J. Math. Kyoto Univ. (JMKYAZ)
23-4 (1983) 649-694

Capillary-gravity waves for an incompressible ideal fluid

By

Hideaki YOSIHARA

(Received Aug. 15, 1982)

§ 1. Introduction

We shall consider nonstationary waves on the free surface of an incompressible ideal fluid when the surface tension is taken into account. The surface tension creates the pressure difference across the surface which is proportional to the mean curvature of the surface (Laplace's formula, [3] Ch. VII).

Matters relevant to the capillarity (capillary phenomena) are treated in [4]. For mathematical problems on capillary surfaces, problems determining the shape of the surface under the action of the surface tension, see, for example, [5] and papers in Pacific J. Math. 88 No. 2 (1980).

We take coordinates $y=(y_1, y_2)$ so that the fluid at rest occupies the domain

$$\{y \mid -\infty < y_1 < +\infty, \quad -h+b(y_1) \leqq y_2 \leqq 0\},$$

where $h=\text{const}>0$ is the mean depth and b is a given function such that $-h+b(y_1)<0$ for all y_1. The gravitational field is equal to $(0, -g)$, where $g=\text{const}$ is not necessarily positive.

The irrotational motion of the fluid is governed by equations and conditions for the density $\rho=\text{const}>0$, the velocity $v=(v_1, v_2)$, the pressure p and F defining the free surface, i.e. the domain

$$\Omega(t)=\{y \mid -\infty < y_1 < +\infty, \quad -h+b(y_1) \leqq y_2 \leqq F(t, y_1)\}$$

which the fluid occupies at time t. ρ, v and p satisfy the equation of motion, continuity and irrotationality, i.e. for t and y such that $y \in \Omega(t)$,

$$v_t+(v\cdot\nabla)v=-\rho^{-1}\nabla p+(0, -g), \tag{1.1}$$

$$\frac{\partial v_1}{\partial y_1}+\frac{\partial v_2}{\partial y_2}=0, \quad \frac{\partial v_2}{\partial y_1}-\frac{\partial v_1}{\partial y_2}=0, \tag{1.2}$$

where $v_t=\dfrac{\partial v}{\partial t}$, $\nabla=\text{grad}$, $v\cdot\nabla=v_1\dfrac{\partial}{\partial y_1}+v_2\dfrac{\partial}{\partial y_2}$. The fluid cannot penetrate the bottom. This means that for y such that $y_2=-h+b(y_1)$,

$$v\cdot N=0 \tag{1.3}$$

where $N=(N_1, N_2)$ is the outer normal to the bottom. The free surface always consists of the same particles of fluid, in other words, if $y(t)$ is a solution of

$$\frac{d}{dt}y(t)=v(t, y(t)), \quad y_2(0)-F(0, y_1(0))=0$$

then $y_2(t)-F(t, y_1(t))=0$. Thus for t and y such that $y_2=F(t, y_1)$,

$$\left(\frac{\partial}{\partial t}+v\cdot\nabla\right)(y_2-F)=0. \tag{1.4}$$

There is a different way to obtain (1.3) and (1.4), that is, we can derive (1.3) and (1.4) from the law of mass conservation (or the equation of continuity) under the assumption that the free surface and the bottom are expressed by smooth functions of t and y_1. The pressure difference across the surface is given by

$$p-p_0=-\alpha\frac{\partial}{\partial y_1}\left(\frac{\partial F}{\partial y_1}\left(1+\left(\frac{\partial F}{\partial y_1}\right)^2\right)^{-1/2}\right) \tag{1.5}$$

where p is the pressure of the fluid just inside the surface, p_0 is the constant external pressure and $\alpha=\text{const}>0$ is the surface tension coefficient. The condition (1.5) follows from the balance of forces acting on a part of the surface:

$$\alpha T(y_1+z)-\alpha T(y_1)+\int_{y_1}^{y_1+z}(p-p_0)N(y_1)ds=0, \quad z>0,$$

where

$$T(y_1)=(1+(F')^2)^{-1/2}(1, F'),$$

$$N(y_1)=(1+(F')^2)^{-1/2}(-F', 1),$$

$$F'=\frac{\partial F}{\partial y_1}, \quad ds=(1+(F')^2)^{1/2}dy_1.$$

Letting $z\to 0$ we obtain

$$\alpha\frac{\partial}{\partial y_1}(1+(F')^2)^{-1/2}-(p-p_0)F'=0,$$

$$\alpha\frac{\partial}{\partial y_1}(F'(1+(F')^2)^{-1/2})+p-p_0=0.$$

Since the first equation follows from the second equation, we obtain the required condition (1.5). Note that the curvature of the surface is equal to

$$\left|\frac{\partial}{\partial y_1}(F'(1+(F')^2)^{-1/2})\right|.$$

Initial conditions at $t=0$ are given in the form

$$\Omega(0)=\Omega, \text{ i.e. } F(0, y_1)=F_0(y_1), \quad v(0, y)=v_0(y) \tag{1.6}$$

where v_0 satisfies (1.2) for $y\in\Omega$ and (1.3).

Exact solutions for progressive waves, stationary nontrivial solutions of (1.1)–(1.5), were found in [6]–[12] under various additional conditions.

There is the trivial solution $F=0$, $v=0$, $p=p_0-\rho g y_2$ of (1.1)–(1.5). For the investigation of nonstationary solutions close to the trivial solution, it is convenient to treat the problem by introducing $X(t, x)=(X_1(t, x), X_2(t, x))$ such that

$$X_2(0, x)-F(0, x+X_1(0, x))=0, \quad 1+X_{1x}(t, x)>0,$$

$$\frac{\partial}{\partial t}((x, 0)+X(t, x))=v(t, (x, 0)+X(t, x)). \tag{1.7}$$

In view of (1.4) we see that $(x, 0)+X(t, x)$, $-\infty<x<+\infty$, represents the free surface at time t. The trivial solution corresponds to $X=0$ (precisely, $X=(\text{const}, 0)$, but in this case we take $x+\text{const}$ in place of x). It follows from (1.2) and (1.3) that $X_{2t}=KX_{1t}$ where $K=K(X)=K(X, b, h)$ is the operator, see [2]. The differentiation of (1.7) with respect to t and (1.1) give $X_{tt}=-\rho^{-1}\nabla p+(0, -g)$, which leads to

$$\begin{aligned}(1+X_{1x}, X_{2x})\cdot(X_{tt}+(0, g))&=(1+X_{1x}, X_{2x})\cdot(-\rho^{-1}\nabla p)\\ &=-\rho^{-1}\frac{\partial}{\partial x}p(t, (x, 0)+X(t, x)).\end{aligned} \tag{1.8}$$

Since $(1+X_{1x}, X_{2x})$ and $(1, F')$, where $y_1=x+X_1(t, x)$, are vectors tangent to the free surface, it holds that

$$F'(1+(F')^2)^{-1/2}=X_{2x}((1+X_{1x})^2+X_{2x}^2)^{-1/2}.$$

Noting that

$$\frac{\partial}{\partial y_1}=\frac{\partial x}{\partial y_1}\frac{\partial}{\partial x}=(1+X_{1x})^{-1}\frac{\partial}{\partial x}$$

we can write (1.5) in the form

$$\begin{aligned}p(t, (x, 0)+X)-p_0&=-\alpha(1+X_{1x})^{-1}(X_{2x}((1+X_{1x})^2+X_{2x}^2)^{-1/2})_x\\ &=-\alpha((1+X_{1x})^2+X_{2x}^2)^{-3/2}(-X_{2x}X_{1xx}+(1+X_{1x})X_{2xx}),\end{aligned} \tag{1.9}$$

which enables us to eliminate p from (1.8). Introducing

$$\begin{cases}\mu=\rho^{-1}\alpha,\\ Q=Q(X_x)=((1+X_{1x})^2+X_{2x}^2)^{1/2},\\ R=R(X_x, X_{xx})\\ \quad=-3Q(X_x)^{-5}((1+X_{1x})X_{1xx}+X_{2x}X_{2xx})(-X_{2x}X_{1xx}+(1+X_{1x})X_{2xx}),\\ S=S(X_x, X_{xxx})=Q(X_x)^{-3}(-X_{2x}X_{1xxx}+(1+X_{1x})X_{2xxx}),\end{cases} \tag{1.10}$$

we see that the problem (1.1)–(1.6) is reduced to the problem

$$(1+X_{1x})X_{1tt}+X_{2x}(g+X_{2tt})=\mu R(X_x, X_{xx})+\mu S(X_x, X_{xxx}), \quad 0\leqq t\leqq T, \tag{1.11}$$

$$X_{2t}=KX_{1t}, \quad 0\leqq t\leqq T, \tag{1.12}$$

$$X(0, x)=X^{(0)}(x), \quad X_{1t}(0, x)=X_1^{(1)}(x). \tag{1.13}$$

Our purpose is to show that if b, T and initial data are small then the problem (1.11)–(1.13) is uniquely solvable. The problem (1.11)–(1.13) with $\mu=0$ was treated in [1] for $h=\infty$ and in [2] for $0<h<\infty$. The method used in this paper is the generalization of that in [1] and [2].

For the investigation of the character of equations (1.11) and (1.12) we consider the equations linearized at $X=0$ (we assume that $b=0$):

$$X_{1tt}+gX_{2x}=\mu X_{2xxx},\quad X_{2t}=-i\tanh(hD)X_{1t},$$

where $D=-i\partial_x$, (note that $K(0, 0, h)=-i\tanh(hD)$). These equations give

$$X_{1ttt}+(gD+\mu D^3)\tanh(hD)X_{1t}=0.$$

(Remember the dispersion relation

$$\omega^2=(gk+\rho^{-1}\alpha k^3)\tanh(hk),$$

cf. [3] Ch. VII. For the agreement of this relation and experiment, see [13].) For the initial value problem

$$(1.14)\qquad u_{tt}+(gD+\mu D^3)\tanh(hD)u=f,\quad 0\leqq t\leqq T,$$

$$(1.15)\qquad u=u_0,\quad u_t=u_1,\quad t=0,$$

it is easily shown by means of Fourier transformation that (i) if

$$u_0\in H^{s+3/2},\quad u_1\in H^s,\quad f\in C^0([0, T], H^s)$$

then there exists the unique solution

$$u\in C^j([0, T], H^{s+3/2-3j/2}),\quad j=0, 1, 2;$$

(ii) moreover, if $g>0$ then the above solution $u=u^{\mu}$ converges to the solution of (1.14), (1.15) with $\mu=0$ when μ tends to zero. For the problem (1.11)–(1.13), theorems corresponding to (i) will be given in §§4–6, (ii)–in §7.

In this paper we use the same notations in [2]. Here we repeat several of them. $H^s=H^s(\boldsymbol{R}^1)$, $-\infty<s<+\infty$, is a Hilbert space with the inner product

$$(u, v)_s=(2\pi)^{-1}\int(1+|\xi|)^{2s}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi,\quad \hat{u}(\xi)=\int u(x)e^{-ix\xi}dx.$$

For $u=(u_1,\dots, u_m)$, $u\in H^s$ means that $u_j\in H^s$ for all j.

$$(u, v)_s=(u_1, v_1)_s+\cdots+(u_m, v_m)_s,\quad \|u\|_s=(u, u)_s^{1/2},\quad (u, v)=(u, v)_0,\quad \|u\|=\|u\|_0.$$

A pseudo-differential operator $P(D)$, $D=-i\dfrac{\partial}{\partial x}$, is given by $\widehat{P(D)u}(\xi)=P(\xi)\hat{u}(\xi)$. Note that $\|\Lambda^s u\|=\|u\|_s$, $\Lambda=1+|D|$. $\mathscr{L}(H^r, H^s)$ is a Banach space consisting of a linear continuous operator A from H^r to H^s, whose norm is given by

$$\|A\|_{r,s}=\sup_u\|Au\|_s,\quad u\in H^r,\quad \|u\|_r=1.$$

If B is a Banach space and u is a B-valued C^j-function on the closed interval $[0, T]$, we say that u belongs to $C^j([0, T], B)$. By A^* we denote the (formal) adjoint operator in H^0 of an operator A: $(Au, v)=(u, A^*v)$. If A and B are operators then we put $[A, B]=AB-BA$.

In proofs we do not distinguish inessential constants and we use the same symbol C. The statement $C=C(a,\dots)>0$ means that C is a positive constant depending on $a,\dots$ The reason for expressing μ explicitly in estimates lies in the need of uniform estimates (with respect to μ), which will be necessary in §7.

§ 2. Quasilinearization

Putting

$$Y=X_{tt},\quad Z=X_x,\quad W=(X, Y, Z),\quad W'=(X, Y_1), \tag{2.1}$$

we shall reduce equations (1.11) and (1.12) to quasilinear equations for W. The method is similar to that used in [2]. The equation for X is

$$X_{tt}=Y. \tag{2.2}$$

Differentiations of (1.12) give

$$\partial_t^j X_{2t}=K(X)\partial_t^j X_{1t}+F_{j0}(X,\dots, \partial_t^j X), \tag{2.3}$$

$$\partial_t^j\partial_x^k X_{2t}=-i\,\mathrm{sgn}\, D\partial_t^j\partial_x^k X_{1t}+F_{jk0}, \tag{2.4}$$

$$F_{jk0}=\{i\,\mathrm{sgn}\, D+K(X)\}\partial_t^j\partial_x^k X_{1t}+F_{jk}(X,\dots, \partial_t^j\partial_x^k X, \partial_t^j X_{1t}),$$

where $j\geqq 0$, $k\geqq 1$ (see [2] §5). From (2.3) with $j=2$ we obtain

$$Y_{2t}=f_2(W, W'_t, b, h)=K(X)Y_{1t}+F_{20}(X, X_t, Y). \tag{2.5}$$

Adding (2.4) with $j=0$, $k=3$ multiplied by $-\mu$ to (2.4) with $j=0$, $k=1$ and applying $(1+\mu D^2)^{-1}$ to this sum we have

$$Z_{2t}=-i\,\mathrm{sgn}\, DZ_{1t}+(1+\mu D^2)^{-1}(F_{010}-\mu F_{030}), \tag{2.6}$$

where

$$\begin{aligned} F_{0k0}&=F_{0k0}(X, Z,\dots, \partial_x^{k-1}Z, X_{1t})\\ &=\{i\,\mathrm{sgn}\, D+K(X)\}(iD)^k X_{1t}+F_{0k}(X, Z,\dots, \partial_x^{k-1}Z, X_{1t}),\quad k=1, 3. \end{aligned} \tag{2.7}$$

Since $X_x=Z$, we obtain from (1.11)

$$\begin{aligned} (1+Z_1)Y_1+Z_2(g+Y_2)&=\mu R(Z, Z_x)+\mu S(Z, Z_{xx})\\ &\quad+g_0(1+\mu D^2)^{-1}i\,\mathrm{sgn}\, D(Z_1-iDX_1), \end{aligned} \tag{2.8}$$

(the constant g_0 will be determined later). Differentiating (2.8) with respect to t and eliminating Z_{2t} by (2.6) we have

$$\begin{aligned} &(g_0(1+\mu D^2)^{-1}i\,\mathrm{sgn}\, D+P_1)Z_{1t}+(-g-\mu D^2+P_2)\{-i\,\mathrm{sgn}\, DZ_{1t}\\ &\quad+(1+\mu D^2)^{-1}(F_{010}-\mu F_{030})\}-(1+Z_1)Y_{1t}-Z_2Y_{2t}\\ &\quad+g_0(1+\mu D^2)^{-1}D\,\mathrm{sgn}\, DX_{1t}=0, \end{aligned} \tag{2.9}$$

where

$$P_j = P_j(Y, Z, Z_x, Z_{xx}) \tag{2.10}$$
$$= -Y_j + \mu\frac{\partial R}{\partial Z_j}(Z, Z_x) + \mu\frac{\partial S}{\partial Z_j}(Z, Z_{xx}) + \mu\frac{\partial R}{\partial Z_{jx}}(Z, Z_x)iD$$
$$+\mu\frac{\partial S}{\partial Z_{jxx}}(Z, Z_{xx})(iD)^2 + (j-1)\mu D^2, \qquad j=1, 2.$$

Noting that $g_0(1+a)^{-1}+a \geqq 2g_0^{1/2}-1$ for $g_0 \geqq 1$ and $a>0$, we put

$$g_0 = \frac{1}{4}(2-g)^2 \quad \text{if} \quad g \leqq 0, \qquad g_0 = 0 \quad \text{if} \quad g>0. \tag{2.11}$$

Then for any $\mu>0$ and any real ξ it holds that

$$g+g_0(1+\mu\xi^2)^{-1}+\mu\xi^2 \geqq \text{const} > 0, \tag{2.12}$$

(const depends only on g). The operator acting on Z_{1t} in (2.9) can be written in the form

$$(1+P_3)(g+g_0(1+\mu D^2)^{-1}+\mu D^2)i \operatorname{sgn} D,$$

where

$$P_3 = -(P_1 i \operatorname{sgn} D + P_2)(g+g_0(1+\mu D^2)^{-1}+\mu D^2)^{-1}. \tag{2.13}$$

Replacing Y_{2t} in (2.9) by f_2 we obtain

$$Z_{1t} = f_3(W, W'_t, \mu, b, g, h) \tag{2.14}$$
$$= i \operatorname{sgn} D(g+g_0(1+\mu D^2)^{-1}+\mu D^2)^{-1}(1+P_3)^{-1}\{-(1+Z_1)Y_{1t} - Z_2 f_2$$
$$+g_0(1+\mu D^2)^{-1}|D|X_{1t} + (-g-\mu D^2+P_2)(1+\mu D^2)^{-1}(F_{010}-\mu F_{030})\}.$$

Replacing Z_{1t} in (2.6) by f_3 we obtain

$$Z_{2t} = f_4(W, W'_t, \mu, b, g, h) \tag{2.15}$$
$$= -i \operatorname{sgn} D f_3 + (1+\mu D^2)^{-1}(F_{010}-\mu F_{030}).$$

Remark 2.16. Assume that functions X, Y, Z satisfy (2.2), (2.5), (2.14) and (2.15). Then in virtue of (2.2) and (2.5),

$$(X_{2t} - K(X)X_{1t})_{tt} = 0. \tag{2.17}$$

From (2.14) and (2.15) it follows that (2.6) holds. The replacement of f_2 in (2.14) by Y_{2t} shows that (2.9) holds. Therefore it holds that

$$\partial_t\{(1+Z_1)Y_1 + Z_2(g+Y_2) - \mu R(Z, Z_x) - \mu S(Z, Z_{xx}) \tag{2.18}$$
$$-g_0(1+\mu D^2)^{-1} i \operatorname{sgn} D(Z_1 - iDX_1)\} = 0.$$

Moreover if $X_{2t} - K(X)X_{1t} = 0$ then by the same procedure that we used to obtain (2.6) from (1.12), we have (2.6) where Z is replaced by X_x. This equation and (2.6) give

$$(2.19)\quad (Z_2 - X_{2x})_t = -i\,\mathrm{sgn}\, D(Z_1 - X_{1x})_t$$
$$+(1+\mu D^2)^{-1}\{F_{01}(X, Z, X_{1t}) - F_{01}(X, X_x, X_{1t})\}$$
$$-\mu(1+\mu D^2)^{-1}\{F_{03}(X, Z, Z_x, Z_{xx}, X_{1t}) - F_{03}(X, X_x, X_{xx}, X_{xxx}, X_{1t})\}.$$

These comments (and also those in Remarks 2.37 and 2.44) will become necessary to show that the unique solvability of the nonlinear system follows from that of the quasilinear system.

From (2.3) with $j=3$ we have

$$(2.20)\qquad Y_{2tt} = KY_{1tt} + F_{30}(X, X_t, Y, Y_t).$$

After the replacement of X_{tt} in (1.11) by Y, differentiating (1.11) two times with respect to t and eliminating Y_{2tt} by means of (2.20) we have

$$(2.21)\quad (1+X_{1x}+X_{2x}K)Y_{1tt} = \mu(R+S)_{tt} - Y_1X_{1xtt} - (g+Y_2)X_{2xtt} - 2Y_t\cdot X_{xt} - X_{2x}F_{30}.$$

Using (2.10) we write the first three terms in the right hand side of (2.21) in the form

$$(2.22)\quad \mu\{R(X_x, X_{xx}) + S(X_x, X_{xxx})\}_{tt} - Y_1X_{1xtt} - (g+Y_2)X_{2xtt}$$
$$= P_1(Y, X_x, X_{xx}, X_{xxx})X_{1xtt} + (-g - \mu D^2 + P_2(Y, X_x, X_{xx}, X_{xxx}))X_{2xtt}$$
$$+\mu I_1(X_x, X_{xt}, X_{xx}, X_{xxt}, X_{xxx}, X_{xxxt}).$$

From (2.4) with $j=k=1$ and $j=1$, $k=3$ we obtain

$$(2.23)\qquad Y_{2x} = -i\,\mathrm{sgn}\, DY_{1x} + (1+\mu D^2)^{-1}(F_{110} - \mu F_{130}),$$

$$(2.24)\qquad F_{1k0} = F_{1k0}(X, X_t, Z, Z_t, \ldots, \partial_x^{k-1}Z, \partial_x^{k-1}Z_t, Y_1)$$
$$= \{i\,\mathrm{sgn}\, D + K(X)\}(iD)^kY_1$$
$$+F_{1k}(X, X_t, Z, Z_t, \ldots, \partial_x^{k-1}Z, \partial_x^{k-1}Z_t, Y_1),\quad k=1, 3.$$

In (2.21) replacing $\partial_x^jX, \partial_x^jX_t, X_{tt}$ by $\partial_x^{j-1}Z, \partial_x^{j-1}Z_t, Y\ (j=1, 2, 3)$ and eliminating Y_{2x} by means of (2.23) we have

$$(2.25)\qquad Y_{1tt} = (1+Z_1+Z_2K)^{-1}\{(P_1 - (-g-\mu D^2+P_2)i\,\mathrm{sgn}\, D)iDY_1 + I_2\},$$

where

$$(2.26)\qquad I_2 = I_2(X, X_t, Y, Y_t, Z, Z_t, Z_x, Z_{tx}, Z_{xx}, Z_{txx})$$
$$= \mu I_1(Z, Z_t, Z_x, Z_{tx}, Z_{xx}, Z_{txx})$$
$$+(-g-\mu D^2 + P_2(Y, Z, Z_x, Z_{xx}))(1+\mu D^2)^{-1}(F_{110} - \mu F_{130})$$
$$-2Y_t\cdot Z_t - Z_2F_{30}(X, X_t, Y, Y_t).$$

Using (see [2] (5.12))

$$(2.27)\quad (1+Z_1+Z_2K)^{-1} = Q(Z)^{-2}(1+Z_1+Z_2i\,\mathrm{sgn}\, D) + P_4,$$

$$(2.28)\quad \begin{aligned} P_4 &= P_4(X, Z) \\ &= -Q(Z)^{-2}Z_2\{i \operatorname{sgn} D + K(X)\} \\ &\quad + Q(Z)^{-2}Z_2\{[K, Z_1]+[K, Z_2]K + Z_2(1+K^2)\}(1+Z_1+Z_2K)^{-1}, \end{aligned}$$

we have

$$(2.29)\quad \begin{aligned} Y_{1tt} &= Q^{-2}(1+Z_1+Z_2 i \operatorname{sgn} D)(P_1-(-g-\mu D^2+P_2) i \operatorname{sgn} D) iDY_1 \\ &\quad + P_4(P_1-(-g-\mu D^2+P_2) i \operatorname{sgn} D) iDY_1 + (1+Z_1+Z_2K)^{-1} l_2. \end{aligned}$$

Noting that

$$(2.30)\quad \begin{aligned} &(1+Z_1+Z_2 i \operatorname{sgn} D)(P_1-(-g-\mu D^2+P_2) i \operatorname{sgn} D) \\ &= (1+Z_1)P_1 + Z_2(-g-\mu D^2+P_2) + \{Z_2P_1 - (1+Z_1)(-g-\mu D^2 \\ &\quad + P_2)\} i \operatorname{sgn} D + iZ_2([\operatorname{sgn} D, P_1] - [\operatorname{sgn} D, P_2] i \operatorname{sgn} D) \end{aligned}$$

and using definitions of R, S and P_j (cf. (1.10) and (2.10)) we have

$$(2.31)\quad \begin{aligned} &Q^{-2}(1+Z_1+Z_2 i \operatorname{sgn} D)(P_1-(-g-\mu D^2+P_2) i \operatorname{sgn} D) iD \\ &= -M - L - Q^{-2}\{(1+Z_1)Y_1 + Z_2(g+Y_2) - \mu(R+S)\} iD \\ &\quad + iZ_2Q^{-2}([\operatorname{sgn} D, P_1] - [\operatorname{sgn} D, P_2] i \operatorname{sgn} D) iD, \end{aligned}$$

where we used notations

$$(2.32)\quad \begin{cases} Q = Q(Z) = \{(1+Z_1)^2 + Z_2^2\}^{1/2}, \\ A_0 = A_0(Z) = Q(Z)^{-3}, \\ A_1 = A_1(Z, Z_x) = -3Q(Z)^{-5}(-Z_2Z_{1x} + (1+Z_1)Z_{2x}), \\ A_2 = A_2(Z, Z_x, Z_{xx}) = Q(Z)^{-2}(4R+3S) \\ \quad = -12Q(Z)^{-7}\{(1+Z_1)Z_{1x} + Z_2Z_{2x}\}\{-Z_2Z_{1x} + (1+Z_1)Z_{2x}\} \\ \quad\quad + 3Q(Z)^{-5}(-Z_2Z_{1xx} + (1+Z_1)Z_{2xx}), \\ A_3 = A_3(Z, Z_x, Z_{xx}) \\ \quad = 3Q(Z)^{-7}\{(-Z_2Z_{1x} + (1+Z_1)Z_{2x})^2 - ((1+Z_1)Z_{1x} + Z_2Z_{2x})^2\} \\ \quad\quad + Q(Z)^{-5}\{(1+Z_1)Z_{1xx} + Z_2Z_{2xx}\}, \\ A_4 = A_4(Y, Z) = Q(Z)^{-2}\{(1+Z_1)(g+Y_2) - Z_2Y_1\}, \\ M = M(W) = M(W, \mu) = \mu(A_0|D|^3 - iA_{0x}D|D| + A_1D^2), \\ L = L(W) = L(W, \mu) = i\mu A_2 D + (\mu A_3 + A_4)|D|. \end{cases}$$

Thus we have

$$(2.33)\quad Y_{1tt} + (M+L)Y_1 + Q^{-2}\{(1+Z_1)Y_1 + Z_2(g+Y_2) - \mu(R+S)\} iDY_1 = f_1,$$

(2.34) $$f_1 = f_1(W, W'_t, \mu, b, g, h)$$
$$= iZ_2Q(Z)^{-2}([\operatorname{sgn} D, P_1] - [\operatorname{sgn} D, P_2] i \operatorname{sgn} D) iDY_1$$
$$+ P_4(P_1 - (-g - \mu D^2 + P_2) i \operatorname{sgn} D) iDY_1 + (1 + Z_1 + Z_2K)^{-1} I_2$$

where Y_{2t}, $\partial_x^j Z_t$ in I_2 are replaced by f_2, $\partial_x^j(f_3, f_4)$.

Remark 2.35. The equation (2.33) can be written in the form

$$u_{tt} + \mu aDaDa|D|u - 3\mu aD\{Q(Z)^{-3}(-Z_2Z_{1x} + (1+Z_1)Z_{2x})\}aDu$$
$$+ 2\mu\{Q(Z)^{-3}(-Z_2Z_{1x} + (1+Z_1)Z_{2x})\}^2 a|D|u$$
$$+ Q(Z)^{-1}\{(1+Z_1)(g+Y_2) - Z_2Y_1\}a|D|u$$
$$+ iQ(Z)^{-1}\{(1+Z_1)Y_1 + Z_2(g+Y_2) - \mu R(Z, Z_x) - \mu S(Z, Z_{xx})\}aDu = f_1$$

where $u = Y_1$ and $a = Q^{-1}$. In this form the meaning of coefficients is clear:

$$-\alpha Q(Z)^{-3}(-Z_2Z_{1x} + (1+Z_1)Z_{2x})$$

is the pressure difference, see (1.9); the curvature of the curve $(x, 0) + X(t, x)$, $-\infty < x < +\infty$, is equal to

$$|Q(Z)^{-3}(-Z_2Z_{1x} + (1+Z_1)Z_{2x})|$$

and the line element of this curve is $ds = Q(Z)dx$, therefore

$$a\partial_x = Q^{-1}\partial_x = \frac{\partial}{\partial s};$$

the normal derivative of the pressure is proportional to A_4, that is,

$$Q(Z)^{-1}\{(1+Z_1)(g+Y_2) - Z_2Y_1\}$$
$$= Q(Z)^{-1}(-Z_2, 1+Z_1)\cdot(Y + (0, g)) = N\cdot(-\rho^{-1}\nabla p).$$

The above form is inadequete to our purpose, that is, H^s-estimates of solutions, because operators of order 2 and 3 are essential to this end.

Putting the third term of (2.33) equal to zero we obtain the required equations for W,

(2.36) $$X_{tt} = Y, \quad Y_{1tt} + (M+L)Y_1 = f_1, \quad Y_{2t} = f_2, \quad Z_t = (f_3, f_4).$$

f_j depends on W, W'_t, μ, b, g and h:

$$f = (f_1, \ldots, f_4), \quad f(W, W'_t) = f(W, W'_t, \mu) = f(W, W'_t, \mu, b),$$

(see (2.5), (2.14), (2.15) and (2.34)).

Remark 2.37. Let $W = (X, Y, Z)$ satisfy (2.36) and (2.8). Then (2.33) holds if f_1 is replaced by

$$f_1 + Q^{-2}Y_{1x}g_0(1+\mu D^2)^{-1} i \operatorname{sgn} D(Z_1 - X_{1x}).$$

Reversing the procedure from (2.25) to (2.33) we have

$$(2.38)\quad (1+Z_1)Y_{1tt}+Z_2(KY_{1tt}+F_{30})+2Y_t\cdot Z_t$$
$$=P_1Y_{1x}+(-g-\mu D^2+P_2)\{-i\operatorname{sgn} DY_{1x}+(1+\mu D^2)^{-1}(F_{110}-\mu F_{130})\}$$
$$+\mu I_1+(1+Z_1+Z_2K)Q^{-2}Y_{1x}g_0(1+\mu D^2)^{-1}i\operatorname{sgn} D(Z_1-X_{1x}).$$

According to Remark 2.16 we see, by differentiations of (2.17) and (2.6) with respect to t, that (2.20) and

$$Z_{2tt}=-i\operatorname{sgn} DZ_{1tt}+(1+\mu D^2)^{-1}(F_{110}-\mu F_{130})$$

hold, where we used definitions of F_{jk0}. In view of (2.22) and (2.10) we have

$$\mu I_1=\mu\{R(Z,Z_x)+S(Z,Z_{xx})\}_{tt}-(Y_1+P_1)Z_{1tt}-(g+Y_2-g-\mu D^2+P_2)Z_{2tt}.$$

Therefore (2.38) is transformed into

$$(1+Z_1+Z_2K)Q^{-2}Y_{1x}g_0(1+\mu D^2)^{-1}i\operatorname{sgn} D(Z_1-X_{1x})$$
$$=\{(1+Z_1)Y_1+Z_2(g+Y_2)-\mu R(Z,Z_x)-\mu S(Z,Z_{xx})\}_{tt}+P_1(Z_{1tt}-Y_{1x})$$
$$+(-g-\mu D^2+P_2)(Z_{2tt}+i\operatorname{sgn} DY_{1x}-Z_{2tt}-i\operatorname{sgn} DZ_{1tt})$$
$$=g_0(1+\mu D^2)^{-1}i\operatorname{sgn} D(Z_1-X_{1x})_{tt}+P_1(Z_1-X_{1x})_{tt}$$
$$-(-g-\mu D^2+P_2)i\operatorname{sgn} D(Z_1-X_{1x})_{tt}$$
$$=(g+g_0(1+\mu D^2)^{-1}+\mu D^2-P_1i\operatorname{sgn} D-P_2)i\operatorname{sgn} D(Z_1-X_{1x})_{tt}.$$

In virtue of (2.13) we obtain

$$(2.39)\quad i\operatorname{sgn} D(Z_1-X_{1x})_{tt}$$
$$=(g+g_0(1+\mu D^2)^{-1}+\mu D^2)^{-1}(1+P_3)^{-1}\times$$
$$(1+Z_1+Z_2K)Q^{-2}Y_{1x}g_0(1+\mu D^2)^{-1}i\operatorname{sgn} D(Z_1-X_{1x}).$$

It remains to determine values of W, W'_t at $t=0$ from $X^{(0)}$, $X_1^{(1)}$ by means of (1.11) and (1.12). Assume that X satisfies (1.11)–(1.13). Initial values of Z and X_{2t} are given by

$$(2.40)\quad Z=X_x,\quad X_{2t}=K(X)X_{1t},\quad t=0.$$

Eliminating X_{2tt} from (1.11) by means of (2.3) with $j=1$ we have

$$(2.41)\quad Y_1=\{1+Z_1+Z_2K(X)\}^{-1}\{-gZ_2-Z_2F_{10}(X,X_t)$$
$$+\mu R(Z,Z_x)+\mu S(Z,Z_{xx})\},\quad t=0.$$

In virtue of (2.3) with $j=1$,

$$(2.42)\quad Y_2=K(X)Y_1+F_{10}(X,X_t),\quad t=0.$$

Differentiating (1.11) with respect to t we obtain

$$(1+X_{1x})Y_{1t}+X_{2x}Y_{2t}-P_1X_{1tx}-(-g-\mu D^2+P_2)X_{2tx}=0,$$

which and (2.3) with $j=2$ lead to

$$(2.43)\quad Y_{1t}=\{1+Z_1+Z_2K(X)\}^{-1}\{-Z_2F_{20}(X, X_t, Y)+P_1(Y, Z, Z_x, Z_{xx})iDX_{1t}$$
$$+(-g-\mu D^2+P_2(Y, Z, Z_x, Z_{xx}))iDX_{2t}\},\quad t=0.$$

Thus (2.40)–(2.43) are the required transformation of X, X_{1t}, $t=0$, into W, W'_t, $t=0$.

Remark 2.44. (i) For initial values W, W'_t, $t=0$, determined in the above way, following relations hold: in virtue of (2.41) and (2.42),

$$(2.45)\qquad (1+Z_1)Y_1+Z_2(g+Y_2)=\mu R(Z, Z_x)+\mu S(Z, Z_{xx}),\quad t=0;$$

$$(2.46)\qquad (1+Z_1)Y_{1t}+Z_2\{K(X)Y_{1t}+F_{20}(X, X_t, Y)\}$$
$$-P_1iDX_{1t}-(-g-\mu D^2+P_2)iDX_{2t}=0,\quad t=0,$$

in virtue of (2.43). (ii) Let W be a solution of (2.36) with initial conditions given by (2.40)–(2.43) and $X_{2t}-KX_{1t}=0$ hold. Then (2.46) minus (2.18) with $t=0$ is

$$P_1(Z_{1t}-iDX_{1t})+(-g-\mu D^2+P_2)(Z_{2t}-iDX_{2t})$$
$$+g_0(1+\mu D^2)^{-1}i\operatorname{sgn} D(Z_{1t}-iDX_{1t})=0,\quad t=0.$$

From (2.19) with $t=0$ it follows that

$$(Z_2-X_{2x})_t=-i\operatorname{sgn} D(Z_1-X_{1x})_t,\quad t=0.$$

Therefore we have

$$(2.47)\qquad 0=(g+g_0(1+\mu D^2)^{-1}+\mu D^2-P_1i\operatorname{sgn} D-P_2)i\operatorname{sgn} D(Z_1-X_{1x})_t$$
$$=(1+P_3)(g+g_0(1+\mu D^2)^{-1}+\mu D^2)i\operatorname{sgn} D(Z_1-X_{1x})_t,\quad t=0.$$

§3. Estimates of functions and operators

In this section following facts (cf. [2] §2) will be used without comment.

$$(3.1)\qquad |u(x)|\leqq\{(2s-1)\pi\}^{-1/2}\|u\|_s,\quad u\in H^s,\quad s>1/2.$$

If u, $v\in H^s$, $s>1/2$, then $uv\in H^s$ and

$$(3.2)\qquad \|uv\|_s\leqq C\|u\|_s\|v\|_s,\quad C=C(s)>0.$$

If $u=(u_1,\dots, u_N)$, $w\in H^s$, $s\geqq 1$, and $u_j(x)$ is real then

$$(3.3)\qquad \|F(u)w\|_s\leqq C\|F\|_{(m)}(1+\|u\|_s)^m\|w\|_s$$

where $C=C(s, N)>0$, m is the integer, $s\leqq m<s+1$, $F(z)$ is a C^m-function on an open set containing

$$G(u)=\{z=(z_1,\dots, z_N)\mid |z_j|\leqq\sup_x|u_j(x)|,\ 1\leqq j\leqq N\}$$

and $\|F\|_{(m)}$ is the supremum of $|\partial_z^\alpha F(z)|$, $|\alpha|\leqq m$, $z\in G(u)$. The difference $F(u)-F(v)$ can be written in the form

$$F(u)-F(v)=\sum_1^N (u_j-v_j)\int_0^1 \frac{\partial F}{\partial z_j}(qu+(1-q)v)dq, \tag{3.4}$$

$$F(u)-F(v)=\sum_1^N \{F(v_1,\dots, v_{j-1}, u_j, u_{j+1},\dots, u_N) - F(v_1,\dots, v_{j-1}, v_j, u_{j+1},\dots, u_N)\}. \tag{3.5}$$

From (3.3) and (3.4) it follows that

$$\|F(u)-F(v)\|_s\leqq C\|F\|_{(m+1)}(1+\|u\|_s+\|v\|_s)^m\|u-v\|_s, \tag{3.6}$$

where $\|F\|_{(m+1)}=\sup_{\alpha,z}|D_z^\alpha F(z)|$, $|\alpha|\leqq m+1$, $z\in G(u)\cup G(v)$. Using

$$\frac{\partial}{\partial t}F(u)=\sum_1^N \frac{\partial F}{\partial z_j}(u)\frac{\partial u_j}{\partial t} \tag{3.7}$$

we see that

$$F(u)-F(0)\in C^k([0, T], H^s) \tag{3.8}$$

if $k\geqq 0$ is an integer, $s\geqq 1$, $u\in C^k([0, T], H^s)$ and F is a C^{m+k+1}-function in a certain open set.

From now on we assume that all functions are real-valued.

By (2.32) and (1.10) we see that Q, $A_0,\dots, A_4$, R and S are singular at $Z_1=-1$, $Z_2=0$. This singularity is avoided if $|Z_1(x)|\leqq\pi^{-1/2}\|Z_1\|_1<1$. Therefore we have the following lemma.

Lemma 3.9. *Let* $0<c_0<\pi^{1/2}$, $s\geqq 1$ *and* $c>0$. *If*

$$Y\in H^s,\quad Z\in H^{s+2},\quad \|Z_1\|_1\leqq c_0,\quad \|Y\|_s+\|Z\|_{s+2}\leqq c \tag{3.10}$$

then (i) $A_0\geqq const>0$ *(const depends only on* c_0*), (ii)*

$$A_0-1\in H^{s+2},\quad A_1\in H^{s+1},\quad A_2,\ A_3,\ A_4-g\in H^s$$

and these are Lipschitz continuous in Y, Z, *i.e.*

$$\|A_0(Z)-A_0(Z^0)\|_{s+2}+\|A_1(Z, Z_x)-A_1(Z^0, Z_x^0)\|_{s+1} + \|A_j(Z, Z_x, Z_{xx})-A_j(Z^0, Z_x^0, Z_{xx}^0)\|_s\leqq C\|Z-Z^0\|_{s+2},$$

$$\|A_4(Y, Z)-A_4(Y^0, Z^0)\|_s\leqq C\|(Y-Y^0, Z-Z^0)\|_s,$$

where Y^0, Z^0 *satisfy* (3.10), $C=C(c_0, c, s, g)>0$ *and* $j=2, 3$.

Lemma 3.11. *Let* $0<c_0<\pi^{1/2}$, $s\geqq 1$, μ, $c>0$. *If*

$$Y\in H^s,\quad Z\in H^{s+2},\quad \|Z_1\|_1\leqq c_0,\quad \|Y\|_s+\|Z\|_s\leqq c \tag{3.12}$$

then for operators P_1, P_2 *and* P_3 *(see* (2.10) *and* (2.13)*) following estimates hold for* $u\in\mathscr{S}$, $j=1, 2$ *and* $r=0, s$*: first,*

$$\|P_j u\|_r \leqq CN\|(1+\mu D^2)u\|_r, \quad \|P_3 u\|_r \leqq CN\|u\|_r,$$

$$\|[\operatorname{sgn} D,\ P_j]u\|_s \leqq CN\|(1+\mu D^2)u\|_1,$$

where $N=\|Y\|_s+\|(1+\mu D^2)Z\|_s+\|(1+\mu D^2)Z\|_s^2$ *and* $C=C(c_0,\ c,\ s,\ g)>0$; *secondly,*

$$\|(P_j-P_j^0)u\|_r \leqq CN_0\|(1+\mu D^2)u\|_r, \quad \|(P_3-P_3^0)u\|_r \leqq CN_0\|u\|_r,$$

$$\|[\operatorname{sgn} D,\ P_j-P_j^0]u\|_s \leqq CN_0\|(1+\mu D^2)u\|_1,$$

where $N_0=\|Y-Y^0\|_s+(1+\|(1+\mu D^2)Z\|_s+\|(1+\mu D^2)Z^0\|_s)^2\|(1+\mu D^2)(Z-Z^0)\|_s$, $P_k^0=P_k(Y^0,\ Z^0,\ Z_x^0,\ Z_{xx}^0)$, $k=1,\ 2,\ 3$, Y^0, Z^0 *satisfy* (3.12) *and* $C=C(c_0,\ c,\ s,\ g)>0$.

Proof. In view of (1.10), (3.2) and (3.3) we have

$$\text{(3.13)} \quad \begin{cases} \left\|\dfrac{\partial(R+S)}{\partial Z_j}\right\|_s \leqq C(\|Z_x\|_s^2+\|Z_{xx}\|_s), \\ \left\|\dfrac{\partial R}{\partial Z_{jx}}\right\|_s \leqq C\|Z_x\|_s, \quad \left\|\dfrac{\partial S}{\partial Z_{jxx}}-(j-1)\right\|_s \leqq C\|Z\|_s, \end{cases}$$

where $C=C(c_0,\ c,\ s)>0$. Using (3.1) if $r=0$ and (3.2) if $r=s$ we have

$$\text{(3.14)} \quad \begin{aligned} \|P_j u\|_r &\leqq C(\|Y\|_s+\mu\|Z_x\|_s^2+\mu\|Z_{xx}\|_s)\|u\|_r \\ &\quad + C\mu^{1/2}\|Z_x\|_s\mu^{1/2}\|Du\|_r+C\|Z\|_s\|\mu D^2 u\|_r \\ &\leqq C(\|Y\|_s+\|(1+\mu D^2)Z\|_s+\|(1+\mu D^2)Z\|_s^2)\|(1+\mu D^2)u\|_r. \end{aligned}$$

In the second inequality we used

$$\text{(3.15)} \quad \mu\|Dv\|_q^2 \leqq \|v\|_q^2+\|\mu D^2 v\|_q^2 \leqq \|(1+\mu D^2)v\|_q^2 \leqq 2(\|v\|_q^2+\|\mu D^2 v\|_q^2),$$

(q is real). These estimates are obtained if we multiply

$$a \leqq 1+a^2 \leqq (1+a)^2 \leqq 2(1+a^2), \quad a=\mu\xi^2,$$

by $(2\pi)^{-1}|(1+|\xi|)^q\hat{v}(\xi)|^2$ and integrate with respect to ξ. Since

$$\|(1+\mu D^2)(g+g_0(1+\mu D^2)^{-1}+\mu D^2)^{-1}u\|_r \leqq C\|u\|_r$$

where $C>0$ depends only on g, (2.13) and (3.14) give the required estimate for $P_3 u$. Using estimates

$$\|[\operatorname{sgn} D,\ a]u\|_s \leqq C\|a\|_s\|u\|_1, \quad C=C(s)>0,$$

(see [2] Lemma 2.14) and (3.13) we obtain

$$\|[\operatorname{sgn} D,\ P_j]u\|_s \leqq CN\|(1+\mu D^2)u\|_1.$$

Taking (3.3) and (3.4) into account we see that

$$\text{(3.16)} \quad \begin{cases} \left\|\dfrac{\partial(R+S)}{\partial Z_j}-\dfrac{\partial(R+S)^0}{\partial Z_j}\right\|_s \leqq C\|Z-Z^0\|_s(\|Z_x\|_s^2+\|Z_x^0\|_s^2+\|Z_{xx}\|_s+\|Z_{xx}^0\|_s) \\ \qquad\qquad + C\|Z_x-Z_x^0\|_s(\|Z_x\|_s+\|Z_x^0\|_s)+C\|Z_{xx}-Z_{xx}^0\|_s, \\ \left\|\dfrac{\partial R}{\partial Z_{jx}}-\dfrac{\partial R^0}{\partial Z_{jx}}\right\|_s \leqq C\|Z-Z^0\|_s(\|Z_x\|_s+\|Z_x^0\|_s)+C\|Z_x-Z_x^0\|_s, \\ \left\|\dfrac{\partial S}{\partial Z_{jxx}}-\dfrac{\partial S^0}{\partial Z_{jxx}}\right\|_s \leqq C\|Z-Z^0\|_s, \end{cases}$$

where $\dfrac{\partial(R+S)^0}{\partial Z_j}=\dfrac{\partial R}{\partial Z_j}(Z^0, Z^0_x)+\dfrac{\partial S}{\partial Z_j}(Z^0, Z^0_{xx})$ and so on. Thus

$$\|(P_j-P^0_j)u\|_r\leqq C\|Y-Y^0\|_s\|(1+\mu D^2)u\|_r+C(1+\|(1+\mu D^2)Z\|_s$$
$$+\|(1+\mu D^2)Z^0\|_s)^2\|(1+\mu D^2)(Z-Z^0)\|_s\|(1+\mu D^2)u\|_r.$$

Remaining estimates are easily obtained, so the proof is finished.

Lemma 3.17. *There exists $c_0>0$ depending only on g such that if $s>3/2$, $\mu>0$, $c>0$ and*

(3.18) $\qquad Y\in H^s,\quad Z\in H^{s+2},\quad \|Y\|_1+\|(1+\mu D^2)Z\|_1\leqq c_0,\quad \|Y\|_s+\|Z\|_s\leqq c$

then the inverse operator $(1+P_3)^{-1}$ exists and this operator is bounded in H^s and Lipschitz continuous in Y, Z:

$$\|(1+P_3)^{-1}\|_{s,s}\leqq C(1+\|Y\|_s+\|(1+\mu D^2)Z\|_s)^{2s},$$
$$\|(1+P_3)^{-1}-(1+P^0_3)^{-1}\|_{s,s}$$
$$\leqq C(1+\|Y\|_s+\|Y^0\|_s+\|(1+\mu D^2)Z\|_s+\|(1+\mu D^2)Z^0\|_s)^{4s+2}\times$$
$$(\|Y-Y^0\|_s+\|(1+\mu D^2)(Z-Z^0)\|_s),$$

where $C=C(c_0, c, s, g)>0$, $P^0_3=P_3(Y^0, Z^0, Z^0_x, Z^0_{xx})$ and Y^0, Z^0 satisfy (3.18).

Proof. Putting $r=0$, $s=1$ in Lemma 3.11 we see that there exists $0<c_0<\pi^{1/2}$ such that

$$\|P_3\|_{0,0}\leqq CN\leqq C(c_0+c_0^2)<1.$$

Therefore $(1+P_3)^{-1}$ exists and $\|(1+P_3)^{-1}u\|\leqq C\|u\|$, $C=C(c_0, g)>0$. Put $v=(1+P_3)^{-1}u$ and $\Lambda=1+|D|$. Then

$$\|\Lambda^s v\|=\|(1+P_3)^{-1}[1+P_3, \Lambda^s]v+(1+P_3)^{-1}\Lambda^s u\|$$
$$\leqq C\|[P_3, \Lambda^s]\Lambda^{1-s}\|_{0,0}\|\Lambda^{s-1}v\|+C\|u\|_s.$$

Since $\{\varepsilon(1+|\xi|)\}^{s-1}\leqq\{\varepsilon(1+|\xi|)\}^s+1$ it holds that

$$\|\Lambda^{s-1}v\|\leqq\varepsilon\|\Lambda^s v\|+\varepsilon^{1-s}\|v\|,\quad \varepsilon>0.$$

Putting $(2\varepsilon)^{-1}=C\|[P_3, \Lambda^s]\Lambda^{1-s}\|_{0,0}+1$ we obtain

(3.19) $$\frac{1}{2}\|v\|_s\leqq(2\varepsilon)^{-1}\varepsilon^{1-s}\|v\|+C\|u\|_s.$$

Noting that

$$[P_3, \Lambda^s]\Lambda^{1-s}=-([P_1, \Lambda^s]i\operatorname{sgn}D+[P_2, \Lambda^s])\Lambda^{1-s}(g+g_0(1+\mu D^2)^{-1}+\mu D^2)^{-1}$$

and using estimates

$$\|[a, \Lambda^s]\Lambda^{1-s}u\|\leqq C\|a\|_s\|u\|,\quad C=C(s)>0,$$

(see [2] Lemma 2.14) and (3.13) we obtain

$$\|[P_3, \Lambda^s]\Lambda^{1-s}\|_{0,0} \leqq C(\|Y\|_s + \|(1+\mu D^2)Z\|_s + \|(1+\mu D^2)Z\|_s^2),$$

which and (3.19) give the required estimate for $(1+P_3)^{-1}$. Using

$$\|(1+P_3)^{-1} - (1+P_3^0)^{-1}\|_{s,s} = \|(1+P_3)^{-1}(P_3^0 - P_3)(1+P_3^0)^{-1}\|_{s,s}$$
$$\leqq \|(1+P_3)^{-1}\|_{s,s}\|P_3^0 - P_3\|_{s,s}\|(1+P_3^0)^{-1}\|_{s,s}$$

and Lemma 3.11, we have the second inequality required in this lemma, so the proof is complete.

Lemma 3.20. *Let $0<c_0<\pi^{1/2}$, $s\geqq 1$, $\mu>0$ and $c>0$. If $W=(0, Y, Z)$ and $W^0=(0, Y^0, Z^0)$ satisfy (3.12) then for the operator $M+L$ (see (2.32)) following estimates hold for any $u\in\mathscr{S}$: first,*

$$\|(M+L)u\|_s \leqq C(1+\|(1+\mu D^2)Z\|_s)^2\|(1+\mu D^2)Du\|_s, \tag{3.21}$$

$$\begin{aligned} &\|(M-M^0+L-L^0)u\|_s \\ &\quad\leqq C\{\|Y-Y^0\|_s + (1+\|(1+\mu D^2)Z\|_s + \|(1+\mu D^2)Z^0\|_s)^2 \times \\ &\qquad \|(1+\mu D^2)(Z-Z^0)\|_s\}\|(1+\mu D^2)Du\|_s \end{aligned} \tag{3.22}$$

where $C=C(c_0, c, s, g)>0$, $M^0=M(W^0)$ and $L^0=L(W^0)$; secondly,

$$\begin{aligned} &\|\{M(W, \varepsilon)-M(W, \delta)+L(W, \varepsilon)-L(W, \delta)\}u\|_s \\ &\quad\leqq C(\delta-\varepsilon)\beta^{-1}(1+\|(1+\beta D^2)Z\|_s)^2\|(1+\beta D^2)Du\|_s \end{aligned} \tag{3.23}$$

where $\beta>0$, $0<\varepsilon<\delta$ and $C=C(c_0, c, s, g)>0$.

Proof. In view of (2.32) it is easily seen that for A_0-1, A_{0x}, A_1, A_2, A_3, A_4-g estimates analogous to (3.13) and (3.16) hold, so in the same way as in the proof of Lemma 3.11 we have (3.21) and (3.22). It is clear that

$$\begin{aligned} &M(W, \varepsilon)-M(W, \delta)+L(W, \varepsilon)-L(W, \delta) \\ &\quad=(\varepsilon-\delta)\beta^{-1}\{M(W, \beta)+L(W, \beta)\}-(\varepsilon-\delta)\beta^{-1}A_4|D|, \end{aligned}$$

which and (3.21) give (3.23). Thus the proof is complete.

For functions $W=(X, Y, Z)$, $W'_t=(X_t, Y_{1t})$ we put

$$\begin{aligned} \|W, W'_t\|_{s,\mu} &= (\|X\|_s^2 + \|X_t\|_s^2)^{1/2} + (\mu\|X_{1t}\|_{s+3/2}^2 + g_0^2\mu^{-1}\|X_{1t}\|_s^2)^{1/2} \\ &\quad + (\|Y_{1t}\|_s^2 + \mu\|Y_1\|_{s+3/2}^2 + \|Y_1\|_s^2)^{1/2} + \|Y_2\|_s + \|(1+\mu D^2)Z\|_s. \end{aligned} \tag{3.24}$$

Since $f_j(W, W'_t)$ does not depend explicitly on t (see §2), there is no need to regard W'_t as the t-derivative of $W'=(X, Y_1)$. In following two lemmas, "t" in W'_t does not mean the t-derivative.

Lemma 3.25. *There exists $c_0=c_0(g, h)>0$ such that if $\mu>0$, $s\geqq 4$, $c>0$ and b, W, W'_t satisfy conditions*

$$b\in H^{s+3},\quad \|b\|_3\leqq c_0,\quad \|b\|_{s+3}\leqq c,$$

$$\text{(3.26)}\qquad \begin{cases} W,\ W'_t,\ \Lambda^{3/2}Y_1,\ \Lambda^2 Z,\ \Lambda^{3/2}X_{1t}\in H^s,\\ \|X\|_3+\|Y\|_1+\|(1+\mu D^2)Z\|_1+\|Z\|_3\leqq c_0,\quad \|(W,\ W'_t)\|_s\leqq c,\end{cases}$$

then f_j, $(1+\mu D^2)f_{2+j}\in H^s$, $j=1, 2$, *and*

$$\text{(3.27)}\qquad \|f_j\|_s+\|(1+\mu D^2)f_{2+j}\|_s\leqq C_1(1+\mu)(1+\|W,\ W'_t\|_{s,\mu})^{4s+7}\|W,\ W'_t\|_{s,\mu}$$

where $C_1=C_1(c_0,\ c,\ s,\ g,\ h)>0$ *and* $f_k=f_k(W,\ W'_t,\ \mu,\ b)$, $1\leqq k\leqq 4$; *moreover, for* W^0, $W_t^{0\prime}$ *satisfying* (3.26) *it holds that*

$$\text{(3.28)}\qquad \|f_j-f_j^0\|_s+\|(1+\mu D^2)(f_{2+j}-f_{2+j}^0)\|_s$$
$$\leqq C(1+\mu)(1+\|W,\ W'_t\|_{s,\mu}+\|W^0,\ W_t^{0\prime}\|_{s,\mu})^{6s+10}\|W-W^0,\ W'_t-W_t^{0\prime}\|_{s,\mu}$$

where $j=1, 2$, $C=C(c_0, c, s, g, h)>0$ *and* $f_k^0=f_k(W^0,\ W_t^{0\prime},\ \mu,\ b)$, $1\leqq k\leqq 4$.

Proof. In this proof we use follwing notations: first, spaces of operators $L(r, s; t)$, $L_0(r, s; t)$ and operators $K_{1,j,k}$, see [2] §4; secondly, functions F_{jk} given by [2] (5.2); by F_{jk}^0, $K_{1,j,k}^0,\ldots$ we denote F_{jk}, $K_{1,j,k},\ldots$ in which W, W'_t are substituted by W^0, $W_t^{0\prime}$; by c_0 we denote all constants depending only on g and h (the minimum of them is the constant c_0 stated in this lemma); by C we denote all constants depending on c_0, c, s, g and h except the case specially stated; lastly,

$$J(p,\ q,\ r)=(1+\mu^{1/2})^p(1+\|W,\ W'_t\|_{s,\mu})^q(1+\|W,\ W'_t\|_{s,\mu}+\|W^0,\ W_t^{0\prime}\|_{s,\mu})^r,$$

$$E=\|W,\ W'_t\|_{s,\mu},\quad E_0=\|W-W^0,\ W'_t-W_t^{0\prime}\|_{s,\mu}.$$

According to [2] Lemmas 4.24 and 4.27 it holds that

$$K_{1,j,k}\in L_0(2+r,\ m+r;\ 3),\quad 0\leqq r<1,\quad m\geqq 3,$$

(m is an integer). By considerations in [2] §4 we see that $K_{1,j,k}$ is rational in b_x, $A_{l,0,0}$ ($1\leqq l\leqq 8$) and polynomial in $\partial_x b_x,\ldots,\partial_x^k b_x$, $A_{l,p,q}$ ($1\leqq l\leqq 8$, $0\leqq p\leqq j$, $0\leqq q\leqq k$, $p+q>0$). By [2] Lemmas 4.14–4.20 we have the following estimates for the operator-norm of $K_{1,j,k}$: there exists $c_0=c_0(h)>0$ such that if $j+k>0$,

$$\text{(3.29)}\qquad X^{00},\ldots,\ X^{jk},\ b,\ldots,\ \partial_x^k b\in H^{m+r},\quad \|X^{00}\|_3+\|b\|_3\leqq c_0,$$

then it holds that

$$\text{(3.30)}\qquad \|K_{1,j,k}(X^{00},\ldots,\ X^{jk},\ b,\ldots,\ \partial_x^k b;\ X^{00},\ b)\|_{2+r,m+r}$$
$$\leqq C\sum_{n,N,p,q}\prod_{i=1}^{n}\|(X^{p_i q_i},\ b^{p_i q_i})\|_{m+r}^{N_i}$$

where $C=C(m, r, j, k, c_0, h)>0$, $b^{\alpha\beta}=\partial_x^\beta b$ if $\alpha=0$, $b^{\alpha\beta}=0$ if $\alpha>0$ and the summation is taken over

$$1\leqq n\leqq j+k,\quad N_i\geqq 1,\quad 0\leqq p_i\leqq j,\quad 0\leqq q_i\leqq k,\quad p_i+q_i\geqq 1,$$

$$N_1p_1+\cdots+N_np_n=j,\quad N_1q_1+\cdots+N_nq_n=k;$$

moreover, for $Y^{00},\dots, Y^{jk}$ satisfying (3.29) it holds that

$$(3.31)\qquad \begin{aligned}&\|K_{1,j,k}(X^{00},\dots, X^{jk}, b,\dots, \partial_x^k b;\ X^{00}, b)\\ &\quad -K_{1,j,k}(Y^{00},\dots, Y^{jk}, b,\dots, \partial_x^k b;\ Y^{00}, b)\|_{2+r,m+r}\\ &\leqq C\|X^{00}-Y^{00}\|_{m+r}\sum_{n,N,p,q}\prod_{i=1}^{n}\|(X^{p_iq_i}, Y^{p_iq_i}, b^{p_iq_i})\|_{m+r}^{N_i}\\ &\quad +C\sum_{n,N,p,q}\sum_{l=1}^{n}\|X^{p_lq_l}-Y^{p_lq_l}\|_{m+r}\|(X^{p_lq_l}, Y^{p_lq_l}, b^{p_lq_l})\|_{m+r}^{N_l-1}\times\\ &\quad \times\prod_{i\neq l}\|(X^{p_iq_i}, Y^{p_iq_i}, b^{p_iq_i})\|_{m+r}^{N_i}\end{aligned}$$

where $C=C(m, r, j, k, c_0, h)>0$.

Estimates for f_2. From (2.5) and the definition of F_{jk} we see that

$$f_2=K(X, b;\ X, b)Y_{1t}+2K_{1,1,0}(X, X_t, b;\ X, b)Y_1+K_{1,2,0}(X, X_t, Y, b;\ X, b)X_{1t}.$$

Therefore we have

$$(3.32)\qquad \begin{cases}\|f_2\|_s\leqq C\|(Y_{1t}, Y_1, X_{1t})\|_s\leqq CE,\\ \|f_2-f_2^0\|_s=\|(K-K^0)Y_{1t}+K^0(Y_{1t}-Y_{1t}^0)+\cdots\|_s\\ \qquad\leqq C\|(X-X^0, Y-Y^0, X_t-X_t^0, Y_{1t}-Y_{1t}^0)\|_s\leqq CE_0.\end{cases}$$

Estimates for f_3 and f_4. From (2.7) it follows that

$$\begin{aligned}F_{0k0}&=\{i\operatorname{sgn} D-i\tanh(hD)+K_{1,0,0}(X, b;\ X, b)\}(iD)^kX_{1t}\\ &\quad+\sum_{q=1}^{k}\binom{k}{q}K_{1,0,q}(X, Z,\dots, \partial_x^{q-1}Z, b,\dots, \partial_x^q b;\ X, b)(iD)^{k-q}X_{1t},\quad k=1, 3.\end{aligned}$$

It is clear that

$$(3.33)\qquad \begin{aligned}&\|F_{010}\|_s\leqq C\|X_{1t}\|_s\leqq CE,\\ &\|F_{010}-F_{010}^0\|_s\leqq C\|(X-X^0, Z-Z^0, X_{1t}-X_{1t}^0)\|_s\leqq CE_0.\end{aligned}$$

Using (3.30) and (3.31) we see that

$$\begin{aligned}\|K_{1,0,3}X_{1t}\|_s&\leqq C(\|(Z_{xx}, b_{xxx})\|_s+\|(Z_x, b_{xx})\|_s\|(Z, b_x)\|_s+\|(Z, b_x)\|_s^3)\|X_{1t}\|_s\\ &\leqq C(1+\|Z_x\|_s+\|Z_{xx}\|_s)\|X_{1t}\|_s\leqq C(1+\|Z_{xx}\|_s)\|X_{1t}\|_s,\end{aligned}$$

(in the last inequality we used (3.15) with $\mu=1$, $q=s$),

$$\begin{aligned}&\|K_{1,0,3}X_{1t}-K_{1,0,3}^0X_{1t}^0\|_s=\|(K_{1,0,3}-K_{1,0,3}^0)X_{1t}+K_{1,0,3}^0(X_{1t}-X_{1t}^0)\|_s\\ &\leqq C\|X-X^0\|_s(\|(Z_{xx}, Z_{xx}^0, b_{xxx})\|_s+\|(Z_x, Z_x^0, b_{xx})\|_s\|(Z, Z^0, b_x)\|_s\\ &\qquad +\|(Z, Z^0, b_x)\|_s^3)\|X_{1t}\|_s\\ &\quad +C(\|Z_{xx}-Z_{xx}^0\|_s+\|Z_x-Z_x^0\|_s\|(Z, Z^0, b_x)\|_s+\|(Z_x, Z_x^0, b_{xx})\|_s\|Z-Z^0\|_s\\ &\quad +\|(Z, Z^0, b_x)\|_s^2\|Z-Z^0\|_s)\|X_{1t}\|_s+C(1+\|Z_{xx}^0\|_s)\|X_{1t}-X_{1t}^0\|_s\\ &\leqq C(1+\|Z_{xx}\|_s+\|Z_{xx}^0\|_s)\|X-X^0\|_s\|X_{1t}\|_s+C(\|Z_{xx}-Z_{xx}^0\|_s+\|Z-Z^0\|_s)\|X_{1t}\|_s\\ &\quad +C(1+\|Z_{xx}^0\|_s)\|X_{1t}-X_{1t}^0\|_s.\end{aligned}$$

The estimates for $K_{1,0,q}(iD)^{3-q}X_{1t}$ is similar to the above. Summing up these estimates we have

$$(3.34)\quad \|\mu F_{030}\|_s \leqq \mu C\|DX_{1t}\|_s + \mu C(1+\|Z_{xx}\|_s)\|X_{1t}\|_s$$
$$\leqq C(\mu\|X_{1t}\|_{s+3/2} + \mu\|Z_{xx}\|_s) \leqq C(1+\mu^{1/2})E,$$

$$(3.35)\quad \|\mu F_{030} - \mu F^0_{030}\|_s \leqq \mu C(\|X - X^0\|_s\|(DX_{1t}, DX^0_{1t})\|_s + \|D(X_{1t} - X^0_{1t})\|_s)$$
$$+\mu C\{(\|X_{1t}\|_s + \|(Z_{xx}, Z^0_{xx})\|_s)\|X - X^0\|_s + \|Z_{xx} - Z^0_{xx}\|_s$$
$$+\|Z - Z^0\|_s\|X_{1t}\|_s + \|X_{1t} - X^0_{1t}\|_s + \|Z^0_{xx}\|_s\|X_{1t} - X^0_{1t}\|_s\}$$
$$\leqq CJ(1, 0, 1)E_0.$$

Noting that

$$\|g_0(1+\mu D^2)^{-1}|D|X_{1t}\|_s \leqq g_0\mu^{-1/2}\|X_{1t}\|_s$$

and using Lemmas 3.11, 3.17 and (3.32)–(3.35) we obtain

$$(3.36)\quad \|(1+\mu D^2)f_3\|_s \leqq \|(1+\mu D^2)(g + g_0(1+\mu D^2)^{-1} + \mu D^2)^{-1}\|_{s,s}\|(1+P_3)^{-1}\|_{s,s}\times$$
$$\times(\|Y_{1t}\|_s + \|f_2\|_s + g_0\mu^{-1/2}\|X_{1t}\|_s$$
$$+\|(-g - \mu D^2 + P_2)(1+\mu D^2)^{-1}\|_{s,s}\|F_{010} - \mu F_{030}\|_s)$$
$$\leqq CJ(1, 2s+2, 0)E,$$

$$(3.37)\quad \|(1+\mu D^2)(f_3 - f^0_3)\|_s$$
$$\leqq C\|(1+P_3)^{-1} - (1+P^0_3)^{-1}\|_{s,s}\| - (1+Z_1)Y_{1t} - Z_2 f_2 + g_0(1+\mu D^2)^{-1}|D|X_{1t}$$
$$+(-g - \mu D^2 + P_2)(1+\mu D^2)^{-1}(F_{010} - \mu F_{030})\|_s$$
$$+C\|(1+P^0_3)^{-1}\|_{s,s}\{\| - (Z_1 - Z^0_1)Y_{1t} - (1+Z^0_1)(Y_{1t} - Y^0_{1t}) - (Z_2 - Z^0_2)f_2$$
$$-Z^0_2(f_2 - f^0_2) + g_0(1+\mu D^2)^{-1}|D|(X_{1t} - X^0_{1t})\|_s$$
$$+\|(P_2 - P^0_2)(1+\mu D^2)^{-1}(F_{010} - \mu F_{030})$$
$$+(-g - \mu D^2 + P^0_2)(1+\mu D^2)^{-1}(F_{010} - F^0_{010} - \mu F_{030} + \mu F^0_{030})\|_s\}$$
$$\leqq CJ(1, 0, 4s+5)E_0.$$

By (2.15) we see that

$$(3.38)\quad \begin{aligned} &\|(1+\mu D^2)f_4\|_s \leqq CJ(1, 2s+2, 0)E,\\ &\|(1+\mu D^2)(f_4 - f^0_4)\|_s \leqq CJ(1, 0, 4s+5)E_0. \end{aligned}$$

Estimates for f_1. We use the decomposition $f_1 = f_{11} + f_{12} + f_{13}$ corresponding to three terms in (2.34). Using Lemma 3.11 and noting that

$$\|(1+\mu D^2)iDY_1\|_1 \leqq \|Y_1\|_2 + \mu\|Y_1\|_4 \leqq \|Y_1\|_s + \mu\|Y_1\|_{s+3/2}$$

we have, in the same way in (3.36) and (3.37), (3.39) $$\|f_{11}\|_s \leqq CJ(1, 2, 0)E, \quad \|f_{11}-f_{11}^0\|_s \leqq CJ(1, 0, 3)E_0.$$

f_{12} and f_{13} contain the operator

$$M = Z_1 + Z_2 K = Z_1 + Z_2\{-i\tanh(hD) + K_{1,0,0}(X, b; X, b)\}.$$

If $\|X\|_3 + \|b\|_3 \leqq c_0$ then

$$\|K_{1,0,0}u\|_q \leqq \|K_{1,0,0}u\|_3 \leqq C\|u\|_2 \leqq C\|u\|_q$$

where $C = C(c_0, h) > 0$ and $2 \leqq q \leqq 3$. Therefore if

$$\|X\|_3 + \|Z\|_3 + \|b\|_3 \leqq c_0, \quad \|X^0\|_3 + \|Z^0\|_3 + \|b\|_3 \leqq c_0$$

($c_0 > 0$ is suitably small) then

(3.40) $$\begin{aligned} &\|(1+M)^{-1}u\|_q \leqq C\|u\|_q, \\ &\|(1+M)^{-1}u - (1+M^0)^{-1}u\|_q \leqq C(\|X - X^0\|_3 + \|Z - Z^0\|_3)\|u\|_q. \end{aligned}$$

Let $s = m + r$ ($0 \leqq r < 1$ and $m \geqq 4$ is an integer). Using (3.40) with $q = 2 + r$ and noting that operators $K_{1,0,0}$, $[K, Z_j]$ and $1 + K^2$ belong to $L_0(2+r, m+r; 3)$ we see that

$$P_4 = P_4(X, Z, b; X, Z, b) \in L_0(2+r, m+r; 3),$$

(cf. the proof of [2] Lemma 5.22). Using the estimate

$$\|(1+\mu D^2)DY_1\|_{2+r} \leqq \|Y_1\|_s + \mu\|Y_1\|_{s+3/2}$$

and Lemma 3.11 we have

(3.41) $$\|f_{12}\|_s \leqq CJ(1, 2, 0)E, \quad \|f_{12} - f_{12}^0\|_s \leqq CJ(1, 0, 3)E_0.$$

Let T_z be the translation operator: $(T_z u)(x) = u(x+z) = u^z(x)$. By definitions in [2] §3 we see that

$$T_z K(X, b; X, b) = K(X^z, b^z; X^z, b^z)T_z = K^z T_z.$$

Applying (3.40) with $q = 3$ to

$$\begin{aligned} &T_z(1+M)^{-1}u - (1+M)^{-1}u \\ &\quad = -(1+M)^{-1}[T_z, 1+M](1+M)^{-1}u + (1+M)^{-1}(T_z u - u), \\ &[T_z, 1+M] = (Z_1^z - Z_1)T_z + (Z_2^z - Z_2)T_z K + Z_2(K^z - K)T_z, \end{aligned}$$

and repeating the same procedure that we used in the proof of [2] Lemma 4.22 we see that

(3.42) $$M_1(X, Z, b; X, Z, b) = (1+M)^{-1} \in L_0(m+r, m+r; 3).$$

Since $f_{13} = (1+M)^{-1}I_2$, it remains to prove estimates for I_2. To simplify notations, from now on, we denote f_2, f_3, f_4 by Y_{2t}, Z_{1t}, Z_{2t} respectively, (where t does not mean t-derivatives). From (2.22) it follows that

$$I_1=\sum_{j,k=1}^{2}\left(\frac{\partial^2 R}{\partial Z_j\partial Z_k}Z_{jt}Z_{kt}+2\frac{\partial^2 R}{\partial Z_{jx}\partial Z_k}Z_{jtx}Z_{kt}+\frac{\partial^2 R}{\partial Z_{jx}\partial Z_{kx}}Z_{jtx}Z_{ktx}\right.$$
$$\left.+\frac{\partial^2 S}{\partial Z_j\partial Z_k}Z_{jt}Z_{kt}+2\frac{\partial^2 S}{\partial Z_{jxx}\partial Z_k}Z_{jtxx}Z_{kt}\right).$$

In view of (1.10), (3.3), (3.15) and (3.36) we obtain

(3.43) $$\|\mu I_1\|_s\leqq \mu C(\|Z_x\|_s^2\|Z_t\|_s^2+\|Z_x\|_s\|Z_t\|_s\|Z_{tx}\|_s+\|Z_{tx}\|_s^2$$
$$+\|Z_{xx}\|_s\|Z_t\|_s^2+\|Z_t\|_s\|Z_{txx}\|_s)$$
$$\leqq C(1+\|(1+\mu D^2)Z\|_s)^2\|(1+\mu D^2)Z_t\|_s^2\leqq CJ(2,\ 4s+7,\ 0)E,$$

moreover, in virtue of (3.4) and (3.37), we have

(3.44) $$\mu\|I_1-I_1^0\|_s\leqq CB(2,\ 2)\|Z-Z^0\|_s+B(1,\ 2)\|(1+\mu D^2)(Z-Z^0)\|_s$$
$$+B(2,\ 1)\|(1+\mu D^2)(Z_t-Z_t^0)\|_s\leqq CJ(2,\ 0,\ 6s+10)E_0,$$
$$B(j,\ k)=(1+\|(1+\mu D^2)Z\|_s$$
$$+\|(1+\mu D^2)Z^0\|_s)^j(\|(1+\mu D^2)Z_t\|_s+\|(1+\mu D^2)Z_t^0\|_s)^k.$$

From (2.24) it follows that ($k=1,\ 3$)

$$F_{1k0}=\{i\ \mathrm{sgn}\ D-i\tanh(hD)+K_{1,0,0}(X,\ b;\ X,\ b)\}(iD)^kY_1$$
$$+K_{1,1,0}(X,\ X_t,\ b;\ X,\ b)(iD)^kX_{1t}$$
$$+\sum_1^k\binom{k}{q}K_{1,0,q}(X,\ Z,\dots,\ \partial_x^{q-1}Z,\ b,\dots,\ \partial_x^q b;\ X,\ b)(iD)^{k-q}Y_1$$
$$+\sum_1^k\binom{k}{q}K_{1,1,q}(X,\ X_t,\ Z,\ Z_t,\dots,\ \partial_x^{q-1}Z,\ \partial_x^{q-1}Z_t,\ b,\dots,\ \partial_x^q b;\ X,\ b)(iD)^{k-q}X_{1t}$$
$$=G_1+\cdots+G_4.$$

For $k=1$ we see by (3.30), (3.31), (3.36) and (3.37) that

(3.45) $$\|F_{110}\|_s\leqq C(1+\|Z_t\|_s)(\|Y_1\|_s+\|X_{1t}\|_s)\leqq CJ(1,\ 2s+3,\ 0)E,$$

(3.46) $$\|F_{110}-F_{110}^0\|_s\leqq C(1+\|Z_t\|_s+\|Z_t^0\|_s)(\|X-X^0\|_s+\|X_t-X_t^0\|_s$$
$$+\|Z-Z^0\|_s+\|Y_1-Y_1^0\|_s)+C\|Z_t-Z_t^0\|_s$$
$$\leqq CJ(1,\ 0,\ 4s+5)E_0.$$

For $k=3$, estimates for $G_1+G_2+G_3$ are similar to those for F_{030}:

$$\mu\|G_1+G_2+G_3\|_s\leqq CJ(1,\ 0,\ 0)E,\ \mu\|G_1+G_2+G_3-G_1^0-G_2^0-G_3^0\|_s\leqq CJ(1,\ 0,\ 1)E_0,$$

cf. (3.34) and (3.35). Using (3.30) and (3.31) we obtain

$$\mu\|K_{1,1,3}X_{1t}\|_s$$
$$\leqq\mu C\{\|Z_{txx}\|_s+\|Z_{tx}\|_s\|(Z,\ b_x)\|_s+\|Z_t\|_s\|(Z_x,\ b_{xx})\|_s+\|X_t\|_s\|(Z_{xx},\ b_{xxx})\|_s$$
$$+\|Z_t\|_s\|(Z,\ b_x)\|_s^2+\|X_t\|_s\|(Z_x,\ b_{xx})\|_s\|(Z,\ b_x)\|_s+\|X_t\|_s\|(Z,\ b_x)\|_s^3\}\ \|X_{1t}\|_s$$
$$\leqq C(1+\mu^{1/2})(1+\|(1+\mu D^2)Z\|_s)(1+\|(1+\mu D^2)Z_t\|_s)(1+\mu^{1/2})\|X_{1t}\|_s$$
$$\leqq CJ(2,\ 2s+4,\ 0)E,$$

$$\mu\|K_{1,1,3}X_{1t}-K^0_{1,1,3}X^0_{1t}\|_s=\mu\|(K_{1,1,3}-K^0_{1,1,3})X_{1t}+K^0_{1,1,3}(X_{1t}-X^0_{1t})\|_s$$
$$\leqq C\|X-X^0\|_s J(2,\ 0,\ 2s+4)E$$
$$+\mu C\{\|Z_{txx}-Z^0_{txx}\|_s+\|Z_{tx}-Z^0_{tx}\|_s\|(Z,\ Z^0,\ b_x)\|_s+\|(Z_{tx},\ Z^0_{tx})\|_s\|Z-Z^0\|_s$$
$$+\|Z_t-Z^0_t\|_s\|(Z_x,\ Z^0_x,\ b_{xx})\|_s+\|(Z_t,\ Z^0_t)\|_s\|Z_x-Z^0_x\|_s$$
$$+\|X_t-X^0_t\|_s\|(Z_{xx},\ Z^0_{xx},\ b_{xxx})\|_s$$
$$+\|(X_t,\ X^0_t)\|_s\|Z_{xx}-Z^0_{xx}\|_s+\|Z_t-Z^0_t\|_s\|(Z,\ Z^0,\ b_x)\|^2_s$$
$$+\|(Z_t,\ Z^0_t)\|_s\|Z-Z^0\|_s\|(Z,\ Z^0,\ b_x)\|_s+\|X_t-X^0_t\|_s\|(Z_x,\ Z^0_x,\ b_{xx})\|_s\|(Z,\ Z^0,\ b_x)\|_s$$
$$+\|(X_t,\ X^0_t)\|_s\|Z_x-Z^0_x\|_s\|(Z,\ Z^0,\ b_x)\|_s+\|(X_t,\ X^0_t)\|_s\|(Z_x,\ Z^0_x,\ b_{xx})\|_s\|Z-Z^0\|_s$$
$$+\|X_t-X^0_t\|_s\|(Z,\ Z^0,\ b_x)\|^3_s+\|(X_t,\ X^0_t)\|_s\|Z-Z^0\|_s\|(Z,\ Z^0,\ b_x)\|^2_s\}\|X_{1t}\|_s$$
$$+CJ(2,\ 0,\ 2s+4)(1+\mu^{1/2})\|X_{1t}-X^0_{1t}\|_s$$
$$\leqq CJ(2,\ 0,\ 2s+5)E_0+C(1+\mu^{1/2})(1+\|(1+\mu D^2)Z\|_s+\|(1+\mu D^2)Z^0\|_s)\times$$
$$(\|X_t-X^0_t\|_s+\|(1+\mu D^2)(Z-Z^0)\|_s+\|(1+\mu D^2)(Z_t-Z^0_t)\|_s)(1+\mu^{1/2})\|X_{1t}\|_s$$
$$+C(\|(1+\mu D^2)Z_t\|_s+\|(1+\mu D^2)Z^0_t\|_s)\|(1+\mu D^2)(Z-Z^0)\|_s\mu^{1/2}\|X_{1t}\|_s$$
$$+CJ(2,\ 0,\ 2s+5)E_0\leqq CJ(2,\ 0,\ 4s+7)E_0.$$

Estimates for remaining terms of G_4 are similar to the above. Thus we have

$$\|\mu F_{130}\|_s\leqq CJ(2,\ 2s+4,\ 0)E,\quad \mu\|F_{130}-F^0_{130}\|_s\leqq CJ(2,\ 0,\ 4s+7)E_0. \tag{3.47}$$

It is easily seen that

$$\|Y_tZ_t\|_s\leqq CJ(1,\ 2s+4,\ 0)E,\quad \|Y_tZ_t-Y^0_tZ^0_t\|_s\leqq CJ(1,\ 0,\ 4s+6)E_0, \tag{3.48}$$

$$\|Z_2F_{30}\|_s\leqq CE,\quad \|Z_2F_{30}-Z^0_2F^0_{30}\|_s\leqq CE_0. \tag{3.49}$$

From (3.43)–(3.49) it follows that

$$\|I_2\|_s\leqq CJ(2,\ 4s+7,\ 0)E,\quad \|I_2-I^0_2\|_s\leqq CJ(2,\ 0,\ 6s+10)E_0,$$

cf. (2.26). Using (3.42) we see that

$$\|f_{13}\|_s\leqq CJ(2,\ 4s+7,\ 0)E,\quad \|f_{13}-f^0_{13}\|_s\leqq CJ(2,\ 0,\ 6s+10)E_0, \tag{3.50}$$

which complete the proof.

Lemma 3.51. *Let $g>0$ (note that $g_0=0$, see (2.11)) and $c_0=c_0(g,\ h)>0$ be the constant in Lemma 3.25. If $s\geqq 4$, $0<\sigma<2$, $c>0$ and b, W, W'_t satisfy conditions*

$$b\in H^{s+\sigma+3},\quad W,\ W'_t,\ \Lambda^{3/2}Y_1,\ \Lambda^2Z,\ \Lambda^{3/2}X_{1t}\in H^{s+\sigma},$$
$$\|b\|_3\leqq c_0,\quad \|X\|_3+\|Y\|_1+\|(1+D^2)Z\|_1+\|Z\|_3\leqq c_0,$$
$$\|b\|_{s+\sigma+3}+\|(W,\ W'_t)\|_{s+\sigma}\leqq c$$

then for $0<\varepsilon<\delta<1$ *it holds that*

$$\|f_j^\varepsilon-f_j^\delta\|_s+\|(1+\varepsilon D^2)(f_{2+j}^\varepsilon-f_{2+j}^\delta)\|_s$$
$$\leqq C(1+\|W,\ W_t'\|_{s+\sigma,\delta})^{6s+2\sigma+9}(\delta-\varepsilon)^{\sigma/2}\|W,\ W_t'\|_{s+\sigma,\delta}$$

where $j=1,\ 2,\ f_k^\mu=f_k(W,\ W_t',\ \mu)$ *and* $C=C(c_0,\ c,\ s,\ \sigma,\ g,\ h)>0$.

Proof. Dependences of operators and functions on μ are expressed by P_j^μ, $F_{jk}^\mu,\dots$ By C we denote all constants depending only on c_0, c, s, σ, g and h. Put

$$N(\mu,\ q)=(1+\|W,\ W_t'\|_{s+\sigma,\mu})^q,\quad \beta=\delta^{(2-\sigma)/2},\quad \lambda=(\delta-\varepsilon)^{\sigma/2}.$$

Since f_2 is independent of μ, we begin with f_3. Put

$$A^\mu=(1+P_3^\mu)(g+\mu D^2)(-i\ \mathrm{sgn}\ D),\quad B^\mu=(-g-\mu D^2+P_2^\mu)(1+\mu D^2)^{-1}.$$

Then

$$\|(1+\varepsilon D^2)(f_3^\varepsilon-f_3^\delta)\|_s\leqq C\|(1+P_3^\varepsilon)^{-1}A^\varepsilon(f_3^\varepsilon-f_3^\delta)\|_s \tag{3.52}$$
$$\leqq C\|(1+P_3^\varepsilon)^{-1}\|_{s,s}\|A^\varepsilon f_3^\varepsilon-A^\delta f_3^\delta+(A^\delta-A^\varepsilon)f_3^\delta\|_s.$$

By (2.14) we see that

$$A^\varepsilon f_3^\varepsilon-A^\delta f_3^\delta=(B^\varepsilon-B^\delta)(F_{010}-\varepsilon F_{030})+B^\delta(-\varepsilon+\delta)F_{030}.$$

Since $(1+\varepsilon D^2)^{-1}-(1+\delta D^2)^{-1}=(1+\varepsilon D^2)^{-1}(\delta-\varepsilon)D^2(1+\delta D^2)^{-1}$ we obtain decompositions

$$B^\varepsilon-B^\delta=B^\varepsilon(\delta-\varepsilon)D^2(1+\delta D^2)^{-1}+(-(\varepsilon-\delta)D^2+P_2^\varepsilon-P_2^\delta)(1+\delta D^2)^{-1},$$
$$A^\varepsilon-A^\delta=\{(1+P_3^\varepsilon)(\varepsilon-\delta)D^2(g+\delta D^2)^{-1}+P_3^\varepsilon-P_3^\delta\}(-i\ \mathrm{sgn}\ D)(g+\delta D^2),$$
$$P_3^\varepsilon-P_3^\delta=P_3^\varepsilon(\delta-\varepsilon)D^2(g+\delta D^2)^{-1}-\{(P_1^\varepsilon-P_1^\delta)i\ \mathrm{sgn}\ D+P_2^\varepsilon-P_2^\delta\}(g+\delta D^2)^{-1},$$

cf. (2.13). By Lemmas 3.11 and 3.17 we see that

$$\|B^\varepsilon\|_{s,s}+\|B^\delta\|_{s,s}+\|P_3^\varepsilon\|_{s,s}\leqq CN(\delta,\ 2),$$
$$\|(1+P_3^\varepsilon)^{-1}\|_{s,s}\leqq CN(\varepsilon,\ 2s)\leqq CN(\delta,\ 2s).$$

Noting that $a^{2-\sigma}\leqq 1+a^2$, $a>0$, we obtain

$$(\delta-\varepsilon)\xi^2(1+\delta\xi^2)^{-1}\leqq(\delta-\varepsilon)^{\sigma/2}|\xi|^\sigma|\delta^{1/2}\xi|^{2-\sigma}(1+\delta\xi^2)^{-1}\leqq\lambda|\xi|^\sigma,$$
$$(1+\beta\xi^2)(1+\delta\xi^2)^{-1}=(1+|\delta^{1/2}\xi|^{2-\sigma}|\xi|^\sigma)(1+\delta\xi^2)^{-1}\leqq 2(1+|\xi|)^\sigma,$$
$$\beta(1+|\xi|)^3=(\delta^{1/2}(1+|\xi|)^{3/2})^{2-\sigma}(1+|\xi|)^{3\sigma/2}\leqq(1+\delta(1+|\xi|)^3)(1+|\xi|)^{2\sigma},$$

which lead us to estimates

$$\begin{cases}\|(\delta-\varepsilon)D^2(1+\delta D^2)^{-1}u\|_s\leqq\lambda\|u\|_{s+\sigma},\\ \|\beta D^2u\|_s\leqq\|(1+\beta D^2)u\|_s\leqq 2\|(1+\delta D^2)u\|_{s+\sigma},\\ \beta\|u\|_{s+3/2}^2\leqq\|u\|_{s+\sigma}^2+\delta\|u\|_{s+\sigma+3/2}^2.\end{cases} \tag{3.53}$$

In view of (3.24) we have

$$\|W,\ W_t'\|_{s,\beta}\leqq 2\|W,\ W_t'\|_{s+\sigma,\delta}. \tag{3.54}$$

From (2.10) it follows that $\beta(\varepsilon-\delta)^{-1}(P_j^\varepsilon-P_j^\delta)$ is equal to P_j with $\mu=\beta$, $Y=0$. By Lemma 3.11 we have

$$\begin{aligned}\|(P_j^\varepsilon-P_j^\delta)u\|_s&\leqq\beta^{-1}(\delta-\varepsilon)C(1+\|(1+\beta D^2)Z\|_s)^2\|(1+\beta D^2)u\|_s\\&\leqq\lambda CN(\delta,\ 2)\|(1+\delta D^2)u\|_{s+\sigma},\qquad j=1,\ 2.\end{aligned} \tag{3.55}$$

The above estimates show that

$$\begin{aligned}\|A^\varepsilon f_3^\varepsilon-A^\delta f_3^\delta\|_s&\leqq CN(\delta,\ 2)\lambda\|F_{010}-\varepsilon F_{030}\|_{s+\sigma}+CN(\delta,\ 2)\lambda\beta\|F_{030}\|_s\\&\leqq CN(\delta,\ 2)\lambda\|W,\ W_t'\|_{s+\sigma,\delta}.\end{aligned}$$

Here we used

$$\|F_{010}-\varepsilon F_{030}\|_{s+\sigma}\leqq C\|W,\ W_t'\|_{s+\sigma,\varepsilon},\quad \beta\|F_{030}\|_s\leqq C\|W,\ W_t'\|_{s,\beta}\leqq C\|W,\ W_t'\|_{s+\sigma,\delta}$$

which follow from (3.33) and (3.34). Since

$$\begin{aligned}\|(A^\varepsilon-A^\delta)f_3^\delta\|_s&\leqq CN(\delta,\ 2)\lambda\|(g+\delta D^2)f_3^\delta\|_{s+\sigma}\\&\leqq CN(\delta,\ 2)\lambda N(\delta,\ 2(s+\sigma)+2)\|W,\ W_t'\|_{s+\sigma,\delta},\end{aligned}$$

cf. (3.36), we obtain

$$\|(1+\varepsilon D^2)(f_3^\varepsilon-f_3^\delta)\|_s\leqq CN(\delta,\ 4s+2\sigma+4)\lambda\|W,\ W_t'\|_{s+\sigma,\delta}. \tag{3.56}$$

By (2.15) we see that

$$\begin{aligned}f_4^\varepsilon-f_4^\delta=&-i\operatorname{sgn} D(f_3^\varepsilon-f_3^\delta)+(1+\varepsilon D^2)^{-1}(\delta-\varepsilon)D^2(1+\delta D^2)^{-1}(F_{010}-\varepsilon F_{030})\\&+(1+\delta D^2)^{-1}(-\varepsilon+\delta)F_{030}.\end{aligned}$$

Noting that $\|(1+\varepsilon D^2)(1+\delta D^2)^{-1}\|_{s,s}\leqq 1$ we obtain

$$\|(1+\varepsilon D^2)(f_4^\varepsilon-f_4^\delta)\|_s\leqq CN(\delta,\ 4s+2\sigma+4)\lambda\|W,\ W_t'\|_{s+\sigma,\delta}. \tag{3.57}$$

Let $f_1=f_{11}+f_{12}+f_{13}$ be the decomposition corresponding to three terms in (2.34). Since P_4 defined by (2.28) is independent of μ we see that

$$\beta(\varepsilon-\delta)^{-1}(f_{11}^\varepsilon+f_{12}^\varepsilon-f_{11}^\delta-f_{12}^\delta)$$

is equal to $f_{11}+f_{12}$ in which $\mu=\beta$ and Y contained in operators P_j is zero. Therefore, using (3.39), (3.41) and (3.54) we have

$$\|f_{11}^\varepsilon+f_{12}^\varepsilon-f_{11}^\delta-f_{12}^\delta\|_s\leqq CN(\delta,\ 2)\lambda\|W,\ W_t'\|_{s+\sigma,\delta}.$$

Note that

$$\begin{aligned}&\|f_{13}^\varepsilon-f_{13}^\delta\|_s\leqq C\|I_2^\varepsilon-I_2^\delta\|_s,\\&I_2^\varepsilon-I_2^\delta=\varepsilon(I_1^\varepsilon-I_1^\delta)+(\varepsilon-\delta)I_1^\delta+B^\varepsilon(F_{110}^\varepsilon-\varepsilon F_{130}^\varepsilon-F_{110}^\delta+\delta F_{130}^\delta)\\&\qquad+(B^\varepsilon-B^\delta)(F_{110}^\delta-\delta F_{130}^\delta)-2Y_t\cdot(Z_t^\varepsilon-Z_t^\delta),\end{aligned}$$

where we denoted f_2, f_3^μ, f_4^μ by Y_{2t}, Z_{1t}^μ, Z_{2t}^μ respectively. In (3.44) putting $\mu=\varepsilon$, $Z^0=Z$, $Z_t=Z_t^\varepsilon$, $Z_t^0=Z_t^\delta$ we see by (3.36), (3.56), (3.57) that

$$\|\varepsilon(I_1^\varepsilon-I_1^\delta)\|_s\leqq C(1+\|(1+\varepsilon D^2)Z\|_s)^2(\|(1+\varepsilon D^2)Z_t^\varepsilon\|_s$$
$$+\|(1+\varepsilon D^2)Z_t^\delta\|_s)\|(1+\varepsilon D^2)(Z_t^\varepsilon-Z_t^\delta)\|_s$$
$$\leqq CN(\delta,\ 6s+2\sigma+9)\lambda\|W,\ W_t'\|_{s+\sigma,\delta},$$

where we used $\|(1+\varepsilon D^2)u\|_s\leqq\|(1+\delta D^2)u\|_s$. By (3.43) we see that

$$\|(\varepsilon-\delta)I_1^\delta\|_s\leqq\lambda\|\beta I_1^\delta\|_s\leqq\lambda C(1+\|(1+\beta D^2)Z\|_s)^2\|(1+\beta D^2)Z_t^\delta\|_s^2$$
$$\leqq\lambda C(1+\|(1+\delta D^2)Z\|_{s+\sigma})^2\|(1+\delta D^2)Z_t^\delta\|_{s+\sigma}^2$$
$$\leqq\lambda CN(\delta,\ 4s+4\sigma+7)\|W,\ W_t'\|_{s+\sigma,\delta}.$$

It is clear that

$$\|F_{110}^\varepsilon-\varepsilon F_{130}^\varepsilon-F_{110}^\delta+\delta F_{130}^\delta\|_s\leqq\|F_{110}^\varepsilon-F_{110}^\delta\|_s+\varepsilon\|F_{130}^\varepsilon-F_{130}^\delta\|_s+\lambda\|\beta F_{130}^\delta\|_s.$$

By (3.46), the first term in the right hand side is smaller than $C\|Z_t^\varepsilon-Z_t^\delta\|_s$. The third term is smaller than

$$\lambda CN(\delta,\ 2s+4)\|W,\ W_t'\|_{s+\sigma,\delta}$$

by (3.47) and (3.54). For the second term it holds that

$$\varepsilon\|F_{130}^\varepsilon-F_{130}^\delta\|_s\leqq C(1+\|(1+\varepsilon D^2)Z\|_s)\|(1+\varepsilon D^2)(Z_t^\varepsilon-Z_t^\delta)\|_s\|W,\ W_t'\|_{s,\varepsilon},$$

cf. the estimate for $K_{1,1,3}-K_{1,1,3}^0$ in the proof of Lemma 3.25. By (3.45) and (3.47) we see that

$$\|(B^\varepsilon-B^\delta)(F_{110}^\delta-\delta F_{130}^\delta)\|_s\leqq CN(\delta,\ 2)\lambda\|F_{110}^\delta-\delta F_{130}^\delta\|_{s+\sigma}$$
$$\leqq CN(\delta,\ 2(s+\sigma)+6)\lambda\|W,\ W_t'\|_{s+\sigma,\delta}.$$

Finally, noting that

$$\|2Y_t\cdot(Z_t^\varepsilon-Z_t^\delta)\|_s\leqq C\|Z_t^\varepsilon-Z_t^\delta\|_s\leqq CN(\delta,\ 4s+2\sigma+4)\lambda\|W,\ W_t'\|_{s+\sigma,\delta}$$

and summing up the above results we have

$$\|f_{13}^\varepsilon-f_{13}^\delta\|_s\leqq CN(\delta,\ 6s+2\sigma+9)\lambda\|W,\ W_t'\|_{s+\sigma,\delta}.$$

Thus the proof is finished.

By (2.40)–(2.43) we determined W, W_t', $t=0$, from $X^{(0)}$, $X_1^{(1)}$. Since $X^{(0)}$, $X_1^{(1)}$ are independent of μ, we see that X, Z, X_t, $t=0$, are independent of μ and Y, Y_{1t}, $t=0$, depend on μ: $Y=Y^\mu$, $Y_{1t}=Y_{1t}^\mu$, $t=0$.

Lemma 3.58. *There exists $c_0=c_0(h)>0$ such that if $\mu>0$, $s\geqq 6$, $c>0$,*

$$(b,\ \Lambda^{3/2}X^{(0)},\ X_1^{(1)})\in H^s,\quad \|b\|_3+\|X^{(0)}\|_4\leqq c_0,\quad \|b\|_s+\|X^{(0)}\|_{s+3/2}+\|X_1^{(1)}\|_s\leqq c$$

then for W, W_t', $t=0$, determined from $X^{(0)}$, $X_1^{(1)}$ by means of (2.40)–(2.43) *the following hold: first,*

$$I(W, W_t') = (\Lambda^{3/2}X, \Lambda^{-3/2}Y, \Lambda^{1/2}Z, X_t, \Lambda^{-3}Y_{1t}) \in H^s,$$

$$\|I(W, W_t')\|_s \leqq C\{(1+\mu)\|X^{(0)}\|_{s+3/2} + \|X_1^{(1)}\|_s\}, \tag{3.59}$$

where $C = C(c_0, c, s, g, h) > 0$; *secondly*,

$$\begin{aligned} &\|X\|_3 + \|Y\|_2 + \|(1+\mu D^2)Z\|_1 + \|Z\|_3 \\ &\quad \leqq C(\|X^{(0)}\|_4 + \|(1+\mu D^2)X^{(0)}\|_3 + \|(1+\mu D^2)X^{(0)}\|_3^2 + \|X_1^{(1)}\|_3^2), \end{aligned} \tag{3.60}$$

where $C = C(c_0, g, h) > 0$; *lastly*,

$$\|Y^\varepsilon - Y^\delta\|_{s-3/2} + \|Y_{1t}^\varepsilon - Y_{1t}^\delta\|_{s-3} \leqq C(\delta - \varepsilon)(\|X^{(0)}\|_{s+3/2} + \|X_1^{(1)}\|_s), \tag{3.61}$$

where $C = C(c_0, c, s, h) > 0$ *and* $0 < \varepsilon < \delta$.

Proof. Since $X = X^{(0)}$ and $Z = X_x$, we can take $c_0 > 0$ so small that if $\|b\|_3 + \|X^{(0)}\|_4 \leqq c_0$ then $K(X, b, h)$ and $(1 + Z_1 + Z_2K)^{-1}$ are bounded operators in H^3, and moreover if $b \in H^s$, $X \in H^{s+1}$ then they are bounded operators in H^s. It is not difficult to prove (3.59) and (3.60) by methods used in the proof of Lemma 3.25. Applying these methods to

$$\begin{aligned} &Y_1^\varepsilon - Y_1^\delta = (1 + Z_1 + Z_2K)^{-1}(\varepsilon - \delta)\{R(Z, Z_x) + S(Z, Z_{xx})\}, \\ &Y_2^\varepsilon - Y_2^\delta = K(Y_1^\varepsilon - Y_1^\delta), \\ &(1 + Z_1 + Z_2K)(Y_{1t}^\varepsilon - Y_{1t}^\delta) = -Z_2\{F_{20}(X, X_t, Y^\varepsilon) - F_{20}(X, X_t, Y^\delta)\} \\ &\quad + \{-(Y_1^\varepsilon - Y_1^\delta) + (\varepsilon - \delta)P_1(0, Z, Z_x, Z_{xx})\} iDX_{1t} \\ &\quad + \{-(\varepsilon - \delta)D^2 - (Y_1^\varepsilon - Y_1^\delta) + (\varepsilon - \delta)P_2(0, Z, Z_x, Z_{xx})\} iDX_{2t}, \end{aligned}$$

we obtain (3.61).

§4. Linear equations

In view of the second equation of (2.36) we consider the problem

$$u_{tt} + \mu(A|D|^3 - imA_xD|D| + BD^2)u + \lambda B_0|D|u = f, \qquad 0 \leqq t \leqq T, \tag{4.1}$$

$$u = u_0, \quad u_t = u_1, \quad t = 0, \tag{4.2}$$

where m is real and $\lambda \geqq 0$. To simplify matters, we reduce this problem (whose H^s-solution we shall need later) to the problem in H^0. We introduce notations

$$\begin{cases} M_m = \mu(A|D|^3 - imA_xD|D| + BD^2) + \lambda B_0|D|, \\ N_m = \mu(-imA_xD|D|), \quad \Lambda = 1 + |D|, \\ A' = A - A^\infty, \ A^\infty = \text{const} > 0, \quad B_0' = B_0 - B_0^\infty, \ B_0^\infty = \text{const} > 0. \end{cases} \tag{4.3}$$

It is clear that the equation (4.1) is equivalent to

$$\Lambda^s u_{tt} + M_{m+s}\Lambda^s u + ([\Lambda^s, M_m] - N_s\Lambda^s)\Lambda^{-s}\Lambda^s u = \Lambda^s f. \tag{4.4}$$

We must show that the operator in the third term is of lower order.

Lemma 4.5. *Let $s>3/2$ and $r\geqq 0$. Then for $a, u\in\mathscr{S}$ it holds that*

$$\|([\Lambda^s, a]D+isa_x\Lambda^s)u\|_r\leqq C\|a\|_{r+s+1}\|u\|_{r+s-1},$$

$$\|[\Lambda^s, a]u\|_r\leqq C\|a\|_{r+s}\|u\|_{r+s-1},$$

where $C=C(r, s)>0$.

Proof. Putting $v=[\Lambda^s, a]Du+isa_x\Lambda^s u$, we have

$$(4.6)\qquad (1+|\xi|)^r\hat{v}(\xi)=(2\pi)^{-1}\int Q(\xi, \eta)\hat{a}(\xi-\eta)\hat{u}(\eta)d\eta,$$

$$Q(\xi, \eta)=(1+|\xi|)^r\{(1+|\xi|)^s\eta-(1+|\eta|)^s\eta-s(\xi-\eta)(1+|\eta|)^s\}.$$

Applying the formula

$$f(y)=f(0)+f'(0)y+\int_0^y(y-t)f''(t)dt$$

to $(1+y|\xi|+(1-y)|\eta|)^s$, we have

$$(1+|\xi|)^s=(1+|\eta|)^s+s(1+|\eta|)^{s-1}(|\xi|-|\eta|)$$
$$+s(s-1)\int_0^1(1-t)(1+t|\xi|+(1-t)|\eta|)^{s-2}(|\xi|-|\eta|)^2dt.$$

By this expansion and the identity

$$(|\xi|-|\eta|)\eta-(\xi-\eta)(1+|\eta|)=\xi\eta(\operatorname{sgn}\xi-\operatorname{sgn}\eta)-(\xi-\eta),$$

we can write Q in the form

$$Q(\xi, \eta)=s(1+|\xi|)^r\{\xi\eta(\operatorname{sgn}\xi-\operatorname{sgn}\eta)-(\xi-\eta)\}(1+|\eta|)^{s-1}$$
$$+s(s-1)(1+|\xi|)^r\eta\int_0^1\cdots dt.$$

Note that $|\xi|\leqq|\xi-\eta|$, $|\eta|\leqq|\xi-\eta|$ if $\operatorname{sgn}\xi-\operatorname{sgn}\eta\neq 0$, i.e. $\operatorname{sgn}\xi=-\operatorname{sgn}\eta$ and that

$$\left|\eta\int_0^1\cdots dt\right|\leqq\int_0^1(1+t|\xi|+(1-t)|\eta|)^{s-1}|\xi-\eta|^2dt\leqq(1+|\eta|+|\xi-\eta|)^{s-1}|\xi-\eta|^2,$$

because $0\leqq 1+|\eta|+t(|\xi|-|\eta|)\leqq 1+|\eta|+|\xi-\eta|$. Therefore

$$|Q|\leqq C(1+|\xi-\eta|)^{r+2}(1+|\eta|)^{s-1}$$
$$+C(1+|\xi|)^r\{|\xi-\eta|(1+|\eta|)^{s-1}+(1+|\eta|+|\xi-\eta|)^{s-1}|\xi-\eta|^2\}$$
$$\leqq C(1+|\xi-\eta|)^{r+2}(1+|\eta|)^{r+s-1}+C(1+|\xi-\eta|)^{r+s+1}$$

Taking L_2-norm of (4.6) and using Hausdorff-Young's inequality we see that

$$\|v\|_r\leqq C\int(1+|\xi|)^{r+2}|\hat{a}(\xi)|d\xi\|u\|_{r+s-1}+C\|a\|_{r+s+1}\int|\hat{u}(\eta)|d\eta$$

$$\leqq C\Big(\int(1+|\xi|)^{-2\delta}d\xi\Big)^{1/2}\|a\|_{r+2+\delta}\|u\|_{r+s-1}$$

$$+C\|a\|_{r+s+1}\|u\|_{\delta}\Big(\int(1+|\eta|)^{-2\delta}d\eta\Big)^{1/2}\leqq C\|a\|_{r+s+1}\|u\|_{r+s-1},$$

where $1/2<\delta<s-1$. Similarly, using

$$|(1+|\xi|)^r\{(1+|\xi|)^s-(1+|\eta|)^s\}|\leqq C\{(1+|\xi-\eta|)^{r+s-1}+(1+|\eta|)^{r+s-1}\}\,|\xi-\eta|$$

we obtain the second estimate. The proof is finished.

Applying the above lemma to

$$([\Lambda^s, M_m]-N_s\Lambda^s)\Lambda^{-s}u$$

$$=\mu([\Lambda^s, A]D+isA_x\Lambda^s-im[\Lambda^s, A_x])D|D|\Lambda^{-s}u$$

$$+(\mu[\Lambda^s, B]D^2+\lambda[\Lambda^s, B_0]|D|)\Lambda^{-s}u,$$

we obtain the following lemma.

Lemma 4.7. *Let* $s>3/2$, $r\geqq 0$ *and* $A'\in H^{r+s+1}$, B, $B_0'\in H^{r+s}$. *Then for* $u\in\mathscr{S}$ *it holds that*

$$\|([\Lambda^s, M_m]-N_s\Lambda^s)\Lambda^{-s}u\|_r$$

$$\leqq C\mu\{(1+|m|)\|A'\|_{r+s+1}+\|B\|_{r+s}\}\,\|u\|_{r+1}+C\lambda\|B_0'\|_{r+s}\|u\|_r,$$

where $C=C(r, s)>0$.

In view of the above lemma we consider the initial value problem

$$u_{tt}+M_mu=f,\qquad 0\leqq t\leqq T, \tag{4.8}$$

$$u=u_0,\quad u_t=u_1,\quad t=0, \tag{4.9}$$

(m is real). Note that $(M_mu)(t, x)$ is real if $u(t, x)$ is real. Therefore from now on in this section we assume that all functions are real.

Assumption 4.10.

1) $A(t, x)\geqq\text{const}>0$, $B_0(t, x)\geqq\text{const}>0$ and $B(t, x)$ is real,

2) $3/2<r\leqq 2$, $A'\in C^j([0, T], H^{r+1/2-3j/2})$,

$$B, B_0'\in C^j([0, T], H^{r-j}),\quad j=0, 1,$$

3) $A'\in C^0([0, T], H^4)$, $B, B_0'\in C^0([0, T], H^3)$,

4) $s\geqq 3$, $A'(t,\cdot)\in H^{s+1}$, $B(t,\cdot)$, $B_0'(t,\cdot)\in H^s$,

$$\|A'(t,\cdot)\|_{s+1}+\|B(t,\cdot)\|_s+\|B_0'(t,\cdot)\|_s\leqq\text{const}<\infty.$$

Under the above assumption we shall prove the estimate for u by means of

$$E^2=(A^qu_t, u_t)+\mu(A^{q+1}|D|^{3/2}u, |D|^{3/2}u)+\mu(A^qBDu, Du)+\mu p(u, u)$$

$$+\lambda(A^qB_0|D|^{1/2}u, |D|^{1/2}u)+\lambda(u, u)=(A^qu_t, u_t)+\mu F^2+\lambda F_0^2, \tag{4.11}$$

where p and q are constants which will be determined later. For this purpose we need the following lemma.

Lemma 4.12. *Let A, B, B_0 satisfy* 1), 2) *of Assumption* 4.10. *Take a, $d>0$ such that*

$$(4.13)\qquad \begin{cases} A(t, x)\geqq a^{-1}, \quad B_0(t, x)\geqq a^{-1}, \\ \|A'(t,\cdot)\|_{r+1/2}+\|B(t,\cdot)\|_r+\|B_0'(t,\cdot)\|_r\leqq a, \\ \|A_t(t,\cdot)\|_{r-1}+\|B_t(t,\cdot)\|_{r-1}+\|B_{0t}(t,\cdot)\|_{r-1}\leqq d. \end{cases}$$

Then for any real q there exists $C_0=C_0(a, q, r, A^\infty, B_0^\infty)>2$ such that

$$(4.14)\qquad \begin{cases} 5C_0^{-1}\leqq A^{q+1}\leqq C_0, \quad C_0^{-1}\leqq A^{q/2}\leqq C_0, \quad C_0^{-1}\leqq A^qB_0\leqq C_0, \\ |A^qB|+|(A^qB)_x|\leqq C_0, \\ |A^{-q}(A^q)_t|+|(A^{q+1})_t|+|(A^qB)_t|+|(A^qB_0)_t|\leqq C_0d, \\ \|[|D|^{3/2}, A^{q+1}]|D|^{3/2}u+\frac{3i}{2}(A^{q+1})_xD|D|u\|\leqq C_0\|u\|_{3/2}, \quad u\in H^{3/2}, \\ \|[|D|^{1/2}, A^qB_0]|D|^{1/2}u\|\leqq C_0\|u\|, \quad u\in H^0. \end{cases}$$

Proof. Let a, $u\in\mathcal{S}$ and put

$$v=[|D|^{3/2}, a]|D|^{3/2}u+\frac{3i}{2}a_xD|D|u.$$

Then

$$\hat{v}(\xi)=(2\pi)^{-1}\int Q(\xi, \eta)\hat{a}(\xi-\eta)\hat{u}(\eta)d\eta,$$

$$Q(\xi, \eta)=(|\xi|^{3/2}-|\eta|^{3/2})|\eta|^{3/2}-\frac{3}{2}(\xi-\eta)\eta|\eta|.$$

From identities

$$|\xi|^{3/2}=|\eta|^{3/2}+\frac{3}{2}|\eta|^{1/2}(|\xi|-|\eta|)+\frac{3}{4}\int_0^1(1-t)(t|\xi|+(1-t)|\eta|)^{-1/2}(|\xi|-|\eta|)^2dt,$$

$$(|\xi|-|\eta|)|\eta|-(\xi-\eta)\eta=\xi(\operatorname{sgn}\xi-\operatorname{sgn}\eta)|\eta|,$$

it follows that

$$Q(\xi, \eta)=\frac{3}{2}\xi(\operatorname{sgn}\xi-\operatorname{sgn}\eta)|\eta|^2+\frac{3}{4}|\eta|^{3/2}\int_0^1\cdots dt$$

$$\left|\int_0^1\cdots dt\right|\leqq\begin{cases}\int_0^1\{t(|\xi|-|\eta|)\}^{-1/2}(|\xi|-|\eta|)^2dt & \text{if } |\xi|\geqq|\eta| \\ \int_0^1\{(1-t)(|\eta|-|\xi|)\}^{-1/2}(|\xi|-|\eta|)^2dt & \text{if } |\eta|\geqq|\xi|\end{cases}$$

$$\leqq||\xi|-|\eta||^{3/2}\int_0^1 t^{-1/2}dt\leqq C|\xi-\eta|^{3/2}.$$

Therefore $|Q|\leqq C|\xi-\eta|^{3/2}|\eta|^{3/2}$, which gives $\|v\|\leqq C\|a\|_{r+1/2}\|u\|_{3/2}$. In view of

(3.1)–(3.8) and [2] Lemma 2.14 we can take C_0 so large that (4.14) holds.

Lemma 4.15. *Let A, B, B_0 satisfy 1), 2) of Assumption 4.10. If*

$$u \in C^j([0, T], H^{3-3j/2}), \quad j=0, 1, 2,$$

is a solution of (4.8) with $f \in C^0([0, T], H^0)$ then it holds that

$$|u(t)|_0 \leqq C_0^5 e^{Ct}|u(0)|_0 + C_0^2 \int_0^t e^{C(t-\tau)}\|f(\tau)\|d\tau \tag{4.16}$$

where $|u(t)|_0 = (\|u_t(t)\|^2 + \mu\|u(t)\|_{3/2}^2 + \lambda\|u(t)\|_{1/2}^2)^{1/2}$,

$$C_0 = C_0(a, (2m-3)/3, r, A^\alpha, B_0^\alpha), \quad C = C_0^2 d + C_0^9 \mu^{1/2} + C_0^4 \lambda^{1/2}.$$

Proof. From inequalities

$$2^{-2}(1+|\xi|)^3 \leqq 1+|\xi|^3 \leqq (1+|\xi|)^3, \quad |\beta\xi|^2 \leqq 1+|\beta\xi|^3$$

it follows that

$$2^{-2}\|u\|_{3/2}^2 \leqq \|u\|^2 + \|\,|D|^{3/2}u\|^2 \leqq \|u\|_{3/2}^2, \quad \|Du\|^2 \leqq \beta^{-2}\|u\|^2 + \beta\|\,|D|^{3/2}u\|^2, \tag{4.17}$$

where $\beta > 0$. Using (4.14) and (4.17) we see that

$$F^2 \geqq (5C_0^{-1} - \beta C_0)\,\|\,|D|^{3/2}u\|^2 + (p - \beta^{-2}C_0)\,\|u\|^2.$$

Putting $\beta = C_0^{-2}$, $p = 4C_0^{-1} + \beta^{-2}C_0 = 4C_0^{-1} + C_0^5$, we have

$$F^2 \geqq 4C_0^{-1}(\|\,|D|^{3/2}u\|^2 + \|u\|^2) \geqq \max\{C_0^{-1}\|u\|_{3/2}^2,\ C_0^{-1}\|Du\|^2\}. \tag{4.18}$$

It is clear that

$$F^2 \leqq C_0\|\,|D|^{3/2}u\|^2 + C_0\|Du\|^2 + p\|u\|^2 \leqq (2C_0 + p)\|u\|_{3/2}^2,$$

$$C_0^{-1}(\|\,|D|^{1/2}u\|^2 + \|u\|^2) \leqq F_0^2 \leqq C_0(\|\,|D|^{1/2}u\|^2 + \|u\|^2).$$

Noting that $C_0 > 2$ and $2C_0 + p \leqq C_0^6$ we obtain

$$C_0^{-1}|u(t)|_0^2 \leqq E(t)^2 \leqq C_0^6|u(t)|_0^2. \tag{4.19}$$

Differentiating (4.11) we see that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}E^2 &= (A^q u_{tt}, u_t) + \frac{1}{2}((A^q)_t u_t, u_t) + \mu(A^{q+1}|D|^{3/2}u, |D|^{3/2}u_t) \\
&\quad + \mu(A^q B Du, Du_t) + \mu p(u, u_t) + \frac{1}{2}\mu((A^{q+1})_t|D|^{3/2}u, |D|^{3/2}u) \\
&\quad + \frac{1}{2}\mu((A^q B)_t Du, Du) + \lambda(A^q B_0|D|^{1/2}u, |D|^{1/2}u_t) + \lambda(u, u_t) \\
&\quad + \frac{1}{2}\lambda((A^q B_0)_t|D|^{1/2}u, |D|^{1/2}u) \\
&= (A^q u_{tt} + A^q M_{3(q+1)/2}u, u_t) + \frac{1}{2}((A^q)_t u_t, u_t)
\end{aligned} \tag{4.20}$$

$$+\frac{1}{2}\mu((A^{q+1})_t|D|^{3/2}u,\ |D|^{3/2}u)+\frac{1}{2}\mu((A^qB)_tDu,\ Du)$$

$$+\frac{1}{2}\lambda((A^qB_0)_t|D|^{1/2}u,\ |D|^{1/2}u)+(Ru,\ u_t),$$

$$R=\mu([|D|^{3/2},\ A^{q+1}]\,|D|^{3/2}+\frac{3}{2}i(A^{q+1})_xD|D|-i(A^qB)_xD+p)$$

$$+\lambda[|D|^{1/2},\ A^qB_0]\,|D|^{1/2}+\lambda.$$

Putting $q=(2m-3)/3$, i.e. $3(q+1)/2=m$ and noting that

$$\|Ru\|\leqq\mu\{C_0\|u\|_{3/2}+C_0\|Du\|+p\|u\|\}+\lambda(C_0+1)\|u\|_{1/2}$$

$$\leqq\mu(2C_0+p)\|u\|_{3/2}+\lambda(C_0+1)\|u\|_{1/2}\leqq\mu C_0^6(4C_0)^{1/2}F+\lambda C_0^2C_0^{1/2}F_0$$

we obtain

$$\frac{1}{2}\frac{d}{dt}E^2\leqq(A^qf,\ u_t)+\frac{1}{2}C_0d(A^qu_t,\ u_t)+\frac{1}{2}\mu C_0d\{\|\,|D|^{3/2}u\|^2+\|Du\|^2\}$$

$$+\frac{\lambda}{2}C_0d\|\,|D|^{1/2}u\|^2+\|Ru\|C_0\|A^{q/2}u_t\|$$

$$\leqq C_0\|f\|E+C_0d(A^qu_t,\ u_t)+\mu C_0^2dF^2+\lambda C_0^2dF_0^2+(C_0^8\mu F+\lambda C_0^3F_0)C_0E$$

$$\leqq C_0\|f\|E+CE^2,$$

$$C=C_0^2d+C_0^9\mu^{1/2}+C_0^4\lambda^{1/2}.$$

Therefore $\frac{d}{dt}E(t)\leqq C_0\|f(t)\|+CE(t)$ holds for t such that $E(t)>0$. Since $E(t)$ is continuous in t we have

$$E(t)\leqq E(0)e^{Ct}+C_0\int_0^t e^{C(t-\tau)}\|f(\tau)\|d\tau.$$

Using (4.19) we obtain the estimate (4.16). The proof is complete.

Lemma 4.21. *Let A, B, B_0 satisfy* 1)–3) *of Assumption* 4.10. *If*

$$u\in C^j([0,\ T],\ H^{3/2-3j/2}),\quad j=0,\ 1,\ 2,$$

is a solution of (4.8) *with $f\in C^0([0,\ T],\ H^0)$ then*

$$|\Lambda^{-r}u(t)|_0\leqq C|\Lambda^{-r}u(0)|_0+C\int_0^t\|\Lambda^{-r}f(t)\|dt \tag{4.22}$$

holds where $C>0$ is independent of u and f.

Proof. The equation (4.8) can be written in the form

$$\Lambda^{-r}u_{tt}+M_{m-r}\Lambda^{-r}u=\Lambda^{-r}f+(-N_r+\Lambda^{-r}[\Lambda^r,\ M_m])\Lambda^{-r}u$$

$$=\Lambda^{-r}f+\mu RD|D|^{1/2}\Lambda^{-r}u+\lambda R_0|D|\Lambda^{-r}u,$$

$$R=(ir A_x+\Lambda^{-r}[\Lambda^r,\ DA-i(m-1)A_x+B\mathrm{sgn}\,D])|D|^{1/2},\quad R_0=\Lambda^{-r}[\Lambda^r,\ B_0].$$

The adjoint of R in H^0 is

$$R^* = |D|^{1/2}(-irA_x\Lambda^r - [\Lambda^r, A]D)\Lambda^{-r} + |D|^{1/2}(-[i(m-1)A_x + \mathrm{sgn}\, DB, \Lambda^r]\Lambda^{-r}).$$

By Lemma 4.5 we see that

$$\|R^*v\| \leqq C(\|A'\|_{r+3/2} + \|B\|_{r+1/2})\|v\|, \quad \|R_0^*v\| \leqq C\|B_0'\|_r\|v\|,$$

which show that $R, R_0 \in C^0([0, T], \mathscr{L}(H^0, H^0))$. Hence we can use (4.16).

$$|\Lambda^{-r}u(t)|_0 \leqq C|\Lambda^{-r}u(0)|_0 + C\int_0^t (\|\Lambda^{-r}f(t)\| + |\Lambda^{-r}u(t)|_0)dt,$$

which leads to (4.22), and the proof is finished.

Theorem 4.23. *Let A, B, B_0 satisfy* 1)–3) *of Assumption* 4.10. *If*

$$u_0 \in H^{3/2}, \quad u_1 \in H^0, \quad f \in C^0([0, T], H^0)$$

then there exists the unique solution

$$u \in C^j([0, T], H^{3/2-3j/2}), \quad j = 0, 1, 2,$$

of (4.8), (4.9). *Moreover* (4.16) *holds for u.*

Proof. The uniqueness of u follows from (4.22). The solution u will be obtained by means of the approximation of M_m by bounded operators. Noting that

$$\Lambda^3u_{tt} + (\Lambda^3M_m\Lambda^{-3})\Lambda^3u = \Lambda^3f, \quad \Lambda^3M_m\Lambda^{-3} = M_{m+3} + ([\Lambda^3, M_m] - N_3\Lambda^3)\Lambda^{-3},$$

we consider the problem

$$v_{tt} + Gv = g, \ 0 \leqq t \leqq T, \qquad v = v_0, \ v_t = v_1, \ t = 0, \tag{4.24}$$

where $G = \Lambda^{-q}Q_\varepsilon\Lambda^q\Lambda^3M_m\Lambda^{-3}Q_\varepsilon$, $Q_\varepsilon = (1 + \varepsilon\Lambda^{3/2})^{-1}$, $0 < \varepsilon < 1$, $q = (2m+3)/3$. Since

$$\|(\Lambda^{-q}Q_\varepsilon\Lambda^q\Lambda^{3/2})^*u\| = \|\Lambda^{3/2}\Lambda^qQ_\varepsilon\Lambda^{-q}u\| \leqq C(1 + \|\Lambda^q - (\Lambda^\infty)^q\|_{3/2})(\sup \Lambda^{-q})\|u\|,$$

$$\|M_mu\|_{3/2} \leqq C(1 + \|A'\|_{5/2} + \|B\|_{3/2} + \|B_0'\|_{3/2})\|u\|_{3+3/2},$$

we see that $G \in C^0([0, T], \mathscr{L}(H^0, H^0))$. Therefore, if

$$v_0 \in H^{3/2}, \quad v_1 \in H^0, \quad g \in C^0([0, T], H^0)$$

then the problem (4.24) has the unique solution

$$v = v^\varepsilon \in C^2([0, T], H^0).$$

Putting

$$E_\varepsilon^2 = (\Lambda^qv_t, v_t) + \mu(\Lambda^{q+1}|D|^{3/2}Q_\varepsilon v, |D|^{3/2}Q_\varepsilon v) + \mu(\Lambda^qBDQ_\varepsilon v, DQ_\varepsilon v) + \mu p(Q_\varepsilon v, Q_\varepsilon v)$$
$$+ \lambda(\Lambda^qB_0|D|^{1/2}Q_\varepsilon v, |D|^{1/2}Q_\varepsilon v) + \lambda(Q_\varepsilon v, Q_\varepsilon v),$$

cf. (4.11), we see that

$$\frac{1}{2}\frac{d}{dt}E_\varepsilon^2=(A^q v_{tt}+Q_\varepsilon A^q M_{m+3}Q_\varepsilon v,\ v_t)+\frac{1}{2}((A^q)_t v_t,\ v_t)$$

$$+\frac{1}{2}\mu((A^{q+1})_t|D|^{3/2}Q_\varepsilon v,\ |D|^{3/2}Q_\varepsilon v)+\frac{1}{2}\mu((A^q B)_t DQ_\varepsilon v,\ DQ_\varepsilon v)$$

$$+\frac{1}{2}\lambda((A^q B_0)_t|D|^{1/2}Q_\varepsilon v,\ |D|^{1/2}Q_\varepsilon v)+(Q_\varepsilon RQ_\varepsilon v,\ v_t).$$

Since

$$G=A^{-q}Q_\varepsilon A^q M_{m+3}Q_\varepsilon+A^{-q}Q_\varepsilon A^q([A^3,\ M_m]-N_3A^3)A^{-3}Q_\varepsilon,$$

$$\|([A^3,\ M_m]-N_3A^3)A^{-3}u\|\leqq C(\|A'\|_4+\|B\|_3+\|B_0'\|_3)\|u\|_1,$$

it holds that

$$\|v_t(t)\|+\|Q_\varepsilon v(t)\|_{3/2}\leqq C(\|v_t(0)\|+\|Q_\varepsilon v(0)\|_{3/2}+\int_0^t\|g(t)\|dt)$$

$$\leqq \text{const (indep. of } t \text{ and } \varepsilon).$$

Using decompositions

$$Q_\varepsilon=1-\varepsilon A^{3/2}Q_\varepsilon,$$

$$G=A^3M_mA^{-3}-A^3M_mA^{-3}\varepsilon A^{3/2}Q_\varepsilon-A^{-q}\varepsilon A^{3/2}Q_\varepsilon A^qA^3M_mA^{-3}Q_\varepsilon,$$

$$=A^3M_mA^{-3}-G',$$

we can write (4.24) in the form

$$v_{tt}+A^3M_mA^{-3}v=g+G'v.$$

Put $u^\varepsilon=A^{-3}v^\varepsilon$. It is clear that

$$\begin{cases} u_{tt}^\varepsilon+M_mu^\varepsilon=A^{-3}g+A^{-3}G'A^3u^\varepsilon, & 0\leqq t\leqq T,\\ u^\varepsilon=A^{-3}v_0,\quad u_t^\varepsilon=A^{-3}v_1, & t=0,\end{cases}$$

$$\sup_{t,\varepsilon}(\|u_t^\varepsilon(t)\|_3+\|Q_\varepsilon u^\varepsilon(t)\|_{3+3/2})<\infty.$$

Noting that

$$\|A^{-3}G'A^3u^\varepsilon\|=\|M_m\varepsilon A^{3/2}Q_\varepsilon u^\varepsilon+(A^{-3}A^{-q}A^{3/2}Q_\varepsilon A^qA^{3/2})\varepsilon A^{3/2}M_mQ_\varepsilon u^\varepsilon\|$$

$$\leqq C\varepsilon\|Q_\varepsilon u^\varepsilon\|_{3+3/2},$$

($C>0$ is independent of ε and t), and using (4.16) we see that

$$\sup_{0\leqq t\leqq T}|u^\varepsilon(t)-u^\delta(t)|_0\longrightarrow 0,\quad (\varepsilon,\ \delta\longrightarrow +0).$$

The limit u of u^ε exists and satisfies (4.16) in which f is replaced by $A^{-3}g$. It is easily seen that u is a solution of

$$u_{tt}+M_mu=A^{-3}g,\quad 0\leqq t\leqq T,$$

$$u=A^{-3}v_0,\quad u_t=A^{-3}v_1,\quad t=0.$$

Therefore by approximations

$$\Lambda^{-3}g \longrightarrow f, \quad \Lambda^{-3}v_0 \longrightarrow u_0, \quad \Lambda^{-3}v_1 \longrightarrow u_1$$

and (4.16) we obtain the required solution. The proof is complete.

Theorem 4.25. *Let A, B, B_0 satisfy Assumption* 4.10. *If*

$$u_0 \in H^{s+3/2}, \quad u_1 \in H^s, \quad f \in C^0([0, T], H^0), \quad f(t) \in H^s, \quad \|f(t)\|_s \leqq \text{const} < \infty$$

then there exists the unique solution u of (4.1), (4.2) *such that*

$$u \in C^j([0, T], H^{s+3/2-3j/2}) \cap C^2([0, T], H^0), \qquad j=0, 1.$$

Moreover for u it holds that

$$(4.26) \quad |u(t)|_s = |\Lambda^s u(t)|_0 \leqq C_0^5 e^{kt}|u(0)|_s + C_0^2 \int_0^t e^{k(t-\tau)}\|f(\tau)\|_s d\tau,$$

$$k = C_0^2 d + C_0^9 \mu^{1/2} + C_0^4 \lambda^{1/2} + C_0^2 \beta, \quad C_0 = C_0(a, (2m+2s-3)/3, r, A^\infty, B_0^\infty),$$

$$\beta = C(0, s) \sup_{0 \leqq t \leqq T} \{\mu^{1/2}(1+|m|)\|A'(t, \cdot)\|_{s+1} + \mu^{1/2}\|B(t, \cdot)\|_s + \lambda^{1/2}\|B_0'(t, \cdot)\|_s\},$$

($C(0, s)$ *is the constant in Lemma* 4.7). *In addition, if*

$$A' \in C^0([0, T], H^{s+1}), \qquad B, B_0', f \in C^0([0, T], H^s)$$

then $u \in C^2([0, T], H^{s-3/2})$.

Proof. The uniqueness of u follows from (4.22). Put

$$Q_\varepsilon = (1+\varepsilon\Lambda^s)^{-1}, \ 0<\varepsilon<1, \qquad R_\varepsilon = ([\Lambda^s, M_m^\varepsilon] - N_s^\varepsilon \Lambda^s)\Lambda^{-s},$$

where M_m^ε, N_s^ε are operators whose coefficients are functions $A^\infty + Q_\varepsilon A'$, $Q_\varepsilon B$, $B_0^\infty + Q_\varepsilon B_0'$ instead of $A = A^\infty + A'$, B, $B_0 = B_0^\infty + B_0'$. In view of (4.4), we consider the problem

$$(4.27) \qquad v_{tt} + M_{m+s} v = Q_\varepsilon \Lambda^s f - R_\varepsilon v, \quad 0 \leqq t \leqq T,$$

$$(4.28) \qquad v = \Lambda^s u_0, \quad v_t = \Lambda^s u_1, \quad t=0.$$

Using Lemma 4.7 and noting that $\|Q_\varepsilon u\|_s \leqq \|u\|_s$ we see that

$$R_\varepsilon \in C^0([0, T], \mathscr{L}(H^1, H^0)), \quad \|R_\varepsilon u\| \leqq \beta|u(t)|_0.$$

By Theorem 4.23, the problem (4.27), (4.28) has the unique solution

$$v = v^\varepsilon \in C^j([0, T], H^{3/2-3j/2}), \quad j=0, 1, 2,$$

and for v^ε (4.16) holds:

$$|v^\varepsilon(t)|_0 \leqq C_0^5 e^{Ct}|v^\varepsilon(0)|_0 + C_0^2 \int_0^t e^{C(t-\tau)}(\|Q_\varepsilon \Lambda^\varepsilon f(\tau)\| + \beta|v^\varepsilon(\tau)|_0) d\tau.$$

From this estimate we obtain

$$(4.29) \qquad |v^\varepsilon(t)|_0 \leqq C_0^5 e^{kt}|v^\varepsilon(0)|_0 + C_0^2 \int_0^t e^{k(t-\tau)}\|Q_\varepsilon \Lambda^s f(\tau)\| d\tau.$$

It is clear that

$$|v^{\varepsilon}(t)-v^{\delta}(t)|_0 \leqq C_0^2 \int_0^t e^{k(t-\tau)}(\|(Q_\varepsilon-Q_\delta)\Lambda^s f(\tau)\|+\|(R_\varepsilon-R_\delta)v^{\delta}(\tau)\|)d\tau.$$

By Lemma 4.7,

$$\|(R_\varepsilon-R_\delta)v^{\delta}(\tau)\| \leqq (\mu+\lambda)C(\|(Q_\varepsilon-Q_\delta)A'(\tau,\cdot)\|_{s+1}+\|(Q_\varepsilon-Q_\delta)B(\tau,\cdot)\|_s$$
$$+\|(Q_\varepsilon-Q_\delta)B_0'(\tau,\cdot)\|_s)\|v^{\delta}(\tau)\|_1.$$

By (4.29), $\|v^{\delta}(\tau)\|_1$ is uniformly bounded in δ and t. Hence we see that

$$\sup_{0\leqq t\leqq T}|v^{\varepsilon}(t)-v^{\delta}(t)|_0 \longrightarrow 0, \quad (\varepsilon,\ \delta \longrightarrow +0).$$

It is not difficult to see that the limit u of $\Lambda^{-s}v^{\varepsilon}$ is the required solution and satisfies (4.26), which is proved if we let $\varepsilon\to 0$ in (4.29). The remaining part of the theorem is trivial. The proof is finished.

Consider the initial value problem for the second equation of (2.36):

$$u_{tt}+(M+L)u=f, \quad 0\leqq t\leqq T, \tag{4.30}$$

$$u=u_0, \quad u_t=u_1, \quad t=0, \tag{4.31}$$

where $M=\mu(A_0|D|^3-iA_{0x}D|D|+A_1D^2)$, $L=i\mu A_2D+(\mu A_3+A_4)|D|$, see (2.32). We put

$$A_4(Y,Z)=A_5(Y,Z)+A_6(Z)$$
$$=Q(Z)^{-2}\{(g+g_0+Y_2)(1+Z_1)-Y_1Z_2\}-g_0Q(Z)^{-2}(1+Z_1).$$

From (2.12) it follows that $g+g_0>0$. By (3.1) we see that if Y, Z are small in H^1 then $1+Z_1\geqq \text{const}>0$, $A_5\geqq \text{const}>0$.

Assumption 4.32. 1) Let $c_0=c_0(g)>0$ be a constant such that $1+Z_1 \geqq \text{const}>0$, $A_5\geqq\text{const}>0$ if $\|Y\|_1+\|Z\|_1\leqq c_0$. 2) Let $s\geqq 3$, $c>0$, $d>0$ and Y, Z satisfy

$$\begin{cases} Y\in C^j([0,T],H^{2-j}), \quad Z\in C^j([0,T],H^{4-2j}), \quad j=0,1, \\ Y(t)\in H^s, \quad Z(t)\in H^{s+2}, \\ \|Y(t)\|_2+\|Z(t)\|_3\leqq c_0, \quad \|Y_t(t)\|_1+\|Z_t(t)\|_2\leqq d, \\ \|Y(t)\|_s+\|Z(t)\|_s\leqq c, \quad \|Z(t)\|_{s+2}\leqq \text{const}<\infty. \end{cases} \tag{4.33}$$

Lemma 4.34. *Under Assumption* 4.32, *the following hold: first,*

$$A_0-1\in C^j([0,T],H^{4-2j}), \quad A_1\in C^j([0,T],H^{3-2j}),$$
$$A_5-(g+g_0)\in C^j([0,T],H^{2-j}),\ j=0,1, \quad A_2,\ A_3,\ A_6-g_0\in C^0([0,T],H^2);$$

secondly, there exists $C_2=C_2(c_0,s,g)>2$ *such that for* $A=A_0$, $B=A_1$, $B_0=A_5$, $q=(2s-1)/3$, *inequalities in* (4.14) *hold and*

$$|A_2|+|A_3|\leqq C_2,\quad |A_6|\leqq g_0C_2;$$

lastly, there exists $C_3=C_3(c_0,\ c,\ s,\ g)>0$ *such that*

$$\mu^{1/2}C(0,\ s)(2\|A_0-1\|_{s+1}+\|A_1\|_s)\leqq C_3(\mu^{1/2}+\|(1+\mu D^2)Z\|_s),$$

$$2\|[A^s,\ L]A^{-s}u\|\leqq C_3(1+\|(1+\mu D^2)Z\|_s)^2\|u\|,\quad u\in H^0,$$

where $C(0,\ s)$ *is the constant in Lemma* 4.7.

Proof. Using (3.1)–(3.8) and Lemma 3.9 we obtain the first assertion, which shows that $A=A_0$, $B=A_1$, $B_0=A_5$ satisfy Assumption 4.10. So the second assertion holds. Using Lemma 4.5, noting that

$$\|A_0-1\|_{s+1}\leqq C\|A_0-1\|_s+C\|A_{0x}\|_s$$

and proceeding as in the proof of Lemma 3.11 we obtain the last assertion.

Theorem 4.35. *Let Assumption* 4.32 *hold. If*

$$u_0\in H^{s+3/2},\quad u_1\in H^s,\quad f\in C^0([0,\ T],\ H^0),\quad f(t)\in H^s,\quad \|f(t)\|_s\leqq \text{const}<\infty$$

then the problem (4.30), (4.31) *has the unique solution*

$$u\in C^j([0,\ T],\ H^{s+3/2-3j/2})\cap C^2([0,\ T],\ H^0),\quad j=0,\ 1.$$

Moreover u satisfies the estimate

$$|u(t)|_{s,\mu}=(\|u_t(t)\|_s^2+\mu\|u(t)\|_{s+3/2}^2+\|u(t)\|_{s+1/2}^2)^{1/2} \tag{4.36}$$

$$\leqq C_2^5e^{Ct}|u(0)|_{s,\mu}+C_2^2\int_0^t e^{C(t-\tau)}\|f(\tau)\|_s d\tau,$$

$$C=C_2^2d+C_2^9\mu^{1/2}+C_2^4 \tag{4.37}$$

$$+C_2^2\{C_2\mu^{1/2}+g_0C_2\mu^{-1/2}+C_3\mu^{1/2}+C_3(1+\sup_{0\leqq t\leqq T}\|(1+\mu D^2)Z(t)\|_s)^2\}.$$

In addition, if

$$f,\ Y\in C^0([0,\ T],\ H^s),\quad Z\in C^0([0,\ T],\ H^{s+2}) \tag{4.38}$$

then $u\in C^2([0,\ T],\ H^{s-3/2})$.

Proof. Theorem 4.25 proves this theorem except (4.36). Assume that (4.38) holds. The equation (4.30) is equivalent to

$$\Lambda^s u_{tt}+(M-is\mu A_{0x}D|D|+A_5|D|)\Lambda^s u=\Lambda^s f-R\Lambda^s u, \tag{4.39}$$

$$R=i\mu A_2D+(\mu A_3+A_6)|D|+([\Lambda^s,\ M]+is\mu A_{0x}D|D|\Lambda^s)\Lambda^{-s}+[\Lambda^s,\ L]\Lambda^{-s}.$$

Noting that

$$\|R\Lambda^s u\|\leqq(\mu(\sup|A_2|+\sup|A_3|)+\sup|A_6|)\|u\|_{s+1}$$

$$+\mu C(2\|A_0-1\|_{s+1}+\|A_1\|_s)\|u\|_{s+1}+\|[\Lambda^s,\ L]u\|$$

$$\leqq \{C_2\mu^{1/2}+g_0C_2\mu^{-1/2}+C_3(\mu^{1/2}+\|(1+\mu D^2)Z\|_s)$$

$$+\frac{1}{2}C_3(1+\|(1+\mu D^2)Z\|_s)^2\}\,|u(t)|_{s,\mu}$$

and using (4.16) we obtain (4.36). If (4.38) does not hold then we obtain (4.36) by the same method used in the proof of Theorem 4.25, that is, by approximations of f and coefficients of R in the right hand side of (4.39). The proof is complete.

§ 5. Quasilinear system

Consider the initial value problem for (2.36):

(5.1) $$X_{tt}+X=X+Y,\quad Y_{1tt}+(M+L)Y_1=f_1,\quad Y_{2t}=f_2,\quad Z_t=(f_3,f_4),\quad 0\leqq t\leqq T,$$

(5.2) $$W=W^{(0)}=(X^{(0)},\ Y^{(0)},\ Z^{(0)}),\quad W'_t=W^{(1)\prime}=(X^{(1)},\ Y_1^{(1)}),\quad t=0.$$

We shall use following notations:

$$W=(X,\ Y,\ Z),\quad W'=(X,\ Y_1),\quad W''=(Y_2,\ Z),\quad \Lambda=1+|D|,$$

$$U(W,\ W'_t)=(\Lambda^{3/2}X_1,\ X_2,\ \Lambda^{3/2}Y_1,\ Y_2,\ \Lambda^2Z,\ \Lambda^{3/2}X_{1t},\ X_{2t},\ Y_{1t}),$$

$$V(W'_{tt},\ W''_t)=(\Lambda^{3/2}X_{1tt},\ X_{2tt},\ \Lambda^{-3/2}Y_{1tt},\ Y_{2t},\ \Lambda^2Z_{1t}),$$

(5.3) $$|W(t)|_s=|W(t)|_{s,\mu}$$
$$=(\|X_t(t)\|_s^2+\|X(t)\|_s^2)^{1/2}+(\mu+g_0^2\mu^{-1})^{1/2}(\|X_{1t}(t)\|_{s+3/2}^2$$
$$+\|X_1(t)\|_{s+3/2}^2)^{1/2}+(\|Y_{1t}(t)\|_s^2+\mu\|Y_1(t)\|_{s+3/2}^2$$
$$+\|Y_1(t)\|_{s+1/2}^2)^{1/2}+\|Y_2(t)\|_s+\|(1+\mu D^2)Z(t)\|_s.$$

By (3.24) we see that

(5.4) $$\|(W(t),\ W'_t(t))\|_s\leqq\|W(t),\ W'_t(t)\|_{s,\mu}\leqq|W(t)|_{s,\mu}.$$

We shall show that the problem (5.1), (5.2) has the unique solution under the following assumption.

Assumption 5.5. The minimum of constants c_0 in Lemma 3.25 and Assumption 4.32 is denoted again by $c_0=c_0(g,\ h)>0$. For b and initial data it holds that

$$s\geqq 4+3/2,\quad b\in H^{s+3},\quad U(W^{(0)},\ W^{(1)\prime})\in H^s,\quad \|b\|_3\leqq c_0,$$

(5.6) $$J=\|X^{(0)}\|_3+\|Y^{(0)}\|_2+\|(1+\mu D^2)Z^{(0)}\|_1+\|Z^{(0)}\|_3<c_0.$$

Theorem 5.7. *Under Assumption 5.5, there exists $T>0$ such that the problem (5.1), (5.2) has the unique solution W satisfying conditions*

(5.8) $$\begin{cases}\Lambda^{3/2}X_1,\ X_2\in C^2([0,\ T],\ H^s),\quad Y_1\in C^j([0,\ T],\ H^{s+3/2-3j/2}),\quad j=0,\ 1,\ 2,\\ Y_2,\ \Lambda^2Z\in C^1([0,\ T],\ H^s),\end{cases}$$

that is, $U(W,\ W'_t),\ V(W'_{tt},\ W''_t)\in C^0([0,\ T],\ H^s)$, (5.9) $$\|X(t)\|_3+\|Y(t)\|_2+\|(1+\mu D^2)Z(t)\|_1+\|Z(t)\|_3\leqq c_0,\quad 0\leqq t\leqq T.$$

Proof. Throughout this proof we denote 4 and s by q, so the conditon "$q=4$, s" is omitted.

Step 1. (Constants). Take J_q, d_q, c_q, d such that

(5.10) $$J_q\geqq C_2^5|W(0)|_q=C_2^5\{(\|X^{(1)}\|_q^2+\|X^{(0)}\|_q^2)^{1/2}+(\mu+g_0^2\mu^{-1})^{1/2}(\|X_1^{(1)}\|_{q+3/2}^2$$
$$+\|X_1^{(0)}\|_{q+3/2}^2)^{1/2}+(\|Y_1^{(1)}\|_q^2+\mu\|Y_1^{(0)}\|_{q+3/2}^2+\|Y_1^{(0)}\|_{q+1/2}^2)^{1/2}$$
$$+\|Y_2^{(0)}\|_q+\|(1+\mu D^2)Z^{(0)}\|_q\},$$

(5.11) $$d_q>J_q,\quad c_q=\max\{d_q,\ \|b\|_{q+3}\},\quad d=d_4+2C_1(1+\mu)(1+d_4)^{23}d_4,$$

where $C_2=C_2(c_0, q, g)>2$ is the constant in Lemma 4.34 and $C_1=C_1(c_0, c_4, 4, g, h)>0$ in Lemma 3.25.

Step 2. (Estimates). We shall prove estimates for W satisfying (5.8), (5.9) and

(5.12) $$\|Y_t(t)\|_1+\|Z_t(t)\|_2\leqq d,\quad |W(t)|_q\leqq d_q,\quad 0\leqq t\leqq T.$$

First of all, from

$$\frac{1}{2}\frac{d}{dt}(\|X_t\|_q^2+\|X\|_q^2)=(X_{tt}+X,\ X_t)_q=(X+Y,\ X_t)_q\leqq(\|X\|_q+\|Y\|_q)\|X_t\|_q$$

we obtain

$$(\|X_t(t)\|_q^2+\|X(t)\|_q^2)^{1/2}\leqq(\|X_t(0)\|_q^2+\|X(0)\|_q^2)^{1/2}+\int_0^t(\|X(t)\|_q+\|Y(t)\|_q)dt.$$

Similarly, (we put $p=q+3/2$),

$$(\|X_{1t}(t)\|_p^2+\|X_1(t)\|_p^2)^{1/2}\leqq(\|X_{1t}(0)\|_p^2+\|X_1(0)\|_p^2)^{1/2}+\int_0^t(\|X_1(t)\|_p+\|Y_1(t)\|_p)dt.$$

Since $\|(W(t),\ W_t'(t))\|_q\leqq|W(t)|_q\leqq c_q$, we see by Lemma 3.25 that

(5.13) $$\|f_j\|_q+\|(1+\mu D^2)f_{2+j}\|_q\leqq C_1(1+\mu)(1+d_q)^{4q+7}|W(t)|_q,$$

where $j=1$, 2 and $C_1=C_1(c_0, c_q, q, g, h)>0$. It is clear that Y, Z satisfy (4.33). So we can use (4.36):

$$|Y_1(t)|_{q,\mu}\leqq C_2^5e^{\beta t}|Y_1(0)|_{q,\mu}+C_2^2\int_0^t e^{\beta(t-\tau)}\|f_1\|_q d\tau,$$

where β is the constant defined by (4.37) in which $\sup\|(1+\mu D^2)Z(t)\|_q$ is replaced by d_q. It is clear that

$$\|Y_2(t)\|_q=\|Y_2(0)+\int_0^t Y_{2t}dt\|_q\leqq\|Y_2(0)\|_q+\int_0^t\|f_2\|_q dt,$$

$$\|(1+\mu D^2)Z(t)\|_q\leqq\|(1+\mu D^2)Z(0)\|_q+\int_0^t\|(1+\mu D^2)(f_3, f_4)\|_q dt.$$

Summing up the above estimates we obtain

$$(5.14)\qquad |W(t)|_q \leqq |W(0)|_q C_2^5 e^{\beta t} + N\int_0^t e^{\beta(t-\tau)}|W(\tau)|_q d\tau,$$

$$N = 2 + (\mu + g_0^2\mu^{-1})^{1/2}\mu^{-1/2} + 2C_2^2 C_1(1+\mu)(1+d_q)^{4q+7}.$$

From this estimate it follows that

$$(5.15)\qquad |W(t)|_q \leqq C_2^5 |W(0)|_q \exp(k_q t) \leqq J_q \exp(k_q t),$$

$$(5.16)\qquad k_q = \beta + N = C_2^3 d + C_2^9 \mu^{1/2} + C_2^4 + C_2^3\{C_2\mu^{1/2} + g_0 C_2 \mu^{-1/2} + C_3\mu^{1/2} + C_3(1+d_q)^2\} + 2 + (1 + g_0^2\mu^{-2})^{1/2} + 2C_2^2 C_1(1+\mu)(1+d_q)^{4q+7},$$

where $C_1 = C_1(c_0, c_q, q, g, h) > 0$ is the constant in Lemma 3.25 and $C_2 = C_2(c_0, q, g) > 2$, $C_3 = C_3(c_0, c_q, q, g) > 0$ in Lemma 4.34.

Step 3. (Iteration). Put

$$(5.17)\qquad T = \min\{(c_0 - J)(d_4 + 2d)^{-1},\ k_4^{-1}\log(d_4 J_4^{-1}),\ k_s^{-1}\log(d_s J_s^{-1})\}.$$

By (5.6) and (5.11) we see that $0 < T < \infty$. By E we denote the totality of W satisfying conditions

$$(5.18)\qquad U(W, W_t'),\ V(W_{tt}', W_t'') \in C^0([0, T], H^s),$$

$$(5.19)\qquad |W(t)|_q \leqq J_q \exp(k_q t),$$

$$(5.20)\qquad \|Y_t(t)\|_1 + \|Z_t(t)\|_2 \leqq d,$$

$$(5.21)\qquad \|X(t) - X^{(0)}\|_3 + \|Y(t) - Y^{(0)}\|_2 + \|(1+\mu D^2)(Z(t) - Z^{(0)})\|_1 + \|Z(t) - Z^{(0)}\|_3 \leqq c_0 - J.$$

Taking $W^0 \in E$ and in (5.1) replacing W contained in $M + L$ and right hand sides by W^0 we obtain the system of linear equations for W. We denote this sytem by (5.1–0) and consider the initial value problem (5.1–0), (5.2). From (5.21) it follows that W^0 satisfies (5.9). Therefore by (5.18) and Lemma 3.25 we see that

$$f_j(W^0, W_t^{0\prime}),\ \Lambda^2 f_{2+j}(W^0, W_t^{0\prime}) \in C^0([0, T], H^s),\quad j = 1, 2,$$

and (5.13) holds in which W is replaced by W^0, if we note that

$$\|(W^0(t), W_t^{0\prime}(t))\|_q \leqq |W^0(t)|_q \leqq J_q \exp(k_q t) \leqq d_q \leqq c_q.$$

Clearly, Y^0, Z^0 satisfy (4.33). Hence by Theorem 4.35 and integrations with respect to t we see that the problem (5.1–0), (5.2) has the unique solution W satisfying (5.18). By the same method used in the step 2 we obtain the estimate (5.14) in which $|W(\tau)|_q$ is replaced by $|W^0(\tau)|_q$. Noting that $|W^0(\tau)|_q \leqq J_q \exp(k_q \tau)$ and $k_q = \beta + N$ we have (5.19). It is easily seen that

$$\begin{aligned}\|Y_t(t)\|_1 + \|Z_t(t)\|_2 &\leqq \|Y_t(t)\|_4 + \|(1+\mu D^2)Z_t(t)\|_4 \\ &\leqq \|Y_{1t}(t)\|_4 + \|f_2\|_4 + \|(1+\mu D^2)(f_3, f_4)\|_4 \\ &\leqq |W(t)|_4 + 2C_1(1+\mu)(1+d_4)^{23}|W^0(t)|_4 \\ &\leqq d_4 + 2C_1(1+\mu)(1+d_4)^{23} d_4 = d;\end{aligned}$$

(5.22) $$\|X(t)-X^{(0)}\|_3+\|Y(t)-Y^{(0)}\|_2+\|(1+\mu D^2)(Z(t)-Z^{(0)})\|_1+\|Z(t)-Z^{(0)}\|_3$$
$$\leqq\int_0^t(\|X_t(t)\|_3+\|Y_t(t)\|_2+2\|(1+\mu D^2)Z_t(t)\|_3)dt\leqq(d_4+2d)t\leqq c_0-J.$$

Thus $W\in E$, which means that G defined by $W=G(W^0)$ is the mapping from E to itself. Put $W^0(t)=W^{(0)}$, $0\leqq t\leqq T$. Since $|W^0(t)|_q\leqq J_q$ by (5.10), for W^0 (5.18)–(5.21) hold, i.e. $W^0\in E$. Hence we can define the sequence $W^j\in E, j\geqq 0$, by $W^{j+1}=G(W^j), j\geqq 0$. The difference $W=W^{j+1}-W^j$, $j\geqq 1$, is a solution of

(5.23) $$\begin{cases} X_{tt}+X=X^j-X^{j-1}+Y^j-Y^{j-1}, \\ Y_{1tt}+(M^j+L^j)Y_1=f_1^j-f_1^{j-1}-(M^j-M^{j-1}+L^j-L^{j-1})Y_1^j, \\ Y_{2t}=f_2^j-f_2^{j-1}, \quad Z_t=(f_3^j-f_3^{j-1}, f_4^j-f_4^{j-1}), \quad 0\leqq t\leqq T, \end{cases}$$
$$W=0, \quad W_t'=0, \quad t=0,$$

where $f^j=f(W^j, W_t^{j\prime})$, $M^j=M(W^j)$, $L^j=L(W^j)$. Since $|W^j(t)|_s\leqq J_s\exp(k_st)$ and $s-3/2\geqq 4$ we see by Lemmas 3.20 and 3.25 that

$$\|(M^j-M^{j-1}+L^j-L^{j-1})Y_1^j\|_r\leqq C(\|Y^j-Y^{j-1}\|_r+\|Z^j-Z^{j-1}\|_{r+2})\|Y_1^j\|_{r+3}$$
$$\leqq C|W^j(t)-W^{j-1}(t)|_r,$$
$$\|f_k^j-f_k^{j-1}\|_r+\|(1+\mu D^2)(f_{2+k}^j-f_{2+k}^{j-1})\|_r\leqq C|W^j(t)-W^{j-1}(t)|_r,$$

where $r=s-3/2$ and $C>0$ is independent of j and t. By the same method used in the step 2 we obtain

(5.24) $$|W^{j+1}(t)-W^j(t)|_r\leqq C\int_0^t|W^j(t)-W^{j-1}(t)|_r dt,$$

where $C>0$ is independent of j and t. Therefore there exists W such that

(5.25) $$\sup_{0\leqq t\leqq T}|W^j(t)-W(t)|_{s-3/2}\longrightarrow 0, \quad j\longrightarrow\infty.$$

It is clear that W is a solution of (5.1), (5.2) and that W satisfies (5.18) where s is replaced by $s-3/2$, (5.19) with $q=4$, (5.20) and (5.21). Since W^j satisfies (5.22) it holds that

(5.26) $$\|X(t)\|_3+\|Y(t)\|_2+\|(1+\mu D^2)Z(t)\|_1+\|Z(t)\|_3<c_0, \quad 0\leqq t<T.$$

The uniqueness of the solution W stated in this theorem is proved by the method used for the derivation of (5.24) from (5.23).

Step 4. (Smoothness). Noting that $|W^j(t)|_s\leqq J_s\exp(k_st)$ and $W^j=G(W^{j-1})$ we see that

(5.27) $$\|U(W^j(t), W_t^{j\prime}(t))\|_s+\|V(W_{tt}^{j\prime}(t), W_t^{j\prime\prime}(t))\|_s\leqq C$$

where $C>0$ is independent of j and t. Since any bounded sequence in a Hilbert space has a weak limit we see, in view of (5.25), that

$$(5.28)\qquad U(W(t),\ W_t'(t))\in H^s$$

for any fixed t and

$$(5.29)\qquad |W(t)|_s \leqq \liminf |W^j(t)|_s \leqq J_s \exp(k_s t).$$

For $0\leqq t_0<t$ it holds that

$$\|X_1(t)-X_1(t_0)\|_p \leqq \liminf \|X_1^j(t)-X_1^j(t_0)\|_p \leqq \liminf \int_{t_0}^t \|X_{1t}^j(t)\|_p dt$$
$$\leqq \mathrm{const}\cdot(t-t_0),$$

where $p=s+3/2$. Therefore $X_1\in C^0([0,T],H^{s+3/2})$. Similarly, using (5.27) we see that

$$\Lambda^{3/2}X_1,\ X_2\in C^1([0,T],H^s),\qquad Y,\ \Lambda^2 Z\in C^0([0,T],H^s).$$

Put $F(t)=f_1(W(t),W_t'(t))$ and consider the problem

$$(5.30)\qquad \begin{cases} u_{tt}+\{M(W)+L(W)\}u=F, & 0\leqq t\leqq T,\\ u=Y_1^{(0)},\quad u_t=Y_1^{(1)}, & t=0.\end{cases}$$

Clearly, $F\in C^0([0,T],H^{s-3/2})$. By (5.28), (5.29) and Lemma 3.25 we see that $F(t)\in H^s$, $\|F(t)\|_s\leqq \mathrm{const}<\infty$ for $0\leqq t\leqq T$. By Theorem 4.35 the problem (5.30) has the unique solution u, on the other hand Y_1 is a solution of (5.30). Therefore

$$Y_1=u\in C^j([0,T],H^{s+3/2-3j/2}),\quad j=0,1.$$

Thus $U(W,W_t')\in C^0([0,T],H^s)$, which guarantees that $F=f_1$, $Y_{2t}=f_2$ and $\Lambda^2 Z_t=(\Lambda^2 f_3,\Lambda^2 f_4)$ are in $C^0([0,T],H^s)$. Again by Theorem 4.35 we see that $Y_{1tt}\in C^0([0,T],H^{s-3/2})$. Thus $V(W_{tt}',W_t'')\in C^0([0,T],H^s)$, which completes the proof.

Remark 5.31. (i) If $U(W^{(0)},W^{(1)\prime})\to 0$ in H^s then putting $d_q=J_q^{1/2}$ and letting $J_q\to 0$ we see by (5.17) that $T\to\infty$.

(ii) Let $\overset{0}{W}$ be the solution of (5.1) whose initial values $\overset{0}{W}{}^{(0)}$, $\overset{0}{W}{}^{(1)\prime}$ satisfy (5.6) and (5.10). By the method used to obtain (5.24) we see that

$$(5.32)\qquad |W(t)-\overset{0}{W}(t)|_{s-3/2}\leqq C|W(0)-\overset{0}{W}(0)|_{s-3/2},\quad 0\leqq t\leqq T,$$

where $C>0$ depends only on s, g, h, μ, c_0, J, J_q, d_q and c_q.

(iii) In the above proof we have shown that W satisfies conditions (5.18)–(5.21). Therefore (5.15) holds for W:

$$(5.33)\qquad |W(t)|_q\leqq C_2^{\S}|W(0)|_q \exp(k_q t)\leqq J_q\exp(k_q t),\quad 0\leqq t\leqq T.$$

§6. Nonlinear equation

In this section we shall show that the problem (1.11)–(1.13) has the unique solution.

Assumption 6.1. The minimum of constants c_0 in Lemmas 3.25, 3.58 and

Assumption 4.32 is denoted again by $c_0=c_0(g, h)>0$. Let $e_0=e_0(g, h)>0$ be a constant such that $C(e_0+e_0^2)<c_0$ where $C=C(c_0, g, h)>0$ is the constant in (3.60). Lastly, let $s\geqq 8+1/2$, $c>0$, $c'>0$ and b, $X^{(0)}$, $X_1^{(1)}$ satisfy conditions

$$(6.2)\qquad \begin{cases} b\in H^{s+3/2}, \quad X^{(0)}\in H^{s+3/2}, \quad X_1^{(1)}\in H^s, \\ \|b\|_3\leqq c_0, \quad \|X^{(0)}\|_4\leqq c_0, \quad \|X^{(0)}\|_4+\|(1+\mu D^2)X^{(0)}\|_3+\|X_1^{(1)}\|_3\leqq e_0, \\ \|b\|_{s+3/2}\leqq c, \quad \|X^{(0)}\|_{s+3/2}+\|X_1^{(1)}\|_s\leqq c'. \end{cases}$$

Theorem 6.3. *Under Assumption* 6.1, *there exists* $T=T(g, h, \mu, c_0, e_0, s, c, c')>0$ such that the problem (1.11)–(1.13) has the unique solution X satisfying conditions

$$(6.4)\qquad X\in C^j([0, T], H^{s+3/2-3j/2}), \quad j=0, 1, 2,$$

$$(6.5)\qquad \|X(t)\|_3\leqq c_0, \quad 0\leqq t\leqq T.$$

Proof. Consider the problem (5.1), (5.2) where $W^{(0)}$, $W^{(1)\prime}$ are determined from $X^{(0)}$, $X_1^{(1)}$ by means of (2.40)–(2.43). Using Lemma 3.58 we see that

$$s-3\geqq 4+3/2, \quad b\in H^{s-3+3}, \quad U(W^{(0)}, W^{(1)\prime})\in H^{s-3},$$

$$(6.6)\qquad J=\|X^{(0)}\|_3+\|Y^{(0)}\|_2+\|(1+\mu D^2)Z^{(0)}\|_1+\|Z^{(0)}\|_3\leqq C(e_0+e_0^2)<c_0,$$

that is, Assumption 5.5 holds if we replace s by $s-3$. By Theorem 5.7 the problem (5.1), (5.2) has the unique solution $W=(X, Y, Z)$ satisfying (5.8) in which s is replaced by $s-3$ and (5.9). By Lemma 3.58, (5.10), (5.11), (5.17) and (6.6) we can take $T>0$ depending only on constants $g,\dots, c'$. Moreover by Remark 5.31 we may assume that $T\to\infty$ if $c'\to 0$, i.e. initial data $X^{(0)}$, $X_1^{(1)}$ tend to zero. By (2.17), (2.40) and (2.42) which shows that $(X_{2t}-KX_{1t})_t=0$, $t=0$, we see that $X_{2t}-KX_{1t}=0$, $0\leqq t\leqq T$, i.e. X satisfies (1.12). By (2.18), (2.40) and (2.45) we see that

$$(6.7)\qquad (1+Z_1)Y_1+Z_2(g+Y_2)=\mu R(Z, Z_x)+\mu S(Z, Z_{xx}) \\ +g_0(1+\mu D^2)^{-1}i\,\mathrm{sgn}\,D(Z_1-X_{1x})$$

for $0\leqq t\leqq T$. From (2.40) and (2.47) it follows that

$$(6.8)\qquad Z_1-X_{1x}=(Z_1-X_{1x})_t=0, \quad t=0,$$

(note that c_0 is so small that the operator in (2.47) is invertible, cf. the definition (2.14) of f_3 and Lemma 3.25). In view of (2.5), (2.14) and (2.15) we see that f_2, f_3 and f_4 are differentiable with respect to t, therefore the derivative Z_{tt} exists. By (6.8) and (2.39) we see that $Z_1-X_{1x}=0$, $0\leqq t\leqq T$. Noting that $Z_2-X_{2x}=0$, $t=0$, by (2.40) and integrating (2.19) we have

$$\|Z_2(t)-X_{2x}(t)\|_{s-4}\leqq C\int_0^t \|Z_2(t)-X_{2x}(t)\|_{s-4}dt,$$

where we used the Lipschitz continuity of F_{0j}. Hence $Z_2-X_{2x}=0$, $0\leqq t\leqq T$. Thus we have proved that $Z=X_x$, $0\leqq t\leqq T$. By (6.7) we see that X satisfies (1.11).

Since X and $\Lambda^2X_x=\Lambda^2Z$ are in $C^1([0, T], H^{s-3})$, $X\in C^1([0, T], H^s)$. Noting that $Y_1 \in C^0([0, T], H^{s-3+3/2})$ and $Y_2 = K(X)Y_1 + F_{10}(X, X_t)$ we see that $X_{tt} = Y \in C^0([0, T], H^{s-3/2})$. Integrating (2.6) we have

$$\mu D^2 Z_2 = (-i \operatorname{sgn} D)\mu D^2 Z_1 + F,$$

$$F = \mu D^2\{Z_2(0) + i \operatorname{sgn} D Z_1(0) + \int_0^t (1+\mu D^2)^{-1}(F_{010} - \mu F_{030})dt\}.$$

The equation (1.11) can be written in the form

$$Q(Z)^3\{(1+Z_1)Y_1 + Z_2(g+Y_2) - \mu R(Z, Z_x)\} = \mu\{-Z_2 Z_{1xx} + (1+Z_1)Z_{2xx}\}$$
$$= \{-Z_2 i \operatorname{sgn} D + (1+Z_1)\}(i \operatorname{sgn} D)\mu D^2 Z_1 - (1+Z_1)F.$$

Consider the integral equation

$$\text{(6.9)} \qquad \begin{cases} \mu u_1 = i \operatorname{sgn} D(1+Z_1 - Z_2 i \operatorname{sgn} D)^{-1}\{G(Y, Z, Z_x) + (1+Z_1)F(u)\}, \\ \mu u_2 = (1+Z_1 - Z_2 i \operatorname{sgn} D)^{-1}\{G(Y, Z, Z_x) + (1+Z_1)F(u)\} - F(u), \end{cases}$$

where

$$G(Y, Z, Z_x) = Q(Z)^3\{(1+Z_1)Y_1 + Z_2(g+Y_2) - \mu R(Z, Z_x)\},$$

$$F(u) = \mu D^2\{Z_2(0) + i \operatorname{sgn} D Z_1(0)\}$$
$$+ \int_0^t \mu D^2(1+\mu D^2)^{-1}\{F_{010}(X, Z, X_{1t}) - \mu F_{030}(X, Z, Z_x, u, X_{1t})\}dt.$$

Since

$$\text{(6.10)} \qquad Z(0) \in H^{s+1/2}, \quad Y \in C^0([0, T], H^{s-3/2}), \quad Z \in C^0([0, T], H^{s-1}),$$

the problem (6.9) has the unique solution $u \in C^0([0, T], H^{s-2})$. On the other hand, Z_{xx} is a solution of (6.9). Hence $Z_{xx} = u$, which shows that

$$\text{(6.11)} \qquad Z \in C^0([0, T], H^s).$$

But under conditions (6.10) and (6.11), the problem (6.9) has the unique solution $u \in C^0([0, T], H^{s-3/2})$. Thus $Z \in C^0([0, T], H^{s+1/2})$ and we have proved (6.4). The condition (6.5) follows from (5.9) or (5.26):

$$\text{(6.12)} \qquad \|X(t)\|_3 + \|X_{tt}(t)\|_2 + \|(1+\mu D^2)X_x(t)\|_1 + \|X_x(t)\|_3 < c_0, \quad 0 \leqq t < T.$$

It remains to show the uniqueness. For a solution $\overset{0}{X}$ of (1.11)–(1.13) we put

$$\overset{0}{Y} = \overset{0}{X}_{tt}, \quad \overset{0}{Z} = \overset{0}{X}_x, \quad \overset{0}{W} = (\overset{0}{X}, \overset{0}{Y}, \overset{0}{Z}).$$

By (6.6) we see that $\overset{0}{W}$ satisfies (6.12) for $0 \leqq t \leqq t_0$, ($t_0 > 0$ is sufficiently small). Using equations (1.11) and (1.12) we see that

$$U(\overset{0}{W}, \overset{0}{W}'_t), \ V(\overset{0}{W}'_{tt}, \overset{0}{W}''_t) \in C^0([0, t_0], H^{s-3}).$$

Moreover, by the reduction in §2, $\overset{0}{W}$ is a solution of (5.1), (5.2). Hence by Theorem

5.7, $W(t)=\overset{0}{W}(t)$, $0\leqq t\leqq t_0$. Taking $t=t_0$ as the initial time instead of $t=0$ and noting that the condition (6.12) guarantees that (6.6) holds for initial values at $t=t_0$ we see that $W(t)=\overset{0}{W}(t)$, $t_0\leqq t\leqq t_1$, ($t_1-t_0>0$ is sufficiently small). Repeating this procedure we see that $W(t)=\overset{0}{W}(t)$, $0\leqq t\leqq T$, i.e. $X(t)=\overset{0}{X}(t)$, $0\leqq t\leqq T$. The proof is complete.

Remark 6.13. Let $\overset{0}{X}$ be a solution of (1.11)–(1.13) whose initial data $\overset{0}{X}{}^{(0)}$, $\overset{0}{X}{}_1^{(1)}$ satisfy (6.2). By (5.32) for $\overset{0}{W}=(\overset{0}{X}, \overset{0}{X}_{tt}, \overset{0}{X}_x)$ it holds that

$$(6.14)\qquad |W(t)-\overset{0}{W}(t)|_{s-3-3/2}\leqq C|W(0)-\overset{0}{W}(0)|_{s-3-3/2}$$

Using (6.14), equations $Y_2=K(X)Y_1+F_{10}(X, X_t)$, $Z_t=(f_3, f_4)$ and Lipschitz continuity of the solution of (6.9) with respect to X, X_{1t}, Y, Z, Z_x we see that

$$\sum_{j=0}^{2}\|\partial_t^j(X(t)-\overset{0}{X}(t))\|_{s-3j/2}\leqq C\|X^{(0)}-\overset{0}{X}{}^{(0)}\|_s+C\|X_1^{(1)}-\overset{0}{X}{}_1^{(1)}\|_{s-3/2},$$

where $C=C(g, h, \mu, c_0, e_0, s, c, c')>0$.

§ 7. Limit of X when μ tends to zero

Throughout this section we assume that $g>0$ is a fixed constant, (therefore $g_0=0$ by (2.11)) and that μ moves in the interval $0<\mu\leqq 1$.

Theorem 7.1. *Let Assumption* 6.1 *hold and*

$$\|X^{(0)}\|_4+\|(1+D^2)X^{(0)}\|_3+\|X_1^{(1)}\|_3\leqq e_0.$$

Then in Theorem 6.3 *we can take T' instead of T such that $T'=T'(g, h, c_0, e_0, s, c, c')>0$, i.e. T' is independent of μ. Moreover, if in Assumption* 6.1 *the condition that $s\geqq 8+1/2$ is replaced by $s\geqq 8+1/2+\sigma$ where $0<\sigma<2$ then the solution $X=X^\mu$ of* (1.11)–(1.13) *converges to the solution of* (1.11)–(1.13) *with $\mu=0$:*

$$(7.2)\qquad \sum_{j=0}^{2}\|\partial_t^j(X^\varepsilon(t)-X^\delta(t))\|_r\leqq C(\delta-\varepsilon)^{\sigma/2}(\|X^{(0)}\|_{s+3/2}+\|X_1^{(1)}\|_s),\quad 0\leqq t\leqq T',$$

where $r=s-3-3/2-\sigma$, $0<\varepsilon<\delta<1$ and $C=C(g, h, c_0, e_0, s, c, c', \sigma)>0$.

Proof. In the proof of Theorem 6.3 we defined T by (5.17), but T depends on μ, so we must show that the infinimum of T (when μ moves in the interval (0, 1]) is positive. For $W^{(0)}$, $W^{(1)\prime}$ determined from $X^{(0)}$, $X_1^{(1)}$ by means of (2.40)–(2.43), note that $X^{(0)}$, $Z^{(0)}$, $X^{(1)}$ are independent of μ and $Y^{(0)}=Y^{(0),\mu}$, $Y_1^{(1)}=Y_1^{(1),\mu}$ depend on μ in such a way that

$$\sup_{0<\mu\leqq 1}(\|X^{(0)}\|_3+\|Y^{(0),\mu}\|_2+\|(1+\mu D^2)Z^{(0)}\|_1+\|Z^{(0)}\|_3)\leqq J'<c_0,$$

$$\sup_{0<\mu\leqq 1}(\|Y^{(0),\mu}\|_{s-3+3/2}+\|Y_1^{(1),\mu}\|_{s-3})<\infty,$$

where $J'=C(e_0+e_0^2)$, cf. (3.59), (3.60) and (6.6). In (5.10) and (5.11) we can take

J_q, d_q, c_q, $(q=4, s-3)$ depending only on g, h, c_0, e_0, s, c, c' and we take $d'=d_4+4C_1(1+d_4)^{23}d_4$ instead of d. Since $g_0=0$ we see by (5.16) that

$$k'_q=\sup_{0<\mu\leqq 1} k_q<\infty.$$

Therefore the minimum T' of constants

$$(c_0-J')(d_4+2d')^{-1}, \quad (k'_4)^{-1}\log(d_4J_4^{-1}), \quad (k'_{s-3})^{-1}\log(d_{s-3}J_{s-3}^{-1}),$$

is independent of μ and $0<T'\leqq T$ for $0<\mu\leqq 1$.

To prove the second part of the theorem, let W^μ be the solution of the problem (5.1), (5.2) corresponding to X^μ and put $W=W^\varepsilon-W^\delta$. Then W is a solution of equations

$$X_{tt}+X=X+Y,$$

$$Y_{1tt}+(M^\varepsilon+L^\varepsilon)Y_1=f_1^\varepsilon-f_1^\delta-(M^\varepsilon-M^\delta+L^\varepsilon-L^\delta)Y_1^\delta,$$

$$Y_{2t}=f_2^\varepsilon-f_2^\delta, \quad Z_t=(f_3^\varepsilon-f_3^\delta, f_4^\varepsilon-f_4^\delta),$$

where $f_j^\mu=f_j(W^\mu, W_t^{\mu\prime}, \mu)$, $M^\mu=M(W^\mu, \mu)$, $L^\mu=L(W^\mu, \mu)$. By (5.4) and (5.33) it holds that

$$\text{(7.3)} \qquad \|W^\mu(t), W_t^{\mu\prime}(t)\|_{\theta,\beta}\leqq|W^\mu(t)|_{\theta,\beta}\leqq|W^\mu(t)|_{s-3,\mu}$$
$$\leqq C_2^5|W^\mu(0)|_{s-3,\mu}\exp(k_{s-3}t)\leqq J_{s-3}\exp(k_{s-3}t),$$

for $0\leqq t\leqq T'$, $0<\theta\leqq s-3$, $0<\beta\leqq\mu\leqq\delta<1$. By the method used to obtain (5.14) we have

$$\text{(7.4)} \quad |W(t)|_{r,\varepsilon}\leqq C|W(0)|_{r,\varepsilon}$$
$$+C\int_0^t (\|(f_1^\varepsilon-f_1^\delta, f_2^\varepsilon-f_2^\delta)\|_r+\|(1+\varepsilon D^2)(f_3^\varepsilon-f_3^\delta, f_4^\varepsilon-f_4^\delta)\|_r)dt$$
$$+C\int_0^t \|(M^\varepsilon-M^\delta+L^\varepsilon-L^\delta)Y_1^\delta\|_r dt,$$

where $r=s-3-3/2-\sigma$ and $C>0$ is independent of ε, δ and t. Applying Lemmas 3.25 and 3.51 to

$$f^\varepsilon-f^\delta=\{f(W^\varepsilon, W_t^{\varepsilon\prime}, \varepsilon)-f(W^\delta, W_t^{\delta\prime}, \varepsilon)\}+\{f(W^\delta, W_t^{\delta\prime}, \varepsilon)-f(W^\delta, W_t^{\delta\prime}, \delta)\}$$

we see that the first integral in (7.4) is smaller than

$$C(1+\varepsilon)\int_0^t (1+\|W^\varepsilon, W_t^{\varepsilon\prime}\|_{r,\varepsilon}+\|W_\delta, W_t^{\delta\prime}\|_{r,\varepsilon})^{6r+10}\|W, W_t'\|_{r,\varepsilon}dt$$
$$+C\int_0^t (1+\|W^\delta, W_t^{\delta\prime}\|_{r+\sigma,\delta})^{6r+2\sigma+9}(\delta-\varepsilon)^{\sigma/2}\|W^\delta, W_t^{\delta\prime}\|_{r+\sigma,\delta}dt$$
$$\leqq C\int_0^t |W(t)|_{r,\varepsilon}dt+C(\delta-\varepsilon)^{\sigma/2}\sup_{0\leqq t\leqq T'}|W^\delta(t)|_{r+\sigma,\delta}.$$

Here we used (7.3) and (5.4). By Lemma 3.20 we have

$$(7.5)\qquad \|(M^{\varepsilon}-M^{\delta}+L^{\varepsilon}-L^{\delta})Y_1^{\delta}\|_r$$
$$\leqq C(1+|W^{\varepsilon}(t)|_{r,\varepsilon}+|W^{\delta}(t)|_{r,\varepsilon})^2|W(t)|_{r,\varepsilon}\|(1+\varepsilon D^2)DY_1^{\delta}\|_r$$
$$+C(\delta-\varepsilon)\beta^{-1}(1+\|(1+\beta D^2)Z^{\delta}\|_r)^2\|(1+\beta D^2)DY_1^{\delta}\|_r$$

where $\beta=\delta^{(2-\sigma)/2}$. Since

$$\|(1+\varepsilon D^2)DY_1^{\delta}\|_r\leqq \|Y_1^{\delta}\|_{r+1}+\delta\|Y_1^{\delta}\|_{r+3}\leqq 2|W^{\delta}(t)|_{r+3/2,\delta}$$

and by (3.53) the second term of the right hand side of (7.5) is smaller than

$$C(\delta-\varepsilon)^{\sigma/2}(1+|W^{\delta}(t)|_{r+\sigma,\delta})^2|W^{\delta}(t)|_{r+\sigma+3/2,\delta}$$

we see that the second integral in (7.4) is smaller than

$$C\int_0^t |W(t)|_{r,\varepsilon}dt+C(\delta-\varepsilon)^{\delta/2}\sup_{0\leqq t\leqq T'}|W^{\delta}(t)|_{r+\sigma+3/2,\delta}.$$

Thus we have

$$(7.6)\qquad |W(t)|_{r,\varepsilon}\leqq\int_0^t |W(t)|_{r,\varepsilon}dt+C|W(0)|_{r,\varepsilon}$$
$$+C(\delta-\varepsilon)^{\sigma/2}\sup_{0\leqq t\leqq T'}(|W^{\delta}(t)|_{r+\sigma,\delta}+|W^{\delta}(t)|_{r+\sigma+3/2,\delta}).$$

By (3.61) we see that

$$(7.7)\qquad |W(0)|_{r,\varepsilon}=(\|Y_{1t}^{\varepsilon}(0)-Y_{1t}^{\delta}(0)\|_r+\varepsilon\|Y_1^{\varepsilon}(0)-Y_1^{\delta}(0)\|_{r+3/2}$$
$$+\|Y_1^{\varepsilon}(0)-Y_1^{\delta}(0)\|_{r+1/2})^{1/2}+\|Y_2^{\varepsilon}(0)-Y_2^{\delta}(0)\|_r$$
$$\leqq C(\delta-\varepsilon)(\|X^{(0)}\|_{r+3+3/2}+\|X_1^{(1)}\|_{r+3}).$$

Using (5.33) and (3.59) we have

$$(7.8)\qquad |W^{\delta}(t)|_{r+\sigma,\delta}+|W^{\delta}(t)|_{r+\sigma+3/2,\delta}\leqq 2|W^{\delta}(t)|_{s-3,\delta}\leqq C|W^{\delta}(0)|_{s-3,\delta}$$
$$\leqq C\|I(W^{\delta}(0),\ W_t^{\delta\prime}(0))\|_s\leqq C(\|X^{(0)}\|_{s+3/2}+\|X_1^{(1)}\|_s).$$

From (7.6)–(7.8) it follows that

$$(7.9)\qquad |W^{\varepsilon}(t)-W^{\delta}(t)|_{r,\varepsilon}=|W(t)|_{r,\varepsilon}\leqq C(\delta-\varepsilon)^{\sigma/2}(\|X^{(0)}\|_{s+3/2}+\|X_1^{(1)}\|_s).$$

Since $|W(t)|_{r,0}\leqq|W(t)|_{r,\varepsilon}$ and $W=(X^{\varepsilon}-X^{\delta},\ X_{tt}^{\varepsilon}-X_{tt}^{\delta},\ X_x^{\varepsilon}-X_x^{\delta})$, (7.9) gives (7.2). Since

$$\|\delta(R(Z^{\delta},\ Z_x^{\delta})+S(Z^{\delta},\ Z_{xx}^{\delta}))\|_r\leqq C\delta^{\sigma/2}\beta(\|Z_x^{\delta}\|_r^2+\|Z_{xx}^{\delta}\|_r)$$
$$\leqq C\delta^{\sigma/2}(1+\|(1+\beta D^2)Z^{\delta}\|_r)^2$$
$$\leqq C\delta^{\sigma/2}(1+\|(1+\delta D^2)Z^{\delta}\|_{r+\sigma})^2,$$

where $\beta=\delta^{(2-\sigma)/2}$, it is not difficult to see that the limit of X^{μ} is a solution of (1.11)–(1.13) with $\mu=0$. The proof is complete.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES,
KYOTO UNIVERSITY

Bibliography

[1] В. И. Налимов, Задача Коши-Пуассона, Динамика сплошной среды, Новосибирск, вып. **18** (1974), 104–210.

[2] H. Yosihara, Gravity waves on the free surface of an incompressible perfect fluid of finite depth, Publ. RIMS Kyoto Univ., **18** (1982), 49–96.

[3] L. D. Landau and E. M. Lifshitz, Fluid mechanics, Pergamon Press, Oxford, Second impression 1963.

[4] E. A. Boucher, Capillary phenomena: properties of systems with fluid/fluid interfaces, Rep. Prog. Phys., **43** (1980), 497–546.

[5] R. Finn, Capillarity phenomena, Russian Math. Surveys, **29**: 4 (1974), 133–153.

[6] N. A. Slioskine, Sur les ondes capillaires permanentes, C. R. Acad. Sci. Paris, **201** (1935), 707–709.

[7] G. D. Crapper, An exact solution for progressive capillary waves of arbitrary amplitude, J. Fluid Mech., **4** (1957), 532–540.

[8] Я. И. Секерж-Зенькович, К теории установившихся капиллярно-гравитационных волн конечной амплитуды, ДАН СССР, **109** (1956), 913–915.

[9] Я. И. Секерж-Зенькович, Об установившихся капиллярно-гравитационных волнах конечной амплитуды на поверхности жидкости над волнистым дном, ПММ, **36** (1972), 1070–1085.

[10] H. Beckert, Existenzbeweis für permanente Kapillarwellen einer schweren Flüssigkeit entlang eines Kanals, Arch. Rational Mech. Anal., **13** (1963), 15–45.

[11] H. Hilbig, Existenzbeweis für Potentialströmungen längs eines Kanals mit welliger Sohle unter Einfluß von Schwerkraft und Oberflächenspannung, Arch. Rational Mech. Anal., **18** (1965), 397–402.

[12] W. Kinnersley, Exact large amplitude capillary waves on sheets of fluid, J. Fluid Mech., **77** (1976), 229–241.

[13] J. C. Scott, The propagation of capillary-gravity waves on a clean water surface, J. Fluid Mech., **108** (1981), 127–131.