d) \\ \qquad\qquad \text{for} \quad q = 2p \\ (\lambda\tilde{\lambda})^{p+1-j} \tilde{a}(x) b_{2j} f(A_{2j}, t - j_\infty(x) - d_{\infty,2j} - (2p+2-2j)d) \\ \qquad\qquad \text{for} \quad q = 2p+1 \end{cases} \tag{5.16}$$

$A_{2j} = X_{2j}(A, \nabla\varphi_0)$, and z_q verifies

$$|z_q|_m(S_{\epsilon(q+1)}(\delta_3)) \le C_m(q-2j)(\lambda\tilde{\lambda}\alpha)^{(q/2-j)} |f|_m(S_1(\delta_0) \times \boldsymbol{R}), \tag{5.17}$$

where C_m is a constant independent of f.

Proof. Let $x \in S_2(\delta_3)$. Using (5.9)

$$v_{2p}(x, t)-w_{2p}(x, t)$$

$$=\lambda^{p+1}\tilde{\lambda}^{p}\left\{\frac{A_{2p}(x)}{\lambda}\frac{A_{2p-1}(X_{-1}(x, \nabla\varphi_{2p}))}{\tilde{\lambda}}\cdots\frac{A_{2j}(X_{-2p+2j}(x, \nabla\varphi_{2p}))}{\lambda}-a(x)b\right\}$$

$$f(X_{-2p+2j-1}(x, \nabla\varphi_{2p}), t-h_{2p,2j}(x))$$

$$+\lambda^{p+1}\tilde{\lambda}^{p}a(x)b\{f(X_{-2p+2j}(x, \nabla\varphi_{2p}), t-h_{2p,2j}(x))$$

$$-f(A_{2j}, t-j_{\infty}(x)-d_{\infty,2j}-(2p-2j)d)\}=I_1+I_2.$$

For $l\leq p$ we have from (3.13)′ and (3.14)′

$$\left\{\begin{array}{l}|A_{2p-2l}(\cdot)-A_{\infty}(\cdot)|_m(S_1(\delta_0))\leq C_m\alpha^{2p-2l},\\ |A_{2p-2l+1}(\cdot)-\tilde{A}_{\infty}(\cdot)|_m(S_2(\delta_0))\leq C_m\alpha^{2p-2l+1}.\end{array}\right. \tag{5.18}$$

Then for $l\leq p$

$$|\partial_{\sigma}^{\beta}(A_{2p-2l}(X_{-2l}(y(\sigma), \nabla\varphi_{2p}))-A_{\infty}(X_{-2l}^{\infty}(y(\sigma))))|$$

$$\leq|\partial_{\sigma}^{\beta}(A_{2p-2l}(X_{-2l}(y(\sigma), \nabla\varphi_{2p}))-A_{\infty}(X_{-2l}(y(\sigma), \nabla\varphi_{2p})))|$$

$$+|\partial_{\sigma}^{\beta}(A_{\infty}(X_{-2l}(y(\sigma), \nabla\varphi_{2p}))-A_{\infty}(X_{-2l}^{\infty}(y(\sigma))))|$$

by (5.18) and (4.12)

$$\leq C_m\alpha^{2p-2l}.$$

By the same way

$$|\partial_{\sigma}^{\beta}(A_{2p-2l+1}(X_{-2l+1}(y(\sigma), \nabla\varphi_{2p}))-A_{\infty}(X_{-2l+1}^{\infty}(y(\sigma))))|\leq C_m\alpha^{2p-2l+1}.$$

Then we have for all $0\leq l\leq p$

$$|A_{2p-2l}(X_{-2l}(\cdot, \nabla\varphi_{2p}))/\lambda-(1+\gamma_{2l}(\cdot))|_m(S_1(\delta_0))\leq C_m\alpha^{2p-2l},$$

$$|A_{2p-2l+1}(X_{-2l+1}(\cdot, \nabla\varphi_{2p}))/\tilde{\lambda}-(1+\gamma_{2l-1}(\cdot))|_m(S_1(\delta_0))\leq C_m\alpha^{2p-2l+1},$$

and

$$\left|\frac{A_{2p}(\cdot)}{\lambda}\frac{A_{2p-1}(X_{-1}(\cdot, \nabla\varphi_{2p}))}{\tilde{\lambda}}\cdots\frac{A_{2l}(X_{-2(p-l)}(\cdot, \nabla\varphi_{2p}))}{\lambda}\right.$$

$$\left.-\prod_{h=0}^{2(p-l)}(1+\gamma_h(\cdot))\right|_m(S_1(\delta_0))\leq C_m2(p-l)\alpha^{\min(2l,2(p-l))}.$$

By combining the above estimate with (5.10) we have

$$\left|\frac{A_{2p}(\cdot)}{\lambda}\frac{A_{2p-1}(X_{-1}(\cdot, \nabla\varphi_{2p}))}{\tilde{\lambda}}\cdots\frac{A_{2l}(X_{-2p+2l}(\cdot, \nabla\varphi_{2p}))}{\lambda}\right. \tag{5.19}$$

$$\left.-a(\cdot)\right|_m(S_1(\delta_0))\leq C_m(2p-2l)\alpha^{\min(2l,2(p-l))}.$$

Suppose that $p>2j$. Then (4.17) implies

$$|\Lambda_{2j+h}(X_{-2p+2j+h}(\cdot, \nabla\varphi_{2p}))-\Lambda_{2j+h}(X_{2j+h}(A, \nabla\varphi_0))|_m(S_1(\delta_0))$$
$$\leq C_m\alpha^p \quad \text{for} \quad 0\leq h\leq p-2j.$$

Then we have for $p\leq 2l\leq p+1$

$$\left|\frac{\Lambda_{2l-1}(X_{-2p+2l-1}(\cdot, \nabla\varphi_{2p}))}{\tilde{\lambda}}\cdots\frac{\Lambda_{2j}(X_{-2p+2j}(\cdot, \nabla\varphi_{2p}))}{\lambda}\right.$$
$$\left.-b_{2j}\right|_m(S_1(\delta_0))\leq C_m(p-2j)\alpha^p.$$

Then by choosing l as $p\leq 2l\leq p+1$ we have from (5.19) and (5.20)

$$|I_1|_m(S_1(\delta_0))\leq C_m(2p-2j)\lambda^{p+1}\tilde{\lambda}^p\alpha^p|f|_m(S_1(\delta_0)\times \boldsymbol{R}).$$

When $p<2j$ we have from (5.19) for $l=j$ and $|b_{2j}-1|\leq C\alpha^{2j}\leq C\alpha^{2(p-j)}$

$$|I_1|_m(S_1(\delta_0))\leq C_m 2(p-j)\alpha^{2(p-j)}.$$

Next consider I_2. Suppose that $p>2j$. Then (ii) of (4.17) shows that

$$|X_{-2p+2j-1}(\cdot, \nabla\varphi_{2p})-A_{2j}|_m\leq C_m\alpha^p \tag{5.21}$$

and (i) of (4.17), (3.13)′ and (3.14)′ imply that

$$|h_\infty(X_{-l}^\infty(\cdot))-h_{2p-l}(X_{-l}(\cdot, \nabla\varphi_{2p}))|_m(S_1(\delta_0))\leq C_m\alpha^p. \tag{5.22}$$

Then

$$|\sum_{q=0}^{p}(j_{\infty,q}(\cdot)+d)-\sum_{l=0}^{p}h_{2p-l}(X_{-l}(\cdot, \nabla\varphi_{2p}))|_m(S_1(\delta_0))\leq C_m p\alpha^p,$$

from which, with the aid of Lemma 5.4,

$$|h_{2p,p}(\cdot)-j_\infty(\cdot)+pd|_m(S_1(\delta_0))\leq C_m p\alpha^p \tag{5.23}$$

follows.
By the same way from (ii) of (4.17) we have

$$|h_{2j+l}(X_{-2p+2j+l}(\cdot, \nabla\varphi_{2p}))-h_{2j+l}(X_{2j+l}(A, \nabla\varphi_0))|_m(S_1(\delta_0))$$
$$\leq C_m\alpha^p \quad \text{for} \quad 0\leq l\leq p-2j.$$

Then by using Lemma 5.5

$$\sum_{l=0}^{p-2j}|h_{2j+l}(X_{-2p+2j+l}(\cdot, \nabla\varphi_{2p}))-d_{\infty,2j}-(p-2j)d|_m(S_1(\delta_0))\leq C_m(2p-2j)\alpha^p. \tag{5.24}$$

And (5.23) and (5.24) show that

$$|h_{2p,2j}(\cdot)-j_\infty(\cdot)-d_{\infty,2j}-2(p-j)d|_m(S_1(\delta_0))\leq C_m 2(p-j)\alpha^p.$$

When $p<2j$ we have from (5.23) and Lemma 5.4

$$|h_{2p,2j}(\cdot)-j_\infty(\cdot)-(2p-2j)d|_m(S_1(\delta_0))\leq C_m 2(p-j)\alpha^p.$$

Taking account of

$$|d_{\infty,2j}|\leq C\alpha^{2j}\leq C\alpha^{p},$$

we see that (5.24) holds for $p<2j$. Note that

$$(5.25)\qquad |X_{-2p+2j}(\cdot,\nabla\varphi_{2p})-A_{2j}|_m(S_1(\delta_0))\leq C_m\alpha^{p-j}$$

holds for all $p\geq j$. Indeed, if $p>2j$, (5.25) is nothing but (5.21). Suppose that $p<2j$.

$$\begin{aligned}&|X_{-2p+2j}(\cdot,\nabla\varphi_{2p})-A_{2j}|_m(S_1(\delta_0))\\&\leq|X_{-2p+2j}(\cdot,\nabla\varphi_{2p})-X^{\infty}_{-2p+2j}(\cdot)|_m(S_1(\delta_0))\\&\quad+|X^{\infty}_{-2p+2j}(\cdot)-a_1|_m(S_1(\delta))+|A_{2j}-a_1|\end{aligned}$$

by (i) of (4.17), (4.13) and Lemma 4.2

$$\leq C_m\alpha^{2j-1}\leq C_m\alpha^{p}.$$

Then it follows from (5.24) and (5.25) that

$$\begin{aligned}&|f(X_{-2p+2j}(\cdot,\nabla\varphi_{2p}),\,t-h_{2p-2j}(x))\\&\quad-f(A_{2j},\,t-j_{\infty}(\cdot)-d_{\infty,2j}-2(p-j)d)|_m(S_1(\delta_0))\leq C_m2(p-j)\alpha^{p}.\end{aligned}$$

Then we have

$$|I_2|_m(S_1(\delta_0))\leq C_m2(p-j)(\lambda\tilde{\lambda}\alpha)^{p-j}|f|_m(S_1(\delta_0)\times \boldsymbol{R}).$$

For q odd we can prove (5.17) by the same way.

Remark 4. A representation of solutions (5.9) of an equation (5.5) for a data (5.4) shows

$$\operatorname{supp} v_q\subset\bigcup_{(x,t)\in\operatorname{supp} f}\mathscr{L}_{q-j}(x,\,t,\,\nabla\varphi_0).$$

Therefore, if

$$\operatorname{supp} f\subset S_1(\delta_3)\times(T_1,\,T_2),$$

it follows that

$$\operatorname{supp} v_q\subset\omega_{q-2j}\times[T_1+(q-2j)d,\,T_2+(j-2j+1)d+d_{\infty}+\sup_{x\in S_1(\delta_3)}j_{\infty}(x)].$$

§6. Transport equation (2)

In order to consider properties of solutions of the transport equation of higher order we introduce some spaces of functions.

Set

$$(6.1)\qquad c_0=\frac{1}{4d}\log\det(I+d\mathscr{K}_{\infty}(0))(I+d\tilde{\mathscr{K}}_{\infty}(0))=-\frac{1}{2d}\log\lambda\tilde{\lambda},$$

(6.2) $$c_1=-\frac{1}{2d}\log((1+d_0C_0)^{-1}+C\delta_0).$$

We set

$$F=\{\boldsymbol{v}=\{v_q,\ \tilde{v}_q\}_{q=0}^{\infty};\ v_q,\ \tilde{v}_q\in C_0^\infty(\bar{\omega}\times(0,\ \infty))\ \text{such that}$$
$$\operatorname{supp} v_q,\ \tilde{v}_q\subset\bar{\omega}\times[2qd,\ (2q+2)d+d+d_{\infty,0}],$$
$$\sup_{0\le q<\infty}\ \sup_{(x,t)\in\omega\times\boldsymbol{R}} e^{c_0t}(|D_{x,t}^\beta v_q(x,\ t)|+|D_{x,t}^\beta\tilde{v}_q(x,\ t)|)<\infty\ \text{for all}\ \beta\},$$

and for $\boldsymbol{v}\in F$

$$\|\boldsymbol{v}\|_{F,m}=\sup_{0\le q<\infty}\ \sup_{(x,t)\in\omega\times\boldsymbol{R}}\ \sum_{|\beta|\le m} e^{c_0t}(|D^\beta v_q(x,\ t)|+|D^\beta\tilde{v}_q(x,\ t)|).$$

For $\boldsymbol{v}_1=\{v_{1,q},\ \tilde{v}_{1,q}\}_{q=0}^{\infty}$ and $\boldsymbol{v}_2=\{v_{2,q},\ \tilde{v}_{2,q}\}_{q=0}^{\infty}$, $a,\ b\in\boldsymbol{C}$ we define $a\boldsymbol{v}_1+b\boldsymbol{v}_2$ by

$$a\boldsymbol{v}_1+b\boldsymbol{v}_2=\{av_{1,q}+bv_{2,q},\ a\tilde{v}_{1,q}+b\tilde{v}_{2,q}\}_{q=0}^{\infty}\in F.$$

For p positive integer

$$F(p)=\{\{v_q,\ \tilde{v}_q\}_{q=0}^{\infty}\in F;\ v_q=\tilde{v}_q=0\quad\text{for}\quad q<p\},$$
$$\mathring{F}(p)=\{\{v_q,\ \tilde{v}_q\}_{q=0}^{\infty}\in F;\ v_q=\tilde{v}_q=0\quad\text{for}\quad q\neq p\},$$
$$K_1(p)=\{\{v_q,\ \tilde{v}_q\}_{q=0}^{\infty}\in F(p);\ v_q(x,\ t)=(\lambda\tilde{\lambda})^q f(x,\ t-2dq),\ q>p$$
$$\text{for some}\quad f\in C_0^\infty(\bar{\omega}\times[0,\ \infty))\quad\text{and}\quad\tilde{v}_q=0\quad\text{for all}\quad q\},$$
$$K_2(p)=\{\{v_q,\ \tilde{v}_q\}_{q=0}^{\infty}\in F(p);\ \tilde{v}_q(x,\ t)=(\lambda\tilde{\lambda})^q\tilde{f}(x,\ t-2dq),\ q>p$$
$$\text{for some}\quad \tilde{f}(x,\ t)\in C_0^\infty(\bar{\omega}\times(0,\ \infty))\quad\text{and}\quad v_q=0\quad\text{for all}\quad q\},$$
$$K(p)=K_1(p)+K_2(p).$$

Since

$$\sup_{0\le q<\infty}\ \sup_{(x,t)\in\omega\times\boldsymbol{R}} e^{c_0t}|D_{x,t}^\beta(\lambda\tilde{\lambda})^q g(c,\ t-2qd)|$$
$$=\sup_{0\le q<\infty} e^{c_0 2qd}(\lambda\tilde{\lambda})^q\sup_{(x,t)\in\omega\times\boldsymbol{R}}|D_{x,t}^\beta(e^{c_0t}g(x,\ t))|$$

we have for $\boldsymbol{v}\in K_1(p)$

(6.3) $$C^{-1}|g|_m(\omega\times\boldsymbol{R})\le\|\boldsymbol{v}\|_{F,m}\le C|g|_m(\omega\times\boldsymbol{R}).$$

Set

$$M_r(p)=\{\boldsymbol{v}=\{v_q,\ \tilde{v}_q\}_{q=0}^{\infty}\in F(p);\ \sup_{0\le q<\infty}\ \sup_{(x,t)}\{(1+t)^{-r}e^{(c_0+c_1)t}$$
$$\cdot(|D_{x,t}^\beta v_q(x,\ t)|+|D_{x,t}^\beta\tilde{v}_q(x,\ t)|)\}<\infty\quad\text{for all}\quad\beta\},$$

and

$$\|v\|_{M_{r},m}=\sup_q \sup_{(x,t)} \{ \sum_{|\beta|\le m} (1+t)^{-r} e^{(c_0+c_1)t}$$

$$\cdot(|D^{\beta}_{x,t}v_q(x,t)|+|D^{\beta}_{x,t}\tilde{v}_q(x,t)|)\} .$$

Remark. From the assumption on the support of v_q, $\tilde{v}_q$ $\sup_q |e^{c_0t}v_q|<\infty$ and $\sup_q |(\lambda\tilde{\lambda})^q v_q|<\infty$ are equivalent. Similarly $\sup_q |(1+t)^{-r}e^{(c_0+c_1)t}v_q|<\infty$ and $\sup_q |q^{-r}(\lambda\tilde{\lambda}\alpha)^q v_q|<\infty$ are equivalent.

Let us set $N_+=\{0, 1,\dots\}$ and $N^s_+=\{J_s=(j_1, j_2,\dots, j_s);\ j_l\in N_+,\ l=1, 2,\dots, s\}$. Denote $j_1+j_2+\cdots+j_s$ by $|J_s|$. We define classes $(CH)_s$, $s=0, 1,\dots$ of sets of functions. $(CH)_0=\{f; f\in C^\infty_0(\bar{\omega}\times(0,\infty))\}$. We say $\{f^{(j)}(x, t)\}_{j\in N_+}$ belongs to $(CH)_1$ when

(i) $f^{(j)}(x, t)\in C^\infty_0(\bar{\omega}\times \boldsymbol{R})$ for all $j\in N_+$,

(ii) there exists $t_1>0$ such that $\operatorname{supp} f^{(j)}\subset\bar{\omega}\times[0, t_1]$ for all j,

(iii) there exists $f^{(\infty)}(x, t)\in C^\infty_0(\bar{\omega}\times \boldsymbol{R})$ such that

$$\sup_j j^{-1}\alpha^{-j}|f^{(j)}-f^{(\infty)}|_m(\bar{\omega}\times \boldsymbol{R})<\infty \qquad \text{for all} \quad m .$$

For $\{f^{(j)}\}_{j\in N_+}\in(CH)_1$ we define semi-norms $|\cdot|_{(CH)_1,m}$, $m=0, 1,\dots$ by

$$|\{f^{(j)}\}_{j\in N_+}|_{(CH)_1,m}$$

$$=|f^{(\infty)}|_m(\omega\times \boldsymbol{R})+\max j^{-1}\alpha^{-j}|f^{(j)}-f^{(\infty)}|_m(\bar{\omega}\times \boldsymbol{R}) .$$

And for $s>1$ we say $\{f^{(J_s)}\}_{J_s\in N^s_+}$ belongs to $(CH)_s$ when

(i) $f^{(J_s)}\in C^\infty_0(\bar{\omega}\times \boldsymbol{R})$ for all $J_s\in N^s_+$,

(ii) there exists $t_s>0$ such that $\operatorname{supp} f^{(J_s)}\subset\bar{\omega}\times[0, t_s]$ for all J_s,

(iii) there exist a linear continous mapping $\boldsymbol{B}_s$ from $C^\infty_0(\bar{\omega}\times \boldsymbol{R})$ into $(CH)_1$ and $\{g^{(J_{s-1})}\}_{J_{s-1}\in N^{s-1}_+}\in(CH)_{s-1}$ such that

$$\{f^{(J_{s-1},l)}\}^\infty_{l=0}=\boldsymbol{B}_s g^{(J_{s-1})} \qquad \text{for all} \quad J_{s-1}.$$

Note that $\{f^{(J_{s-1},\infty)}\}_{J_{s-1}\in N^{s-1}_+}\in(CH)_{s-1}$ and for each j

$$\{f^{(J_{s-1},j)}-f^{(J_{s-1},\infty)}\}_{J_{s-1}\in N^{s-1}_+}\in(CH)_{s-1}.$$

We define $|\cdot|_{(CH)_s,m}$ by

$$|\{f^{(J_s)}\}|_{(CH)_s,m}=|\{f^{(J_{s-1},\infty)}\}|_{(CH)_{s-1},m}$$

$$+\sup_j j^{-1}\alpha^{-j}|\{f^{(J_{s-1},j)}-f^{(J_{s-1},\infty)}\}|_{(CH)_{s-1},m}.$$

Definition 6.1. We say $W=\{w^{(J_s)}\}_{J_s\in N^s_+}$ belongs to $\mathscr{H}_s(l)$ when

(i) $w^{(J_s)}\in K(|J_s|+l)$ for all $J_s\in N^s_+$,

(ii) if we set

$$w^{(J_s)}=\{(\lambda\tilde{\lambda})^q f^{(J_s)}(x, t-2dq), (\lambda\tilde{\lambda})^q \tilde{f}^{(J_s)}(x, t-2dq)\}_{q\ge|J_s|+l}$$

then $\{f^{(J_s)}\}_{J_s\in N^s_+}$, $\{\tilde{f}^{(J_s)}\}_{J_s\in N^s_+}\in(CH)_s$. And define semi-norms in $\mathscr{H}_s(l)$ by

$$\|W\|_{\mathscr{H}_{s,m}}=|\{f^{(J_s)}\}_{J_s\in N_+^s}|_{(CH)_{s,m}}+|\{\tilde{f}^{(J_s)}\}_{J_s\in N_+^s}|_{(CH)_{s,m}}.$$

Definition 6.2. Let $\boldsymbol{f}=\{f_q, \tilde{f}_q\}_{q=0}^{\infty}$ be a sequence such that $f_q\in C_0^\infty(S_1(\delta_0)\times(0, \infty))$, $\tilde{f}_q\in C_0^\infty(S_2(\delta_0)\times(0, \infty))$ and let $\boldsymbol{g}=\{g_q, \tilde{g}_q\}_{q=0}^{\infty}$ be a sequence such that $g_q, \tilde{g}_q\in C_0^\infty(\bar{\omega}\times(0, \infty))$. We say that $\boldsymbol{v}=\{v_q, \tilde{v}_q\}_{q=0}^{\infty}$ is a solution of

$$\begin{cases} T\boldsymbol{v}=\boldsymbol{g} & \text{in} \quad \omega\times\boldsymbol{R} \\ \boldsymbol{v}=\boldsymbol{f} & \text{on} \quad S(\delta_2)\times\boldsymbol{R} \end{cases}$$

when

$$T_{2q}v_q=g_q \quad \text{in} \quad \omega\times \boldsymbol{R}$$
$$v_q=v_{1,1}\tilde{v}_{q-1}+f_q \quad \text{on} \quad S_1(\delta_0)\times\boldsymbol{R}$$

and

$$T_{2q+1}\tilde{v}_q=\tilde{g}_q \quad \text{in} \quad \omega\times\boldsymbol{R}$$
$$\tilde{v}_q=v_{2,1}v_q+\tilde{f}_q \quad \text{on} \quad S_2(\delta_0)\times\boldsymbol{R}.$$

Remark that the definitions 5.1 and 6.2 have only a difference in assigning a number to elements of sequences. Hereafter we will use $T\boldsymbol{v}=\boldsymbol{g}$, $\boldsymbol{v}=\boldsymbol{f}$ in the sense of Definition 6.2.

Lemma 6.1. *Let $\boldsymbol{g}\in\mathring{F}(p)$ and let $\boldsymbol{v}$ be a solution of*

$$\begin{cases} T\boldsymbol{v}=\boldsymbol{g} & \text{in} \quad \omega\times\boldsymbol{R} \\ \boldsymbol{v}=0 & \text{on} \quad S(\delta_2)\times\boldsymbol{R}. \end{cases}$$

Then $\boldsymbol{v}$ is decomposed as

$$\boldsymbol{v}=\boldsymbol{w}+\boldsymbol{z}, \quad \boldsymbol{w}\in K(p), \quad \boldsymbol{z}\in M_1(p).$$

Moreover it holds that

$$\|\boldsymbol{w}\|_{F,m}\leq C_m\|\boldsymbol{g}\|_{F,m}$$
$$\|\boldsymbol{z}\|_{M_1,m}\leq C_m\|\boldsymbol{g}\|_{M_0,m}$$

where C_m is a constant independent of $\boldsymbol{g}$ and p.

Proof. Set $\boldsymbol{g}=\{g_q, \tilde{g}_q\}_{q=0}^{\infty}$, $\boldsymbol{v}=\{v_q, \tilde{v}_q\}_{q=0}^{\infty}$. Evidently $v_q=\tilde{v}_q=0$ for $q<p$,

$$\text{(6.4)} \qquad \begin{cases} T_{2p}v_p=g_p & \text{in} \quad \omega\times\boldsymbol{R} \\ v_p=0 & \text{on} \quad \Gamma_1\times\boldsymbol{R}, \end{cases}$$

$$\text{(6.5)} \qquad \begin{cases} T_{2p+1}\tilde{v}_p=\tilde{g}_p & \text{in} \quad \omega\times\boldsymbol{R} \\ \tilde{v}_p=v_{2,1}(x)v_p & \text{on} \quad S_2(\delta_2)\times\boldsymbol{R} \end{cases}$$

and for $q\geq p+1$

(6.6)
$$\begin{cases} T_{2q}v_q=0 & \text{in} \quad \omega\times \boldsymbol{R} \\ v_q=v_{1,1}(x)\tilde{v}_{q-1} & \text{on} \quad S_1(\delta_2)\times \boldsymbol{R} \end{cases}$$

and

(6.7)
$$\begin{cases} T_{2q+1}\tilde{v}_q=0 & \text{in} \quad \omega\times \boldsymbol{R} \\ \tilde{v}_q=v_{2,1}(x)v_q & \text{on} \quad S_2(\delta_2)\times \boldsymbol{R} \end{cases}$$

Then we have from (6.4) and (6.5)

(6.8)
$$(\lambda\tilde{\lambda})^{-p}\{|v_p|_m(\omega\times \boldsymbol{R})+|\tilde{v}_p|_m(\omega\times \boldsymbol{R})\}$$
$$\leq C_m(\lambda\tilde{\lambda})^{-p}\{|g_p|_m(\omega\times \boldsymbol{R})+|\tilde{g}_p|_m(\omega\times \boldsymbol{R})\}\leq C_m\|\boldsymbol{g}\|_{F,m}.$$

Applying Proposition 5.6 we have for $q\geq p+1$

$$v_q=w_q+z_q,\ \tilde{v}_q=\tilde{w}_q+\tilde{z}_q$$

where

$$w_q=\lambda^{q+1-p}\tilde{\lambda}^{q-p}a(x)b_{2p}\tilde{v}_p(A,\ t-j_\infty(x)-d_{\infty,2p}-(2q+1-2p)d)$$
$$\tilde{w}_q=(\lambda\tilde{\lambda})^{q-p}\tilde{a}(x)b_{2p}\tilde{v}_p(A,\ t-j_\infty(x)-d_{\infty,2p}-2(q-p)d),$$

and

$$|z_q|_m(\omega\times \boldsymbol{R})+|\tilde{z}_q|_m(\omega\times \boldsymbol{R})\leq C_m2(q-p)(\lambda\tilde{\lambda}\alpha)^{q-p}|v_{1,1}v_p|_m(S_1(\delta_0)\times \boldsymbol{R}).$$

Then by using (6.8) we have

$$\|\boldsymbol{w}\|_{F,m}\leq C_m(\lambda\tilde{\lambda})^{-p}|\tilde{v}_p|_m(S_1(\delta_2)\times \boldsymbol{R})\leq C_m\|\boldsymbol{g}\|_{F,m}.$$

Similarly we have for all q

$$q^{-1}(\lambda\tilde{\lambda}\alpha)^{-q}\{|z_q|_m(\omega\times \boldsymbol{R})+|\tilde{z}_q|_m(\omega\times \boldsymbol{R})\}$$
$$\leq 2C_m(\lambda\tilde{\lambda}\alpha)^{-p}\{|g_p|_m(\omega\times \boldsymbol{R})+|\tilde{g}_p|_m(\omega\times \boldsymbol{R})\}$$
$$\leq C_m\|\boldsymbol{g}\|_{M_0,m},$$

which implies $\|\boldsymbol{z}\|_{M_1,m}\leq C_m\|\boldsymbol{g}\|_{M_0,m}$. Q. E. D.

Lemma 6.2. *Let $\boldsymbol{g}=\{g_q,\ \tilde{g}_q\}_{q=0}^\infty\in M_r(p)$. Then a solution of*

$$\begin{cases} \boldsymbol{T}\boldsymbol{v}=\boldsymbol{g} & \text{in} \quad \omega\times \boldsymbol{R} \\ \boldsymbol{v}=0 & \text{on} \quad S(\delta_2)\times \boldsymbol{R} \end{cases}$$

can be decomposed as

$$\boldsymbol{v}=\sum_{j=p}^{\infty}\boldsymbol{w}^{(j)}+\boldsymbol{z},\quad \boldsymbol{w}^{(j)}\in K(j),\quad \boldsymbol{z}\in M_{r+2}(p).$$

And they satisfy

(6.9)
$$\|\boldsymbol{w}^{(j)}\|_{F,m}\leq C_mj^r\alpha^j\|\boldsymbol{g}\|_{M_r,m}$$

(6.10) $$\|z\|_{M_{r+2},m}\leq C_{m,r}\|g\|_{M_r,m}\,.$$

Proof. Set

$$g^{(j)}=\{g_q^{(j)},\ \tilde{g}_q^{(j)}\}_{q=0}^{\infty},$$

where $g_j^{(j)}=g_j$, $\tilde{g}_j^{(j)}=\tilde{g}_j$ and $g_q^{(j)}=\tilde{g}_q^{(j)}=0$ for $q\neq j$. Evidently it holds that

$$g=\sum_{j=p}^{\infty}g^{(j)}$$

and

$$\|g^{(j)}\|_{F,m}\leq j^r\,\alpha^j\,\|g\|_{M_r,m}.$$

Let $v^{(j)}$ be a solution of

$$\begin{cases} Tv^{(j)}=g^{(j)} & \text{in} \quad \omega\times R \\ v^{(j)}=0 & \text{on} \quad S(\delta_2)\times R. \end{cases}$$

Applying the previous lemma to each $v^{(j)}$ we have

$$v^{(j)}=w^{(j)}+z^{(j)},\ w^{(j)}\in K(j),\ z^{(j)}\in M_1(j),$$

$$\|w^{(j)}\|_{F,m}\leq C_m\|g^{(j)}\|_{F,m}\leq C_m j^r\alpha^j\,\|g\|_{M_r,m},$$

$$\|z^{(j)}\|_{M_1,m}\leq C_m\|g^{(j)}\|_{M_0,m}\leq C_m j^r\ \|g\|_{M_0,m}.$$

Set $z=\sum_{j=p}^{\infty}z^{(j)}$. Then

$$\sup_{(x,t)\in\omega\times R}\ \sum_{|\beta|\leq m}(\lambda\tilde{\lambda}\alpha)^{-q}\ q^{-r-2}|D_{x,t}^{\beta}z_q(x,t)|$$

$$\leq q^{-r-1}\sum_{j=p}^{q}\ \sup_{(x,t)\in\omega\times R}\ \sum_{|\beta|\leq m}(\lambda\tilde{\lambda}\alpha)^{-q}q^{-1}|D_{x,t}^{\beta}z_q^{(j)}(x,t)|$$

$$\leq q^{-r-1}\sum_{j=p}^{q}\|z^{(j)}\|_{M_1,m}\leq q^{-r}\sum_{j=p}^{q}\|g^{(j)}\|_{M_0,m}.$$

And we have

$$\sup_{q\geq 1}q^{-r-1}\sum_{j=p}^{q}\|g^{(j)}\|_{M_0,m}$$

$$\leq\sup_{q\geq 1}q^{-r-1}\sum_{j=1}^{q}(\lambda\tilde{\lambda}\alpha)^{-j}\,j^{-r}\ \|g\|_{M_r,m}$$

$$\leq C_m\|g\|_{M_r,m}.$$

Thus we have (6.10). Q. E. D.

Lemma 6.3. *Let $g\in K(p)$ and v be a solution of*

$$\begin{cases} T_{\infty}v=g & \text{in} \quad \omega\times R \\ v=0 & \text{on} \quad S(\delta_2)\times R. \end{cases}$$

Then $\boldsymbol{v}$ is decomposed as

$$\boldsymbol{v}=\sum_{j=0}^{\infty} \boldsymbol{w}^{(j)}, \quad \boldsymbol{w}^{(j)}\in K(p+j).$$

If we set

$$\boldsymbol{g}=\{g_q(x, t), \tilde{g}_q(x, t)\}_{q\geq p}=\{(\lambda\tilde{\lambda})^q g(x, t-2pd), (\lambda\tilde{\lambda})^q\tilde{g}(x, t-2qd)\}_{q\geq p}$$

$$\boldsymbol{w}^{(j)}=\{(\lambda\tilde{\lambda})^q f_j(x, t-2qd), (\lambda\tilde{\lambda})^q\tilde{f}_j(x, t-2qd)\}_{q\geq p+j},$$

then there exist $\boldsymbol{h}=\{(\lambda\tilde{\lambda})^l h(x, t-2ld), (\lambda\tilde{\lambda})^l\tilde{h}(x, t-2ld)\}_{l=0}^{\infty}\in K(0)$ and $\boldsymbol{z}\in M_1(0)$ such that

$$\{(\lambda\tilde{\lambda})^l f_l(x, t-2ld), (\lambda\tilde{\lambda})^l\tilde{f}_l(x, t-2ld)\}_{l=0}^{\infty}=\boldsymbol{h}+\boldsymbol{z}.$$

Moreover there exist linear continuous mappings $\mathscr{A}$ from $(C_0^\infty(\omega\times\boldsymbol{R}))^2$ into $(C_0^\infty(\omega\times\boldsymbol{R}))^2$ and A form $(C_0^\infty(\omega\times\boldsymbol{R}))^2$ into $M_1(0)$ such that

$$\{h, \tilde{h}\}=\mathscr{A}\{g, \tilde{g}\}$$

$$\boldsymbol{z}=A\{g, \tilde{g}\}.$$

Proof. Set $\boldsymbol{g}_0=\{g_{0q}, \tilde{g}_{0q}\}_{q=0}^{\infty}$ where $g_{0,0}=g(x, t)$, $\tilde{g}_{0,0}=\tilde{g}(x, t)$ and $g_{0,q}=\tilde{g}_{0q}=0$ for $q\geq 1$. Let $\boldsymbol{v}_0=\{v_{0q}, \tilde{v}_{0q}\}_{q=0}^{\infty}$ be a solution of

$$\begin{cases} \boldsymbol{T}_\infty \boldsymbol{v}_0=\boldsymbol{g}_0 & \text{in} \quad \omega\times\boldsymbol{R} \\ \boldsymbol{v}_0=0 & \text{on} \quad S(\delta_2)\times\boldsymbol{R}. \end{cases}$$

From Lemma 6.1 we have

$$\boldsymbol{v}_0=\boldsymbol{w}_0+\boldsymbol{z}, \ \boldsymbol{w}_0\in K(0), \ \boldsymbol{z}\in M_1(0). \tag{6.11}$$

Set

$$\boldsymbol{w}_0=\{(\lambda\tilde{\lambda})^q h(x, t-2qd), (\lambda\tilde{\lambda})^q\tilde{h}(x, t-2qd)\}_{q=0}^{\infty}.$$

Denote by τ_j a mapping from $F(p)$ onto $F(p+j)$ defined by

$$\tau_j\boldsymbol{v}=(\lambda\tilde{\lambda})^j\{v_{q-j}(x, t-2jd), \tilde{v}_{q-j}(x, t-2jd)\}_{q\geq p+j}$$

for $\boldsymbol{v}=\{v_q(x, t), v_q(x, t)\}_{q\geq p}$.

If we set

$$\boldsymbol{g}^{(l)}=\{g_q^{(l)}, \tilde{g}_q^{(l)}\}_{q=0}^{\infty}, \ g_{p+l}^{(l)}=g_{p+l}, \ \tilde{g}_{p+l}^{(l)}=\tilde{g}_{p+l}, \ g_q^{(l)}=\tilde{g}_q^{(l)}=0 \qquad \text{for} \quad q\neq p+l$$

we have

$$\boldsymbol{g}^{(l)}=\tau_{p+l}\boldsymbol{g}_0, \tag{6.12}$$

and

$$\boldsymbol{g}=\sum_{l=0}^{\infty}\boldsymbol{g}^{(l)}. \tag{6.13}$$

Let $\boldsymbol{v}^{(l)}=\{v_q^{(l)}, \tilde{v}_q^{(l)}\}_{q=0}^{\infty}$ be a solution of

$$\begin{cases} T_\infty v^{(l)} = g^{(l)} & \text{in} \quad \omega \times R \\ v^{(l)} = 0 & \text{on} \quad S(\delta_2) \times R. \end{cases}$$

Since $T_\infty \tau_j v = \tau_j T_\infty v$ for all $v \in F$, we have from (6.12) and (6.13)

$$v^{(l)} = \tau_{p+l} v_0 ,$$

$$v = \sum_{l=0}^{\infty} v^{(l)} .$$

Namely

$$\text{(6.14)} \quad \begin{cases} v_q^{(l)}(x, t) = (\lambda\tilde{\lambda})^{p+l} v_{0,q-(p+l)}(x, t-2(p+l)d) & \text{for} \quad q \geq p+l \\ \tilde{v}_q^{(l)}(x, t) = (\lambda\tilde{\lambda})^{p+l} \tilde{v}_{0,q-(p+l)}(x, t-2(p+l)d) & \text{for} \quad q \geq p+l . \end{cases}$$

If we set

$$w^{(l)} = \{w_q^{(l)}, \tilde{w}_q^{(l)}\}_{q=p+l}^{\infty} = \{v_q^{(q-p-l)}, \tilde{v}_q^{(q-p-l)}\}_{q=p+l}^{\infty},$$

it follows from (6.14) that for $q \geq p+l$

$$w_q^{(l)}(x, t) = (\lambda\tilde{\lambda})^q v_{0,l}(x, t-2(q-l)d)$$

$$\tilde{w}_q^{(l)}(x, t) = (\lambda\tilde{\lambda})^q \tilde{v}_{0,l}(x, t-2(q-l)d) .$$

Then $w^{(l)} \in K(p+l)$. Since $v_q^{(l)} = 0$ for $q < p+l$ we have

$$\sum_{l=0}^{\infty} w_q^{(l)} = \sum_{l=0}^{q-l} v_q^{(q-p-l)} = \sum_{l=0}^{q-p} v_q^{(l)} = \sum_{l=0}^{\infty} v_q^{(l)} .$$

Similarly we have

$$\sum_{l=0}^{\infty} \tilde{w}_q^{(l)} = \sum_{l=0}^{\infty} \tilde{v}_q^{(l)} .$$

These equalities imply

$$\sum_{l=0}^{\infty} w^{(l)} = \sum_{l=0}^{\infty} v^{(l)} = v .$$

The linearity and the continuity of mappings g_0 to v_0, w_0 and z show the existence of $\mathscr{A}$ and A with the properties mentioned in our lemma. Q. E. D.

Lemma 6.4. *Let $G = \{g^{(J_s)}\}_{J_s \in N_+^s} \in \mathscr{H}_s(p)$ and let v be a solution of*

$$\begin{cases} T_\infty v = \sum_{J_s \in N_+^s} g^{(J_s)} & \text{in} \quad \omega \times R \\ v = 0 & \text{on} \quad S(\delta_2) \times R. \end{cases}$$

Then v is represented as

$$\text{(6.15)} \qquad v = \sum_{J_{s+1} \in N_+^{s+1}} w^{(J_{s+1})}, \quad W = \{w^{(J_{s+1})}\}_{J_{s+1} \in N_+^{s+1}} \in \mathscr{H}_{s+1}(p).$$

Moreover

(6.16) $$\|W\|_{\mathscr{H}_{s+1},m}\le C_{s,m}\|G\|_{\mathscr{H}_{s},m}.$$

Proof. Denote by $v^{(J_s)}$ a solution of

$$\begin{cases} T_\infty v=g^{(J_s)} & \text{in} \quad \omega\times R \\ v=0 & \text{on} \quad S(\delta_2)\times R. \end{cases}$$

Lemma 6.3 shows that

$$v^{(J_s)}=\sum_{j=0}^{\infty} w^{(J_s,j)}, \quad w^{(J_s,j)}\in K(|J_s|+p+j).$$

Then

$$v=\sum_{J_s\in N_+^s} v^{(J_s)}=\sum_{J_s\in N_+^s}\sum_{j=0}^{\infty} w^{(J_s,j)}=\sum_{J_{s+1}\in N_+^{s+1}} w^{(J_{s+1})}.$$

If we set

$$g^{(J_s)}=\{(\lambda\tilde{\lambda})^q g^{(J_s)}(x,\ t-2qd),\ (\lambda\tilde{\lambda})^q\tilde{g}^{(J_s)}(x,\ t-2qd)\}_{q\ge|J_s|+p}$$

$$w^{(J_s,j)}=\{(\lambda\tilde{\lambda})^q f^{(J_s,j)}(x,\ t-2qd),\ (\lambda\tilde{\lambda})^q\tilde{f}^{(J_s,j)}(x,\ t-2qd)\}_{q\ge|J_s|+p+j}$$

a mapping $\{g^{(J_s)},\ \tilde{g}^{(J_s)}\}$ to $\{f^{(J_s,j)},\tilde{f}^{(J_s,j)}\}_{j=0}^{\infty}$ is linear and continuous from $(C_0^\infty(\omega\times R))^2$ into $(CH)_1$. This shows that

$$\{f^{(J_{s+1})}\}_{J_{s+1}\in N_+^{s+1}}, \quad \{\tilde{f}^{(J_{s+1})}\}_{J_{s+1}\in N_+^{s+1}}\in(CH)_{s+1},$$

which implies $W=\{w^{(J_{s+1})}\}_{J_{s+1}\in N_+^{s+1}}\in\mathscr{H}_{s+1}(p)$. Q. E. D.

Lemma 6.5. *Let $G=\{g^{(J_s)}\}_{J_s\in N_+^s}\in\mathscr{H}_s(p)$ and let v be a solution of*

$$\begin{cases} Tv=\sum_{J_s\in N_+^s} g^{(J_s)} & \text{in} \quad \omega\times R \\ v=0 & \text{on} \quad S(\delta_2)\times R. \end{cases}$$

Then v is decomposed as

(6.18) $$v=\sum_{J_{s+1}\in N_+^{s+1}} w^{(J_{s+1})}+\sum_{j=0}^{\infty} u^{(j)}+z$$

where

$$W=\{w^{(J_{s+1})}\}_{J_{s+1}\in N_+^{s+1}}\in\mathscr{H}_{s+1}(p),$$

$$u^{(j)}\in K(p+j), \quad z\in M_{s+2}(p),$$

and the following estimates hold:

(6.18) $$\begin{cases} \|W\|_{\mathscr{H}_{s+1},m}+|z|_{M_{s+1},m}\le C_m\|G\|_{\mathscr{H}_s,m} \\ |u^{(j)}|_{F,m}\le C_m j^{s+1}\alpha^j\|G\|_{\mathscr{H}_s,m}. \end{cases}$$

Proof. Let w be a solution of

$$\begin{cases} T_\infty w = \sum_{J_s \in N_+^s} g^{(J_s)} & \text{in} \quad \omega \times R \\ w = 0 & \text{on} \quad S(\delta_2) \times R . \end{cases}$$

Then the previous lemma shows that

$$w = \sum_{J_{s+1} \in N_+^{s+1}} w^{(J_{s+1})}, \quad W = \{w^{(J_{s+1})}\}_{J_{s+1} \in N_+^{s+1}} \in \mathscr{H}_{s+1}(p) .$$

Note that

$$|w_q|_m(\omega \times R) \leq \sum_{J_{s+1} \in N_+^{s+1}} |w_q^{(J_{s+1})}|_m(\omega \times R)$$

$$\leq (\lambda\tilde{\lambda})^q \ \sharp\{J_{s+1} ; |J_{s+1}| \leq q\} \| W \|_{\mathscr{X}_{s+1},m}$$

$$\leq C_m (\lambda\tilde{\lambda})^q q^{s+1} \| W \|_{\mathscr{X}_{s+1},m} .$$

Thus

$$(T - T_\infty) w \in M_{s+1}(p) .$$

Since v satisfies

$$\begin{cases} T(v-w) = -(T-T_\infty) w & \text{in} \quad \omega \times R \\ v - w = 0 & \text{on} \quad S(\delta_2) \times R \end{cases}$$

we have from Lemma 6.2

$$v - w = \sum_{j=0}^{\infty} u^{(j)} + z, \quad u^{(j)} \in K(p+j), \quad z \in M_{r+2}(p) .$$

Thus (6.17) is proved. Estimates (6.18) follows from

$$|(T - T_\infty) w|_{M_{r+1},m} \leq C_m \|G\|_{\mathscr{X}_s,m}$$

and the estimates of solutions in Lemma 6.2. Q. E. D.

Proposition 6.6. *Let* $f(x, t) \in C_0^\infty(S_1(\delta_2) \times R)$. *Set* $f = \{f_q, 0\}_{q=0}^\infty$ $f_0 = f$, $f_q = 0$ *for* $q > 1$. *Define* v_r *successively by*

$$\begin{cases} T v_0 = 0 & \text{in} \quad \omega \times R \\ v_0 = f & \text{on} \quad S(\delta_2) \times R \end{cases}$$

and for $r > 0$

$$\begin{cases} T v_r = \frac{1}{i} \square v_{r-1} & \text{in} \quad \omega \times R \\ v_r = 0 & \text{on} \quad S(\delta_2) \times R. \end{cases}$$

Then v_r, $r \geq 1$ *are decomposed as*

$$(6.19) \qquad v_r = \sum_{J_r \in N_+^r} w_r^{(J_r)} + \sum_{h=1}^{r} \sum_{l=0}^{\infty} \sum_{J_{r-h} \in N_+^{r-h}} w_{r,h,l}^{(J_{r-h})} + z_r$$

where

(6.20) $$W_r=\{w_r^{(J_r)}\}_{J_r\in N_+^r}\in\mathscr{H}_r(0),$$

(6.21) $$W_{r,h,l}=\{w_{r,h,l}^{(J_{r-h})}\}_{J_{r-h}\in N_+^{r-h}}\in\mathscr{H}_{r-h}(l),$$

(6.22) $$z_r\in M_{2r}(0),$$

and it holds that

(6.23) $$\|W_r\|_{\mathscr{H}_{r,m}},\ |z_r|_{M_{2r,m}}\le C_{r,m}|f|_{m+2r}(S_1(\delta_2)\times\boldsymbol{R}).$$

(6.24) $$\|W_{r,h,l}\|_{\mathscr{H}_{r-h,m}}\le C_{r,m}\alpha^l l^{r-h}|f|_{m+2r}(S_1(\delta_2)\times\boldsymbol{R}).$$

Proof. Proposition 5.6 shows that v_0 is represented as

$$v_0=w_0+z_0$$

where $w_0\in K(0)$, $z_0\in M(0)$. Let $v_{1,0}$ be a solution of

$$\begin{cases} Tv=\frac{1}{i}\Box w_0 & \text{in}\quad \omega\times\boldsymbol{R}\\ v=0 & \text{on}\quad S(\delta_2)\times\boldsymbol{R}.\end{cases}$$

Taking account of $\frac{1}{i}\Box w_0\in K(0)$ and

$$\left|\frac{1}{i}\Box w_0\right|_{F,m}\le C_m|w_0|_{F,m+2}\le C_m|f|_{m+2}(S_1(\delta_2)\times\boldsymbol{R})$$

we have from Lemma 6.5

$$v_{1,0}=\sum_{j=0}^{\infty}w_1^{(j)}+z_{1,0},\quad \{w_1^{(j)}\}_{j\in N_+}\in\mathscr{H}_1(0)$$

and $z_{1,0}\in M_1(0)$. Let $v_{1,1}$ be a solution of

$$\begin{cases} Tv=\frac{1}{i}\Box z_0\\ v=0.\end{cases}$$

Then Lemma 6.2 shows that

$$v_{1,1}=\sum_{l=0}^{\infty}w_{1,l}+z_{1,1},\quad w_{1,l}\in K(l),\ z_{1,1}\in M_2(0),$$

and

$$|w_{1,l}|_{F,m}\le C_m\alpha^l l|f|_{m+2}(S_1(\delta_2)\times\boldsymbol{R}).$$

Thus Proposition is proved for $r=1$. Suppose that (6.19) (6.24) holds for $r=s$. Let $v_{s+1,0}$ be a solution of

$$\begin{cases} Tv = \dfrac{1}{i} \displaystyle\sum_{J_s \in N_+^s} \Box w^{(J_s)} & \text{in} \quad \omega \times R \\ v = 0 & \text{on} \quad S(\delta_2) \times R. \end{cases}$$

Since $\{\Box w^{(J_s)}\}_{J_s \in N_+^s} \in \mathscr{H}_s(0)$ Lemma 6.5 shows that

$$v_{s+1,0} = \sum_{J_{s+1} \in N_+^{s+1}} w_{s+1}^{(J_{s+1})} + \sum_{j=0}^{\infty} u_{s+1}^{(j)} + z_{s+1,0},$$

$$W_{s+1} = \{w_{s+1}^{(J_{s+1})}\}_{J_{s+1} \in N_+^{s+1}} \in \mathscr{H}_{s+1}(0),$$

$$u_{s+1}^{(j)} \in \mathscr{H}_0(j), \; z_{s+1,0} \in M_{2s+2}(0),$$

$$|u_{s+1}^{(j)}|_{F,m} \leq C_m \alpha^j j^{s+1} |f|_{m+2s+2}.$$

Denote by $v_{s+1,h,l}$ a solution of

$$\begin{cases} Tv = \dfrac{1}{i} \displaystyle\sum_{J_{s-h} \in N_+^{s-h}} \Box w_{s,h,l}^{(J_{s-h})} & \text{in} \quad \omega \times R \\ v = 0 & \text{on} \quad S(\delta_2) \times R \end{cases}$$

and we have

$$v_{s+1,h,l} = \sum_{J_{s+1-h} \in N_+^{s+1-h}} w_{s+1,h,l}^{(J_{s+1-h})} + z_{s+1,h},$$

$$W_{s+1,h,l} = \{w_{s+1,h,l}^{(J_{s+1-h})}\}_{J_{s+1-h} \in N_+^{s+1-h}} \in \mathscr{H}_{s+1-h}(l)$$

$$\| W_{s+1,h,l} \|_{\mathscr{H}_{s+1-h},m} \leq C_{s+1,m} \alpha^l l^{s+1-h} |f|_{m+2(s+1-h)}(S_1(\delta_2) \times R)$$

$$|z_{s+1,h,l}|_{M_{2+2s},m} \leq C_{s+1,m} \alpha^l l^{s+1-h} |f|_{m+2(s+1-h)}(S_1(\delta_2) \times R).$$

Let $\tilde{v}_{s+1}$ be a solution of

$$\begin{cases} Tv = \dfrac{1}{i} \Box z_s & \text{in} \quad \omega \times R \\ v = 0 & \text{on} \quad S(\delta_2) \times R. \end{cases}$$

Then we have from Lemma 6.2

$$v_{s+1} = \sum_{l=0}^{\infty} w_{s+1}^{(l)} + \tilde{z}_{s+1}, \quad w_{s+1}^{(l)} \in K(l), \quad \tilde{z}_{s+1} \in M_{2s+2}(0)$$

$$|w_{s+1}^{(l)}|_{F,m} \leq C_{s+1,m} \alpha^l l^{2s+2} |f|_{m+2s+2}(S_1(\delta_2) \times R).$$

Since

$$v_{s+1} = v_{s+1,0} + \sum_{h=0}^{\infty} \sum_{l=0}^{\infty} v_{s+1,h,l} + \tilde{v}_{s+1}$$

we have the required properties from the decomposion and estimates of each element.

§ 7. Asymptotic solutions of (1.1)

Let us set

(7.1) $$u(x, t; k)=e^{ik(\varphi(x)-t)}v(x, t; k),$$

$$v(x, t; k)=\sum_{j=0}^{N} v_j(x, t)k^{-j}.$$

Apply $\square$ to u of (7.1) and we have

$$\square u(x, t; k)=-e^{ik(\varphi-t)}\sum_{j=0}^{N+2} k^{-(j-2)}\Big\{(-(\nabla\varphi)^2+1)v_j$$

$$+i\Big(2\frac{\partial v_{j-1}}{\partial t}+2\nabla\varphi\nabla v_{j-1}+\triangle\varphi v_{j-1}\Big)-\square v_{j-2}\Big\},$$

where we set $v_{-1}=v_{-2}=v_{N+1}=v_{N+2}=0$. Then, if

(7.2) $$|\nabla\varphi|^2=1,$$

(7.3) $$2\frac{\partial v_j}{\partial t}+2\nabla\varphi\cdot\nabla v_j+\triangle\varphi v_j=\frac{1}{i}\square v_{j-1},\ j=0, 1,\dots, N$$

hold, we have

(7.4) $$\square u=e^{ik(\varphi-t)}k^{-N}\square v_N.$$

Let

(7.5) $$m(x, t; k)=e^{ik(\varphi(x)-t)}f(x, t),\quad f(x, t)\in C_0^\infty(S_1(\delta_2)\times \boldsymbol{R})$$

be an oscillatory boundary data given on $\Gamma_1\times\boldsymbol{R}$. Suppose that $\varphi(x)$ satisfies conditions (2.2), (2.3), (2.4) and (2.13). Let $\varphi_0, \varphi_1, \varphi_2,\dots$ be a sequence of phase functions constructed in §2, and let $\boldsymbol{v}_r=\{v_{r,q}(x, t), \tilde{v}_{r,q}(x, t)\}_{q=0}^\infty$, $r=0, 1, 2,\dots$ be solutions of transport equations constructed in Proposition 6.6. Set

(7.6) $$\begin{cases} u_q(x, t; k)=e^{ik(\varphi_{2q}(x)-t)}\sum_{r=0}^{N} v_{r,q}(x, t)k^{-r} \\ \tilde{u}_q(x, t; k)=e^{ik(\varphi_{2q+1}(x)-t)}\sum_{r=0}^{N} \tilde{v}_{r,q}(x, t)k^{-r}, \end{cases}$$

(7.7) $$\boldsymbol{u}(x, t; k)=\{u_q(x, t; k), \tilde{u}_q(x, t; k)\}_{q=0}^\infty.$$

Taking account of the above remark and the equations which $\boldsymbol{v}_r$ satisfy we have

(7.8) $$\square\boldsymbol{u}(x, t; k)=k^{-N}\{e^{ik(\varphi_{2q}-t)}\square v_{N,q},\ e^{ik(\varphi_{2q+1}-t)}\square\tilde{v}_{N,q}\}_{q=0}^\infty.$$

From (3.13) we have

$$|\varphi_{2q}(a_2)-\varphi_{2q}(a_1)-(\varphi_\infty(a_2)-\varphi_\infty(a_1))|\le C\alpha^{2q}.$$

On the other hand $\varphi_\infty(a_2)-\varphi_\infty(a_1)=d$ follows from (3.15). Then

$$|\varphi_{2q}(a_2)-\varphi_{2q}(a_1)-d|\leq C\alpha^{2q}.$$

Similarly we have

$$|\varphi_{2q+1}(a_1)-\varphi_{2q+1}(a_2)-d|\leq C\alpha^{2q}.$$

Recall that $\varphi_{2q}(a_2)=\varphi_{2q+1}(a_2)$ and $\varphi_{2q+2}(a_1)=\varphi_{2q+1}(a_1)$. Set

$$\sum_{q=0}^{\infty}(\varphi_{2q+2}(a_1)-\varphi_{2q}(a_1)-2d)=d_0.$$

Then it holds that

$$|\varphi_{2q+2}(a_1)-\varphi_0(a_1)-2(q+1)d-d_0|\leq C\alpha^{2q}.$$

Combining this inequality with (3.13) we have

$$|\varphi_{2q+2}(\cdot)-(\varphi_\infty(\cdot)+2(q+1)d+d_0)|_m(S_1(\delta_0))\leq C_m\alpha^{2q}. \tag{7.9}$$

Similarly we have

$$|\varphi_{2q+1}(\cdot)-(\tilde{\varphi}_\infty(\cdot)+2qd+\tilde{d}_0)|_m(S_2(\delta_0))\leq C_m\alpha^{2q} \tag{7.10}$$

for some constant $\tilde{d}_0$.

By using (6.19) we have

$$\boldsymbol{u}(x,t;k)=\sum_{r=0}^{N}k^{-r}\{\sum_{J_r\in N_+^r}\boldsymbol{u}_r^{(J_r)}+\sum_{h=1}^{r}\sum_{l=0}^{\infty}\sum_{J_{r-h}\in N_+^{r-h}}\boldsymbol{u}_{r,h,l}^{(J_{r-h})}+\tilde{\boldsymbol{u}}_r\} \tag{7.11}$$

where

$$\boldsymbol{u}_r^{(J_r)}=\{e^{ik(\varphi_\infty+2qd+d_0-t)}w_{r,q}^{(J_r)},e^{ik(\tilde{\varphi}_\infty+2qd+\tilde{d}_0-t)}\tilde{w}_{r,q}^{(J_r)}\}_{q\geq|J_r|},$$

$$\boldsymbol{u}_{r,h,l}^{(J_{r-h})}=\{e^{ik(\varphi_\infty+2qd+d_0-t)}w_{r,h,l,q}^{(J_{r-h})},\ e^{ik(\tilde{\varphi}_\infty+2qd+\tilde{d}_0-t)}\tilde{w}_{r,h,l,q}^{(J_{r-h})}\}_{q\geq|J_{r-h}|}$$

and

$$\tilde{\boldsymbol{u}}_r=\{e^{ik(\varphi_{2q}-t)}z_{r,q},\ e^{ik(\varphi_{2q+1}-t)}\tilde{z}_{r,q}\}_{q=0}^{\infty}$$
$$+\{(e^{ik(\varphi_{2q}-t)}-e^{ik(\varphi_\infty+2qd+d_0-t)})v_{r,q},\ (e^{ik(\varphi_{2q+1}-t)}-e^{ik(\tilde{\varphi}_\infty+2qd+\tilde{d}_0-t)})\tilde{v}_{r,q}\}_{q=0}^{\infty}.$$

Then (6.20) and (6.23) imply

$$\begin{cases} \boldsymbol{U}_r=\{\boldsymbol{u}_r^{(J_r)}\}_{J_r\in N_+^r}\in\mathscr{H}_r(0), \\ \|\boldsymbol{U}_r\|_{\mathscr{H}_{r,m}}\leq C_{r,m}k^mB_{m+2r}, \end{cases} \tag{7.12}$$

where B_m denotes $|f|_m(S_1(\delta_2)\times\boldsymbol{R})$. Similarly we have from (6.21) and (6.24)

$$\begin{cases} \boldsymbol{U}_{r,h,l}=\{\boldsymbol{u}_{r,h,l}^{(J_{r-h})}\}_{J_{r-h}\in N_+^{r-h}}\in\mathscr{H}_{r-h}(l), \\ \|\boldsymbol{U}_{r,h,l}\|_{\mathscr{H}_{r-h,m}}\leq C_{r,m}\alpha^l l^{r-h}k^mB_{m+2r}. \end{cases} \tag{7.13}$$

Concerning $\boldsymbol{u}_r$ we have from (6.22)

$$|e^{ik(\varphi_{2q}-t)}z_{r,q}|_m(\omega\times\boldsymbol{R})+|e^{ik(\varphi_{2q+1}-t)}\tilde{z}_{r,q}|_m(\omega\times\boldsymbol{R})$$
$$\leq C_{r,m}k^m\alpha^q q^{2r}\,B_{m+2r}.$$

Since

$$|e^{ik(\varphi_{2q}-t)}-e^{ik(\varphi_\infty+2qd+d_0-t)}|_m(\omega\times\boldsymbol{R})\leq C_m k^{m+1}\alpha^{2q}$$

follows from (7.9), an estimate

$$|v_{r,q}|_m(\omega\times\boldsymbol{R})\leq C_m(\lambda\tilde{\lambda})^q q^{2r}B_{m+2r}$$

which is proved in Proposition 6.6 implies

$$|(e^{ik(\varphi_{2q}-t)}-e^{ik(\varphi_\infty+2qd+d_0-t)})v_{r,q}|_m(\omega\times\boldsymbol{R})\leq C_m k^{m+1}(\lambda\tilde{\lambda})^q\alpha^{2q}.$$

By the same way we have

$$|(e^{ik(\varphi_{2q+1}-t)}-e^{ik(\tilde{\varphi}_\infty+2qd+\tilde{d}_0-t)})\tilde{v}_{r,q}|_m(\omega\times\boldsymbol{R})\leq C_m k^{m+1}(\lambda\tilde{\lambda})^q\,\alpha^{2q}.$$

Then

$$\text{(7.14)}\qquad \begin{cases} \tilde{\boldsymbol{u}}_r\in M_{2r}(0), \\ |\tilde{\boldsymbol{u}}_r|_{M_{2r},m}\leq C_{m,r}k^{m+1}B_{m+2r}. \end{cases}$$

Thus we have the following

Lemma 7.1. *$\boldsymbol{u}(x, t; k)$ defined by* (7.7) *and* (7.8) *is decomposed as* (7.11) *where* (7.12), (7.13) *and* (7.14) *hold.*

Corollary. *$\square\boldsymbol{u}$ is decomposed as*

$$\square\boldsymbol{u}=k^{-N}\{\sum_{J_N}\boldsymbol{g}_N^{(J_N)}+\sum_{h=1}^{N}\sum_{l=0}^{\infty}\sum_{J_{N-h}}\boldsymbol{g}_{N,h,l}^{(J_{N-h})}+\tilde{\boldsymbol{g}}_N\}$$

where

$$\text{(7.15)}\qquad \begin{cases} \boldsymbol{G}_N=\{\boldsymbol{g}^{(J_N)}\}_{J_N\in N_+^N}\in\mathscr{H}_N(0) \\ \|\boldsymbol{G}_N\|_{\mathscr{H}_N,m}\leq C_{N,m}k^{m+1}B_{m+2N}, \end{cases}$$

$$\text{(7.16)}\qquad \begin{cases} \boldsymbol{G}_{N,h,l}=\{\boldsymbol{g}_{N,h,l}^{(J_{N-h})}\}_{J_{N-h}\in N_+^{N-h}}\in\mathscr{H}_{N-h}(l) \\ \|\boldsymbol{G}_{N,h,l}\|_{\mathscr{H}_{N-h},m}\leq C_{N,m}k^{m+1}\alpha^l l^{N-h}, \end{cases}$$

and

$$\text{(7.17)}\qquad \begin{cases} \tilde{\boldsymbol{g}}\in M_{2N}(0)| \\ |\tilde{\boldsymbol{g}}|_{M_{2N},m}\leq C_{N,m}k^{m+1}B_{m+2N}. \end{cases}$$

Extend all the elements of $\boldsymbol{g}^{(J_N)}=\{(\lambda\tilde{\lambda})^q g^{(J_N)}(x, t-2qd),\ (\lambda\tilde{\lambda})^q\tilde{g}^{(J_N)}(x, t-2qd)\}_{q\geq|J_N|}$ by a fixed manner in to $\mathscr{O}$ and denote them as

$$g'^{(J_N)}=\{(\lambda\tilde{\lambda})^q g'^{(J_N)}(x,\ t-2qd),\ (\lambda\tilde{\lambda})^q \tilde{g}'^{(J_N)}(x,\ t-2qd)\}_{q\geq|J_N|}.$$

Let $u_q'^{(J_N)}$, $\tilde{u}_q'^{(J_N)}$ be solutions of

$$\Box u_q'^{(J_N)}=(\lambda\tilde{\lambda})^q g'^{(J_N)}(x,\ t-2qd) \qquad \text{in} \quad \boldsymbol{R}^3\times\boldsymbol{R}$$

$$\Box \tilde{u}_q'^{(J_N)}=(\lambda\tilde{\lambda})^q \tilde{g}'^{(J_N)}(x,\ t-2qd) \qquad \text{in} \quad \boldsymbol{R}^3\times\boldsymbol{R}$$

such that the supports $\subset \boldsymbol{R}^3\times\{t\geq 0\}$.

Denote by $\mathscr{H}_r^{\Omega_R}(p)$ and $\mathscr{H}_r^{\Gamma}(p)$ the spaces defined by the procedure of Definition 6.1 replacing ω by Ω_R and Γ respectively. Then if we set

$$\boldsymbol{u}'^{(J_N)}=\{u_q'^{(J_N)}(x,\ t),\ \tilde{u}_q'^{(J_N)}(x,\ )\}_{q\geq|J_N|},$$

it follows that

$$\text{(7.18)} \qquad \begin{cases} \boldsymbol{U}'=\{\boldsymbol{u}'^{(J_N)}\}_{J_N\in N_+^N}\in\mathscr{H}_N^{\Omega_R}(0) \\ \|\boldsymbol{U}'\|_{\mathscr{H}_N^{\Omega_R},m}\leq C_{N,m,R}k^{m+1}B_{2N+m}. \end{cases}$$

Construct $\boldsymbol{u}_{N,h,l}'^{(J_{N-h})}$ for $\boldsymbol{g}_{N,h,l}^{(J_{N-h})}$ and $\tilde{\boldsymbol{u}}_N'$ for $\tilde{\boldsymbol{g}}_N$ by the above manner. Then we have

$$\text{(7.19)} \qquad \begin{cases} \boldsymbol{U}'_{N,h,l}=\{\boldsymbol{u}_{N,h,l}'^{(J_{N-h})}\}_{J_{N-h}\in N_+^{N-h}}\in\mathscr{H}_{N-h}^{\Omega_R}(l) \\ \|\boldsymbol{U}'_{N,h,l}\|_{\mathscr{H}_N^{\Omega_R},m}\leq C_{N,m,R}k^{m+1}\alpha^l l^{N-h}B_{m+2N} \end{cases}$$

and

$$\text{(7.20)} \qquad \begin{cases} \tilde{\boldsymbol{u}}'\in M_{2N}^{\Omega_R}(0) \\ |\tilde{\boldsymbol{u}}'|_{M_{2N}^{\Omega_R},m}\leq C_{N,m,R}k^{m+1}B_{m+2N}. \end{cases}$$

Then, setting

$$\boldsymbol{u}'=k^{-N}\{\sum_{J_N}u_N'^{(J_N)}+\sum_{h=1}^{N}\sum_{l=0}^{\infty}\sum_{J_{N-h}}\boldsymbol{u}_{N,h,l}'^{(J_{N-h})}+\tilde{\boldsymbol{u}}_N'\}$$

we have

$$\text{(7.21)} \qquad \Box(\boldsymbol{u}-\boldsymbol{u}')=0 \qquad \text{in} \quad \omega\times\boldsymbol{R}.$$

Set

$$\text{(7.22)} \qquad u(x,\ t;\ k)=\sum_{q=0}^{\infty}(u_q(x,\ t;\ k)-\tilde{u}_q(x,\ t;\ k))$$

$$=\sum_{q=0}^{\infty}\sum_{r=0}^{N}k^{-r}(e^{ik(\varphi_{2q}-t)}v_{r,q}-e^{ik(\varphi_{2q+1}-t)}\tilde{v}_{r,q}).$$

Note that

$$\operatorname{supp} u\,|_{\Gamma\times\boldsymbol{R}}\subset S(\delta_0)\times\boldsymbol{R}$$

follows from Remark 4 of §5 and we have from Proposition 6.6

$$(7.23)\qquad u(x, t; k)=\begin{cases} m(x, t; k)-\sum_{q=0}^{\infty}\sum_{r=0}^{N} k^{-r}e^{ik(\varphi_{2q}-t)}v_{1,2}\tilde{v}_{r,q} & \text{on} \quad S_1(\delta_0)\times \boldsymbol{R} \\ \sum_{q=0}^{\infty}\sum_{r=0}^{N} k^{-r}e^{ik(\varphi_{2q+1}-t)}v_{2,2}v_{r,q} & \text{on} \quad S_2(\delta_0)\times \boldsymbol{R} \end{cases}$$

and

$$(7.24)\qquad u(x, t; k)=m(x, t; k) \quad \text{on} \quad S(\delta_2)\times \boldsymbol{R}.$$

Set

$$\boldsymbol{f}=\{f_q, \tilde{f}_q\}_{q=0}^{\infty}=\{-e^{ik(\varphi_{2q}-t)}\sum_{r=0}^{N} k^{-r}v_{1,2}\tilde{v}_{r,q},\ e^{ik(\varphi_{2q+1}-t)}\sum_{r=0}^{N} k^{-r}\, v_{2,2}v_{r,q}\}$$

Recall that Corollary of Lemma 3.3 of [5] shows that

$$\#\mathscr{X}(x, \nabla\varphi_{2q})\le K+1 \qquad \text{for all} \quad x\in \text{Proj}_x(\text{supp} f_q)$$

$$\#\mathscr{X}(x, \nabla\varphi_{2q+1})\le K+1 \qquad \text{for all} \quad x\in \text{Proj}_x(\text{supp} \tilde{f}_q)$$

hold for all q. Let $u''_q(x, t; k)$ be an asymptotic solution constructed for an oscillatory data f_q on $\Gamma_1\times \boldsymbol{R}$ following the procedure in §7 of [5], and let $\tilde{u}''_q(x, t; k)$ be an asymptotic solution of an oscillatory data $\tilde{f}_q$ on $\Gamma_2\times \boldsymbol{R}$. Set

$$\boldsymbol{u}''(x, t; k)=\{u''_q(x, t; k), \tilde{u}''_q(x, t; k)\}_{q=0}^{\infty}.$$

With the aid of considerations of Corollary of Proposition 8.1 of [5] $u''_q(x, t; k)$ satisfies

$$|u''_q(\cdot\,; k)|_m(\Omega_R\times \boldsymbol{R})\le C_{m,R}k^{m+1}\sum_{r=0}^{N} k^{-r}|v_{r,q}|_{m+N+N'}(S_1(\delta_2)\times \boldsymbol{R})$$

$$\Box u''_q=0 \qquad \text{in} \quad \omega\times \boldsymbol{R}$$

$$|u''(\cdot\,; k)-f_q|_m(\Gamma\times \boldsymbol{R})\le C_m k^{-N+1+m}\sum_{r=0}^{N} k^{-r}|v_{r,q}|_{m+N+N'}(S_1(\delta_0)\times \boldsymbol{R})$$

$$\text{supp}\, u''_q|_{\Gamma\times R}\subset \Gamma\times[2qd, 2qd+d_1] \qquad \text{for some} \quad d_1.$$

Estimates of the same type hold for $\tilde{u}''_q$. Taking account of Proposition 6.6 and the continuity of a correspondences of f_q to u''_q and $\tilde{f}_q$ to $\tilde{u}''_q$ we see that $\boldsymbol{u}''$ can be decomposed as

$$\boldsymbol{u}''=\sum_{r=0}^{N} k^{-r}\,\{\sum_{J_r}\boldsymbol{u}_r''^{(J_r)}+\sum_{h=1}^{r}\sum_{l=0}^{\infty}\sum_{J_{r-h}}\boldsymbol{u}''^{(J_{r-h})}_{r,h,l}+\tilde{\boldsymbol{u}}''_r\}$$

where

$$\begin{cases} \boldsymbol{U}''_r=\{\boldsymbol{u}_r''^{(J_r)}\}_{J_r\in N_+^r}\in\mathscr{H}_r^{\Omega_R}(0) \\ \|\boldsymbol{U}''\|_{\mathscr{H}_r^{\Omega_R},m}\le C_{r,m}k^{m+1}B_{m+2(N+N')}, \end{cases}$$

$$\begin{cases} \boldsymbol{U}''_{r,h,l}=\{\boldsymbol{u}''^{(J_{r-h})}_{r,h,l}, \tilde{\boldsymbol{u}}''^{(J_{r-h})}_{r,h,l}\}_{J_{r-h}\in N_+^{r-h}}\in\mathscr{H}_{r-h}^{\Omega_R}(l) \\ \|\boldsymbol{U}''_{r,h,l}\|_{\mathscr{H}_{r-h}^{\Omega_R},m}\le C_{r,m,R}k^{m+1}\alpha^l l^{r-h}B_{m+2(N+N')}, \end{cases}$$

$$\begin{cases} \tilde{u}'' \in M_{2r}(0) \\ |\tilde{u}_r''|_{M_{2r}^{\Omega_R},m} \le C_{r,m,R} k^{m+1} B_{m+2(N+N')}. \end{cases}$$

Now denote $u - u' - u''$ by u again. Then we have

Proposition 7.2. *For an oscillatory data $m(x, t; k)$ on $\Gamma_1 \times \boldsymbol{R}$ given by* (7.5), *there exists an asymptotic solution*

$$u(x, t; k) = \sum_{q=0}^{\infty} (u_q(x, t; k) - \tilde{u}_q(x, t; k))$$

with the following properties:

(i) $\boldsymbol{u}(x, t; k) = \{u_q(x, t; k), \tilde{u}_q(x, t; k)\}_{q=0}^{\infty}$ *is decomposed as*

$$\boldsymbol{u} = \sum_{r=0}^{N} k^{-r} \{ \sum_{J_r} \boldsymbol{u}_r^{(J_r)} + \sum_{h=1}^{r} \sum_{l=0}^{\infty} \sum_{J_{r-h}} \boldsymbol{u}_{r,h,l}^{(J_{r-h})} + \tilde{\boldsymbol{u}}_r \}$$

where it holds that for all $R > 0$

$$\begin{cases} \boldsymbol{U}_r = \{\boldsymbol{u}_r^{(J_r)}\}_{J_r \in N_+^r} \in \mathscr{H}_r^{\Omega_R}(0) \\ \|\boldsymbol{U}_r\|_{\mathscr{H}_r^{\Omega_R},m} \le C_{r,m,R} k^{m+1} B_{m+2(N+N')}, \end{cases}$$

$$\begin{cases} \boldsymbol{U}_{r,h,l} = \{\boldsymbol{u}_{r,h,l}^{(J_{r-h})}\}_{J_{r-h} \in N_+^{r-h}} \in \mathscr{H}_{r-h}^{\Omega_R}(l) \\ \|\boldsymbol{U}_{r,h,l}\|_{\mathscr{H}_{r-h}^{\Omega_R},m} \le C_{r,m,R} k^{m+1} \alpha^l l^{r-h} B_{m+2(N+N')}, \end{cases}$$

$$\tilde{\boldsymbol{u}}_r \in M_{2r}^{\Omega_R}(0)$$

$$|\tilde{\boldsymbol{u}}_r|_{M_{2r}^{\Omega_R},m} \le C_{r,m,R} k^{m+1} B_{m+2(N+N')},$$

here B_m denotes $|f|_m(S_1(\delta_2) \times \boldsymbol{R})$.

(ii) $\Box u(x, t; k) = 0$ *in* $\omega \times \boldsymbol{R}$.

(iii) $\operatorname{supp} u \subset \Omega \times \{t;\ t \ge 0\}$.

(iv) *If we set*

$$u - m = \begin{cases} \sum_{q=0}^{\infty} f_q & \text{on } \Gamma_1 \times \boldsymbol{R} \\ \sum_{q=0}^{\infty} \tilde{f}_q & \text{on } \Gamma_2 \times \boldsymbol{R}, \end{cases}$$

$\boldsymbol{f} = \{f_q, \tilde{f}_q\}_{q=0}^{\infty}$ *is decomposed as*

$$\boldsymbol{f} = k^{-N} \{ \sum_{J_N} \boldsymbol{f}_N^{(J_N)} + \sum_{h=1}^{N} \sum_{l=0}^{\infty} \sum_{J_{N-h}} \boldsymbol{f}_{N,h,l}^{(J_{N-h})} + \tilde{\boldsymbol{f}}_N \}$$

where

$$\begin{cases} \boldsymbol{F}_N = \{\boldsymbol{f}_N^{(J_N)}\}_{J_N \in N_+^N} \in \mathscr{H}_N^{\Gamma}(0) \\ \|\boldsymbol{F}_N\|_{\mathscr{H}_N^{\Gamma},m} \le C_{N,m} k^{m+1} B_{m+2(N+N')}, \end{cases}$$

$$\begin{cases} F_{N,h,l} = \{f_{N,h,l}^{(J_{N-h})}\}_{J_{N-h} \in N_+^{N-h}} \in \mathscr{H}_{N-h}^{\Gamma}(l) \\ \|F_{N,h,l}\|_{\mathscr{H}_{N-h,m}^{\Gamma}} \leq C_{N,m} k^{m+1} \alpha^l l^{N-h} B_{m+2(N+N')} \end{cases}$$

$$\begin{cases} \tilde{f}_N \in M_{2N}^{\Gamma}(0) \\ |f_N|_{M_{2N,m}^{\Gamma}} \leq C_{N,m} k^{m+1} B_{m+2(N+N')}. \end{cases}$$

§8. Laplace transformation of functions in $\mathscr{H}_r$

Denotes by S a mapping from F into $C_0^\infty(\omega \times \boldsymbol{R})$ defined by

$$Sw = \sum_{q=0}^{\infty} (w_q - \tilde{w}_q) \qquad \text{for} \quad w = \{w_q, \tilde{w}_q\}_{q=0}^{\infty}.$$

Note that $w(x, t) = Sw$ satisfies

$$|w|_m(\omega, t) \leq C_m e^{-c_0 t} |w|_{F,m}.$$

Then the Laplace transformation of $w(x, t)$

$$\hat{w}(x, \mu) = \int_{-\infty}^{\infty} e^{-\mu t} w(x, t) dt$$

is defined for $\operatorname{Re}\mu > -c_0$, and we have for any $\varepsilon > 0$

$$|\hat{w}(\cdot, \mu)|_m(\omega) \leq C_{m,\varepsilon} |w|_{F,m} \qquad \text{for all} \quad \operatorname{Re}\mu > -c + \varepsilon.$$

Let $w = \{(\lambda\tilde{\lambda})^q f(x, t-2qd), (\lambda\tilde{\lambda})^q \tilde{f}(x, t-2qd)\}_{q \geq p} \in K(p)$. Since

$$\int_{-\infty}^{\infty} e^{-\mu t} (\lambda\tilde{\lambda})^q f(x, t-2qd) dt = (\lambda\tilde{\lambda} e^{-2\mu d})^q \hat{f}(x, \mu) \qquad \text{for all} \quad \mu \in \boldsymbol{C}$$

we have for $w(x, t) = Sw$ and $\operatorname{Re}\mu > -c_0$

$$\begin{aligned} \hat{w}(x, \mu) &= \sum_{q=p}^{\infty} (\lambda\tilde{\lambda} e^{-2\mu d})^q (\hat{f}(x, \mu) - \hat{\tilde{f}}(x, \mu)) \\ &= (\lambda\tilde{\lambda} e^{-2\mu d})^p (1 - \lambda\tilde{\lambda} e^{-2\mu d})^{-1} (\hat{f}(x, \mu) - \hat{\tilde{f}}(x, \mu)). \end{aligned}$$

Since the right hand side is meromorphic in the whole complex plane we have the following

Lemma 8.1. *Let $w(x, t) = Sw$, $w \in K(p)$. Then the Laplace transformation of w*

$$\hat{w}(x, \mu) = \int_{-\infty}^{\infty} e^{-\mu t} w(x, t) dt$$

converges in $\operatorname{Re}\mu > -c_0$. And it is prolonged analytically to a meromorphic function in $\boldsymbol{C}$ of the form

$$\hat{w}(x, \mu) = (\lambda\tilde{\lambda} e^{-2\mu d})^p (1 - \lambda\tilde{\lambda} e^{-2\mu d})^{-1} F(x, \mu), \tag{8.1}$$

where $F(x, \mu)$ is holomorphic in the whole complex plane. And a mapping from $\{f, \tilde{f}\}$ to $F(x, \mu)$ is linear, and continuous in the following sense

$$|F(\cdot, \mu)|_m(\omega) \leq C_m\{|e^{(c_0+c)t}f|_m(\omega\times \boldsymbol{R}) + |e^{(c_0+c)t}\tilde{f}|_m(\omega\times \boldsymbol{R})\} \tag{8.2}$$

for all $\operatorname{Re}\mu > -(c_0+c)$, $c\in \boldsymbol{R}$.

Lemma 8.2. *Let $W=\{w^{(j)}\}_{j\in N_+}\in \mathscr{H}_1(p)$. Set*

$$w(x, t) = \sum_{j\in N_+} Sw^{(j)}.$$

Then the Laplace transformation of $w(x, t)$ converges in $\operatorname{Re}\mu > -c_0$ *and it is prolonged analytically to a meromorphic function in* $\{\mu;\ \operatorname{Re}\mu > -c_0-c_1\}$ *of the form*

$$\hat{w}(x, \mu) = (\lambda\tilde{\lambda}e^{-2\mu d})^p\{(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-1}F_1(x, \mu) + (1-\lambda\tilde{\lambda}e^{-2\mu d})^{-2}F_2(x, \mu)\} \tag{8.3}$$

where $F_1(x, \mu)$ and $F_2(x, \mu)$ are holomorphic in $\{\mu;\ \operatorname{Re}\mu > -c_0-c_1\}$. *Moreover correspondences $\boldsymbol{W}$ to F_1 and F_2 are linear, and continuous in the following sense;*

$$\sup_{\operatorname{Re}\mu > -(c_0+c_1)+\varepsilon} |F_j(\cdot, \mu)|_m(\omega) \leq C_{m,\varepsilon}\|W\|_{\mathscr{X}_{1,m}}. \tag{8.4}$$

Proof. Let us set $\boldsymbol{w}^{(j)} = \{(\lambda\tilde{\lambda})^q f^{(j)}(x, t-2dq), (\lambda\tilde{\lambda})^q\tilde{f}^{(j)}(x, t-2qd)\}_{q\geq p+j}$ and $w^{(j)} = S\boldsymbol{w}^{(j)}$. By using the result of the previous lemma we have

$$\hat{w}^{(j)}(x, \mu) = (\lambda\tilde{\lambda}e^{-2\mu d})^{p+j}(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-1}F^{(j)}(x, \mu).$$

From the property (iii) of the definition of $(CH)_1$ we have $f^{(\infty)}(x, t), \tilde{f}^{(\infty)}(x, t)\in C_0^\infty(\omega\times \boldsymbol{R})$ and

$$|F^{(j)}(\cdot, \mu) - F^{(\infty)}(\cdot, \mu)|_m(\omega) \leq C_{m,\varepsilon}\{|e^{(c_0+c_1)t}(f^{(j)}-f^{(\infty)})|_m(\omega\times \boldsymbol{R}) + |e^{(c_0+c_1)t}(\tilde{f}^{(j)}-\tilde{f}^{(\infty)})|_m(\omega\times \boldsymbol{R})\}$$

for $\operatorname{Re}\mu \geq -c_0-c_1+\varepsilon$. Note that

$$\sum_{j=0}^{\infty}(\lambda\tilde{\lambda}e^{-2\mu d})^j F^{(j)}(x, \mu) = \sum_{j=0}^{\infty}(\lambda\tilde{\lambda}e^{-2\mu d})^j F^{(\infty)}(x, \mu) + \sum_{j=0}^{\infty}(\lambda\tilde{\lambda}\alpha e^{-2\mu d})^j\,\frac{F^{(j)}(x, \mu)-F^{(\infty)}(x, \mu)}{\alpha^j}.$$

Then for $\operatorname{Re}\mu \geq -c_0-c_1+\varepsilon$

$$\begin{aligned}&\sum_{j=0}^{\infty}|\lambda\tilde{\lambda}\alpha e^{-2\mu d}|^j\left|\frac{F^{(j)}-F^{(\infty)}}{\alpha^j}\right| \\ &\leq \sup_j|(F^{(j)}-F^{(\infty)})\alpha^{-j}|\sum_{j=0}^{\infty}|\lambda\tilde{\lambda}\alpha e^{-2\mu d}|^j \\ &\leq (1-|\lambda\tilde{\lambda}\alpha e^{-2\mu d}|)^{-1}(|\{f^{(j)}\}_{j\in N_+}|_{(CH)_1,m} + |\{\tilde{f}^{(j)}\}_{j\in N_+}|_{(CH)_1,m}) \\ &\leq C_{m,\varepsilon}\|W\|_{\mathscr{X}_{1,m}}.\end{aligned}$$

Therefore

$$\sum_{j=0}^{\infty}(\lambda\tilde{\lambda}\alpha e^{-2\mu d})^j(F^{(j)}-F^{(\infty)})\alpha^{-j}=F_1(x,\mu)$$

is holomorphic in $\{\mu;\ \mathrm{Re}\,\mu\geq -c_0-c_1+\varepsilon\}$. On the other hand

$$\sum_{j=0}^{\infty}(\lambda\tilde{\lambda}e^{-2\mu d})^j F^{(\infty)}(x,\mu)=(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-1}F^{(\infty)}(x,\mu).$$

Then setting $F_2(x,\mu)=F^{(\infty)}(x,\mu)$ we have (8.3). The linearity and the continuity of mapping W to F_1 and F_2 already shown. Q.E.D.

Proposition 8.3. *Let* $W=\{w^{(J_r)}\}_{J_r\in N_+^r}\in\mathscr{H}_r(p)$. *Set*

$$w(x,t)=\sum_{J_r}Sw^{(J_r)}.$$

Then the Laplace transformation

$$\hat{w}(x,\mu)=\int_{-\infty}^{\infty}e^{-\mu t}w(x,t)dt$$

converges for all $\mathrm{Re}\,\mu>-c_0$ *and it can be prolonged analytically to a meromorphic function in* $\mathrm{Re}\,\mu>-c-c_1$ *of the form*

$$\hat{w}(x,\mu)=(\lambda\tilde{\lambda}e^{-2\mu d})^p\sum_{j=0}^{r}(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-(r-j)-1}F_j(x,\mu) \tag{8.5}$$

where F_j, $j=0,1,\dots,r$, *are* $C^\infty(\omega)$*-valued holomorphic function in* $\{\mu;\ \mathrm{Re}\,\mu>-c_0-c_1\}$. *Moreover a mapping* $W\in\mathscr{H}_r(p)$ *to* $\{F_j(x,\mu)\}_{j=0}^r$ *is linear, and continuous in the following sense;*

$$\sup_{\mathrm{Re}\,\mu>-c_0-c_1+\varepsilon}\sum_{j=0}^{r}|F_j(\cdot,\mu)|_m(\omega)\leq C_{m,\varepsilon}\|W\|_{\mathscr{H}_r,m}. \tag{8.6}$$

Proof. First admit the following

Assertion. Let $\tilde{B}$ be a linear continuous mapping from $C_0^\infty(\omega\times\boldsymbol{R})$ into a set of a $C^\infty(\omega)$ valued holomorphic functions in $\{\mu;\ \mathrm{Re}\,\mu>-c_0-c_1\}$ such that

$$\sup_{\mathrm{Re}\,\mu>-c_0-c_1+\varepsilon}|(\tilde{B}f)(\cdot,\mu)|_m(\omega)\leq C_{m,\varepsilon}|f|_m(\omega\times\boldsymbol{R}).$$

Then for $\{g^{(J_s)}\}_{J_s\in N_+^s}\in(CH)_s$ we have

$$\sum_{J_s}(\lambda\tilde{\lambda}e^{-2\mu d})^{|J_s|}(B\tilde{g}^{(J_s)})(x,\mu)$$

$$=\sum_{j=0}^{s}(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-(s-j)}G_j(x,\mu),$$

where $G_j(x,\mu)$ are holomorphic in $\{\mu;\ \mathrm{Re}\,\mu>-c_0-c_1\}$ and

$$\sup_{\mathrm{Re}\,\mu>-c_0-c_1+\varepsilon}|G_j(\cdot,\mu)|_m(\omega)\leq C_{m,\varepsilon}|\{g^{(J_s)}\}|_{(CH)_s,m}$$

holds for $j=0,1,\dots,s$.

Let us prove Proposition by using the above assertion. Note that the assertion of Proposition for $r=1$ is nothing but Lemma 8.2. Suppose that $r\geq 2$ and $\boldsymbol{W}=\{w^{(J_s)}\}_{J_r\in N_+^r}\in\mathscr{H}_r(p)$ such that

$$w^{(J_r)}=\{(\lambda\tilde{\lambda})^q f^{(J_r)}(x,\ t-2qd),\ 0\}_{q\geq|J_s|+p}.$$

From (iii) of definition of $(CH)_r$ there exist B and $\{g^{(J_{r-1})}\}\in(CH)_{r-1}$ such that

$$\{f^{(J_{r-1},j_r)}\}_{j_r=0}^{\infty}=\boldsymbol{B}g^{(J_{r-1})}\qquad\text{for all}\quad J_{r-1}.$$

Since we have from Lemma 8.2

$$\begin{aligned}&\sum_{j_r=0}^{\infty}(Sw^{(J_{r-1},j_r)})^{\wedge}(x,\ \mu)\\&\quad=(\lambda\tilde{\lambda}e^{-2\mu d})^{|J_{r-1}|+p}\sum_{j=1}^{2}(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-j}F_j^{(J_{r-1})}(x,\ \mu),\end{aligned}$$

it follows that

$$\begin{aligned}(8.7)\quad \hat{w}(x,\ \mu)&=\sum_{J_r}(Sw^{(J_r)})^{\wedge}(x,\ \mu)\\&=(\lambda\tilde{\lambda}e^{-2\mu d})^p\sum_{j=1}^{2}(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-j}\sum_{J_{r-1}}(\lambda\tilde{\lambda}e^{-2\mu d})^{|J_{r-1}|}F_j^{(J_{r-1})}(x,\ \mu).\end{aligned}$$

Taking account of the linearity and the continuity of a mapping $\{f^{(J_{r-1},j_r)}\}_{j_r\in N_+}\in(CH)_1$ to $\{F_j^{(J_{r-1})}\}_{j=1,2}$, which we denote by $\boldsymbol{B}'$, we can write

$$\{F_j^{(J_{r-1})}\}_{j=1,2}=\boldsymbol{B}'\{f^{(J_{r-1},j_r)}\}_{j_r=0}^{\infty}=\boldsymbol{B}'\boldsymbol{B}g^{(J_{r-1})}=\tilde{\boldsymbol{B}}g^{(J_{r-1})}$$

from which the linearity and the continuity of $\tilde{\boldsymbol{B}}$ follow. Then Applying Assertion we have

$$\begin{aligned}&\sum_{J_{r-1}}(\lambda\tilde{\lambda}e^{-2\mu d})^{|J_{r-1}|}F_j^{(J_{r-1})}(x,\ \mu)\\&\quad=\sum_{l=0}^{r}(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-(r-l)}F_{j,l}(x,\ \mu).\end{aligned}$$

Substitute his relation into (8.7) and the assertion of Proposition follows. For $\boldsymbol{W}=\{w^{(J_s)}\}\in\mathscr{H}_s(p)$ such that

$$w^{(J_s)}=\{0,\ (\lambda\tilde{\lambda})^q\tilde{f}^{(J_s)}(x,\ t-2qd)\}_{q\geq|J_s|+p}$$

Proposition is proved by the same way.

We turn to the proof of Assertion. For $s=1$ it is proved already in Lemma 8.2. Suppose that Assertion is ture for $s=r$. Let $\{g^{(J_{r+1})}\}\in(CH)_{r+1}$. Then the property (iii) of $(CH)_{r+1}$ assures the existence of $\boldsymbol{B}$ and $\{g^{(J_r)}\}\in(CH)_r$ such that

$$\{g^{(J_r,j_{r+1})}\}_{j_{r+1}=0}^{\infty}=\boldsymbol{B}g^{(J_r)}\qquad\text{for all}\quad J_r.$$

The assertion in the case of $s=1$ shows

$$\sum_{j_{r+1}=0}^{\infty}(\lambda\tilde{\lambda}e^{-2\mu d})^{j_{r+1}}(\tilde{B}g^{(J_r,j_{r+1})})(x,\mu)$$
$$=(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-1}G_1^{(J_r)}(x,\mu)+G_0^{(J_r)}(x,\mu)\,.$$

Therefore

$$\sum_{J_{r+1}}(\lambda\tilde{\lambda}e^{-2\mu d})^{|J_{r+1}|}(\tilde{B}g^{(J_{r+1})})(x,\mu)$$
$$=(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-1}\sum_{J_r}(\lambda\tilde{\lambda}e^{-2\mu d})^{|J_r|}G_1^{(J_r)}(x,\mu)$$
$$+\sum_{J_r}(\lambda\tilde{\lambda}e^{-2\mu d})^{|J_r|}G_0^{(J_r)}(x,\mu)\,. \tag{8.8}$$

The linearities and the continuities of $\boldsymbol{B}$ and $\{g^{(J_r,j_{r+1})}\}_{j_{r+1}=0}^{\infty}$ to $\{G\}_j^{(J_r)}{}_{j=1,2}$ imply that a mapping $g^{(J_r)}$ to $\{G_j^{(J_r)}\}_{j=1,2}$ is linear and continuous. Therefore we can apply Assertion in the case of $s=r$ to $\{G^{(J_r)}\}$ and we have

$$\sum_{J_r}G_j^{(J_r)}(x,\mu)(\lambda\tilde{\lambda}e^{-2\mu d})^{|J_r|}$$
$$=\sum_{l=0}^{r}(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-(r-l)}G_{j,l}(x,\mu)\,.$$

Substitution of the above relation into (8.8) derives that Assertion is also true for $s=r+1$. By the induction Assertion is true for all $s\geq 1$. Thus the proof of Proposition is completed.

§9. Proof of Theorem 2

Let $h(x,t)\in C_0^{\infty}(S_1(\delta_2)\times(0,d/2))$. Then it holds that

$$h(y(\sigma),t)=\omega(y(\sigma),t)\int\cdots\int e^{-ik(t-t')}e^{i\xi(\sigma-\sigma')}\cdot h(y(\sigma'),t')d\sigma'dt'd\xi dk,$$

where $\omega\in C_0^{\infty}(S_1(\delta_0)\times\boldsymbol{R})$ verifying

$$\omega(x,t)=\begin{cases}1 & \text{for } (x,t)\in S_1(\delta_0)\times(0,d/2)\\ 0 & \text{for } (x,t)\notin S_1(\delta_3)\times(-d/2,d)\,.\end{cases}$$

Set

$$(\mathscr{V}_1h)(y(\sigma),t)=\omega(y(\sigma),t)\int\cdots\int_{|k|\geq 1}e^{-ik(t-t')}e^{i\xi(\sigma-\sigma')}$$
$$\chi_1\left(\frac{|\xi|}{\beta_0|k|}\right)h(y(\sigma'),t')d\sigma'dt'dkd\xi,$$

$$\mathscr{V}_2h=h-\mathscr{V}_1h,$$

where $\chi(l)\in C_0^{\infty}(\boldsymbol{R})$ such that

$$\chi(l)=\begin{cases}1 & |l|\leq 1\\ 0 & |l|\geq 2,\end{cases}$$

and β_0 is a constant which will be fixed later. Then we have

$$(\mathscr{V}_1 h)(y(\sigma), t)=\omega(y(\sigma), t)\int_{|k|\geq 1} k^2 dk\int_{\eta\in S^1} d\eta\int d\beta\iint dt'd\sigma'$$
$$e^{ik\{\beta\langle\eta,\sigma-\sigma'\rangle-(t-t')\}}\chi_1(\beta/\beta_0)h(y(\sigma'), t').$$

Let $-2\beta_0\leq\beta\leq 2\beta_0$, $\eta\in S^1=\{(\eta_1, \eta_2); \eta_1^2+\eta_2^2=1\}$ and let $\varphi(x; \beta, \eta)$ be a real valued C^∞ function verifying

$$(9.1)\qquad \begin{cases} |\nabla\varphi|=1 \\ \varphi(y(\sigma); \beta, \eta)=\beta<\sigma, \eta> \\ \dfrac{\partial\varphi}{\partial n}>0. \end{cases}$$

Fix $\beta_0>0$ so small that $\varphi(x; \beta, \eta)$ satisfies (2.14) and (2.17). Define a mapping $\mathscr{U}_1$ from $C_0^\infty(S_1(\delta_2)\times(0, d/2))$ into $C^\infty(\bar{\Omega}\times\boldsymbol{R})$ by

$$(9.2)\qquad (\mathscr{U}_1 h)(x, t)=\int_{|k|\geq 1} k^2 dk\int d\eta\int d\beta\, u(x, t; k, \beta, \eta)\chi_1(\beta/\beta_0)\tilde{h}(k, \beta, \eta),$$

where

$$\tilde{h}(k, \beta, \eta)=\iint e^{ik(t'-\beta\langle\sigma',\eta\rangle)}h(y(\sigma'), t')d\sigma'dt'$$

and $u(x, t; \beta, \eta)$ denotes an asymptotic solution in Proposition 7.2 for an oscillatory data

$$m(x, t; k, \beta, \eta)=e^{ik(\varphi(x;\beta,\eta)-t)}\omega(x, t).$$

Concerning $\mathscr{V}_2 h$, if we choose $\delta_3>\delta_2>0$ sufficiently small, we can construct following Corollary of Lemma 3.3 and Proposition 7.5 of [5] an operator $\mathscr{U}_2$ from $C_0^\infty(S_1(\delta_2)\times(0, d/2))$ into $C^\infty(\bar{\Omega}\times\boldsymbol{R})$ with the following properties:

$$(9.3)\qquad \operatorname{supp}\mathscr{U}_2 h\subset\{(x, t); t-t_1\leq|x|\leq t+t_2\}$$

where t_1, t_2 are positive constants.

$$(9.4)\qquad |\mathscr{U}_2 h|_m(\Omega_R\times\boldsymbol{R})\leq C_{m,R}|h|_{m+5}(\Gamma\times\boldsymbol{R})\qquad \text{for}\quad R>0.$$

$$(9.5)\qquad \square\mathscr{U}_2 h=0\qquad \text{in}\quad \Omega\times\boldsymbol{R}.$$

$$(9.6)\qquad |\mathscr{U}_2 h-\mathscr{V}_2 h|_m(\Gamma\times\boldsymbol{R})\leq C_m|h|_0(\Gamma\times\boldsymbol{R})\qquad \text{for}\quad m\leq N-1.$$

Therefore if we set

$$U_2(\mu)h=\int_{-\infty}^{\infty} e^{-\mu t}(\mathscr{U}_2 h)(x, t)dt$$

we have from (9.3) and (9.4)

$$(9.7)\qquad U_2(\mu)h \text{ is holomorphic in } \boldsymbol{C},$$

and for any $R>0$ and $m\leq N-1$

(9.8) $$\sum_{j=0}^{m}|\mu|^{j}|U_{2}(\mu)h|_{m-j}(\Omega_{R})\leq C_{m,R}|h|_{m+5}e^{-(t_{1}+R)\mathrm{Re}\mu},$$

and from (9.5) we have for all $\mu\in\boldsymbol{C}$

(9.9) $$(\mu^{2}-\triangle)U_{2}(\mu)h=0 \quad \text{in} \quad \Omega,$$

from (9.6)

(9.10) $$\sum_{j=0}^{m}|\mu|^{j}|U_{2}(\mu)h-(\mathscr{V}_{2}h)(\mu)|_{m-j}(\Gamma)\leq C_{m}|h|_{0}(\Gamma\times\boldsymbol{R})e^{-\rho\mathrm{Re}\mu}.$$

Moreover we see easily, with the aid of the energy inequality of (P), from (9.5) and (9.6) that

(9.11) $$U_{2}(\mu)h\in\bigcap_{m\geq0}H^{m}(\Omega) \qquad \text{if} \quad \mathrm{Re}\,\mu>0.$$

Now we turn to consideration of Laplace transformation of $\mathscr{U}_{1}h$. Note that it follows from Proposition 7.2 that

$$|u(\cdot\,;\beta,\eta)|_{m}(\Omega_{R},t)\leq C_{R,N,m}t^{N}e^{-c_{0}t}k^{m+1}.$$

Then the Laplace transform

$$U_{1}(\mu)h=\int_{-\infty}^{\infty}e^{-\mu t}(\mathscr{U}_{1}h)(x,t)dt$$

converges absolutely for $\mathrm{Re}\,\mu>-c_{0}$. Therefore

(9.12) $U_{1}(\mu)h$ is holomorphic in $\mathrm{Re}\,\mu>-c_{0}$.

Next consider an analytic continuation of $U_{1}(\mu)h$. Let us set

$$Q_{r}(\mu)h=\int\cdots\int e^{-\mu t}u_{r}(x,t;\beta,\eta)k^{-r}h(k,\beta,\eta)\chi_{1}(\beta/\beta_{0})k^{2}dkd\beta d\eta dt$$

$$Q_{r,h,l}(\mu)h=\int\cdots\int e^{-\mu t}u_{r,h,l}(x,t;k,\beta,\eta)k^{-r}\tilde{h}(k,\beta,\eta)\chi_{1}(\beta/\beta_{0})k^{2}dkd\beta d\eta dt$$

$$\tilde{Q}_{r}(\mu)h=\int\cdots\int e^{-\mu t}\tilde{u}_{r}(x,t;\beta,\eta)k^{-r}\tilde{h}(k,\beta,\eta)\chi_{1}(\beta/\beta_{0})k^{2}dkd\beta d\eta dt$$

where

$$u_{r}(x,t;k,\beta,\eta)=\sum_{J_{r}\in N_{+}^{r}}S\boldsymbol{u}_{r}^{(J_{r})}(x,t;\beta,\eta)$$

$$u_{r,h,l}(x,t;k,\beta,\eta)=\sum_{J_{h-r}}S\boldsymbol{u}_{r,h,l}^{(J_{r-h})}(x,t;k,\beta,\eta)$$

$$\tilde{u}_{r}(x,t;\beta,\eta)=S\tilde{\boldsymbol{u}}_{r}(x,t;k,\beta,\eta).$$

We have from (i) of Proposition 7.2

$$|\tilde{u}_{r}(\cdot\,;k,\beta,\eta)|_{m}(\Omega_{R},t)\leq C_{N,m,R,\varepsilon}e^{-(c_{0}+c_{1}-\varepsilon)t}t^{r}k^{m+1},$$

from which it follows that

(9.13) $\tilde{Q}_r(\mu)h$ is holomorphic in $\mathrm{Re}\,\mu > -c_0 - c_1$,

and

(9.14) $|\mu|^r|\tilde{Q}_r(\mu)h|_m(\Omega_R) \leq C_{N,m,R,\varepsilon}|h|_{m+5}(\Gamma\times\boldsymbol{R})$ for $\mathrm{Re}\,\mu \geq -c_0 - c_1 + \varepsilon$.

By applying Proposition 8.3 to $\int e^{-\mu t}u_r(x, t; \beta, \eta)dt$ we have

(9.15) $$Q_r(\mu)h = (1-\lambda\tilde{\lambda}e^{-2\mu d})^{-r-1}(\mathscr{F}_r h)(x, \mu)$$

where

$$\mathscr{F}_r h(x, \mu) = \sum_{l=0}^{r}(1-\lambda\tilde{\lambda}e^{-2\mu d})^l\int k^2dk\int d\eta\int d\beta\, F_{j,l}(x, \mu; \beta, \eta)$$
$$k^{-r}\chi_1(\beta/\beta_0)\tilde{h}(k, \beta, \eta).$$

(8.5) and (8.6) imply that $\mathscr{F}_r h$ is holomorphic in $\mathrm{Re}\,\mu \geq -c_0 - c_1 + \varepsilon$ and

(9.16) $$|\mu|^r\sum_{j=0}^{m}|\mu|^j|(\mathscr{F}_r h)(\cdot, \mu)|_{m-j}(\Omega_R) \leq C_{N,m,R,\varepsilon}|h|_{m+5}(\Gamma\times\boldsymbol{R})$$

for $\mathrm{Re}\,\mu \geq -c_0 - c_1 + \varepsilon$. Similarly we have for $\mathrm{Re}\,\mu \geq -c_0 - c_1 + \varepsilon$

(9.17) $$Q_{r,h,l}(\mu)h = (\lambda\tilde{\lambda}e^{-2\mu d})^l(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-(r-h)-1}\mathscr{F}_{r,h,l}h(x, \mu)$$

where $\mathscr{F}_{r,h,l}h(x, \mu)$ is holomorphic in $\mathrm{Re}\,\mu \geq -c_0 - c_1 + \varepsilon$ and

(9.18) $$|\mu|^r\sum_{j=0}^{m}|\mu|^j|\mathscr{F}_{r,h,l}h(\cdot, \mu)|_{m-j}(\Omega_R)$$
$$\leq C_{N,m,R,\varepsilon}\alpha^l l^{r-h}|h|_{m+5}(\Gamma\times\boldsymbol{R}).$$

Note that

$$U_1(\mu)h = \sum_{r=0}^{N}\{Q_r(\mu)h + \sum_{h=1}^{r}\sum_{l=0}^{\infty}Q_{r,h,l}(\mu)h + \tilde{Q}_r(\mu)h\}$$

and for $\mathrm{Re}\,\mu \geq -c_0 - c_1 + \varepsilon$

$$\sum_{l=0}^{\infty}\sum_{j=0}^{m}|\mu|^j|\lambda\tilde{\lambda}e^{-2\mu d}|^l|(\mathscr{F}_{r,h,l}h)(\cdot, \mu)|_{m-j}(\Omega_R)$$
$$\leq C_{N,m,R,\varepsilon}(1+|\mu|)^{-r}\sum_{l=0}^{\infty}|\alpha\lambda\tilde{\lambda}e^{-2\mu d}|^l l^{r-h}|h|_{m+5}(\Gamma\times\boldsymbol{R}) < \infty.$$

Thus by setting

$$\tilde{U}(\mu)h = U_1(\mu)h + U_2(\mu)h,$$

we have

Lemma 9.1. *A linear mapping $\tilde{U}(\mu)$ from $C_0^\infty(S_1(\delta_2)\times(0, d/2))$ into $C^\infty(\bar{\Omega}\times\boldsymbol{R})$ is of the form*

(9.19) $$\tilde{U}(\mu)h=\sum_{r=0}^{N}(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-r-1}\tilde{\mathscr{F}}_r h(x,\mu)$$

where $\tilde{\mathscr{F}}_r h(x,\mu)$ *is* $C^\infty(\bar{\Omega})$ *valued holomorphic function in* $\{\mu;\ \mathrm{Re}\,\mu>-c_0-c_1\}$ *satisfying an estimate for* $\mathrm{Re}\,\mu\geq -c_0-c_1+\varepsilon$

(9.20) $$\sum_{j=0}^{m}|\mu|^j|\tilde{\mathscr{F}}_r h(\cdot,\mu)|_{m-j}(\Omega_R)\leq C_{N,m,R,\varepsilon}(1+|\mu|)^{-r}|h|_{m+5}(\Gamma\times \boldsymbol{R}).$$

Next consider the boundary value of $\tilde{U}(\mu)h$. We have from (iv) of Proposition 7.2

$$U_1(\mu)h-(\mathscr{V}_1\hat{h})(x,\mu)=(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-N-1}\mathscr{G}h(x,\mu)$$

where $\mathscr{G}h(x,\mu)$ is $C^\infty(\Gamma)$ valued holomorphic function in $\mathrm{Re}\,\mu>-c_0-c_1$ and satisfies for $\mathrm{Re}\,\mu\geq -c_0-c_1+\varepsilon$

$$\sum_{j=0}^{N-5}|\mu|^j|\mathscr{G}h(\cdot,\mu)|_{N-j-5}(\Gamma)\leq C_{N,\varepsilon}|1-\lambda\tilde{\lambda}e^{-2\mu d}|^{-N-1}|h|_0(\Gamma\times\boldsymbol{R}).$$

Combining this estimate with (9.6) we have

Lemma 9.2. *It holds that for all* $h\in C_0^\infty(S_1(\delta_2)\times(0,d/2))$

$$\sum_{j=0}^{N-5}|\mu|^j|\tilde{U}(\mu)h-\hat{h}(\cdot,\mu)|_{N-5-j}(\Gamma)$$
$$\leq C_{N,\varepsilon}|1-\lambda\tilde{\lambda}e^{-2\mu d}|^{-N-1}|h|_0(\Gamma\times\boldsymbol{R})\qquad \text{for}\quad \mathrm{Re}\,\mu\geq -c_0-c_1+\varepsilon.$$

Until now we restricted boundary data to be in $C_0(S_1(\delta_2)\times(0,d/2))$. But as remarked in the proof of Proposition 8.1 of [5] we can remove this restriction, that is, from Lemmas 9.1 and 9.2

Lemma 9.3. *There exists a linear mapping* $\tilde{U}(\mu)$ *from* $C_0^\infty(\Gamma\times(0,d/2))$ *into* $C^\infty(\bar{\Omega})$ *with a parameter* $\mu\in D=\{\mu;\ \mathrm{Re}\,\mu>-c_0-c_1\}$ *with the following properties*:

(i) $\tilde{U}(\mu)h=\sum_{r=0}^{N}(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-r-1}\tilde{\mathscr{F}}_r h(x,\mu)$

where $\tilde{\mathscr{F}}_r h(x,\mu)$ *is holomorphic in D and has an estimate*

$$\sum_{j=0}^{m}|\mu|^j|\tilde{\mathscr{F}}_r h(\cdot,\mu)|_{m-j}(\Omega_R)\leq C_{N,m,R,\varepsilon}(1+|\mu|)^{-r}|h|_{m+5}(\Gamma\times\boldsymbol{R})$$

for $D_\varepsilon=\{\mu;\ \mathrm{Re}\,\mu>-c_0-c_1+\varepsilon\}$.

(ii) $(\mu^2-\triangle)\tilde{U}(\mu)h=0$ *in* Ω.

(iii) $\tilde{U}(\mu)h-\hat{h}(x,\mu)=(1-\lambda\tilde{\lambda}e^{-2\mu d})^{-N-1}\mathscr{G}h(x,\mu)$ *on* Γ

where $\mathscr{G}h(x,\mu)$ *is holomorphic in D and has an estimate*

$$\sum_{j=0}^{N-5}|\mu|^j|\mathscr{G}h(\cdot,\mu)|_{N-5-j}(\Gamma)\leq C_{N,\varepsilon}|h|_0(\Gamma\times\boldsymbol{R})\qquad \text{for}\quad \mu\in D_\varepsilon.$$

(iv) $\tilde{U}(\mu)h\in\bigcap_{m>0}H^m(\Omega)$ *for all* $\mathrm{Re}\,\mu>0$.

Let $m(t)$ be a function in $C_0^\infty(0, d/2)$ such that

$$m(t)\geq 0 \quad \text{and} \quad \int e^{-t}m(t)dt=2.$$

Set for k real

$$m_k(t)=e^{ikt}m(t).$$

Since $\hat{m}_k(\mu)=\hat{m}(\mu-ik)$, there exists $a_0>0$ such that

(9.21) $|m_k(ik'+\zeta)|\geq 1 \qquad \text{for all} \quad |k'-k|\leq a_0 \qquad \text{and} \quad 1\geq\zeta\geq -c_0-c_1.$

Set $\tilde{D}_k=\{\zeta+ik';\ 1\geq\zeta\geq -c_0-c_1,\ |k'-k|\leq a_0\}$. For each $k\in \boldsymbol{R}$ define an operator $\tilde{U}_k(\mu)$ from $C^\infty(\Gamma)$ into $C^\infty(\bar{\Omega})$ with a parameter $\mu\in\tilde{D}_k$ by

$$\tilde{U}_k(\mu)g=\frac{1}{\hat{m}_k(\mu)}\tilde{U}(\mu)h \qquad \text{for} \quad g\in C^\infty(\Gamma)$$

where $h(x, t)=g(x)m_k(t)$. Since $|h|_m(\Gamma\times\boldsymbol{R})\leq C_m k^m|g|_m(\Gamma)$ we have from (i) of Lemma 9.3 for $\mu\in D_{k,\varepsilon}=\tilde{D}_k\cap D_\varepsilon$

$$\text{(9.22)} \qquad \sum_{j=0}^{m}|\mu|^j|\tilde{U}_k(\mu)g|_{m-j}(\Omega_R)$$
$$\leq C_{N,m,R,\varepsilon}\sum_{j=0}^{N}|1-\lambda\tilde{\lambda}e^{-2\mu d}|^{-j-1}(1+|\mu|)^{-j}k^{m+5}|g|_{m+5}(\Gamma).$$

Similarly we have from (iii) of Lemma 9.3 and (9.21)

$$\text{(9.23)} \qquad \sum_{j=0}^{N-5}|\mu|^j|\tilde{U}_k(\mu)g-g|_{N-5-j}(\Gamma)$$
$$\leq C_{N,m,\varepsilon}|1-\lambda\tilde{\lambda}e^{-2\mu p}|^{-N-1}|g|_0(\Gamma) \qquad \text{for} \quad \mu\in D_{k,\varepsilon}.$$

Take $N=24$. It follows from (9.23) that

$$(1+|\mu|)^{12}|\tilde{U}_k(\mu)g-g|_7(\Gamma)\leq C_\varepsilon|1-\lambda\tilde{\lambda}e^{-2\mu d}|^{-24}|g|_0(\Gamma).$$

Then for

$$\text{(9.24)} \qquad \mu\in\left\{\mu;\frac{1}{2}((1+|\mu|)|1-\lambda\tilde{\lambda}e^{-2\mu d}|^2)^{12}\geq C_\varepsilon\right\}\cap D_{k,\varepsilon}=\tilde{D}_{k,\varepsilon}$$

we have

$$\|\tilde{U}_k(\mu)-I\|_{\mathscr{L}(C^7(\Gamma),C^7(\Gamma))}\leq\frac{1}{2}.$$

Set

$$V_k(\mu)=\sum_{j=0}^{\infty}(\tilde{U}_k(\mu)-I)^j.$$

Then we have

$$\text{(9.25)} \qquad \|V_k(\mu)\|_{\mathscr{L}(C^7(\Gamma),C^7(\Gamma))}\leq 2$$

and

(9.26) $$\tilde{U}_k(\mu)\cdot V_k(\mu)=I \quad \text{in } C^7(\Gamma) \quad \text{for all } \mu\in\tilde{D}_{k,\varepsilon}.$$

Set for $\mu\in\tilde{D}_{k,\varepsilon}$

$$U_k(\mu)g=\tilde{U}_k(\mu)\cdot V_k(\mu)g \qquad \text{for} \quad g\in C^7(\Gamma).$$

Then it holds that

$$U_k(\mu)g=g \qquad \text{on} \quad \Gamma.$$

From (9.25) and (9.22) we have $U_k(\mu)g\in C^2(\bar{\Omega})$ and

$$|U_k(\mu)g|_2(\Omega_R)\le C_{R,\varepsilon}|k|^7|g|_7(\Gamma) \qquad \text{for} \quad \mu\in\tilde{D}_{k,\varepsilon}.$$

Evidently it holds that

$$(\mu^2-\triangle)U_k(\mu)g=0 \qquad \text{in} \quad \Omega.$$

From Lemma 9.3 and the definition of $U_k(\mu)$ we see that $U_k(\mu)g$ is holomorphic in $\bigcup_{\varepsilon>0}\tilde{D}_{k,\varepsilon}$ and $U_k(\mu)g\in H^2(\Omega)$ for $\mathrm{Re}\,\mu>0$. The uniqueness of solutions of the problem

$$\begin{cases} (\mu^2-\triangle)u=0 & \text{in} \quad \Omega \\ u=g & \text{on} \quad \Gamma \end{cases}$$

in $H^2(\Omega)$ for $\mathrm{Re}\,\mu>0$ implies

$$U_k(\mu)=U_{k'}(\mu) \qquad \text{for} \quad \mu\in D_{k,\varepsilon}\cap D_{k',\varepsilon}.$$

Set

$$\tilde{D}_\varepsilon=\bigcup_{k\in\mathbf{R}}\tilde{D}_{k,\varepsilon}, \quad \tilde{D}=\bigcup_{\varepsilon>0}\tilde{D}_\varepsilon$$

and define $U(\mu)$ for $\mu\in\tilde{D}$ by

$$U(\mu)=U_k(\mu) \qquad \text{for} \quad \mu\in\tilde{D}_{k,\varepsilon}.$$

Set

$$\mu_j=-c_0+i\frac{\pi}{d}j, \quad j=0,\ \pm1,\ \pm2,\dots.$$

Then we have

$$\tilde{D}_\varepsilon\supset\mathscr{D}_\varepsilon=\{\mu;\ 1\ge\mathrm{Re}\,\mu\ge-c_0-c_1+\varepsilon\}-\bigcup_{j=-\infty}^{\infty}\{\mu;\ |\mu-\mu_j|\le C(1+|j|)^{-1/2}\}$$

for some $C>0$. Thus we have

Theorem 9.4. *For $g\in C^7(\Gamma)$, $U(\mu)g$ is $C^2(\bar{\Omega})$-valued holomorphic function in $\mathscr{D}=\bigcup_{\varepsilon>0}\mathscr{D}_\varepsilon$ satisfying*

$$(\mu^2-\triangle)U(\mu)g=0 \qquad \text{in} \quad \Omega$$

$$U(\mu)g=g \qquad \text{on} \quad \Gamma.$$

And it holds that

$$|U(\mu)g|_2(\Omega_R)\leq C_{R,\varepsilon}(1+|\mu|)^7|g|_7(\Gamma) \quad \text{for} \quad \mu\in\mathscr{D}_\varepsilon,$$

$$U(\mu)g\in H^2(\Omega) \quad \textit{for} \quad \mathrm{Re}\,\mu>0.$$

Remark. The regularity theorem for $\triangle$ derives from the above theorem that for $g\in C^\infty(\Gamma)$

$$U(\mu)g\in C^\infty(\bar{\Omega})$$

and

$$|U(\mu)g|_m(\Omega_R)\leq C_{R,m,\varepsilon}\sum_{j=0}^{m+7}|\mu|^j|g|_{m+7-j}(\Gamma) \quad \text{for} \quad \mu\in\mathscr{D}_\varepsilon.$$

§ 10. Existence of an infinite number of poles of $U(\mu)$

To prove Theorem 3 it suffices to show that for any $\varepsilon>0$ a region $\{\mu;\ \mathrm{Re}\,\mu>-c_0-\varepsilon\}$ contains an infinite number of poles of $U(\mu)$. Suppose the contrary:

(A) There exists $\varepsilon_0>0$ such that a region $D_0=\{\mu;\ \mathrm{Re}\,\mu>-c_0-\varepsilon_0\}$ contains only a finite number of poles.

By exchanging ε_0 a smaller one if necessary we may assume that there are no poles on a line $\{\mu;\ \mathrm{Re}\,\mu=-c_0-\varepsilon_0\}$. Let $\mathscr{C}$ be a simple closed curve in $D_0\cap\{\mathrm{Re}\,\mu<0\}$ containing all the poles of $U(\mu)$ with $\mathrm{Re}\,\mu>-c_0-\varepsilon_0$.

Consider a mixed problem

$$(10.1) \qquad \begin{cases} \square w=0 & \text{in} \quad \Omega\times \boldsymbol{R} \\ w(x,t)=m(x,t) & \text{on} \quad \Gamma\times \boldsymbol{R} \\ \mathrm{supp}\, w\subset\bar{\Omega}\times[0,\infty) & \end{cases}$$

for a boundary data $m(x,t)\in C_0^\infty(\Gamma\times(0,d/2))$. Then the solution $w(x,t)$ of (10.1) is represented as

$$(10.2) \qquad w(x,t)=\int_{-\infty}^{\infty}e^{(a+ik)t}(U(a+ik)\hat{m}(\cdot,a+ik))(x)dk$$

where a is a positive constant and

$$\hat{m}(x,\mu)=\int e^{-\mu t}m(x,t)dt.$$

Note that the integral of the right hand side of (10.2) is independent of $a>0$. By using an estimate of $U(\mu)$ in Theorem 9.4 we can obtain from the assumption of the finiteness of poles

$$(10.3) \quad |U(\mu)g|_2(\Omega_R)\leq C_{R,\varepsilon}(1+|\mu|)^7|g|_7(\Gamma)$$

$$\text{for all } |\mu| \text{ sufficiently large and } \mathrm{Re}\,\mu>-c_0-c_1+\varepsilon.$$

Since

$$|\hat{m}(\cdot,\mu)|_p(\Gamma)\leq C_{p,l}|m|_{p+l}(\Gamma\times \boldsymbol{R})(1+|\mu|)^{-l}e^{-\mathrm{Re}\mu d/2} \tag{10.4}$$

holds for $p,\ l=0,\ 1,\ 2,\dots$, we can change the path of integration of (10.2) as

$$\begin{aligned} w(x,t) &= \int_{-\infty}^{\infty} e^{(-(c_0+\varepsilon_0)+ik)t}U(-c_0-\varepsilon_0+ik)\hat{m}(\cdot,\ -c_0-\varepsilon_0+ik)dk \\ &\quad +\int_{\mathscr{C}} e^{\mu t}U(\mu)\hat{m}(\cdot,\mu)d\mu \\ &= w_1(x,t)+w_2(x,t). \end{aligned}$$

With the aid of (10.3) and (10.4) we have

Lemma 10.1. *It holds that*

$$|w_1|_2(\Omega_R,\ t)\leq C_R e^{-(c_0+\varepsilon_0)t}|m|_{16}(\Gamma\times \boldsymbol{R}).$$

Let $w(x,t;k)$ be a solution of (10.1) for a boundary data

$$m(x,t;k)=e^{ik(\varphi_\infty(x)-t)}f(x)p(t) \tag{10.5}$$

where $f\in C^\infty(\Gamma)$ and $p(t)\in C_0^\infty(0,\ d/2)$. Then

$$\hat{m}(x,\mu;k)=e^{ik\varphi_\infty(x)}f(x)\hat{p}(\mu-ik).$$

Since

$$|\hat{p}(\mu)|\leq C_N(1+|\mu|)^{-N} \qquad \text{for all} \quad \mathrm{Re}\,\mu\geq -c_0-c_1$$

we have

$$\max_{\mu\in\mathscr{C}}|\hat{p}(\mu-ik)|\leq C_N k^{-N} \qquad \text{for} \quad k\geq 1.$$

Taking account of $\mathscr{C}\subset\{\mathrm{Re}\,\mu\leq 0\}$ we have

$$|w_2(\cdot\,;k)|_2(\Omega_R\ t)\leq C_{R,N}k^{-N} \qquad \text{for all} \quad t\geq 0.$$

Thus combining this estimate with Lemma 10.1 we have

Lemma 10.2. *Assume that (A) holds. For an oscillatory* data $m(x,t;k)$ *of* (10.5) *a solution* $w(x,t;k)$ *of* (10.1) *satisfies*

$$|w(\cdot\,;k)|_2(\Omega_R,\ t)\leq C_R e^{-(c_0+\varepsilon_0)t}k^{16}+C_{R,N}k^{-N} \tag{10.6}$$

for all $t\geq 0$, *where* C_R *and* $C_{R,N}$ *depend on* $f(x)$ *and* $p(t)$, R *and* N, *but independent of* k.

Let $f\in C_0^\infty(S_1(\delta_2))$ such that

$$f(a_1)=1 \tag{10.7}$$

and let $p(t)\in C_0^\infty(0,\ d/2)$ such that

$$p(d/4)=1. \tag{10.8}$$

Construct an asymptotic solution $u(x, t; k)$ for $m(x, t; k)$ of (10.5) with (10.7) and (10.8) following the procedure in the previous sections. In this case u_q, $\tilde{u}_q$ in Proposition 7.2 are of the form

$$u_q(x, t; k) = e^{ik(\varphi_\infty(x)+2qd-t)} \sum_{r=0}^{N} v_{r,q}(x, t)k^{-r}$$

$$\tilde{u}_q(x, t; k) = e^{ik(\tilde{\varphi}_\infty(x)+2qd-t)} \sum_{r=0}^{N} \tilde{v}_{r,q}(x, t)k^{-r}.$$

Remark that

$$\operatorname*{supp}_t v_{0,q}(a_0, \cdot) \subset \{t;\ (2q+1/2)d \leq t \leq (2q+1)d\}$$

$$\operatorname*{supp}_t \tilde{v}_{0,q}(a_0, \cdot) \subset \{t;\ (2q+3/2)d \leq t \leq (2q+2)d\}.$$

and

$$v_{0,q}(a_2, (2q+5/4)d) = \lambda^{q+1}\tilde{\lambda}^q$$

where a_0 denotes the middle point of a_1 and a_2. Then we have

$$v_{0,q}(a_0, (2q+3/4)d) > v_{0,q}(a_2, (2q+5/4)d) = \lambda^{q+1}\lambda\tilde{\lambda}^q$$

$$v_{0,s}(a_0, (2q+3/4)d) = 0 \qquad s \neq q$$

$$\tilde{v}_{0,s}(a_0, (2q+3/4)d) = 0 \qquad \text{for all} \quad s.$$

Thus $u(x, t; k)$ in Proposition 7.2 satisfies

$$|u(a_0, (2q+3/4)d; k)| \geq (\lambda\tilde{\lambda})^{q+1} - C(\lambda\tilde{\lambda})^q \sum_{r=1}^{N} k^{-r}q^r \tag{10.9}$$

$$\Box u = 0 \quad \text{in} \quad \Omega \times \boldsymbol{R} \tag{10.10}$$

$$|u(\cdot\,; k) - m(\cdot\,; k)|_m(\Gamma, t) \leq C_m k^{-N} t^{2N} e^{-c_0 t}. \tag{10.11}$$

Denote by $z(x, t; k)$ a solution of

$$\begin{cases} \Box z = 0 & \text{in} \quad \Omega \times \boldsymbol{R} \\ z = -(u(x, t; k) - m(x, t; k)) & \text{on} \quad \Gamma \times \boldsymbol{R} \\ \operatorname{supp} z \subset \bar{\Omega} \times \{t \geq 0\}. \end{cases}$$

Then from (10.11) we have

$$|z(a_0, t; k)| \leq C_N k^{-N} t^{2N} \qquad \text{for all} \quad t \geq 0. \tag{10.12}$$

Evidently we have $w(x, t; k) = u(x, t; k) + z(x, t; k)$. From (10.9) and (10.12) it follows that for all q and k

$$|w(a_0, (2q+3/4)d; k)| \geq (\lambda\tilde{\lambda})^q (1 - C_N \sum_{r=1}^{N} k^{-r}q^r) - C_N k^{-N} q^{2N}.$$

Combining this estimate with (10.6) we have

$$C_R e^{-(c_0+\varepsilon_0)(2q+3/4)d}\, k^{16} + C_{R,N} k^{-N}$$

$$\geq (\lambda\tilde{\lambda})^q (1 - C_N \sum_{r=1}^{N} k^{-r} q^r) - C_N k^{-N} q^{2N}.$$

Recall that $e^{-2c_0 d} = \lambda\tilde{\lambda}$. Choose k as

$$k^{16} = e^{\varepsilon_0(2q+3/4)d/2}.$$

And take $N = [2c_0/\varepsilon_0] + 3$. Then $(1 - C_N \sum_{r=1}^{N} k^{-r} q^r) \geq 1/2$ holds for large q. Thus we have

$$C_R e^{-(c_0+\varepsilon_0/2)(2q+3/4)d}$$

$$\geq \frac{1}{2} e^{-c_0(2q+3/4)d} - C_N q^{2N} e^{-(c_0+\varepsilon_0/2)(2q+3/4)d}.$$

Letting $q \to \infty$ we have a contradiction. Thus our assertion is proved.

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