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Structure of open algebraic surfaces, I

By

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§0. Introduction

In this paper, we shall study the structure of algebraic surfaces which may not be complete. The main results were announced in the note [11], which will serve as an introduction to this paper.

Let X be a nonsingular surface over $\boldsymbol{C}$ and let $\bar{P}_m(X)$, $\bar{\kappa}(X)$ denote the logarithmic m-genus of X, the logarithmic Kodaira dimension of X, respectively (see Iitaka [3]). It is an important problem to find the smallest one among those positive integers m with $\bar{P}_m(X)>0$. If X is complete, $\bar{\kappa}(X)=-\infty$ if and only if $\bar{P}_{12}(X)=0$ by virtue of the classification theory. Our results, which extends the above result to the case of open algebraic surfaces, are summarized as follows: Take a smooth completion $\bar{X}$ of X such that $D:=\bar{X}-X$ is a divisor on X with simple normal crossings.

(1) (Theorem 2.1 of §2). If $\bar{\kappa}(X)=0$, then $\bar{P}_i(X)=1$ for some $1\leqq i\leqq 66$.

(2) (Theorem 3.3 of §3). If $\bar{\kappa}(X)\geqq 0$, and if D is connected, then $\bar{P}_{12}(X)>0$.

In particular, by virtue of Miyanishi-Sugie-Fujita's cancellation theorem [2], we deduce from (2) the following theorem:

Theorem. *Assume that D is connected. Then $\bar{P}_{12}(X)=0$ if and only if X contains an open set U of the form $U\cong \boldsymbol{A}^1\times C$, where C is an open curve.*

In a forthcoming paper, entitled "Structure of open algebraic surface II, An application to plane curves", we apply the results obtained in this article to projective plane curves.

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Notation and Coventions

1. We use the following notations. A triple $(X, \bar{X}, D)$ is said to be nonsingular if $\bar{X}$ is a complete nonsingular algebraic surface and D is a reduced divisor with only simple normal crossings (i.e., D consists of nonsingular irreducible components crossing normally) such that $X=\bar{X}-D$.
2. Let L be a free $\boldsymbol{Z}$-module generated by all irreducible curves on X. Each element of $L\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ is called a $\boldsymbol{Q}$-divisor. Let D be a $\boldsymbol{Q}$-divisor. If $D=\sum a_iD_i$ is a de-

composition into irreducible components, we define $[D]$ to be $\sum[a_i]D_i$, where $[a_i]$ is the Gauss symbol of a_i.

3. Let D be a divisor on $\bar{X}$. Suppose that $H^0(\bar{X}, nD)\neq 0$ for some integer $n>0$. Then there exist an integer κ and positive numbers α, β and m_0 such that

$$\alpha m^{\kappa}\leqq \dim H^0(\bar{X}, mm_0D)\leqq \beta m^{\kappa}$$

for all $m\gg 0$. We define $\kappa(D, \bar{X})$ to be the integer κ. If $H^0(\bar{X}, nD)=0$ for all $n>0$, then we set $\kappa=-\infty$. If D is a $\boldsymbol{Q}$-divisor, we define $\kappa(D, \bar{X})$ to be $\kappa(mD, \bar{X})$, where mD is a divisor in the usual sense.

4. If $(X, \bar{X}, D)$ is a nonsingular triple, we define $\bar{P}_m(X)$ (resp. $\bar{\kappa}(X)$) to be $\dim H^0(\bar{X}, m(K(\bar{X})+D))$ (resp. $\kappa(K(\bar{X})+D, X)$), where $K(\bar{X})$ is a canonical divisor of $\bar{X}$.

5. If D is a reduced connected divisor, we write $p_a(D)=\frac{1}{2}(D, K+D)+1$ and $\omega_D=(K+D)|_D$. Note that $p_a(D)\geqq 0$ and $p_a(D)=0$ if and only if D consists of nonsingular rational curves whose dual graph is a tree.

6. Let D_1, D_2 be divisors on $\bar{X}$. We write $D_1\sim D_2$ when D_1 is linearly equivalent to D_2.

7. Let $(X, \bar{X}, D)$ and $(Y, \bar{Y}, C)$ be nonsingular triples. Let $f\colon \bar{X}\to\bar{Y}$ be a surjective morphism such that $f(X)\subset Y$. Then there is an effective divisor B on $\bar{X}$ such that

$$K(\bar{X})+D\sim f^*(K(\bar{Y})+C)+B.$$

We call B the logarithmic ramification divisor and denote it by $\bar{R}_f$ (cf. Iitaka [3]). In particular, if $D=C=0$, B is called the ramification divisor and is denoted by R_f.

Denote by $f^{-1}(A)$ the set-theoretical inverse image of an algebraic set A of $\bar{Y}$. If A is a reduced divisor on $\bar{Y}, f^{-1}(A)$ becomes a reduced divisor on $\bar{X}$.

8. Let $f\colon \bar{X}\to\bar{Y}$ be a birational morphism between nonsingular complete algebraic surfaces. For a divisor Γ on $\bar{X}$, $f_*\Gamma$ denotes the direct image Γ on $\bar{Y}$. Let C be a curve on $\bar{Y}$. Then the proper transform $f'(C)$ of C on $\bar{X}$ is usually abbreviated as C'.

9. Let $\mathcal{O}\oplus\mathcal{O}(e)$ $(e\geqq 0)$ be a vector bundle of rank 2 on $\boldsymbol{P}^1$. We set $\Sigma_e:=\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(e))$ and call it the Hirzebruch surface.

§1. Almost minimal triples

We shall introduce the notion of almost minimal triple and construct an almost minimal triple from a given triple $(X, \bar{X}, D)$ with $\bar{\kappa}(X)\geqq 0$. Note that our definition of almost minimal triple is closely related to the notion of relatively minimal model by Kawamata [5].

First of all, we recall the following general notion and fact due to Zariski [14]. Let $\bar{X}$ be a nonsingular complete surface. A divisor D on $\bar{X}$ is said to be *semipositive* (or *arithmetically effective*, after the terminology of Zariski) if $(D, C)\geqq 0$ for every irreducible curve C on $\bar{X}$. Furthermore, a $\boldsymbol{Q}$-divisor D is said to be semipositive whenever some positive multiple mD is a semipositive divisor.

Theorem 1.1. *Let D be a $\boldsymbol{Q}$-divisor on $\bar{X}$. Suppose that $\kappa(D, \bar{X}) \geqq 0$. Then there exists a unique effective $\boldsymbol{Q}$-divisor N such that*:

(1) $N=0$ *or the intersection matrix of N is negative-definite*;

(2) $D-N$ *is a semipositive $\boldsymbol{Q}$-divisor*;

(3) $(D-N, N)=0$.

Proof. By hypothesis, some positive multiple mD is a divisor such that $|mD| \neq \emptyset$. Applying Theorem 7.7 in Zariski [14] to a member D' of $|mD|$, we find a $\boldsymbol{Q}$-divisor N' which has the properties (1), (2), (3) for D'. Then $N=N'/m$ has the required properties.

Denoting $D-N$ and N by D^+ and D^-, respectively, we say that D^+ and D^- are *the semipositive* and *negative components* of D, respectively. The decomposition $D=D^+ + D^-$ is called *the Zariski decomposition* of D.

Proposition 1.2. (1) *For every $\boldsymbol{Q}$-divisor D and every positive integer n, $(nD)^+ = n(D^+)$ and $(nD)^- = n(D^-)$.*

(2) *If D is a usual divisor, then $H^0(\bar{X}, D) \cong H^0(\bar{X}, [D^+])$.*

Proof. See Kawamata [5; (1.4)].

Let $(X, \bar{X}, D)$ be a nonsingular triple such that $\bar{\kappa}(X) \geqq 0$. Then, by Theorem 1.1, we have the effective $\boldsymbol{Q}$-divisor $(K+D)^-$, where K denotes a canonical divisor of $\bar{X}$. We say that the triple $(X, \bar{X}, D)$ is almost minimal if $(K+D)^-$ contains no exceptional curves of the first kind.

Now we state the existence theorem of almost minimal triple as follows:

Theorem 1.3. *Given a nonsingular triple $(X, \bar{X}, D)$ with $\bar{\kappa}(X) \geqq 0$, there exist an almost minimal triple $(Z, \bar{Z}, B)$ and a birational morphism $f: \bar{X} \to \bar{Z}$ having the following properties*:

(1) $B=f_*(D)$,

(2) $(K+D)^+ = f^*((K(\bar{Z})+B)^+)$,

(3) $R_f \subseteq \mathrm{supp}\,(K+D)^-$, *where* $K=K(\bar{X})$.

Proof. Step (1). To prove this, we have to introduce the following simple notions concerning the boundary of X.

Let $(X, \bar{X}, D)$ be a nonsingular triple. An irreducible component C of D is said to be *an edge component*, if $(D-C, C) \leqq 1$. A connected reduced divisor $\sum_{j=1}^{r} C_j$ is said to be *a linear chain*, if each C_j is an edge component of $C_j + \cdots + C_r + (D - \sum_j C_j)$. Moreover, a linear chain is said to be *rational*, if each component is a nonsingular rational curve. Hence a rational linear chain C satisfies $(K+C, C) = -2$. Furthermore,

$$(K+D, C)=(K+C, C)+(D-C, C)=-2+(D-C, C)=-2 \quad \text{or} \quad -1,$$

according as $(D-C, C)=0$ or 1. A maximal rational linear chain means a rational linear chain which is not contained in a larger rational linear chain. Let $D(1), \ldots, D(s)$ be all the maximal linear chains contained in D. For each $D(i)$, let

$\sum_{j=1}^{r(i)} D(i)_j$ be the decomposition of $D(i)$ into irreducible components such that the first component $D(i)_1$ is an edge component and $(D(i)_j, D(i)_{j-1})=1$ for $2\leqq j\leqq r(i)$.

Step (2). Assume that some $D(i)_j$ is an exceptional curve of the first kind and denote it by E. Let $\mu\colon \bar{X}\to\bar{Y}$ be the contraction of E, under which $C:=\mu_*(D)$ is a divisor with simple normal crossings on $\bar{Y}$. Then we have

$$K+D=\mu^*(K(\bar{Y})+C)+aE$$

for some non-negative integer a. By the projection formula, we know that

$$\kappa(K(\bar{Y})+C,\ \bar{Y})=\kappa(\mu^*(K(\bar{Y})+C)+aE,\ \bar{X})=\bar{\kappa}(X)\geqq 0.$$

We shall show that

$$(K+D)^+=\mu^*((K(\bar{Y})+C)^+).$$

Set $\varepsilon_+=\mu^*((K(\bar{Y})+C)^+)$ and $\varepsilon_-=\mu^*(K(\bar{Y})+C)^-+aE$. For every irreducible curve Γ on $\bar{X}$, we have

$$(\varepsilon_+,\Gamma)=(\mu^*((K(\bar{Y})+C)^+),\Gamma)=((K(\bar{Y})+C)^+,\ \mu_*(\Gamma))\geqq 0,$$

because $(K(\bar{Y})+C)^+$ is semipositive. Let E' be an irreducible component of ε_-. Then $\mu(E')$ is either a point or a component of $(K(\bar{Y})+C)^-$. Hence

$$(\mu^*((K(\bar{Y})+C)^+),\ E')=((K(\bar{Y})+C)^+,\ \mu_*(E'))=0.$$

(cf. (3) of Theorem 1.1). Let $(K(\bar{Y})+C)^-=\sum_{i=1}^{p} r_iN_i$ be the decomposition into irreducible components with $r_i\in\boldsymbol{Q}$ and $r_i>0$. For integers x_i $(i=1,\dots,p)$ and y $(\neq 0)$, we obtain that

$$(\textstyle\sum_{i=1}^{p} x_i\mu^*N_i+yE)^2=(\sum_{i=1}^{p} x_i\mu^*N_i)^2+y^2E^2=(\sum_{i=1}^{p} x_iN_i)^2-y^2<0.$$

This implies that the intersection matrix of ε_- is negative-definite. Therefore, by the uniqueness of the Zariski decomposition, we have

$$(K+D)^+=\mu^*((K(\bar{Y})+C)^+).$$

By contracting all exceptional curves of the first kind in $\sum_{i,j} D(i)_j$ successively, we may assume that every $D(i)_j$ is not an exceptional curve of the first kind.

Step (3). We claim that

$$D(i)_j\subseteqq \mathrm{supp}\,(K+D)^-.$$

For simplicity, we write D_j for $D(i)_j$. Thus D_1 is an edge component. As was remarked before, $(K+D,\ D_1)<0$. Since $\kappa(K+D,\ X)=\bar{\kappa}(X)\geqq 0$, we have some positive integer m such that $|m(K+D)|\neq\emptyset$; hence $(D_1^2)<0$. For $\Gamma\in|m(K+D)|$, we have $\Gamma=kD_1+\Gamma_0$, where k is a positive integer, Γ_0 is an effective divisor and D_1 is not an irreducible component of Γ_0. Then we have

$$(K+D,\ D_1)=1/m(\Gamma,\ D_1)=1/m(kD_1+\Gamma_0,\ D_1)\geqq k/m(D_1^2).$$

So, $k/m\geqq a:=(K+D,\ D_1)/(D_1^2)>0$. Hence we know that

$$m(K+D-aD_1) \sim kD_1+\Gamma_0-maD_1=(k-ma)D_1+\Gamma_0$$

and $k \geqq ma$. Thus $\kappa(K+D-aD_1, \bar{X}) \geqq 0$. Let $\varepsilon_+=(K+D-aD_1)^+$ and $\varepsilon_-=(K+D-aD_1)^- +aD_1$, where $\varepsilon_+ +\varepsilon_- =K+D$. If D_1 is contained in $\operatorname{supp}(K+D-aD_1)^-$, then $(\varepsilon_+, D_1)=0$ and the intersection matrix of ε_- is negative-definite. If D_1 is not contained in $\operatorname{supp}(K+D-aD_1)^-$, then $((K+D-aD_1)^-, D_1) \geqq 0$. Since $(K+D-aD_1)^+$ is semipositive, it follows that $((K+D-aD_1)^+, D_1) \geqq 0$. On the other hand, $(K+D-aD_1, D_1)=0$ by the choice of a. Hence we have

$$((K+D-aD_1)^+, D_1)=((K+D-aD_1)^-, D_1)=0.$$

In both cases, $\varepsilon_+ +\varepsilon_-$ gives rise to the Zariski decomposition of $K+D$. Therefore, D_1 is a component of $(K+D)^-$. Furthermore, we have

$$\begin{aligned}(K+D-aD_1, D_2)&=(K+D_2+D-D_2-aD_1, D_2)\\ &=(K+D_2, D_2)+(D-D_2, D_2)+(-aD_1, D_2) \leqq -a<0.\end{aligned}$$

Thus, replacing $K+D$ and D_1 by $K+D-aD_1$ and D_2, respectively, in the above argument, we see that D_2 is a component of $(K+D)^-$. Repeating the above argument, we see that each $D(i)_j$ is a component of $(K+D)^-$.

Step (4). Let $F_r(X_1,\ldots, X_r)$ be the polynomial in $X_1,\ldots, X_r$ defined by

$$F_r(X_1,\ldots, X_r)=\det\begin{vmatrix} X_1 & -1 & & & 0 \\ -1 & X_2 & & & \\ & & \ddots & & \\ 0 & & & X_{r-1} & -1 \\ & & & -1 & X_r \end{vmatrix},$$

where $\det(*)$ denotes the determinant of a matrix $(*)$. Note that $F_r(X_1,\ldots, X_r)=X_1F_{r-1}(X_2,\ldots, X_r)-F_{r-2}(X_3,\ldots, X_r)$.

Setting $a_{ij}=-(D(i)_j^2)$, we have a matrix

$$\begin{vmatrix} -a_{i1} & 1 & & & 0 \\ 1 & -a_{i2} & & & \\ & & \ddots & & \\ 0 & & & \ddots & 1 \\ & & & 1 & -a_{ir(i)} \end{vmatrix} \quad (i=1,\ldots, s),$$

which is the intersection matrix of $\sum_{j=1}^{r(i)} D(i)_j$. Since this matrix is negative-definite, it follows that $F_{r(i)}(a_{i1},\ldots, a_{ir(i)}) \neq 0$. Set

$$d_{ij}=\begin{cases} 1-\dfrac{F_{r(i)-j}(a_{ij+1},\ldots, a_{ir(i)})}{F_{r(i)}(a_{i1},\ldots, a_{ir(i)})}, & \text{if } D(i)_{r(i)} \text{ is not an edge component of } D, \\[2ex] 1-\dfrac{F_{j-1}(a_{i1},\ldots, a_{ij-1})+F_{r(i)-j}(a_{ij+1},\ldots, a_{ir(i)})}{F_{r(i)}(a_{i1},\ldots, a_{ir(i)})}, & \text{otherwise.}\end{cases}$$

Here, we set $F_0=F_{-1}=1$

We claim that

$$(K+D)^{+}=(K+D'+\sum_{p,q} d_{pq}D(p)_q)^{+},$$

where D' denotes $D-\sum_{p,q} D(p)_q$. First, we shall show that

$$(K+D'+\sum_{p,q} d_{pq}D(p)_q,\ D(i)_j)=0,$$

for all i, j. If $D(i)_{r(i)}$ is not an edge component of D, then

$$\begin{aligned}&(K+D'+\sum_{p,q} d_{pq}D(p)_q,\ D(i)_j)\\ &=(K,\ D(i)_j)+d_{ij-1}+d_{ij}(D(i)_j^2)+d_{ij+1}\\ &=a_{ij}-2+1-\frac{F_{r(i)-j+1}(a_{ij},\dots,\ a_{ir(i)})}{F_{r(i)}(a_{i1},\dots,\ a_{ir(i)})}\\ &\quad -a_{ij}\left(1-\frac{F_{r(i)-j}(a_{ij+1},\dots,\ a_{ir(i)})}{F_{r(i)}(a_{i1},\dots,a_{ir(i)})}\right)+1-\frac{F_{r(i)-j-1}(a_{ij+1},\dots,\ a_{ir(i)})}{F_{r(i)}(a_{i1},\dots,\ a_{ir(i)})}\\ &=0.\end{aligned}$$

Here, we set $d_{i0}=0$ and $d_{ir(i)+1}=1$. If $D(i)_{r(i)}$ is an edge component of D, then

$$\begin{aligned}&(K+D'+\sum_{p,q} d_{pq}D(p)_q,\ D(i)_j)=(K+\sum_{p,q} d_{pq}D(p)_q,\ D(i)_j)\\ &=a_{ij}-2+1-\frac{F_{j-2}(a_{i1},\dots,\ a_{ij-2})+F_{r(i)-j+1}(a_{ij},\dots,\ a_{ir(i)})}{F_{r(i)}(a_{i1},\dots,\ a_{ir(i)})}\\ &\quad -a_{ij}\left(1-\frac{F_{j-1}(a_{i1},\dots,\ a_{ij-2})+F_{r(i)-j}(a_{ij+1},\dots,\ a_{ir(i)})}{F_{r(i)}(a_{i1},\dots,\ a_{ir(i)})}\right)\\ &\quad +1-\frac{F_j(a_{i1},\dots,\ a_{ij})+F_{r(i)-j-1}(a_{ij-2},\dots,\ a_{ir(i)})}{F_{r(i)}(a_{i1},\dots,\ a_{ir(i)})}\\ &=0.\end{aligned}$$

Here, we set $d_{i0}=d_{ir(i)+1}=0$. Secondly, we shall show that

$$\kappa(K+D'+\sum_{p,q} d_{pq}D(p)_q,\ \bar{X})\geqq 0.$$

By hypothesis, there exist a positive integer n and an effective divisor Γ such that $\Gamma\sim n(K+D)$. Write $\Gamma=\Gamma_0+\sum_{p,q}\alpha_{pq}D(p)_q$, where α_{pq}'s are non-negative integers and Γ_0 is an effective divisor which contains none of $D(p)_q$. Then, it suffices to show that $\alpha_{pq}/n\geq 1-d_{pq}$, for every p and q. Let $\beta_{pq}=(\alpha_{pq}/n)-(1-d_{pq})$. We define a $\boldsymbol{Q}$-divisor C to be $\sum_{p,q}\beta_{pq}D(p)_q$. We shall show that C is effective. Note that

$$(K+D,\ D(i)_j)=1/n(\Gamma_0+\sum_{p,q}\alpha_{pq}D(p)_q,\ D(i)_j)\geqq\left(\sum_{p,q}\frac{\alpha_{pq}}{n}D(p)_q,\ D(i)_j\right)$$

and

$$(K+D-\sum_{p,q}(1-d_{pq})D(p)_q,\ D(i)_j)=(K+D'+\sum_{p,q} d_{pq}D(p)_q,\ D(i)_j)=0$$

for every i and j. Thus, we obtain

$$(C,\ D(i)_j)=\left(\sum_{p,q}\frac{\alpha_{pq}}{n}D(p)_q,\ D(i)_j\right)-(\sum_{p,q}(1-d_{pq})D(p)_q,\ D(i)_j)$$

$$\leqq (K+D,\ D(i)_j) - (K+D,\ D(i)_j) = 0$$

for every i and j. Setting $C_0 = \sum_{\beta_{pq} \geqq 0} \beta_{pq} D(p)_q$ and $C_1 = -\sum_{\beta_{pq} < 0} \beta_{pq} D(p)_q$, we have $(C_0 - C_1,\ C_1) = \sum_{-\beta_{pq} < 0} \beta_{pq}(C_0 - C_1,\ D(p)_q) \leqq 0$. This implies that $0 \leqq (C_0,\ C_1) \leqq (C_1^2)$. On the other hand, since the intersection matrix of C is negative-definite (cf. Step (3)), we have $C_1 = 0$. Therefore,

$$\frac{\alpha_{pq}}{n} - (1 - d_{pq}) = \beta_{pq} \geqq 0$$

for every p and q. This implies that

$$\kappa(K + D' + \textstyle\sum_{p,q} d_{pq} D(p)_q,\ \bar{X}) \geqq 0$$

as required.

Now let $\Delta_+ = (K + D' + \sum_{p,q} d_{pq} D(p)_q)^+$ and $\Delta_- = (K + D' + \sum_{p,q} d_{pq} D(p)_q)^- + \sum_{p,q} (1 - d_{pq}) D(p)_q$. We can verify, by the same argument as in the previous case (cf. Step (3)), that $\Delta_+ + \Delta_-$ is the Zariski decomposition of $K + D$. Hence we obtain that

$$(K+D)^+ = (K + D' + \textstyle\sum_{p,q} d_{pq} D(p)_q)^+.$$

Step (5). Let D_0 be an irreducible component of D such that $(K + D' + \sum_{p,q} d_{pq} D(p)_q,\ D_0) < 0$. Then $D_0 \subsetneqq \sum_{i,j} D(i)_j$, because $(K + D' + \sum_{p,q} d_{pq} D(p)_q,\ D(i)_j) = 0$. Hence D_0 is a rational curve, i.e., $p_a(D_0) = 0$. Now, we claim that

$$(\textstyle\sum_{i,j} D(i)_j,\ D_0) \geqq 1.$$

Indeed, supposing that $(\sum_{i,j} D(i)_j,\ D_0) \leqq 0$, we shall derive a contradiction. Since $D_0 \not\subseteq \sum_{i,j} D(i)_j$, we then $(\sum_{i,j} D(i)_j,\ D_0) = 0$. Thus we have

$$(K+D,\ D_0) = (K + D' + \textstyle\sum_{i,j} D(i)_j,\ D_0) = (K + D' + \sum_{i,j} d_{ij} D(i)_j,\ D_0) < 0.$$

Since we have, by the adjunction formula,

$$0 > (K+D,\ D_0) = (K + D_0,\ D_0) + (D - D_0,\ D_0) \geqq -2,$$

it follows that $(D - D_0,\ D_0) \leqq 1$, which implies that D_0 is a rational edge component. This contradicts the fact that $D_0 \not\subseteq \sum_{i,j} D(i)_j$.

Let $C_1, \dots, C_l, D(1)_{r(1)}, \dots, D(t)_{r(t)}$ be all components of D which meet D_0, where C_i's denote the components which are not contained in $\sum_{p,q} D(p)_q$. If $l \geqq 2$, we have

$$(K + D' + \textstyle\sum_{p,q} d_{pq} D(p)_q,\ D_0) \geqq (K + C_1 + C_2 + D_0,\ D_0) \geqq 0,$$

which contradicts the assumption. If $l = 1$, then $t \geqq 2$ by the definition of a maximal rational linear chain. It is easily checked by induction on $r(i)$ that

$$1 - \frac{1}{F_{r(i)}(a_{i1}, \dots, a_{ir(i)})} \geqq \frac{1}{2},$$

because $a_{ij} \geqq 2$ for $1 \leqq j \leqq r(i)$. Then

$$(K+D'+\textstyle\sum_{p,q} d_{pq}D(p)_q,\ D_0)\geqq(K+D_0+C_1+\sum_{i=1}^{t} d_{ir(i)}D(i)_{r(i)},\ D_0)\geqq 0,$$

which is a contradiction. Thus, this case can not occur. If $l=0$, then $t\geqq 3$ and

$$(K+D'+\textstyle\sum_{p,q} d_{pq}D(p)_q,\ D_0)<0$$

if and only if

$$\textstyle\sum_{i=1}^{t} d_{ir(i)}=\sum_{i=1}^{t}\left(1-\dfrac{1}{F_{r(i)}(a_{i1},\dots,a_{ir(i)})}\right)<2.$$

Therefore, we conclude that $t=3$ and

$$\{F_{r(1)}(a_{11},\dots,a_{1r(1)}),\ F_{r(2)}(a_{21},\dots,a_{2r(2)}),\ F_{r(3)}(a_{31},\dots,a_{3r(3)})\}=\{2, 2, n\},$$
$$\{2, 3, 3\},\ \{2, 3, 4\},\ \{2, 3, 5\},\ \text{up to a suitable permutation,}$$

where n is an integer$\geqq 2$. Letting $a_1,\dots,a_r$ be integers$\geqq 2$, we obtain

$$F_r(a_1,\dots,a_r)=2 \Longleftrightarrow r=1,\ a_1=2,$$

$$F_r(a_1,\dots,a_r)=3 \Longleftrightarrow r=1,\ a_1=3$$
$$\text{or } r=2,\ a_1=a_2=2,$$

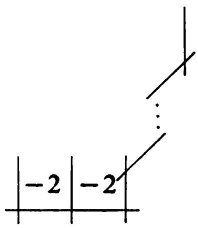
$$F_r(a_1,\dots,a_r)=4 \Longleftrightarrow r=1,\ a_1=4,$$
$$\text{or } r=3,\ a_1=a_2=a_3=2,$$

$$F_r(a_1,\dots,a_r)=5 \Longleftrightarrow r=1,\ a_1=5,$$
$$r=2,\ a_1=3,\ a_2=2,$$
$$r=2,\ a_1=2,\ a_2=3,$$
$$\text{or } r=4,\ a_1=a_2=a_3=a_4=2,$$

$$F_r(a_1,\dots,a_r)=6 \Longleftrightarrow r=1,\ a_1=6$$
$$\text{or } r=5,\ a_1=a_2=a_3=a_4=a_5=2$$

(cf. Proposition 2.2). Therefore the configuration of the connected component of D containing D_0 is one of the following:

Type D



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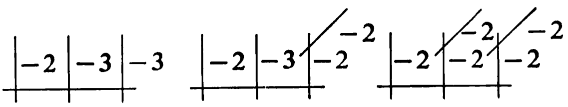


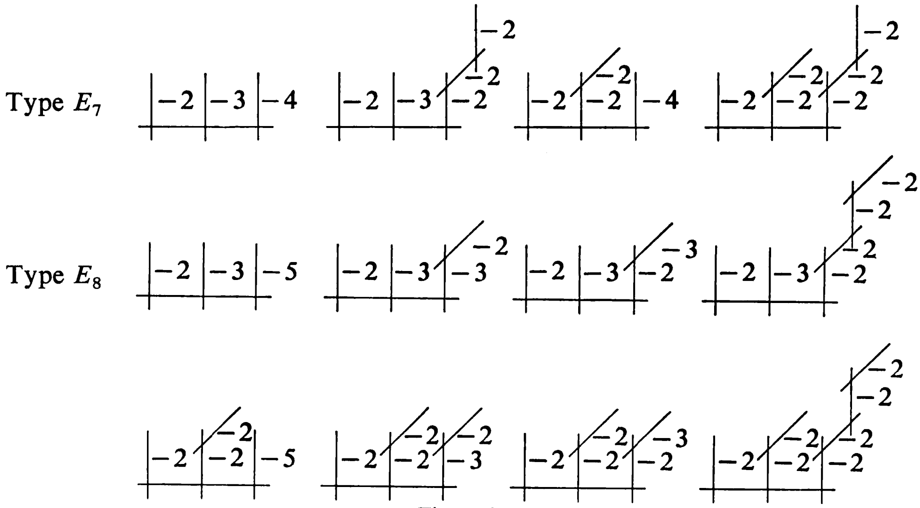


Figure 1.

Here, each line represents a nonsingular rational curve, an integer attached to each line stands for the self-intersection number of the curve corresponding to the line and each horizontal line represents D_0.

We shall prove that D_0 is not an exceptional curve of the first kind. Suppose, on the contrary, that D_0 is an exceptional curve of the first kind. Then, by examining separately each configuration shown above, we can check that the intersection matrix of the connected component B of D containing D_0 is not negative-definite. On the other hand, we see that D_0 is a component of $(K+D)^-$, because $(K+D'+\sum_{p,q} d_{pq}D(p)_q, D_0)<0$. Since the other irreducible components of B are contained in $(K+D)^-$ by construction, B should have the negative-definite intersection matrix. This is a contradiction. Hence D_0 is not an exceptional curve of the first kind.

Let $B(i)\,(i=1,\dots, t)$ be a connected component of D of which configuration is one of Types D, E_6, E_7, E_8 in the above table and let $B(i)=\sum_{j=1}^{s(i)} B(i)_j$ be the decomposition of $B(i)$ into irreducible components. Since the intersection matrix of $\sum_i B(i)$ is negative-definite, we have the uniquely determined positive rational numbers b_{pq} such that

$$(K+\textstyle\sum_{p,q} b_{pq}B(p)_q,\ B(i)_j)=0$$

for every i, j. It is easily checked that each b_{pq} is smaller than one. Writing $D''=D-\sum_p D(p)-\sum_i B(i)$ we have

$$(K+D''+\textstyle\sum'_{p,q} d_{pq}D(p)_q+\sum_{n,m} b_{nm}B(n)_m,\ \Gamma)\geqq 0$$

for every irreducible component Γ of D, where $\sum'_{p,q} d_{pq}D(p)_q$ denotes the sum of the $D(p)_q$'s such that $D(p)_q\subsetneqq B(i)$. It can be shown, by the same argument as above, that the divisor $D^*:=D''+\sum'_{p,q} d_{pq}D(p)_q+\sum_{n,m} b_{nm}B(n)_m$ satisfies $\kappa(K+D^*, X)\geqq 0$ and that $(K+D)^+=(K+D^*)^+$.

Step (6). If $K+D^*$ is semipositive, then the triple $(X, \bar{X}, D)$ is almost minimal by definition. Hence we may assume that $(K+D^*, \Gamma)\leqq 0$ for some curve $\Gamma\not\subset D$.

Then $(\Gamma^2)<0$ because $\kappa(K+D^*, \bar{X})\geqq 0$, and $(K, \Gamma)<0$ because $\Gamma\not\subseteq D$. This implies that Γ is an exceptional curve of the first kind, whence $(\Gamma^2)=-1$. Let $\mu\colon \bar{X}\to\bar{Y}$ be the contraction of Γ and let $\Delta=\mu_*(D)$.

We shall show that Δ has only simple normal crossings. Let $C_1,\dots, C_l$ be all irreducible components of D which meet Γ. Let c_i denote the coefficient of C_i in D^*. Then $0\leqq c_i<1$ if C_i is contained in $\sum'_{p,q} D(p)_q+\sum_{n,m} B(n)_m$ and $c_i=1$, otherwise. Note that

$$0>(K+D^*, \Gamma)=-1+(D^*, \Gamma)=-1+\textstyle\sum_i c_i(C_i, \Gamma).$$

This implies that all $C_1,\dots, C_l$ are contained in $\sum'_{p,q} D(p)_q+\sum_{n,m} B(n)_m$. We claim that $c_i\geqq c'_i:=1+2/(C_i^2)$. Indeed, since $(K+c'_iC_i, C_i)=0$, we have

$$0=(K+D^*, C_i)-(K+c'_iC_i, C_i)=(D^*-c_iC_i, C_i)+(c_i-c'_i)(C_i^2).$$

Since $(D^*-c_iC_i, C_i)\geqq 0$ and $(C_i^2)<0$, it follows that $c_i\geqq c'_i$. Hence we have

$$(*)\qquad 1>\textstyle\sum_i c_i(C_i, \Gamma)\geqq\sum_i c'_i(C_i, \Gamma).$$

Without losing generality, we may assume $(C_1^2)\geqq\cdots\geqq(C_l^2)$. First, assume that $(C_l^2)\leqq-6$. Then $(C_1^2)=\cdots=(C_{l-1}^2)=-2$ and $(C_l, \Gamma)=1$ by $(*)$. On the other hand, the intersection matrix of $C_1+\cdots+C_l+\Gamma$ is negative-definite, because $(K+D^*, \Gamma)<0$ implies that $\Gamma\subseteq\operatorname{supp}(K+D)^-$. From this, we infer readily that $l\leqq 2$ and $(C_1, \Gamma)=1$ and $(C_1, C_2)\leqq 1$. If $C_1\cap C_2\cap\Gamma=\emptyset$ then Δ has simple normal crossings. So, suppose that $C_1\cap C_2\cap\Gamma\neq\emptyset$. We put

$$d_1:=\frac{2+a}{1+2a},\ d_2:=2d_1,$$

where $a:=(C_2^2)$. Then, we have

$$(K+d_1C_1+d_2C_2, C_i)=0\quad(i=1, 2),$$

where we note that $(C_1, C_2)=1$ and $(C_1^2)=-2$. Since

$$(K+c_1C_1+c_2C_2, C_i)\leqq(K+D^*, C_i)=0,$$

we have

$$(**)\qquad (c_1C_1+c_2C_2-d_1C_1-d_2C_2, C_i)$$
$$=(K+c_1C_1+c_2C_2, C_i)-(K+d_1C_1+d_2C_2, C_i)\leqq 0.$$

We set $c_1C_i+c_2C_2-d_1C_1-d_2C_2=A-B$, where A, B are effective $\boldsymbol{Q}$-divisors with no common components. Then, by $(**)$, we have $(A-B, B)\leqq 0$. This implies that $B=0$ because the intersection matrix of B is negative-definite and $(A, B)\geqq 0$. Therefore, we have $c_i\geqq d_i$. On the other hand, by a direct computation, we have $d_1+d_2\geqq 1$, which is a contradiction. Hence, $C_1\cap C_2\cap\Gamma=\emptyset$ and Δ has simple normal crossings if $(C_l^2)\leqq-6$.

The case in which $(C_l^2)=-2,\ -3,\ -4$ or -5 is treated in a similar fashion.

We write $K+D=\mu^*(K(\bar{Y})+\Delta)+a'\Gamma$ for some integer a'. Setting $b=|a'|$, we have

$$0 \leqq \kappa(K+D, \bar{X}) \leqq \kappa(\mu^*(K(\bar{Y})+\Delta)+(b+a')\Gamma, \bar{X}) = \kappa(K(\bar{Y})+\Delta, Y).$$

We shall prove

$$(K+D)^+ = \mu^*((K(\bar{Y})+\Delta)^+)$$

by examining separately each of the following cases.
Case 1. $a' \geqq 0$: We obtain $(K+D)^+ = \mu^*((K(\bar{Y})+\Delta)^+)$ by the same argument as in Step (2).
Case 2. $a' < 0$: It is clear that $(K+D)^+ + ((K+D)^- - a'\Gamma)$ gives rise to the Zariski decomposition of $\mu^*(K(\bar{Y})+\Delta)$, because Γ is a component of $(K+D)^-$. If $(Y-\Delta, \bar{Y}, \Delta)$ is not almost minimal, we repeat the above argument all again for $(Y-\Delta, \bar{Y}, \Delta)$ and finally we obtain an almost minimal triple $(Z, \bar{Z}, B)$ having the required properties. This completes the proof of Theorem 1.3.

Proposition 1.4. *Let $(X, \bar{X}, D)$ be a nonsingular triple with $\bar{\kappa}(X) \geqq 0$. Let $(Z, \bar{Z}, B)$ and $f: \bar{X} \to \bar{Z}$ be as in Theorem* 1.3. *If $(Y, \bar{Y}, C)$ and $g: \bar{X} \to \bar{Y}$ are an arbitrary almost minimal triple and a birational morphism, respectively, satisfying the conditions* (1), (2) *of Theorem* 1.3, *then $g \cdot f^{-1}$ becomes a morphism.*

Proof. Let E be an exceptional curve of the first kind on $\bar{X}$ such that $f(E)$ is a point on $\bar{Z}$. We claim that E is contained in the ramification divisor R_g of g. We have

$$K+D+g^*C-D+R_g = g^*(K(\bar{Y})+C)+2R_g.$$

Since $(K+D)^+ = g^*((K(\bar{Y})+C)^+)$, it follows that

$$(K+D)^- + g^*C - D + R_g = g^*((K(\bar{Y})+C)^-)+2R_g.$$

Note that $g^*((K(\bar{Y})+C)^-)+2R_g$ has the negative-definite intersection matrix. Since $g^*C-D \geq 0$ (cf. the condition (1) of Theorem 1.3) and $E \subseteq \operatorname{supp}(K+D)^-$ by the condition (3) of Theorem 1.3, the intersection matrix of $E+R_g$ is negetive-definite. This implies that $E \subseteq R_g$ or $E \cap R_g = \emptyset$. Assume $E \cap R_g = \emptyset$. Then $E_0 := g(E)$ is an exceptional curve of the first kind on $\bar{Y}$. On the other hand, since

$$E \subseteq \operatorname{supp}(K+D)^- \subseteq \operatorname{supp}(g^*((K(\bar{Y})+C)^-)+2R_g),$$

we have $E_0 \subseteq \operatorname{supp}((K(\bar{Y})+C)^-)$. This contradicts the almost-minimality of $(Y, \bar{Y}, C)$. Therefore, $E \subseteq R_g$. Since g is birational, $g(E)$ is also a point. This implies that $g \cdot f^{-1}$ is a morphism. Q.E.D.

Let $(X, \bar{X}, D)$ be a nonsingular triple with $\bar{\kappa}(X) \geqq 0$. An almost minimal triple $(Z, \bar{Z}, B)$ satisfying the condition (1), (2), (3) of Theorem 1.3 is called *an almost minimal model of* $(X, \bar{X}, D)$.

We recall the definition of a "*relatively minimal model*" due to Kawamata [5]. Let $(X, \bar{X}, D)$ be a nonsingular triple with $\bar{\kappa}(X) \geqq 0$. A pair $(\bar{Y}, C)$ is said to be *a relatively minimal model of* $(X, \bar{X}, D)$ if there exists a birational morphism

$f:\bar{X}\to\bar{Y}$ such that

(1) $\bar{Y}$ is a nonsingular complete surface and C is an effective $\boldsymbol{Q}$-divisor with coefficients not greater than one,

(2) $(K+D)^+=f^*((K(\bar{Y})+C)^+)=f^*(K(\bar{Y})+C)$.

Now, we prove the following:

Proposition 1.5. *Let $(X, \bar{X}, D)$ be an almost minimal triple with $\bar{\kappa}(X)\geqq 0$. Then $D-(K+D)^-$ is effective and $(\bar{X}, D-(K+D)^-)$ is a relatively minimal model of $(X, \bar{X}, D)$.*

Proof. By the construction of $(K+D)^-$ in the Step (4) of the proof of Theorem 1.3, it is clear that $D-(K+D)^-$ is effective. Then, since $K+D-(K+D)^-=(K+D)^+$, this implies that $(\bar{X}, D-(K+D)^-)$ is a relatively minimal model of $(X, \bar{X}, D)$.

Proposition 1.6. *Let the notations and the assumptions be the same as in Theorem 1.3. Then we have $\bar{P}_n(X)=\bar{P}_n(Z)$ for each positive n.*

Proof. $\bar{P}_n(X)=\dim H^0(\bar{X}, n(K+D))=\dim H^0(\bar{X}, [n(K+D)^+])$

$=\dim H^0(\bar{X}, [f^*(n(K(\bar{Z})+B)^+)])$. On the other hand, $\bar{P}_n(Z)$

$=\dim H^0(\bar{Z}, n(K(\bar{Z})+B))=\dim H^0(\bar{Z}, [n(K(\bar{Z})+B)^+])$

$=\dim H^0(\bar{X}, f^*([n(K(\bar{Z})+B)^+]))$. Set $B_m:=B-(K(\bar{Z})+B)^-$.

Then there is an effective divisor F on $\bar{X}$ such that $[f^*nB_m]=f^*[nB_m]+F$ and codim $f(F)\geqq 2$. Noting that $K(\bar{Z})+B_m=(K(\bar{Z})+B)^+$, we have

$$\begin{aligned}\bar{P}_n(X)&=\dim H^0(\bar{X}, [f^*(n(K(\bar{Z})+B)^+)])\\&=\dim H^0(\bar{X}, [f^*(n(K(\bar{Z})+B_m))])\\&=\dim H^0(\bar{X}, f^*nK(\bar{Z})+f^*[nB_m]+F)\\&=\dim H^0(\bar{X}, f^*(nK(\bar{Z})+[nB_m])+F)\\&=\dim H^0(\bar{Z}, n(K(\bar{Z})+B))\\&=\bar{P}_n(Z).\end{aligned}$$

Remark.

(1) Let $(X, \bar{X}, D)$ be an almost minimal triple. Then the configuration of a connected component of $(K+D)^-$ is a linear chain, or has one of Type D, Type E_6, Type E_7, Type E_8 in the Figure 1.

(2) Let C be a connected component of $(K+D)^-$. If C is not a rational linear chain, then C is a connected component of D.

§2. Triples $(X, \bar{X}, D)$ with $\bar{\kappa}(X)=0$

Theorem 2.1. *Let* $(X, \bar{X}, D)$ *be a nonsingular triple with* $\bar{\kappa}(X)=0$. *Then* $\bar{P}_i(X)=1$ *for some integer* i, $1 \leqq i \leqq 66$.

Proof. By Proposition 1.6, we may assume that $(X, \bar{X}, D)$ is almost minimal. Set $D_m := D-(K+D)^-$. By Theorem (2.2) of Kawamata [5], there exists some positive integer r such that $r(K+D_m)$ is integral and trivial. To find the smallest integer among such integers r, we shall construct a ramified cyclic cover of $\bar{X}$ by the following argument. Choose an affine open covering $U=\{U_i\}$ of $\bar{X}$ such that $\mathcal{O}(K+D)$, identified with the associated line bundle, is defined by suitable transition functions $\{\phi_{ij}\}$ with respect to U. Take a member F of $|r(K+D)|$. Then $F\sim r(K+D)\sim r(K+D)^-$; hence $F=r(K+D)^-$. We take a set of regular functions $\{s_i\}$ on $\{U_i\}$ which represents the section of $\mathcal{O}(r(K+D))$ defining F; thus $s_i=\phi_{ij}^r s_j$ on $U_i \cap U_j$. Setting $V_i=\{(x, t)\in U_i\times \boldsymbol{C} \mid t^r=s_i(x)\}$, $\{V_i\}$ can be patched together to form an algebraic subset S of the total space of the line bundle associated with $K+D$. Choose an irreducible component $\bar{X}'$ of S and denote by $\pi'\colon \bar{X}'\to\bar{X}$ the morphism induced by the canonical projection $\mathcal{O}(K+D)\to\bar{X}$. Since the cyclic group of order r acts naturally on S, a cyclic subgroup G acts on $\bar{X}'$ in such a way that the quotient $\bar{X}'/G$ is birationally equivalent to $\bar{X}$. The morphism π' is étale outside $\pi'^{-1}(F)$. Thus, we have a nonsingular complete surface $\bar{\mathscr{X}}$ and a birational morphism $\mu\colon \bar{\mathscr{X}}\to\bar{X}'$ such that μ is isomorphic outside $\pi'^{-1}(F)$ and $\mathscr{D}:=\mu^{-1}(\pi'^{-1}(F))$ has only simple normal crossings. Moreover, we may assume that the action of G on $\bar{\mathscr{X}}$ is regular. Setting $\pi=\pi'\cdot\mu$, we have

$$K(\bar{\mathscr{X}})+\mathscr{D}=\pi^*(K(\bar{X})+D)+\bar{R}_\pi$$

and supp $\bar{R}_\pi\subseteq\pi^{-1}(F)$; hence $\bar{\kappa}(\mathscr{X})=0$, where $\mathscr{X}=\bar{\mathscr{X}}-\mathscr{D}$. By construction, $\bar{P}_g(\mathscr{X})=1$. Such a triple $(\mathscr{X}, \bar{\mathscr{X}}, \mathscr{D})$ has been studied by Iitaka [4] and can be classified in the following three cases.

Let $\bar{S}$ be a relatively minimal model of $\mathscr{X}$, let $\rho\colon\bar{\mathscr{X}}\to\bar{S}$ be the associated birational morphism and let $C=\rho_*(\mathscr{D})$.

Case 1. $\kappa(\bar{\mathscr{X}})=0$. $\bar{S}$ is a $K3$ surface or an abelian surface. Then either C is a zero divisor or C consists of nonsingular rational curves.

Case 2. $\bar{S}$ is a ruled surface of genus 1. Then C consists of two disjoint regular sections.

Case 3. $\bar{S}$ is a rational surface. Then C is one of the following;

(1) an elliptic curve,
(2) a disjoint union of an elliptic curve and a nonsingular rational curve,
(3) a reduced divisor consisting of nonsingular rational curves.

Let σ be a generator of G. Then σ gives rise to an automorphism σ^* of the vector space $H^0(\bar{\mathscr{X}}, K(\bar{\mathscr{X}})+\mathscr{D})$ of dimension 1. For a nonzero element $\omega\in H^0(\bar{\mathscr{X}}, K(\bar{\mathscr{X}})+\mathscr{D})$, we have $\sigma^*\omega=\alpha\omega$. Here, α is a primitive n-th root of unity for some integer $n>0$, because σ^* has finite order. We shall show that $\bar{P}_n(X)=1$. Take a nonzero element $\omega_0\in H^0(\bar{\mathscr{X}}, n(K(\bar{\mathscr{X}})+\mathscr{D}))$. Then ω_0 is σ-invariant. Regaining the previous situation, denote supp F by N. Since N is the union of the zero loci of s_j's on U_j's, $\pi^{-1}(N)$ is σ-invariant. Hence σ acts on $\mathscr{X}-\pi^{-1}(N)$, and $\mathscr{X}-\pi^{-1}(N)\to(\mathscr{X}-\pi^{-1}(N))/G\cong X-N$ is an étale covering. If one regards ω_0 as an element of $H^0(\bar{\mathscr{X}}-\pi^{-1}(N), n(K(\bar{\mathscr{X}})+\mathscr{D}))$, then ω_0 is σ-invariant and so it is derived from an element $\omega_1\in H^0(\bar{X}-N, n(K(\bar{X})+D))$. Hence we have $H^0(\bar{X}, n(K(\bar{X})+D)+aN)\neq 0$ for some integer $a\gg 0$. Noting that $n(K(\bar{X})+D)^+ +(n(K(\bar{X})+D)^- +aN)$ is the Zariski decomposition of $n(K+D)+aN$, we have

$$\bar{P}_n(X)=\dim H^0(\bar{X}, n(K+D))=\dim H^0(\bar{X}, [n(K+D)^+)])$$
$$=\dim H^0(\bar{X}, n(K+D)+aN)\neq 0$$

(cf. Proposition 1.2).
Therefore, for the proof of Theorem 2.1, it suffices to show that n is not larger than 66. We consider three cases separately.
Case 1. $\bar{S}$ is a $K3$ surface or an abelian surface. In this case, since $\bar{S}$ is absolutely minimal, σ induces an automorphism of $\bar{S}$, denoted by the same letter σ, and we have isomorphisms of one-dimensional vector spaces compatible with the canonical actions of σ, $H^0(\bar{\mathscr{X}}, K(\bar{\mathscr{X}})+\mathscr{D})\cong H^0(\bar{S}, K(\bar{S})+C)\cong H^0(\bar{S}, K(\bar{S}))$. By the Hodge theory, α is an eigenvalue of the automorphism σ^* of $H^2(\bar{S}, \boldsymbol{Q})/L$, induced by σ, where L is the subspace generated by divisors. The second Betti number $b_2(\bar{S})$ is 6 if $\bar{S}$ is an abelian surface and $b_2(\bar{S})$ is 22 if $\bar{S}$ is a $K3$ surface. Furthermore, dim $L\geqq 1$. Therefore, counting the dimension of a σ^*-stable subspace of $H^2(\bar{S}, Q)/L$, we know that $\phi(n)\leqq 21$, where $\phi(n)$ denotes the Euler function. By a straightforward computation, we have $n\leqq 66$.
Cases 2 and 3. $\bar{S}$ is a ruled surface of genus 1 or a rational surface. Let $\mathscr{D}=\sum_i\mathscr{D}_i$ be the decomposition into irreducible components. There exist at most two non-rational components, which are, in fact, elliptic curves; hence $\sum_i g(\mathscr{D}_i)\leqq 2$. By Deligne [1], we have the following commutative diagram;

$$\begin{array}{l} H^2(\bar{\mathscr{X}}, \boldsymbol{C})\simeq H^1(\bar{\mathscr{X}}, \Omega^1) \\ \quad\downarrow j^* \\ H^2(\mathscr{X}, \boldsymbol{C})\simeq H^1(\bar{\mathscr{X}}, \Omega^1(\log\mathscr{D}))\oplus H^0(\bar{\mathscr{X}}, K(\bar{\mathscr{X}})+\mathscr{D}), \end{array}$$

where j^* is the canonical homomorphism induced by the inclusion $j\colon \mathscr{X}\to\bar{\mathscr{X}}$. From an exact sequence (cf. Iitaka [15; the proof of Lemma 1]),

$$0\longrightarrow\Omega^1\longrightarrow\Omega^1(\log\mathscr{D})\longrightarrow\oplus_j\mathscr{O}_{\mathscr{D}_j}\longrightarrow 0,$$

we have a long exact sequence

$$\cdots\longrightarrow H^1(\bar{\mathscr{X}}, \Omega^1)\longrightarrow H^1(\bar{\mathscr{X}}, \Omega^1(\log\mathscr{D}))\longrightarrow\oplus_j H^1(\mathscr{D}_j, \mathscr{O}_{\mathscr{D}_j})\longrightarrow\cdots$$

Note that dim $\oplus_j H^1(\mathscr{D}_j, \mathscr{O}_{\mathscr{D}_j})=\sum_j g(\mathscr{D}_j)\leqq 2$. Thus,

$$\dim H^2(\mathscr{X}, \boldsymbol{C})/\mathrm{Im}\, j^*\leqq 3.$$

On the other hand, the homomorphism j^* is, in fact, defined over $\boldsymbol{Q}$. Let $L' = \operatorname{Im}(j^*\colon H^2(\bar{\mathscr{X}}, \boldsymbol{Q}) \to H^2(\mathscr{X}, \boldsymbol{Q}))$. Note that $\sigma^*(L') = L'$, $\dim_{\boldsymbol{Q}} H^2(\mathscr{X}, \boldsymbol{Q})/L' \leqq 3$ and $\omega \notin \operatorname{Im} j^*$, because $\omega \in H^0(\bar{\mathscr{X}}, K(\bar{\mathscr{X}}) + \mathscr{D})$. Hence we have $\phi(n) \leqq 3$, whence $n \leqq 6$.

Q. E. D.

Proposition 2.2. *Let $(X, \bar{X}, D)$ be an almost minimal triple with $\bar{\kappa}(X) = 0$. Assume that $\bar{X}$ is rational, D is connected and $\bar{P}_g(X) = 0$. Then $\bar{P}_2(X) = 1$, $\bar{P}_3(X) = 1$, $\bar{P}_4(X) = 1$ or $\bar{P}_6(X) = 1$. Furthermore, the configuration of D is one of the following*:

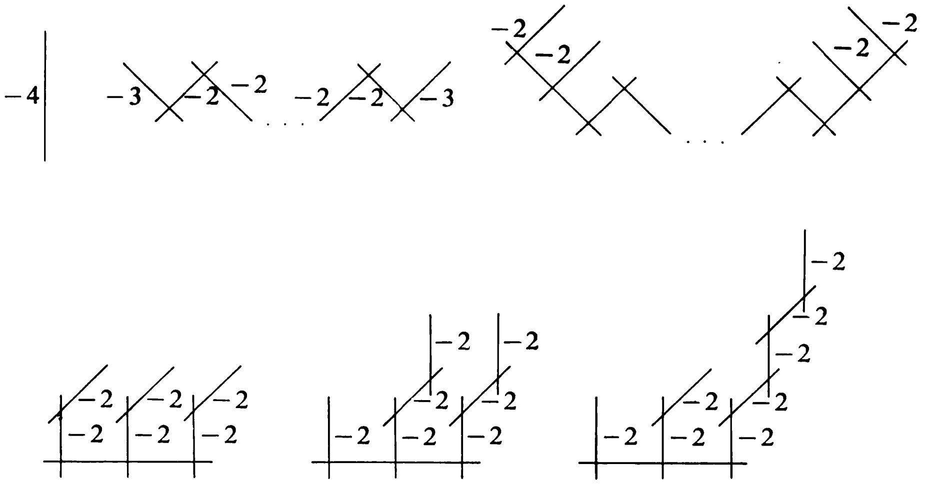


Figure 2.

Here each line represents a nonsingular rational curve and each number indicates the self-intersection number of the corresponding curve.

Proof. We shall prove that $\bar{P}_i(X) = 1$, where $i = 2$, 3, 4 or 6. We consider separately the following two cases.

Case 1: $[D_m] = 0$. Then $\operatorname{supp} D = \operatorname{supp}(K+D)^-$, because $D_m = D - (K+D)^-$ is an effective $\boldsymbol{Q}$-divisor with every coefficient < 1. Thus we have $(K + D_m, C) = 0$ for each irreducible component C of D. Note that $(K + D_m)^2 = 0$ because $K + D_m$ is semipositive and $\bar{\kappa}(X) = 0$. Hence we have $0 = (K + D_m)^2 = (K + D_m, K) + (K + D_m, D_m) = (K + D_m, K)$.

On the other hand, since the triple is almost minimal and $\operatorname{supp} D = \operatorname{supp}(K+D)^-$, D does not contain any exceptional curve of the first kind. Let C be an irreducible component of D. Then $(C^2) < 0$ because $C \subseteq \operatorname{supp}(K+D)^-$. If $(C, K) < 0$, then C is an exceptional curve of the first kind, which contradicts the assumption. Hence $(C, K) \geqq 0$. Thus

$$(K + D, K) \geqq (K + D_m, K) = 0.$$

Note that $H^2(\bar{X}, 2K + D) = H^0(\bar{X}, -K - D) = 0$ because $\bar{X}$ is rational and $\bar{p}_g(X) = 0$. By the Riemann-Roch Theorem,

$$h^0(\bar{X}, 2K+D) \geqq \frac{1}{2}(2K+D, K+D)+1=(K, K+D) \geqq 0,$$

because $\bar{p}_g(X)=0$ and the connectedness of D imply $(D, D+K)=-2$ (cf. Miyanishi [8; Lemma 2.1.3.]). Assume that $\bar{P}_2(X)=0$. Then we have $(K, K+D)=0$ and also $(K, C)=0$ for each irreducible component C of D. Then $(K^2)=0$. Since $(K+D_m)^2=0$ and $(K, D_m)=0$, we have $(D_m^2)=0$. This implies that $D_m=0$ because $\operatorname{supp} D_m \subseteq \operatorname{supp}(K+D)^-$. Since $nK \sim n(K+D_m) \sim 0$ for some integer n and since $\bar{X}$ is rational, we have $K \sim 0$, which is a contradiction. Hence $\bar{P}_2(X)=1$.

Case 2: $[D_m] \neq 0$. We set $D_0=[D_m]$ and $D'_m=D_m-D_0$. The Q-divisor $(K+D)^-$ is obtained by the method explained in the Step (4) of the proof of Theorem 1.3. In particular, D_0 is connected because D is connected and if $C_1, \ldots, C_l$ are all the irreducible components of D'_m which meet D_0 (if such components exist at all), then every C_i is a component of the form $D(j)_{r(j)}$ according to the previous notations. Hence the coefficient of C_i in D'_m is of the form $1-1/a_i$ with $a_i \geqq 2$. Since $r(K+D_m) \sim 0$, it follows that $(K+D_m, D_0)=0$ and so, we have

$$(K+D_m, D_0)=(K, D_0)+(D_0^2)+(D'_m, D_0).$$

However, $(K, D_0)+(D_0^2)=-2$ because D_0 is connected and $|K+D_0|=\emptyset$. Thus

$$-2+\sum_{i=1}\left(1-\frac{1}{a_i}\right)=0.$$

This implies that, if we assume $a_1 \leqq a_2 \leqq \cdots \leqq a_l$, we have

$$l=3 \quad \text{and} \quad a_1=a_2=a_3=3,$$

$$\text{or} \quad l=3 \quad \text{and} \quad a_1=2,\ a_2=a_3=4,$$

$$\text{or} \quad l=3 \quad \text{and} \quad a_1=2,\ a_2=3,\ a_3=6,$$

$$\text{or} \quad l=4 \quad \text{and} \quad a_1=a_2=a_3=a_4=2.$$

By recalling again the construction of $K+D_m=(K+D)^+$ in Theorem 1.3, we know that $a(K+D_m)$ is an integral divisor, where $a:=L.C.M.(a_1, \ldots, a_l)$. Since $\bar{X}$ is rational, $\bar{P}_a(X)=1$ for $a=2$, 3, 4, or 6.

Secondly, we shall determine the configuration of D. If $[D_m]=0$, then every irreducible component of D appears in D_m with positive coefficient (<1) and $2(K+D_m)$ is an integral divisor, which is, in fact, a trivial divisor. Hence we infer that $D_m=\frac{1}{2}D$. We shall show that D is a linear chain. Assume that the configuration of D has Type D, E_6, E_7 or E_8 (cf. Remarks in §1). By a simple computation (cf. Step (3) in Theorem 1.3), we know that the coefficient of an edge component C with $(C^2)=-2$ in D_m is less than $\frac{1}{2}$. This is a contradiction. Hence D is a linear chain. Let $D=\sum_{i=1}^r D_i$ be the decomposition into irreducible components, where D_1 is an edge component and $(D_i, D_{i-1})=1$ for $i=1, \ldots, r-1$. Set $a_i=-(D_i^2)$. Then, by Step (3) of Theorem 1.3, we have $D_m=\frac{1}{2}D$ if and only if

$$1-\frac{F_{j-1}(a_1,\dots,a_{j-1})+F_{r-j}(a_{j+1},\dots,a_r)}{F_r(a_1,\dots,a_r)}=\frac{1}{2}$$

for all j. The solutions of these equations are as follows:

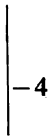
(1) $r=1$ and $a_1=4$,

(2) $a_1=a_r=3,\ a_2=\cdots=a_{r-1}=2$.

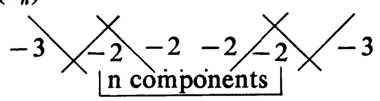
If $[D_m]\neq 0$, a connected component of $(K+D)^-$ is a linear chain (cf. Remarks in §1). Set $D_0:=[D_m]$. Since $(K+D_0, D_0)=-2$, an edge component of D_0 meets at least two irreducible component of $(K+D)^-$ (cf. Step (1) of Theorem 1.3). If D_0 has only one edge components, then D_0 is irreducible. If D_0 has just two edge components, then D_0 is a linear chain. From these facts and Step (3) of Theorem 1.3, we know that the configuration of D is one of the following, where the first two configurations appear in the case $[D_m]=0$:

(I)

(1)

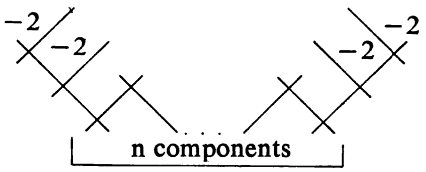


(2_n)



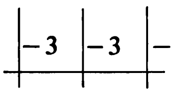
(II)

(1_n)

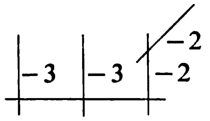


(III)

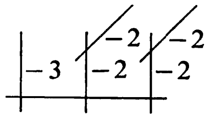
(1)



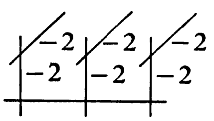
(2)



(3)



(4)



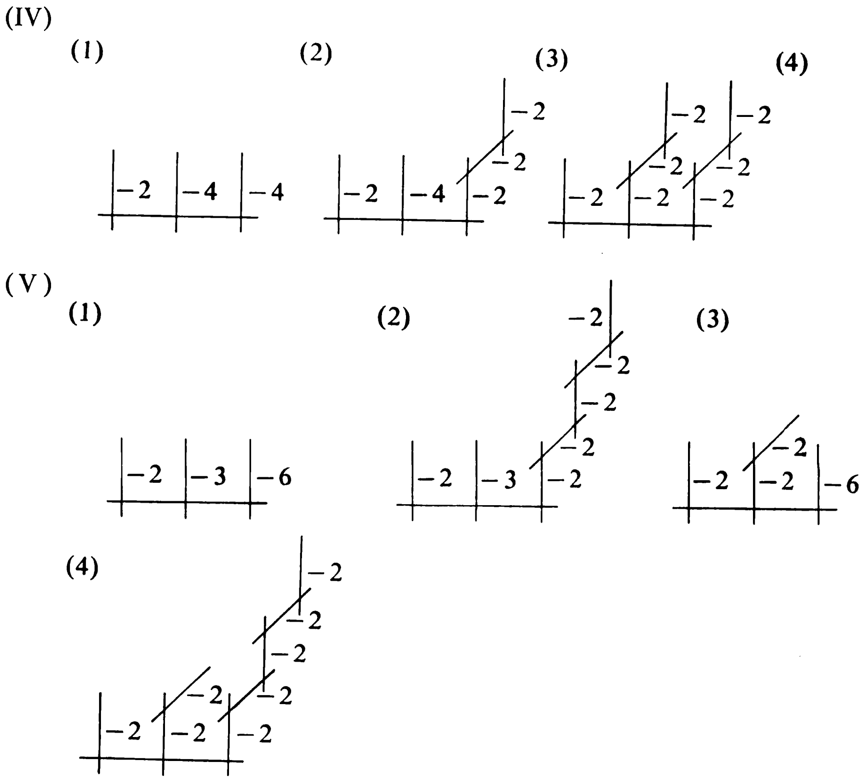


Figure 3.

We shall prove that the cases III–(1), (2) can not occur. We can show in a similar fashion that the other cases except those listed in the statement of the proposition do not occur. We assume that D has such a configuration.

Case III–(1)

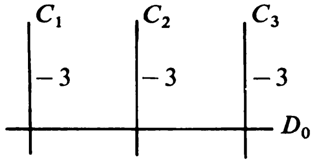


Figure 4.

Let $D-D_0=C_1+C_2+C_3$ be the decomposition into irreducible components. Then

$$D_m=D_0+\frac{2}{3}(C_1+C_2+C_3).$$

Noting that $(K+D_m, K)=0$, we have $(K^2)+(D_0, K)+2=0$. If $\bar{X}$ has no exceptional curve of the first kind then $\bar{X}$ is either $\boldsymbol{P}^2$ or a Hirzebruch surface Σ_n ($n=0, 2, 3,\ldots$). Such a divisor D does not exist on $\boldsymbol{P}^2$ or Σ_n. Hence $\bar{X}$ has an exceptional curve E of the first kind. Then we have

$$0=(K+D_m, E)=-1+(D_0, E)+\frac{2}{3}((C_1, E)+(C_2, E)+(C_3, E)).$$

Therefore either (a) $(D_0, E)=1$ and $(C_i, E)=0$ for $i=1, 2, 3$ or (b) $D_0=E$. By contracting E, the case (a) can be reduced to the case (b). Suppose $D_0=E$. Let μ: $\bar{X}\to\bar{Y}$ be the contraction of E. Then $D'=\mu_*(D)$ has the following configuration:

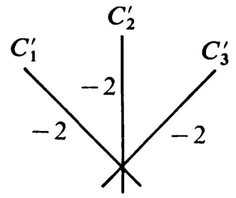


Figure 5.

In this case, we have $K(\bar{Y})^2=0$ because $\mu^*\left(K(\bar{Y})+\frac{2}{3}D'\right)=K(\bar{X})+D_m$ and $(K(\bar{X})+D_m)^2=0$. Thus there exists an exceptional curve of the first kind E' on $\bar{Y}$. Letting $C'_i=\mu(C_i)$, one has $\left(K(\bar{Y})+\sum_i \frac{2}{3}C'_i, E'\right)=0$. But this is a contradiction.
Case III–(2).

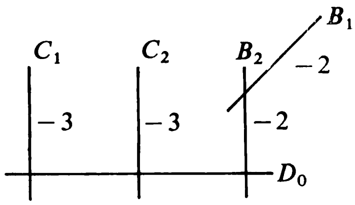


Figure 6.

Let C_i, B_j be the irreducible components as shown in the above configuration. Then

$$D_m=D_0+\frac{2}{3}(C_1+C_2+B_2)+\frac{1}{3}B_1.$$

In this case,

$$0=(K+D_m, K)=K^2+(D_0, K)+\frac{2}{3}+\frac{2}{3}=K^2+(D_0, K)+\frac{4}{3},$$

which is impossible. Q.E.D.

Remark The configuration of D is I–(1) or (2_n) or II–(1_n) (resp. III–(4), resp. IV–(3), resp. V–(4)) in Figure (3) if and only if $\bar{P}_1(X)=0$ and $\bar{P}_2(X)=1$ (resp. $\bar{P}_1(X)=\bar{P}_2(X)=0$ and $\bar{P}_3(X)=1$, resp. $\bar{P}_2(X)=\bar{P}_3(X)=0$ and $\bar{P}_4(X)=1$, resp. $\bar{P}_3(X)=\bar{P}_4(X)=\bar{P}_5(X)=0$ and $\bar{P}_6(X)=1$).

Now we shall give several examples.

Example 1. Let C_1 be a nonsingular conic on $\boldsymbol{P}^2$ and let C_2 be an irreducible cubic on $\boldsymbol{P}^2$ such that

(1) $\{p\}=C_1 \cap C_2$,

(2) C_2 has only one singular point $q \neq p$ (see Figure 7–(i)). We resolve the singularity of C_1+C_2. Let μ: $\bar{Y} \to \boldsymbol{P}^2$ be the composite of blowing-ups such that the configuration of $D'=\mu^{-1}(C_1+C_2)$ is as shown in Figure 7–(ii). Let C_i' be the proper transform of C_i for $i=1, 2$. Let μ_1: $\bar{X} \to \bar{Y}$ be the blowing-up of one of two points in $\mu^{-1}(q) \cap C_2'$ and let D be the proper transform of D'. Then the configuration of D is as shown in Figure 7–(iii). Putting $X=\bar{X}-D$, we have

$$\bar{\kappa}(X)=0,\ \bar{P}_3(X)=\bar{P}_4(X)=\bar{P}_5(X)=0 \quad \text{and} \quad \bar{P}_6(X)=1.$$

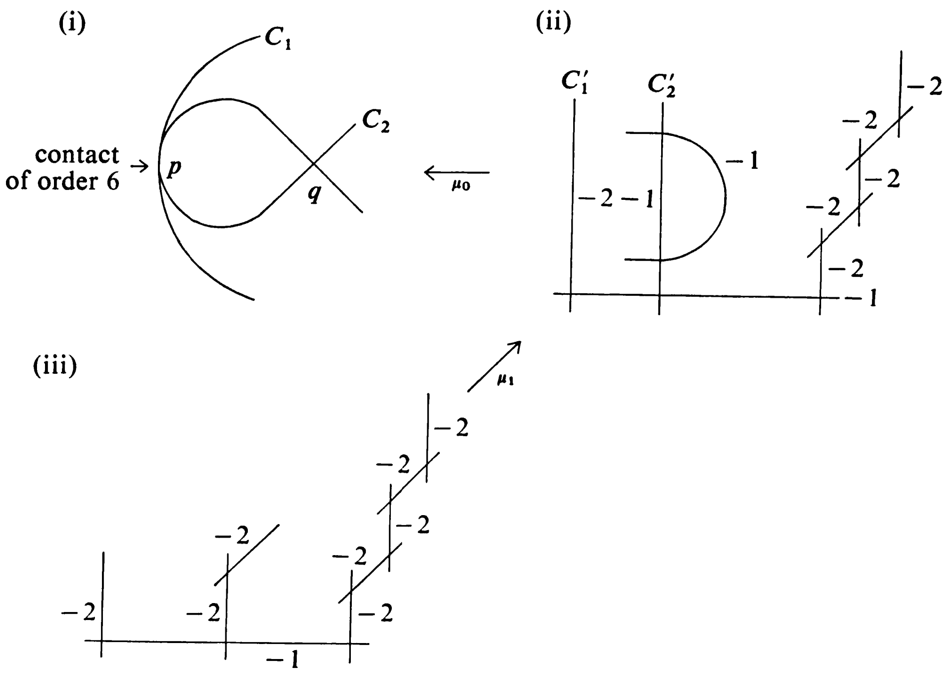


Figure 7.

Example 2. Let M be the minimal section of the $\boldsymbol{P}^1$-bundle morphism of ψ: $\Sigma_1 \to \boldsymbol{P}^1$ and l a fiber $\psi^{-1}(u)$. Let C_1 (resp. C_2) be an irreducible curve linearly equivalent to $M+l$ (resp. $M+2l$) such that $D_0:=M+l+C_1+C_2$ is as shown in Figure 8–(i). Let μ_0: $\bar{Y} \to \Sigma_1$ be the composite of blowing-ups of $p_0:=C_1 \cap C_2$ and its infinitely near points p_1, p_2 of order 1, 2 lying on the curve C_1 and the point $q_0:=C_2 \cap M$. Then we obtain the configuration of $\mu_0^{-1}(D_0)$ as shown in Figure 8–(ii). Let μ_1: $\bar{X} \to \bar{Y}$ be the composite of blowing-ups of the point $q_1:=C_2' \cap E_3$ and its infinitely near point q_2 on C_2'. Let D be $(\mu_0\mu_1)^{-1}(D_0)$ with $\mu_1'(E_2)$ and E_5 deleted off, where E_5 is the exceptional curve arising from the blowing-up of q_2. Then the configuration of D is as shown in Figure 8–(ii). Let $X:=\bar{X}-D$. Then we have

$$\bar{\kappa}(X)=0,\ \bar{P}_2(X)=\bar{P}_3(X)=0 \quad \text{and} \quad \bar{P}_4(X)=1.$$

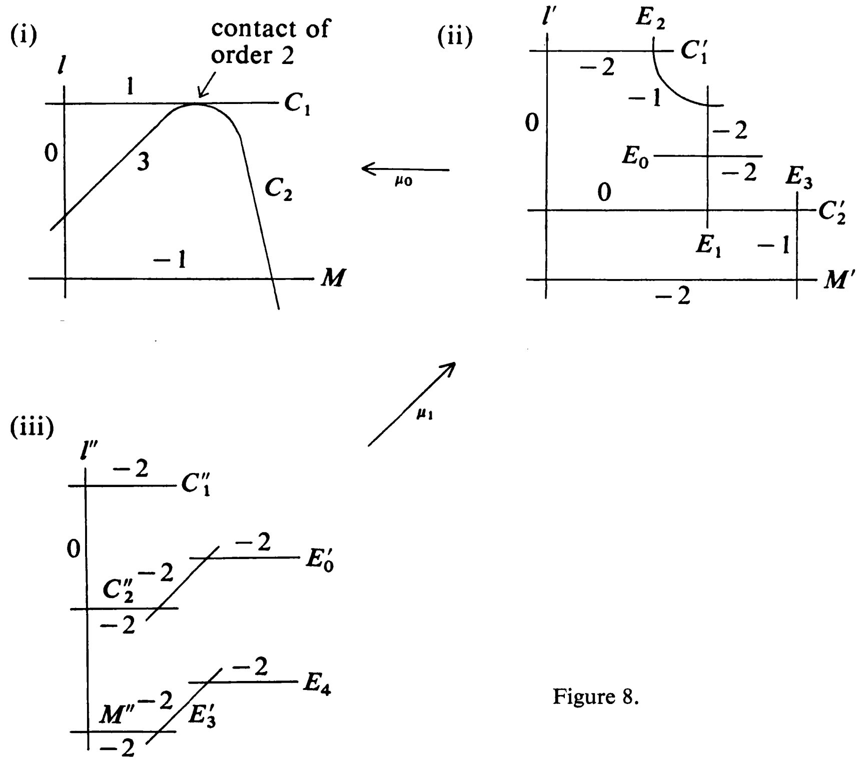


Figure 8.

Example 3. Let M be the section of Σ_2 with $(M^2)=-2$ and let l be a fiber. Let C_1 (resp. C_2) be an irreducible curve linearly equivalent to $M+2l$ (resp. $2M+4l$). Suppose that the configuration of $D_0=C_1+C_2$ is as shown in Figure 9–(i). We resolve the singularity of D_0. Let μ_0: $\bar{Y}_0\to\Sigma_2$ be a composite of suitable blowing-ups by which $\mu_0^{-1}(D_0)$ becomes as shown in Figure 9–(ii). Let $\mu_1: \bar{X}\to\bar{Y}$ be the blowing-up of one point q of $\mu_0^{-1}(p)\cap C_2'$ and its infinitely near point of order one on C_2'. Let D be $\mu_0^{-1}\mu_1^{-1}(D_0)$ with the exceptional curve of the first kind appearing in the last stage deleted off and let $X=\bar{X}-D$. Then we have

$$\bar{\kappa}(X)=0,\ \bar{P}_2(X)=\bar{P}_3(X)=0 \quad\text{and}\quad \bar{P}_4(X)=1.$$

Example 4. Let C_1 (resp. C_2) be an irreducible curve of Σ_2 linearly equivalent to $M+2l$ (resp. $2M+4l$) as in Example 3 and let $D_0=C_1+C_2$, whose configuration is, however, as shown in Figure 10–(i). Let μ_0: $\bar{Y}\to\Sigma_2$ be a composite of blowing-ups by which $\mu_0^{-1}(D_0)$ becomes as shown in Figure 10–(ii). Let μ_1: $\bar{Y}_1\to\bar{Y}_0$ be the composite of blowing-ups at $p'=C_1'\cap C_2'$ and one point q of $C_2'\cap\mu_0^{-1}(p)$, where p is the singular point of C_2. Let μ_2: $\bar{X}\to\bar{Y}_1$ be the blowing-up of $p'':=C_2''\cap\mu_1^{-1}(p')$. Let

$$D=(\mu_0\mu_1\mu_2)^{-1}(C_1+C_2)-(\mu_2^{-1}(\mu_1^{-1}(q))\cup\mu_2^{-1}(p''))$$

and let $X=\bar{X}-D$. Then we have

$$\bar{\kappa}(X)=0,\ \bar{P}_2(X)=0 \quad \text{and} \quad \bar{P}_3(X)=1.$$

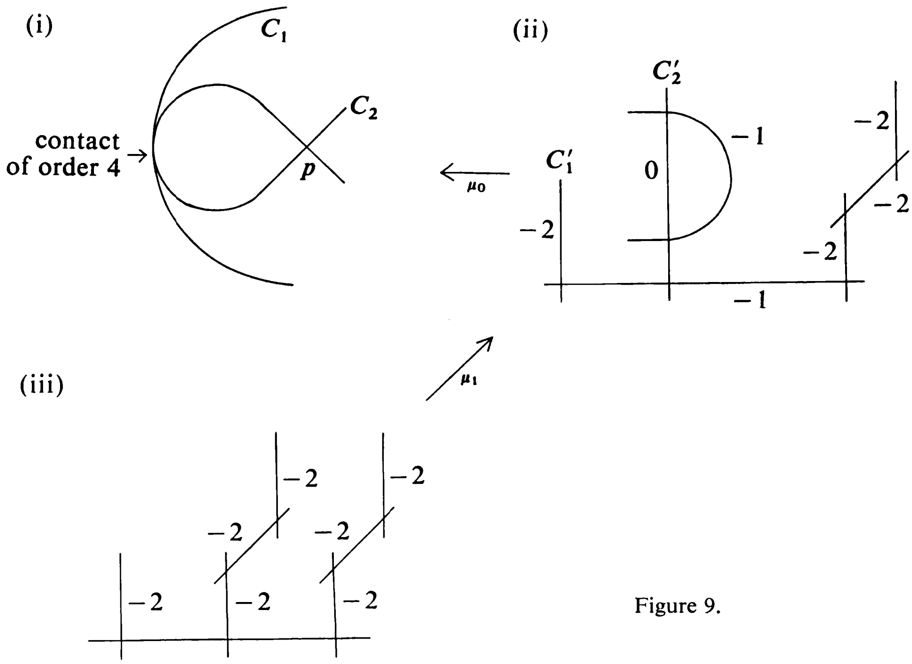


Figure 9.

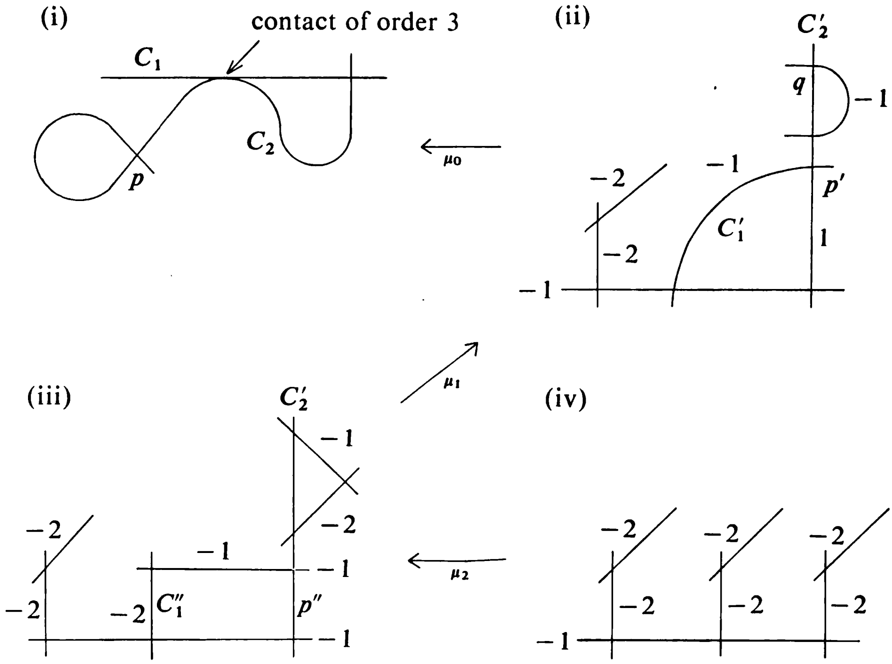


Figure 10.

Example 5. Let M be the minimal section of Σ_2 and let l be a fiber. Let C_1, C_2 be irreducible curves linearly equivalent to $M+2l$. We assume that $D_0:=M+l+C_1+C_2$ has only simple normal crossings as shown in Figure 11–(i). Let $p_0=C_2\cap l$ and $\{p_1, p_2\}=C_1\cap C_2$. Let μ_0: $\bar{Y}_0\to\Sigma_2$ be the composite of blowing-ups of p_0 and p_1. Let μ_1: $\bar{Y}\to\bar{Y}_0$ be the blowing-up of $q_1:=C_2'\cap\mu_0^{-1}(p_1)$. Let μ_2: $\bar{X}\to\bar{Y}_2$ be the composite of blowing-ups of $l''\cap(\mu_0\mu_1)^{-1}(p_0)$ and $C_2''\cap\mu_1^{-1}(q_1)$. Let D be the proper transform $\mu_2'((\mu_0\mu_1)^{-1}(D_0))$ and let $X=\bar{X}-D$. Then we have

$$\bar{\kappa}(X)=0,\ \bar{P}_2(X)=0 \quad\text{and}\quad \bar{P}_3(X)=1.$$

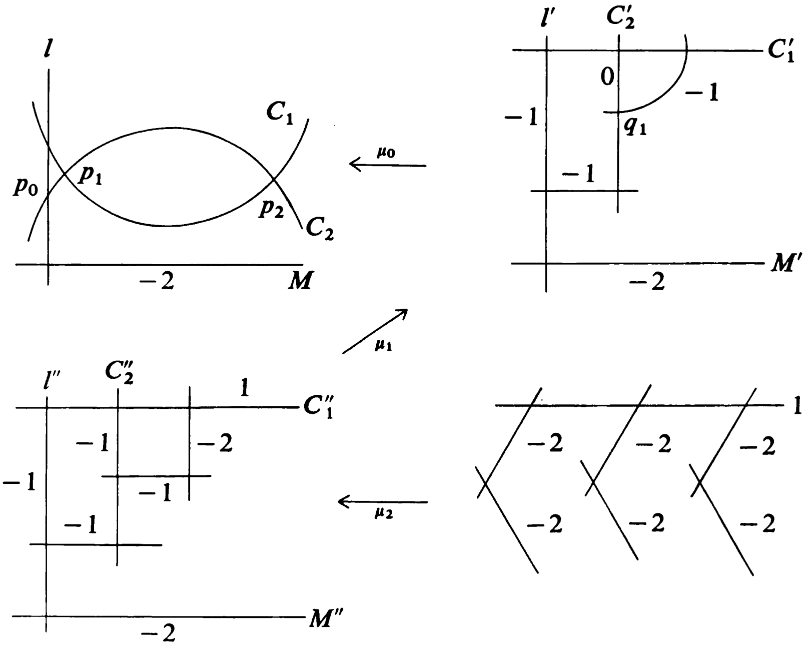


Figure 11.

Proposition 2.3. *Let $(X, \bar{X}, D)$ be an almost minimal triple such that $\bar{\kappa}(X)=\bar{P}_2(X)=0$, $\bar{X}$ is rational and D is connected. Assume that there are no exceptional curves E of the first kind with $(D, E)=1$. If the intersection matrix of D is not negative-semidefinite, then $(X, \bar{X}, D)$ is isomorphic to one of the triples enumerated in the above examples.*

Proof. We shall give a proof in the case where $\bar{P}_3(X)=\bar{P}_4(X)=0$ and $\bar{P}_6(X)=1$. The other cases are proved in a similar fashion. Then, since $\bar{P}_6(X)=1$, we know, by Proposition 2.2, that D has the following configuration:

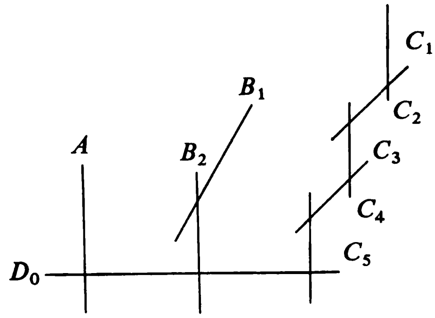


Figure 12.

where all curves (possibly except D_0) have self-intersection number -2. Since D is not negative-semidefinite, $(D_0^2)\geqq -1$. Suppose that $(D_0^2)=0$. Noting that

$$0=(K+D_m, K)=(K+D_0, K),$$

we have $(K^2)=2$. Then there exist a complete nonsingular surface $\bar{Y}$ and a birational morphism $\mu: \bar{X}\to\bar{Y}$ such that $\bar{Y}$ is isomorphic to a Hirzebruch surface Σ_n and $\mu(D_0)$ is a fiber; consider the $\boldsymbol{P}^1$-fibration on $\bar{X}$ induced by the linear system $|D_0|$. Let l be a fiber of Σ_n. Since

$$(\mu(A), l)=(\mu(B_2), l)=(\mu(C_5), l)=1,$$

it follows that $\mu(A)$, $\mu(B_2)$, $\mu(C_5)$ are nonsingular. Note that $\mu(B_1+C_1+\cdots+C_5)$ is contained in a union of several fibers. Let E be an exceptional curve of the first kind contracted by μ. Noting that

$$D_m=D_0+\frac{1}{2}A+\frac{1}{3}(B_1+2B_2)+\frac{1}{6}(C_1+2C_2+3C_3+4C_4+5C_5),$$

$(K+D_m, E)=0$, and

$$(F, E)\leqq 1$$

for $F=A$, B_i, C_j, where $i=1, 2$ and $j=1,\dots, 5$, we have one of the following five cases:

(1) $(A, E)=(C_3, E)=1$,

(2) $(B_2, E)=(C_2, E)=1$,

(3) $(B_1, E)=(C_4, E)=1$,

(4) $(C_1, E)=(C_5, E)=1$,

(5) $(C_2, E)=(C_4, E)=1$.

We consider separately each of the above cases.

Case (1). Let $\mu_0: \bar{X}\to\bar{Z}_0$ be the contraction of $E+C_3+C_2+C_1$. Then

$(\mu_0(C_4)^2)=1$. On the other hand, since we may assume that μ_0 factors μ, $\mu(C_4)$ is contained in some fiber of $\bar{Y}$. This is a contradiction.

Case (2). Let $\mu_0: \bar{X}\to\bar{Z}_0$ be the contraction of $E+C_2+C_1$. Then $\mu_0(D)$ is given as follows:

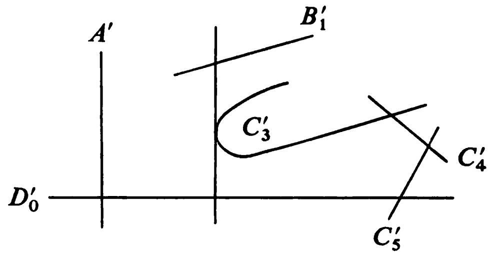


Figure 13.

where $A' := \mu_0(A)$, $B'_1 := \mu_0(B_1)$, etc. Then $(C_3'^2)=0$ and, since we may assume that μ_0 factors μ, the image of C'_3 by $\mu\cdot\mu_0^{-1}$ is a fiber on $\bar{Y}$. But $(B'_2, C'_3)=2$ and B'_2 becomes a section of the $\boldsymbol{P}^1$-fibration of $\bar{Y}$. This is a contradiction.

Case (3). Let $\mu_0: \bar{X}\to\bar{Z}_0$ be the contraction of $E+C_4+C_3+C_2+C_1$. Then $(\mu_0(B_1)^2)=3>0$. This is a contradiction because B'_1 is contained in a fiber of the $\boldsymbol{P}^1$-fibration.

Case (4). Let $\mu_0: \bar{X}\to\bar{Z}_0$ be the contraction of $E+C_1+C_2+C_3+C_4$. Then $C'_5 := \mu_0(C_5)$ is singular and a section of the $\boldsymbol{P}^1$-fibration. This is a contradiction.

Case (5). By the same reasoning as in Case (4), we have a contradiction.

Therefore we obtain $(D_0^2)\neq 0$. Suppose $n := (D_0^2)>0$. Let $p_1,\dots, p_n$ be general points of D_0. Let $\nu: \bar{Y}\to\bar{X}$ be the composite of the blowing-ups of $p_1,\dots, p_n$ and let D' (resp. D'_0) be the proper transform of D (resp. D_0) by ν. Then we have another triple $(\bar{Y}-D', \bar{Y}, D')$ with $(D_0'^2)=0$. But this case does not take place. (Note that we do not use the assumption that there are no exceptional curves E of the first kind with $(D, E)=1$ in the case where $(D_0^2)=0$.) Hence $(D_0^2)<0$ and then $(D_0^2)=-1$. Let $\mu_0: \bar{X}\to\bar{Z}_0$ be the contraction of $D_0+C_5+\cdots+C_1$, which gives the configuration:

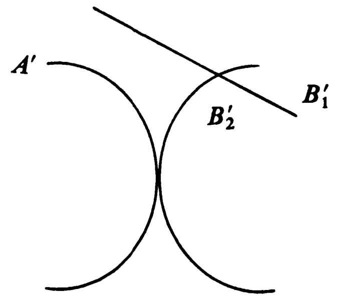


Figure 14.

Note that $(A'^2)=4$ and $(B_2'^2)=4$. Since $(K+D_0, K)=0$ and $(D_0, K)=-1$, $(K(\bar{X})^2)=1$ and $K(\bar{Z}_0)^2=7$. Let E' be an exceptional curve of the first kind on $\bar{Z}_0$. Then we have one of the following three cases:

(1) $(A', E')=2$,

(2) $(B'_1, E')=3$,

(3) $(B'_2, E')=(B'_1, E')=1$.

First, we shall show that the case (2) does not occur. Assume that the case (2) occurs. By the contraction of E', B'_1 becomes singular. Since every irreducible singular curve on a relatively minimal rational surface meets all curves except a minimal section, we have a contradiction because the image of A' has a positive self-intersection number. Hence the case (2) can not occur. Second, assume case (1). Then we shall prove that there exists another exceptional curve of the first kind E'' on $\bar{Z}_0$ such that $(B'_2, E'')=(B'_1, E'')=1$. Let $\sigma: \bar{Z}_0 \to \bar{W}$ be the contraction of E'. Since $(K(\bar{W})^2)=(K(\bar{Z}_0)^2)+1=8$, $\bar{W}$ is a Hirzebruch surface. Then $(\sigma(B'_1)^2)=-2$ implies that $\sigma(B'_1)$ is a minimal section. Let l be the fiber of $\boldsymbol{P}^1$-bundle structure of $\bar{W}$ such that $\sigma(E')\in l$ and let l' be the proper transform of l by σ. Note that l' is an exceptional curve of the first kind and that $(l', B'_1)=1$ because $(\sigma(B'_1), l)=1$. Therefore, putting $l'=E''$, E'' has required properties.

Hence we may assume that the case (3) occurs, if necessary, changing E for another exceptional curve of the first kind. Then, by contracting E' and B'_1, we obtain the case considered in Example 1. Q.E.D.

§3. Triples $(X, \bar{X}, D)$ with $\bar{\kappa}(X)=2$

Let $(X, \bar{X}, D)$ be an almost minimal triple with $\bar{\kappa}(X)=2$. We shall introduce some definitions concerning D. Let C be a connected component of D. C is said to be a 1-elliptic component of D if C is either a nonsingular elliptic curve or a cycle of nonsingular rational curves. Exlcuding these cases, suppose that C consists of nonsingular rational curves. The connected component C is said to be $\frac{1}{2}$-elliptic (resp. $\frac{1}{3}$-elliptic, resp. $\frac{1}{4}$-elliptic, resp. $\frac{1}{6}$-elliptic) if C has one of the configurations in Figure 3-(II) (resp. 3-(III), resp. 3-(IV), resp. 3-(V)). Given a positive integer n and a divisor D with only simple normal crossings, we define $\varepsilon_i(n, D)$ by

$$\varepsilon_i(n, D)=\begin{cases} \#\left\{\frac{1}{i}\text{-elliptic components of } D\right\}, & \text{if } n\equiv 1 \pmod{i} \\ 0\ , & \text{otherwise.}\end{cases}$$

We abbreviate $\varepsilon_i(n, D)$ as $\varepsilon_i(D)$ if there is no danger of confusion. Then we have the following:

Proposition 3.1. *With notations and assumptions as above, we have*

$$\bar{P}_n(X)=\frac{1}{2}(nK-[-(n-1)D_m]+[D_m], (n-1)K-[-(n-1)D_m]+[D_m])$$
$$+\chi(O_{\bar{X}})+\varepsilon_1(D)+\varepsilon_2(D)+\varepsilon_3(D)+\varepsilon_4(D)+\varepsilon_6(D), \qquad \text{if } n\geqq 2,$$

where $D_m = D-(K+D)^-$.

Proof. The assumption $\bar{\kappa}(X)=2$ implies that $|r(K+D_m)|$ is a linear system of integral divisors free from base points for an integer $r \gg 0$ and that $(K+D_m)^2>0$ (cf. Kawamata [5; (2.9)]). By Kawamata's vanishing theorem [6], we have

$$H^1(\bar{X}, [-(n-1)(K+D_m)])=0 \qquad \text{for} \quad n \geqq 2.$$

By the Serre duality,

$$H^1(\bar{X}, nK-[-(n-1)D_m])=0 \qquad \text{if} \quad n \geqq 2.$$

On the other hand, it is easy to verify the relations:

$$nK+nD \geq nK-[-(n-1)D_m]+[D_m] \geq [n(K+D_m)].$$

Since

$$H^0(\bar{X}, [n(K+D_m)]) \cong H^0(X, n(K+D)),$$

this implies that

$$\bar{P}_n(X)=h^0(\bar{X}, nK-[-(n-1)D_m]+[D_m]).$$

We shall compute $h^1(\bar{X}, nK-[-(n-1)D_m]+[D_m])$. First of all, note that

$$h^2(\bar{X}, nK-[-(n-1)D_m]+[D_m])=h^0(\bar{X}, (1-n)K+[-(n-1)D_m]-[D_m])$$
$$\leqq h^0(\bar{X}, [(1-n)(K+D_m)])=0 \qquad \text{if} \quad n \geqq 2,$$

and that

$$h^2(\bar{X}, nK-[-(n-1)D_m])=h^0(\bar{X}, (1-n)K+[-(n-1)D_m])$$
$$=h^0(\bar{X}, [(1-n)(K+D_m)])=0 \qquad \text{if} \quad n \geqq 2.$$

From an exact sequence with $n \geqq 2$,

$$0 \longrightarrow \mathcal{O}(nK-[-(n-1)D_m]) \longrightarrow \mathcal{O}(nK-[-(n-1)D_m]+[D_m])$$
$$\longrightarrow \mathcal{O}_{[D_m]}((nK-[-(n-1)D_m]+[D_m])|_{[D_m]}) \longrightarrow 0,$$

we have a long exact sequence

$$\cdots \longrightarrow H^1(\bar{X}, nK-[-(n-1)D_m]) \longrightarrow H^1(\bar{X}, nK-[-(n-1)D_m]+[D_m])$$
$$\longrightarrow H^1([D_m], (nK-[-(n-1)D_m]+[D_m])|_{[D_m]}) \longrightarrow 0.$$

It follows that

$$h^1(\bar{X}, nK-[-(n-1)D_m]+[D_m])=h^1([D_m], (nK-[-(n-1)D_m]+[D_m])|_{[D_m]}),$$
$$\text{for} \quad n \geqq 2.$$

Put $D_0=[D_m]$ and $D'_m=D_m-D_0$. Take a connected component C of D_0. Then, by the Serre duality, we have

$$h^1(C, (nK-[-(n-1)D_m]+D_0)|_C)=h^1(C, (nK+nD_0-[-(n-1)D'_m]|_C)$$
$$=h^1(C, n\omega_C-[-(n-1)D'_m]|_C)=h^0(C, (1-n)\omega_C+[-(n-1)D'_m]|_C),$$

where $\omega_C=(K+C)|_C$. and $n\geqq 2$. Suppose $h^0(C, (1-n)\omega_C+[-(n-1)D'_m]|_C)\neq 0$. Then we have

$$(*)\quad \deg((1-n)\omega_C+[-(n-1)D'_m]|_C)=(1-n)(K+C, C)+([-(n-1)D'_m], C)\geqq 0,$$

where $n\geqq 2$.
Since C and D'_m have no common components, $([-(n-1)D'_m], C)\leqq 0$. It follows that $(K+C, C)\leqq 0$. Suppose that $(K+C, C)=0$. Then $([-(n-1)D'_m], C)=0$. We shall then show that C is a connected component of D. Assume the contrary. Let E be an irreducible component of $D-C$ with $C\cap E\neq\emptyset$. By the definition of C and D_0, we have $E\subsetneqq D_0$. Thus the coefficient of E in D'_m is smaller than one. But since $E\cap D_0\supseteq E\cap C\neq\emptyset$, the coefficient of E in D'_m is nonzero (cf. Step (4) in the proof of Theorem 1.3). Hence, $E\subseteq \operatorname{supp}[-(n-1)D'_m]$ for $n\geqq 2$, which is a contradiction. Therefore, C is a connected component of D. Since $h^0(C, (1-n)\omega_C)\neq 0$ by the assumption and $\deg\omega_C=0$, $\omega_C\sim\mathcal{O}_C$ and hence C is a 1-elliptic component of D.

Suppose that $(K+C, C)<0$. Then $(K+C, C)=-2$ and every irreducible component of C is a nonsingular rational curve. From $(*)$, we have

$$(**)\qquad 2(n-1)\geqq(-[-(n-1)D'_m], C), \qquad (n\geqq 2).$$

Let $C_1,\dots, C_l$ exhaust irreducible components of D'_m which meet C and let $c_i=1-\frac{1}{a_i}$ be the coefficient of C_i in D'_m (cf. the proof of Proposition 2.2.), where we note that $a_i\geqq 2$ for all i. By $(**)$, we have

$$2(n-1)\geqq\textstyle\sum_i -\left[-(n-1)\left(1-\frac{1}{a_i}\right)\right]. \qquad (n\geqq 2)$$

Under the additional assumption $a_1\leqq a_2\leqq\cdots\leqq a_l$, such a system of integers $(n, a_1,\dots, a_l)$ can be enumerated as follows:

(1) $n\equiv 1\ (2)$, $l=4$, $a_1=a_2=a_3=a_4=2$,

(2) $n\equiv 1\ (3)$, $l=3$, $a_1=a_2=a_3=3$,

(3) $n\equiv 1\ (4)$, $l=3$, $a_1=2$, $a_2=a_3=4$,

(4) $n\equiv 1\ (6)$, $l=3$, $a_1=2$, $a_2=3$, $a_3=6$.

In each case, we have

$$2(n-1)=\textstyle\sum_i -\left[-(n-1)\left(1-\frac{1}{a_i}\right)\right] \qquad (n\geqq 2),$$

whence

$$((1-n)(K+C)+[-(n-1)D'_m])|_C\sim\mathcal{O}_C \qquad (n\geqq 2).$$

This implies that

$$(***) \qquad ((1-n)(K+C)+[-(n-1)D'_m],\ E)=0 \qquad (n\geqq 2)$$

for every irreducible component E of C. First of all, assume that C is reducible. Since the configuration of C is a tree because $(K+C,\ C)=-2$, C has at least two edge components. Each edge component meets at least two distinct irreducible components of supp D'_m (cf. the proof of Step (4) of Theorem 1.3). From these facts, we know that every connected component of D containing C is a $\frac{1}{2}$-elliptic component.

Secondly, assume that C is irreducible. It is easy to verify that every connected component of D containing C is a $\frac{1}{3}$-elliptic component, a $\frac{1}{4}$-elliptic component or a $\frac{1}{6}$-elliptic component (cf. the proof of Proposition 2.2). In each of the above cases, it is also clear that

$$(1-n)\omega_C+([-(n-1)D'_m])|_C\sim\mathcal{O}_C$$

and that

$$h^0(C,\ (1-n)\omega_C+[-(n-1)D'_m]|_C)=1\,.$$

Therefore we have shown that

$$\begin{aligned}&h^1([D_m],\ (nK-[-(n-1)D_m]+[D_m])|_{[D_m]})\\&\quad=\varepsilon_1(D)+\varepsilon_2(D)+\varepsilon_3(D)+\varepsilon_4(D)+\varepsilon_6(D)\,.\end{aligned}$$

Therefore we obtain the stated estimation of $\bar{P}_n(X)$. Q. E. D.

Proposition 3.2. *Let $(X,\ \bar{X},\ D)$ be an almost minimal triple with $\bar{\kappa}(X)=2$. If $[D_m]\neq 0$, then $\bar{P}_{12}(X)>0$.*

Proof. We shall show that the assumption $\bar{P}_2(X)=\bar{P}_3(X)=\bar{P}_4(X)=\bar{P}_6(X)=0$ leads to a contradiction. By Proposition 3.1, we have

$$\begin{aligned}(*)\quad 0=\bar{P}_n(X)=&\frac{1}{2}(nK-[-(n-1)D_m]+[D_m],\ (n-1)K-[-(n-1)D_m]+[D_m])\\&+\chi(\mathcal{O}_{\bar{X}})+\varepsilon_1(D)+\varepsilon_2(D)+\varepsilon_3(D)+\varepsilon_4(D)+\varepsilon_6(D)\,,\end{aligned}$$

for $n=2$, 3, 4 and 6. On the other hand, by Kawamata's vanishing theorem [6], we have

$$\begin{aligned}(**)\qquad &h^0(nK-[-(n-1)D_m])\\&=\frac{1}{2}(nK-[-(n-1)D_m],\ (n-1)K-[-(n-1)D_m])+\chi(\mathcal{O}_{\bar{X}})\end{aligned}$$

for $n\geqq 2$ (cf. the proof of Proposition 3.1). From $(*)$ and $(**)$, we have

$$(***_n)\qquad \frac{1}{2}([D_m],\ (2n-1)K-2[-(n-1)D_m]+[D_m])+\varepsilon_1(D)+\varepsilon_2(D)$$
$$+\varepsilon_3(D)+\varepsilon_4(D)+\varepsilon_6(D)=0$$

for $n=2$, 3, 4 and 6. Let C be a connected component of $[D_m]$ and let $D_1,\dots, D_l$ be all irreducible components of $D-[D_m]$ which meet C. Then the coefficient of D_i in D_m is $1-\frac{1}{a_i}$ for some integer $a_i\geqq 2$ (cf. the proof of Proposition 2.2). Since $K+D_m$ is semipositive, we have

$$\left(K+C+\sum\left(1-\frac{1}{a_i}\right)D_i,\ C\right)\geqq 0.$$

Noting that $(2n-1)\left(1-\frac{1}{a_i}\right)\leqq -2\left[-(n-1)\left(1-\frac{1}{a_i}\right)\right]$, this inequality implies

$$(C,\ (2n-1)K-2[-(n-1)D_m]+C)\geqq 0$$

for $n=2$, 3, 4 and 6. Thus, the relation $(***_n)$ implies

$$(C,\ (2n-1)K-2[-(n-1)D_m]+C)=0$$

for every connected component C of $[D_m]$ and for $n=2$, 3, 4 and 6; moreover, we have $\varepsilon_i(D)=0$ for $i=1$, 2, 3, 4, 6. From this we have

(1) $3(C,\ K+C)+2l=0$

(2) $5(C,\ K+C)-2\sum\left[-2\left(1-\frac{1}{a_i}\right)\right]=0$

(3) $7(C,\ K+C)-2\sum\left[-3\left(1-\frac{1}{a_i}\right)\right]=0$

(4) $11(C,\ K+C)-2\sum\left[-5\left(1-\frac{1}{a_i}\right)\right]=0.$

If $l=0$, then $(C,\ K+C)=0$. This implies that C is a 1-elliptic component. Hence, $\varepsilon_1(D)\neq 0$, which is a contradiction. So we may assume $l\neq 0$. From (1), we have $(C,\ K+C)=-2$, whence $l=3$. We may assume that $a_1\leqq a_2\leqq a_3$. From (2), we have $a_1=2$ and $a_3\geqq 3$. From (3), we have $a_2=3$. On the other hand, note that $-2+\sum\left(1-\frac{1}{a_i}\right)\geqq 0$ because $(K+D_m,\ C)\geqq 0$. Hence $a_3\geqq 6$. Then, we have

$$0=-22-2\left[-\frac{5}{2}\right]-2\left[-\frac{10}{3}\right]-2\left[-\frac{25}{6}\right]>0,$$

which contradicts (4). Q.E.D.

Theorem 3.3. *Let $(X, \bar{X}, D)$ be an almost minimal triple with $\bar{\kappa}(X)\geqq 0$. Assume that D is connected. Then $\bar{P}_{12}(X)>0$.*

Proof. By Kuramoto [7], we know that $\bar{P}_{12}(X)>0$ if $\bar{X}$ is not a rational surface. Hence, we may assume that $\bar{X}$ is rational. First of all, assume that

$[D_m]=0$ and $\bar{P}_2(X)=0$. Then we have

$$0=h^0(\bar{X},\ 2K+D)\geqq(K,\ K+D)$$

by virtue of the Riemann-Roch theorem and the fact that $(K+D,\ D)=-2$. On the other hand,

$$0\leqq(K+D_m)^2=(K,\ K+D_m)\leqq(K,\ K+D)$$

because each irreducible component C of D satisfies $(C,\ K)\geqq 0$, which is a consequence of the assumption that $(X,\ \bar{X},\ D)$ is almost minimal and $[D_m]=0$. Hence $\bar{P}_2(X)=0$ and $[D_m]=0$ imply that $(C,\ K)=0$ for all irreducible components C of D and that

$$K^2=(K,\ K+D_m)=(K+D_m)^2=0.$$

Hence $(D_m^2)=0$. Since either $\operatorname{supp} D=\operatorname{supp}(K+D)^-=\emptyset$ or the intersection matrix of D_m is negative-definite, we have $D_m=0$. Therefore, $\bar{\kappa}(X)\geqq 0$ implies that $\kappa(\bar{X})\geqq 0$, which is a contradiction because $\bar{X}$ is rational.

Secondly, we assume $[D_m]\neq 0$. If $\bar{\kappa}(X)=0$, we proved in Proposition 2.2 that $\bar{P}_{12}(X)=1$. If $\bar{\kappa}(X)=1$, we can show that $\bar{P}_{12}(X)>0$ by making use of formulas (2.5) and (2.8) of Kawamata [5] (or Miyanishi [8; Lemma 4.1]). If $\bar{\kappa}(X)=2$, we have $\bar{P}_{12}(X)>0$ by virtue of Proposition 3.2. Q.E.D.

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