J. Math. Kyoto Univ. (JMKYAZ)
23-2 (1983) 219-224

Metrical Finsler structures and metrical Finsler connections

Dedicated to Professor Makoto Matsumoto on the occasion of his sixtieth birthday

By

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(Communicated by Prof. Toda, February 1, 1982)

A Finsler space is sometimes adopted as a basic concept in the theoretical physics. However, the fact that the fundamental tensor field of a Finsler space is provided from a positively homogeneous function is not always desirable for physicists, as pointed out by several authors. In fact, recently Matsumoto [8] showed an unexpected result (Corollary of Theorem 2) on four-dimensional Finsler spaces, which may be a direct consequence of such an origin of the fundamental tensor field. It seems to the author that Kern's Lagrange geometry [6] is noteworthy in this aspect. As to physical viewpoint, see Ingarden's lecture [5] and Takano's lecture [11]. Further it is suggestive that Horváth and Moór [4] again developed their theory based on a generalized metric in Moór's terminology, after their Finsler-geometric treatment of the same subject [3].

In the present paper we first define a metrical structure on a differentiable manifold as a Finsler tensor field g of type (0, 2) in Matsumoto's terminology [7] and establish the existence of a set of connections $F\Gamma$ of Finsler type which are metrical with respect to g. Based on the notion of absolute energy associated to g, we define regular Finsler structures. From a regular Finsler structure a metrical Finsler connection, called canonical by us, is uniquely determined. This Finsler connection is regarded as a generalization of the Cartan connection in case of Finsler geometry.

Almost all the theorems in this paper are proved applying the methods given by the present author and Hashiguchi [9, 10]; so the proofs are omitted.

Throughout the present paper we suppose that the contents of Matsumoto's monograph [7] are known.

§1. The metrical Finsler structures and metrical Finsler connections

Let M be an n-dimensional differentiable manifold, TM its tangent bundle and π: $TM \to M$ the natural projection. If $U \subset M$ is the coordinate neighborhood of a coordinate system (x^i), then $\pi^{-1}(U) \subset TM$ is a coordinate neighborhood, too. Let (x^i, y^i) be the coordinate system of a point $y \in \pi^{-1}(U)$, $x = (x^i) = \pi(y)$.

Definition 1.1. A Finsler tensor field g of type (0, 2) which is symmetric and nondegenerate is called a *metrical Finsler structure* on the differentiable manifold M.

If $g_{ij}(x, y)$ are local components of g on $\pi^{-1}(U)$, the above conditions for g are written as

$$g_{ij}(x, y)=g_{ji}(x, y), \qquad \det(g_{ij})\neq 0. \tag{1.1}$$

Let $g^{ij}(x, y)$ be the reciprocal tensor field of g: $g_{ij}g^{kj}=\delta^k_i$, and let

$$\Omega^{ir}_{sj}=(\delta^i_s\delta^r_j-g_{sj}g^{ir})/2, \quad \Omega^{*ir}_{sj}=(\delta^i_s\delta^r_j+g_{sj}g^{ir})/2 \tag{1.2}$$

be the Obata operators of g.

Definition 1.2. A Finsler connection $F\Gamma=(N, F, C)$ is called *metrical* if it satisfies the conditions:

$$g_{ij|k}=0, \qquad g_{ij}|_k=0. \tag{1.3}$$

Evidently we have $\Omega^{ir}_{sj|k}=\Omega^{ir}_{sj}|_k=0$ and $\Omega^{*ir}_{sj|k}=\Omega^{*ir}_{sj}|_k=0$ from (1.3). Using the Ricci identities we easily obtain

Theorem 1.1. *The curvature tensor fields R^i_{jkl}, P^i_{jkl} and S^i_{jkl} of a metrical Finsler connection $F\Gamma$ have the property that the Finsler tensor fields $\Omega^{*ir}_{sj}R^s_{rkl}$, $\Omega^{*ir}_{sj}P^s_{rkl}$, $\Omega^{*ir}_{sj}S^s_{rkl}$ and their h-and v-covariant derivatives of any order vanish.*

On a similar way to the proof used in the papers [9, 10] we get

Theorem 1.2. *Let $F\mathring{\Gamma}$ be a fixed Finsler connection. Then any metrical Finsler connection $F\Gamma$ is given by*

$$\begin{aligned} N^i_k &= \mathring{N}^i_k - X^i_k, \\ F^i_{jk} &= \mathring{F}^i_{jk} + \mathring{C}^i_{jm}X^m_k + g^{im}(g_{mj\mathring{|}k} + g_{mj}\mathring{|}_p X^p_k)/2 + \Omega^{ir}_{sj}X^s_{rk}, \\ C^i_{jk} &= \mathring{C}^i_{jk} + g^{im}g_{mj}\mathring{|}_k/2 + \Omega^{ir}_{sj}Y^s_{rk}, \end{aligned} \tag{1.4}$$

where $\mathring{|}$ and $\mathring{|}$ denote the h- and v-covariant derivatives with respect to $F\mathring{\Gamma}$ and X^i_j, X^i_{jk}, Y^i_{jk} are arbitrary Finsler tensor fields.

As a particular case:

Theorem 1.3. *Let $F\mathring{\Gamma}$ be a given Finsler connection. Then the following Finsler connection $F\Gamma$ is metrical*:

$$N^i_k=\mathring{N}^i_k, \quad F^i_{jk}=\mathring{F}^i_{jk}+g^{im}g_{mj\mathring{|}k}/2, \quad C^i_{jk}=\mathring{C}^i_{jk}+g^{im}g_{mj}\mathring{|}_k/2. \tag{1.5}$$

The last two theorems show the existence and arbitrariness of the metrical Finsler connections.

Remark 1. Applying the method in the paper [9, 10] we can study the metrical Finsler connections, the transformation group of metrical Finsler connections and its invariants. The theory of semi-symmetric metrical Finsler connections is very interesting, too. See [2].

§ 2. Regular metrical Finsler structures

To a metrical Finsler structure g we associate the function

$$L(x, y)=g_{00}=g_{ij}(x, y)y^i y^j, \tag{2.1}$$

which is called the *absolute energy*.

Consider the Finsler tensor field

$$\mathring{C}^i_{jk}=g^{im}(\partial g_{jm}/\partial y^k+\partial g_{km}/\partial y^j-\partial g_{jk}/\partial y^m)/2, \tag{2.2}$$

which is symmetric in the indices j and k, and we put

$$\mathring{C}_{ijk}=g_{jm}\mathring{C}^m_{ik}, \tag{2.3}$$

$$\mathring{C}_{i00}=\mathring{C}_{ijk}y^j y^k=(\partial g_{jk}/\partial y^i)y^j y^k/2, \tag{2.4}$$

$$A^i_j=\delta^i_j+g^{im}(\partial g_{hm}/\partial y^j)y^h. \tag{2.5}$$

Definition 2.1. The metrical Finsler structure g is called *regular* if

(2.6) $\quad \mathring{C}_{i00}=0, \qquad$ (2.7) $\quad \det(\partial^2 L/\partial y^j\partial y^k)\neq 0$

are satisfied.

Proposition 2.1. *The metrical Finsler structure g is regular if and only if*

(a) $\quad g_{ij}y^j=(\partial L/\partial y^i)/2, \qquad$ (b) $\quad \det(A^i_j)\neq 0$

are satisfied.

Indeed, the condition (2.6) is equivalent to (a) and, because of $(\partial^2 L/\partial y^j\partial y^k)/2=g_{jk}+(\partial g_{hj}/\partial y^k)y^h$, we have $(g^{im}\partial^2 L/\partial y^m\partial y^j)/2=A^i_j$, so that (2.7) is equivalent to (b).

Remark 2. (1) If g is a regular metrical Finsler structure, we get $(\partial g_{hj}/\partial y^k)y^h=(\partial g_{hk}/\partial y^j)y^h$.

(2) If we are concerned with the characteristic polynomial of the matrix $(g^{im}\{\partial g_{hm}/\partial y^j\}y^h)$, the determinant of the matrix (A^i_j) can be easily computed.

Let B^i_j be the reciprocal tensor field of A^i_j:

$$B^i_h A^h_j=\delta^i_j, \tag{2.8}$$

and put

$$\gamma^i_{jk}=g^{im}(\partial g_{jm}/\partial x^k+\partial g_{km}/\partial x^j-\partial g_{jk}/\partial x^m)/2, \tag{2.9}$$

$$\gamma^i_{00}=\gamma^i_{jk}y^j y^k. \tag{2.10}$$

Then we have

Theorem 2.1. *For any regular metrical Finsler structure g, $\mathring{N}^i_j(x, y)$ given by*

$$\mathring{N}^i_j(x, y)=\{\partial(B^i_h\gamma^h_{00})/\partial y^j\}/2 \tag{2.11}$$

are coefficients of a non-linear connection $\overset{\circ}{N}$ determined by the structure g only.

Proof. A coordinate transformation on the tangent bundle *TM*, namely,

$$\bar{x}^i = \bar{x}^i(x^1, \dots, x^n), \quad \bar{y}^i = \bar{X}^i_p y^p, \quad (\bar{X}^i_p = \partial \bar{x}^i / \partial x^p),$$

implies the transformation $(\partial/\partial x^p, \partial/\partial y^p) \to (\partial/\partial \bar{x}^i, \partial/\partial \bar{y}^i)$ given by

$$\partial/\partial \bar{x}^i = \underline{X}^p_i \partial/\partial x^p + (\partial \underline{X}^p_h / \partial \bar{x}^i) \bar{y}^h \partial/\partial y^p, \quad (\underline{X}^p_i = \partial x^p / \partial \bar{x}^i), \; \partial/\partial \bar{y}^i = \underline{X}^p_i \partial/\partial y^p.$$

From these equations, direct computation leads to

$$\bar{\overset{\circ}{N}}{}^i_j = \bar{X}^i_p \underline{X}^q_j \overset{\circ}{N}{}^p_q + \bar{X}^i_p (\partial \underline{X}^p_j / \partial \bar{x}^k) \bar{y}^k,$$

which shows that $\overset{\circ}{N} = \{\overset{\circ}{N}{}^i_j\}$ certainly is a non-linear connection.

The non-linear connection $\overset{\circ}{N}$ is considered as a distribution: $y \in TM \mapsto \overset{\circ}{N}_y \subset TM_y$ having the property $TM_y = \overset{\circ}{N}_y \oplus TM^v_y$, and the vector fields

$$\delta/\delta x^i = \partial/\partial x^i - \overset{\circ}{N}{}^m_i \partial/\partial y^m, \qquad i = 1, \dots, n, \tag{2.12}$$

provide a local basis of the (horizontal) distribution $\overset{\circ}{N}$.

§ 3. Canonical Finsler connections

From the non-linear connection $\overset{\circ}{N}$ given by Theorem 2.1 we can introduce

$$\overset{\circ}{F}{}^i_{jk} = g^{im}(\delta g_{jm}/\delta x^k + \delta g_{km}/\delta x^j - \delta g_{jk}/\delta x^m)/2. \tag{3.1}$$

Then we get

Theorem 3.1. *Let g be a regular metrical Finsler structure. The triad* $F\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$, *where* $\overset{\circ}{N}$, $\overset{\circ}{F}$ *and* $\overset{\circ}{C}$ *are given by* (2.11), (3.1) *and* (2.2) *respectively, is a metrical Finsler connection.*

Proof. It is easy to see that $F\overset{\circ}{\Gamma}$ is a Finsler connection. A straightforward calculation shows that $g_{ij\overset{\circ}{|}k} = 0$ and $g_{ij}\overset{\circ}{|}_k = 0$.

The above metrical Finsler connection $F\overset{\circ}{\Gamma}$ has the properties:

(a) It is determined by the regular metrical Finsler structure g only.

(b) Its torsion tensor fields $\overset{\circ}{T}$ and $\overset{\circ}{S}{}^1$ vanish.

For these reasons $F\overset{\circ}{\Gamma}$ is called the *canonical metrical Finsler connection* of the regular metrical Finsler structure g.

In the formulas (1.4), taking the canonical metrical Finsler connection of g as the fixed Finsler connection $F\overset{\circ}{\Gamma}$, we obtain

Theorem 3.2. *Let g be a regular metrical Finsler structure and let* $F\overset{\circ}{\Gamma}$ *be its canonical metrical Finsler connection. Then, the set of all the metrical Finsler connection* $F\Gamma$ *is given by*

$$N^i_k = \overset{\circ}{N}{}^i_k - X^i_k,$$

(3.2)
$$F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{jm} X^m_k + \Omega^{ir}_{sj} X^s_{rk},$$

$$C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \Omega^{ir}_{sj} Y^s_{rk},$$

where X^i_j, X^i_{jk}, Y^i_{jk} *are arbitrary Finsler tensor fields.*

We denote by $F\Gamma(N)=(N, F, C)$ any Finsler connection which has a fixed non-linear connection N. Then the last theorem has the following consequence:

Theorem 3.3. *Let* g *be a regular metrical Finsler structure and* $F\overset{\circ}{\Gamma}=(\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ *be its canonical metrical Finsler connection. Then the set of all the metrical Finsler connections* $F\Gamma(\overset{\circ}{N})$ *is given by*

(3.3)
$$F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \Omega^{ir}_{sj} X^s_{rk},$$

$$C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \Omega^{ir}_{sj} Y^s_{rk},$$

where X^i_{jk}, Y^i_{jk} *are arbitrary Finsler tensor fields.*

Now, applying the method used in the papers [9, 10], we obtain

Theorem 3.4. *Let* g *be a regular metrical Finsler structure and* $F\overset{\circ}{\Gamma}$ *be its canonical metrical Finsler connection. Further, let two alternate Finsler tensor fields* T^i_{jk}, S^i_{jk} *be given. Then there exists an unique metrical Finsler connection* $F\Gamma(\overset{\circ}{N})$ *having torsion tensor fields* $T=(T^i_{jk})$ *and* $S^1=(S^i_{jk})$, *which is given by*

$$F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + g^{im}(g_{mh}T^h_{jk} - g_{jh}T^h_{mk} + g_{kh}T^h_{jm})/2,$$

$$C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + g^{im}(g_{mh}S^h_{jk} - g_{jh}S^h_{mk} + g_{kh}S^h_{jm})/2.$$

Consequently we get

Theorem 3.5. *Let* g *be a regular metrical Finsler structure and* $F\overset{\circ}{\Gamma}=(\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ *be its canonical metrical Finsler connection. Then there exists an unique metrical Finsler connection* $F\Gamma(\overset{\circ}{N})$ *with* $T=S^1=0$. *This is the canonical metrical Finsler connection* $F\overset{\circ}{\Gamma}$.

It is easy to particularise the above results to the Finsler spaces. If $F(x, y)$ denotes the fundamental function of a Finsler space F_n, the absolute energy $L(x, y)$ is $F^2(x, y)$. We have $\overset{\circ}{C}_{j00}=0$ and $A^i_j=\delta^i_j=B^i_j$. Thus $g_{ij}=(\partial^2 F^2/\partial y^i \partial y^j)/2$ is a regular metrical Finsler structure. In this case the canonical metrical Finsler connection $F\overset{\circ}{\Gamma}$ is obviously nothing but Cartan's connection $C\Gamma$. So we have

Theorem 3.6. *If* g *is the metrical Finsler structure* $g_{ij}=(\partial^2 F^2/\partial y^i \partial y^j)/2$ *obtained from a fundamental function* $F(x, y)$ *of a Finsler space, it is regular and its canonical metrical Finsler connection* $F\overset{\circ}{\Gamma}$ *coincides with the Cartan connection* $C\Gamma$.

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