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Some results on the piston problem related with fluid mechanics

By

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§1. Introduction

We shall discuss in this paper firstly the so-called piston problem of the form

$$(1.1)\quad \begin{cases} \rho(x,t)\left[\dfrac{\partial}{\partial t}v(x,t)+v\dfrac{\partial}{\partial x}v\right]=\mu\dfrac{\partial^2}{\partial x^2}v-K\dfrac{\partial}{\partial x}\rho^{\gamma}, \\ \dfrac{\partial\rho}{\partial t}+\dfrac{\partial}{\partial x}(\rho v)=0,\ (t\geqq 0,\ x\in[0,\bar{l}(t)]), \end{cases}$$

[$\bar{l}(t)\equiv l+\int_0^t V(\tau)\,d\tau$ (>0) (l, positive const.; $V(t)$, suitably smooth function to be clarified later); μ, K, positive constants; γ, const. $(\geqq 1)$; ρ and v are model functions for density and 1-dimensional velocity, resp.],

$$(1.1)'\quad \begin{cases} \rho(x,0)=\rho_0(x)\ (>0),\quad v(x,0)=v_0(x)\,; \\ v(0,t)=0,\quad v(\bar{l}(t),t)=V(t),\ (t\geqq 0)\,. \end{cases}$$

As a result, we shall demonstrate the unique existence of a temporally global solution for (1.1)-(1.1)′ or, rather, for a system of partial differential equations equivalent to (1.1)-(1.1)′, with conventional complementary conditions added.

Secondly, we shall consider the piston problem of the form which is the accurate 1-dimensional model of the fundamental system of differential equations for fluid, i.e.,

$$(1.2)\quad \begin{cases} \rho(x,t)\left[\dfrac{\partial}{\partial t}v(x,t)+v\dfrac{\partial}{\partial x}v\right]=\mu^*\dfrac{\partial^2}{\partial x^2}v-\dfrac{\partial}{\partial x}(R\rho\theta), \\ C_V\rho\left[\dfrac{\partial}{\partial t}\theta(x,t)+v\dfrac{\partial}{\partial x}\theta\right]=-R\rho\theta\dfrac{\partial}{\partial x}v+\mu^*\left(\dfrac{\partial}{\partial x}v\right)^2, \\ \dfrac{\partial}{\partial t}\rho+\dfrac{\partial}{\partial x}(\rho v)=0,\ (t\geqq 0,\ x\in[0,\bar{l}(t)]), \end{cases}$$

[θ, absolute temperature (>0); R, gas constant; C_V, specific heat at constant volume (>0); $\mu^* = 4\mu/3$ (μ, viscosity coefficient (>0))],

$$(1.2)' \quad \begin{cases} \rho(x, 0) = \rho_0(x)\ (>0),\ v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x)\ (>0); \\ v(0, t) = 0, \quad v(\bar{l}(t), t) = V(t), \quad \theta_x(0, t) = \theta_x(l(t), t) = 0, \\ (t \geqq 0), \end{cases}$$

where we assume for simplicity that C_V and μ^* are constants. We shall treat (1.2)–(1.2)′ by transforming our problem into another equivalent one in the same way as in the first case and obtain some a priori estimates for it.

The notation is the same as in [3], [4]. For example, $H^{2+\alpha}_{(I)}$ and $H^{2+\alpha}_{T(I)}$ ($\alpha \in (0, 1)$, $T \in (0, \infty)$) denote Hölder spaces $H^{2+\alpha}$ and $H^{2+\alpha}_T$, resp., as R^1 in their definitions is resplaced by $I = [0, l]$ (cf. [2]). For reference, the definitions of $H^{2+\alpha}$ and $H^{2+\alpha}_T$ are:

$$(1.3) \qquad H^{2+\alpha} \equiv \{f(x), \text{ defined on } R^1: \sum_{m=0}^{2} |D_x^m f|^{(0)} + |D_x^2 f|^{(\alpha)} < \infty\},$$

($|f|^{(0)}$, the sup-norm; $|f|^{(\alpha)}$, the Hölder coefficient of exponent α of f);

$$(1.3)' \quad H^{2+\alpha}_T \equiv \{g(x, t), \text{ defined on } R^1 \times [0, T]:$$

$$\sum_{2r+m=0}^{2} |D_x^m D_t^r g|_T^{(0)} + \sum_{2r+m=1}^{2} |D_x^m D_t^r g|_T^{(\alpha)} < \infty\},$$

($|g|_T^{(0)} \equiv \sup_{(x,t) \in R^1 \times [0,T]} |g(x, t)|$; $|g|_T^{(\alpha)} \equiv |g|_{t,T}^{(\alpha/2)} + |g|_{x,T}^{(\alpha)}$; $|g|_{t,T}^{(\alpha/2)}$, the Hölder coefficient of exponent $\alpha/2$ concerning the time variable; $|g|_{x,T}^{(\alpha)}$, the Hölder coefficient of exponent α concerning the spatial variable).

§2. The first problem

We express the problem (1.1)-(1.1)′ with additional conditions in the v-characteristic coordinates (x_0, t_0) as follows:

$$(2.1) \quad \begin{cases} \hat{v}_{t_0}(x_0, t_0) = \dfrac{\mu}{\rho_0(x_0)} \left(\dfrac{\hat{v}_{x_0}}{1 + \omega(x_0, t_0)} \right)_{x_0} - \dfrac{K}{\rho_0} \left(\left(\dfrac{\rho_0}{1 + \omega} \right)^{\gamma} \right)_{x_0}, \\ (\hat{\rho}^{-1})_{t_0} = \rho_0^{-1} \hat{v}_{x_0}, \quad (x_0 \in I = [0, l],\ t_0 \geqq 0), \end{cases}$$

$$(\omega(x_0, t_0) \equiv \int_0^{t_0} \hat{v}_{x_0}(x_0, \tau)\, d\tau, \text{ therefore } \hat{\rho} = \rho_0/(1 + \omega)),$$

$$(2.1)' \quad \begin{cases} \hat{v}(x_0, 0) = v_0(x_0) \in H^{2+\alpha}_{(I)},\ \hat{\rho}(x_0, 0) = \rho_0(x_0) \in H^{1+\alpha}_{(I)} \\ \qquad (0 < \underline{\rho}_0 \equiv \inf \rho_0 \leqq \rho_0 \leqq \bar{\rho}_0 \equiv |\rho_0|^{(0)}_{(I)} < \infty); \\ \hat{v}(0, t_0) = 0,\ \hat{v}(l, t_0) = V(t_0) \text{ (defined on } [0, \infty) \text{ and belonging to } H^{1+\alpha/2}_{([0,T])} \text{ for} \\ \text{an arbitrary } T \in (0, \infty)); \\ \mu v_0''(0) - K\gamma \rho_0^{\gamma-1} \rho_0'(0) = 0,\ \mu \rho_0^{-1} v_0''(l) - K\gamma \rho_0^{\gamma-2} \rho_0'(l) = V'(0). \end{cases}$$

We note that there is essentially no difference between the study of (1.1)–(1.1)′ and that of (2.1)–(2.1)′ as an independent system of partial differential equations. Therefore, we shall here deal with the temporally global problem of (2.1)–(2.1)′. It goes without saying that, for some $T\in(0,\infty)$, there exists a unique solution $(\hat{v},\hat{\rho})\in H^{2+\alpha}_{T(I)}\times B^{1+\alpha}_{T(I)}$ of (2.1)–(2.1)′. We give without proof:

Lemma 2.1. *Take an arbitrary* $T\in(0,\infty)$ *and fix it. Let* $(\hat{v},\hat{\rho})\in H^{2+\alpha}_{T(I)}\times B^{1+\alpha}_{T(I)}$ *satisfy* (2.1)–(2.1)′. *Then,* $\|\hat{v}\|^{(2+\alpha)}_{T(I)}$ *is estimated from the upper by* $C(|1+\omega|^{(0)}_T, |(1+\omega)^{-1}|^{(0)}_T, T)$, *where C is monotonically increasing in each argument.*

In the proof of the above lemma, remark the relation

$$(2.2)\qquad \omega_{x_0x_0}=\exp\left[-\int_0^{t_0}\frac{k\rho_0^\gamma-1}{1+\omega}dt_0'\right]\times \int_0^{t_0}\left\{\exp\left[\int_0^{t_0'}\frac{k\rho_0^\gamma-1}{1+\omega}dt_0''\right]\right\}\left[\frac{k\rho_0^{\gamma-1}\rho_0'}{(1+\omega)^{\gamma-1}}+\frac{\rho_0}{\mu}\hat{v}_{t_0'}(1+\omega)\right]dt_0'$$

$$(k\equiv K/\mu).$$

Let T and $\hat{v}$ be the same as stated in the above lemma. Then,

$$(2.3)\quad \int_I\frac{1}{2}\rho_0\hat{v}^2dx_0=\int_I\frac{1}{2}\rho_0v_0^2dx_0+\int_0^{t_0}\left[\mu\frac{\hat{v}_{x_0}}{1+\omega}-\left(K\left(\frac{\rho_0}{1+\omega}\right)^\gamma-Ka_0^\gamma\right)\right]_{x_0=l} V(\tau)d\tau-\int_I\int_0^{t_0}\mu\frac{\hat{v}^2_{x_0}}{1+\omega}dx_0dt_0+K\int_I\int_0^{t_0}\left[\left(\frac{\rho_0}{1+\omega}\right)^\gamma-a_0^\gamma\right]\hat{v}_{x_0}dx_0d\tau,$$

$$\left(0\leqq t_0\leqq T,\ a_0\equiv l^{-1}\int_I\rho_0dx_0\right).$$

Now, we define $\psi(s)$ $(s>0)$ by

$$(2.4)\qquad \psi(s)\equiv\int_{a_0^{-1}}^{s}(a_0^\gamma-\lambda^{-\gamma})d\lambda =\begin{cases} a_0^\gamma(s-a_0^{-1})+\dfrac{1}{\gamma-1}(s^{-\gamma+1}-a_0^{\gamma-1})\ (\gamma>1)\\ a_0(s-a_0^{-1})-\log(a_0s)\ (\gamma=1),\end{cases}$$

[*N.B.*: $\psi(a_0^{-1})=0$; $\psi'(s)=a_0^\gamma-s^{-\gamma}>0(s>a_0^{-1}),\ =0(s=a_0^{-1}),\ <0(s<a_0^{-1})$].

We note that the following relations hold:

$$(2.5)\qquad \frac{\partial}{\partial t_0}\left(\bar{\rho}_0\psi\left(\frac{1+\omega}{\rho_0}\right)\right)=\left[a_0^\gamma-\left(\frac{\rho_0}{1+\omega}\right)^\gamma\right]\hat{v}_{x_0},$$

$$(2.5)'\quad \int_I\int_0^{t_0}\left[\left(\frac{\rho_0}{1+\omega}\right)^\gamma-a_0^\gamma\right]\hat{v}_{x_0}dx_0dt_0=-\int_I dx_0\int_0^{t_0}\frac{\partial}{\partial\tau}\left(\rho_0\psi\left(\frac{1+\omega}{\rho_0}\right)\right)d\tau =\int_I\rho_0\psi(\rho_0^{-1})dx_0-\int_I\rho_0\psi\left(\frac{1+\omega}{\rho_0}\right)dx_0.$$

Thus, from (2.3) we have

$$\int_I \frac{1}{2}\rho_0 \hat{v}^2 \, dx_0 + K\int_I \rho_0 \psi(\rho_0^{-1}(1+\omega))\, dx_0 + \mu \int_I \int_0^{t_0} \frac{\hat{v}_{x_0}^2}{1+\omega}\, d\, x_0 \, d\tau = \tag{2.6}$$

$$= E_0 + K\psi_0 + \mu \int_0^{t_0} \left\{ \frac{\hat{v}_{x_0}}{1+\omega} - K\left[\left(\frac{\rho_0}{1+\omega}\right)^{\gamma} - a_0^{\gamma}\right\}\right]_{x_0=l} \cdot V(\tau)\, d\tau,$$

$$\left(\Psi_0 \equiv \int_I \rho_0 \psi(\rho_0^{-1})\, d\, x_0,\ E_0 \equiv \int_I \frac{1}{2}\rho_0 v_0^2 \, dx_0\right).$$

Next, we define $y(x_0, t_0)$ by

$$y(x_0, t_0) = -\int_0^{t_0} \left[\frac{\hat{v}_{x_0}}{1+\omega} - k\left(\frac{\rho_0}{1+\omega}\right)^{\gamma}\right]_{x_0=l} d\tau \tag{2.7}$$

$$-\int_0^{x_0} \frac{\rho_0}{\mu}\, [\hat{v}(x_0', t_0) - v_0(x_0')] dx_0'.$$

Note the following relations concerning $y(x_0, t_0)$:

$$\begin{cases} y_{t_0} = -\dfrac{\hat{v}_{x_0}}{1+\omega} + k\left(\dfrac{\rho_0}{1+\omega}\right)^{\gamma}, \\ (1+\omega)_{t_0} + y_{t_0}(1+\omega) = \begin{cases} k\rho_0^{\gamma}(1+\omega)^{-\gamma+1} & (\gamma > 1) \\ k\rho_0 & (\gamma = 1), \end{cases} \\ ((1+\omega)^{\gamma})_{t_0} + \gamma y_{t_0}(1+\omega)^{\gamma} = \gamma k \rho_0^{\gamma} & (\gamma \geqq 1). \end{cases} \tag{2.8}$$

Hence, we have

$$(1+\omega)^{\gamma} = e^{-\gamma y(x_0, t_0)}\{1 + \gamma k \rho_0^{\gamma}(x_0) \int_0^{t_0} e^{\gamma y(x_0, \tau)}\, d\tau\}. \tag{2.9}$$

This shows that, in order to solve the global problem of (2.1)–(2.1)$'$, it suffices by Lemma 2.1 to have a priori estimates for $|y|_{T(l)}^{(0)}$.

Now, according to the definition of $y(x_0, t_0)$, we obtain, for the 3rd term of the right-hand side of (2.6),

$$\mu \int_0^{t_0} \left\{\frac{\hat{v}_{x_0}}{1+\omega} - k\left[\left(\frac{\rho_0}{1+\omega}\right)^{\gamma} - a_0^{\gamma}\right\}\right]_{x_0=l} V(\tau)\, d\tau = \tag{2.10}$$

$$= \mu \int_0^{t_0} k a_0^{\gamma} V(\tau)\, d\tau - \mu \int_0^{t_0} y_{\tau}(l, \tau) V(\tau)\, d\tau =$$

$$= \mu k a_0^{\gamma} \int_0^{t_0} V(\tau)\, d\tau - \mu y(l, t_0) V(t_0) + \mu \int_0^{t_0} y(l, \tau) V'(\tau)\, d\tau.$$

Moreover, we note that

$$|y(x_0, t_0)| \leqq |y(l, t_0)| + |y(l, t_0) - y(x_0, t_0)|, \tag{2.11}$$

and that, by (2.6),

$$|y(l, t_0) - y(x_0, t_0)| = \left|\int_{x_0}^{l} \frac{\rho_0}{\mu} \hat{v}\, dx_0'\right| \leqq \mu^{-1}(2\bar{\rho}_0 l)^{1/2}\left(\int_I \frac{1}{2}\rho_0 \hat{v}^2 \, dx_0\right)^{1/2} \leqq \tag{2.11$'$}$$

$$\leqq A_1[E_0+K\Psi_0+\mu k a_0^\gamma \int_0^{t_0} V(\tau)\,d\tau-\mu y(l,t_0)V(t_0)+$$

$$+\mu\int_0^{t_0} y(l,\tau)V'(\tau)\,d\tau]^{1/2} \quad (A_1\equiv\mu^{-1}(2\bar{\rho}_0 l)^{1/2}).$$

Thus, in order to estimate $y(x_0, t_0)$, we have only to estimate $y(l, t_0)$. Directly from the 2nd equality of (2.8), we have by integration and by the definition of $\psi(s)$

$$(2.12)\quad \int_0^{t_0}\int_I (1+\omega)_{t_0}\,dx_0\,dt_0+\int_0^{t_0}\int_I y_{t_0}(1+\omega)\,dx_0\,dt_0=$$

$$=\int_0^{t_0}\int_I \rho_0 k\left(\frac{\rho_0}{1+\omega}\right)^{\gamma-1} dx_0\,d\,t_0=k(\gamma-1)\int_0^{t_0}\int_I \rho_0\Psi\left(\frac{1+\omega}{\rho_0}\right)dx_0\,dt_0+$$

$$+k\gamma a_0^\gamma l t_0-\int_0^{t_0} k(\gamma-1)a_0^\gamma \bar{l}(\tau)\,d\tau \quad (\gamma\geqq 1),$$

where it is to be noted that, for $\gamma=1$, the extreme right-hand side gives ka_0lt_0. Also, we have,

$$(2.13)\qquad \text{the extreme left-hand side of (2.12)}=\int_0^{t_0} V(\tau)\,d\tau+$$

$$+\bar{l}(t_0)y(l,t_0)+\int_0^l (x_0+\int_0^{t_0}\hat{v}(x_0,\tau)\,d\tau)\frac{\rho_0}{\mu}\hat{v}\,dx_0-$$

$$-\int_0^{t_0} V(\tau)y(l,\tau)\,d\tau+\int_0^{t_0}\int_0^l \frac{\rho_0}{\mu}\hat{v}^2 dx_0\,dt_0.$$

Thus, noting the relation

$$0\leqq x_0+\int_0^{t_0}\hat{v}(x_0,\tau)\,d\tau\leqq\bar{l}(t_0),$$

we obtain from (2.12)

$$(2.14)\quad \bar{l}(t_0)|y(l,t_0)|\leqq\int_0^{t_0}|V(\tau)|\,d\tau+\bar{l}(t_0)\int_I \frac{\rho_0}{\mu}|\hat{v}|\,dx_0+$$

$$+\int_0^{t_0}|V(\tau)|\,|y(l,\tau)|\,d\tau+\int_0^{t_0}\int_I \frac{\rho_0}{\mu}\hat{v}^2\,dx_0\,dt_0+k\gamma a_0^\gamma l t_0+$$

$$+a_0^\gamma k(\gamma-1)\int_0^{t_0}(\bar{l}(\tau)\,d\tau+k(\gamma-1)\int_0^{t_0}\int_I \rho_0\Psi\left(\frac{1+\omega}{\rho_0}\right)dx_0\,dt_0$$

$$(0\leqq t_0\leqq T;\ \gamma\geqq 1).$$

Hence,

$$(2.15)\quad |y(l,t_0)|\leqq\frac{|V|_T}{L(T)}\left[T+\int_0^{t_0}|y(l,\tau)|\,d\tau\right]+A_1\left(\int_I \frac{1}{2}\rho_0\hat{v}^2\,dx_0\right)^{1/2}+$$

$$+L(T)^{-1}\left[2\int_0^{t_0}\int_I \frac{\rho_0}{\mu}\frac{\hat{v}^2}{2}dx_0\,dt_0+ka_0^\gamma(\gamma l+(\gamma-1)|\bar{l}|_T)T+\right.$$

$$+k(\gamma-1)\int_0^{t_0}\int_I \rho_0 \Psi\left(\frac{1+\omega}{\rho_0}\right) dx_0\, dt_0\Big]_1,$$

$(|V|_T \equiv \max_{0 \leqq t_0 \leqq T} |V(t_0)|$, $|\bar{l}|_T$ is similarly defined;

$$L(T) \equiv \min_{0 \leqq t_0 \leqq T} \bar{l}(t_0)).$$

Furthermore, it holds by (2.6) that

$$[\cdots]_1 \leqq \left(\frac{2}{\mu}+\frac{\gamma-1}{\mu}\right)\left[E_0 T + K\Psi_0 T + K a_0^{\gamma} |V|_T \frac{T^2}{2}\right] + \tag{2.16}$$

$$+\mu(|V|_T + |V'|_T)\int_0^{t_0}\left[|y(l, \tau)| + \int_0^{\tau} |y(l, \tau')|\, d\tau'\right] d\tau +$$

$$+ k a_0^{\gamma}[\gamma l + (\gamma-1)|\bar{l}|_T] T.$$

Also, we have

$$\int_I \frac{1}{2}\rho_0 \hat{v}^2\, dx_0 \leqq \Big[E_0 + K\Psi_0 + K a_0^{\gamma} |V|_T T + \tag{2.17}$$

$$+\mu(|V|_T |y(l, t_0)| + |V|_T \int_0^{t_0} |y(l, \tau)|\, d\tau)\Big]_2 \leqq$$

$$\leqq \left\{\beta[\cdots]_2 + \frac{1}{4\beta}\right\}^2 \quad (\forall \beta > 0).$$

Now, take $\beta = [A_1 \mu(1+|V|_T)]^{-1}$. Then, (cf. (2.12))

$$A_1\left(\int_I \frac{1}{2}\rho_0 \hat{v}^2\, dx_0\right)^{1/2} \leqq A_1 \beta[\cdots]_2 + \frac{A_1}{4\beta} = A_1 \beta \mu |V|_T |y(l, t_0)| \tag{2.18}$$

$$+ A_1 \beta\Big[E_0 + K\Psi_0 + K a_0^{\gamma} |V|_T T + |V|_T \int_0^{t_0} |y(l, \tau)|\, d\tau\Big]_3 + \frac{A_1^2}{4}\mu(1+|V|_T) =$$

$$= \frac{|V|_T}{1+|V|_T}|y(l, t_0)| + A_1 \beta [\cdots]_3 + \frac{A_1^2}{4}\mu(1+|V|_T).$$

Hence, from (2.15) it follows that

$$|y(l, t_0)| \leqq \mu^{-1}[\cdots]_3 + \frac{1}{4}A_1^2 \mu(1+|V|_T)^2 + L(T)^{-1}(1+|V|_T)[\cdots]_1 + \tag{2.19}$$

$$+ L(T)^{-1}(1+|V|_T)|V|_T(T + \int_0^{t_0} |y(l, \tau)|\, d\tau)$$

$$\leqq C_1(T) + C_2(T)\left[\int_0^{t_0}(|y(l, \tau)| + \int_0^{t_0} |y(l, \tau')|\, d\tau')\, d\tau\right],$$

$$(0 \leqq t_0 \leqq T),$$

where $C_1(T)$ (>0) and $C_2(T)$ $(\geqq 0)$; remark the case of $V(\tau) \equiv 0$) are monotonically increasing in T. Thus, it follows that

(2.20) $$|y(l,\ t_0)|_T^{(0)}\leqq C_3(T)\equiv C_1(T)e^{(1+C_2(T))T}\quad (\nearrow \text{ as } T\nearrow).$$

Hence, we have:

Lemma 2.2. *If* $(\hat{v},\ \hat{\rho})\in H_{T(\bar{I})}^{2+\alpha}\times B_{T(\bar{I})}^{1+\alpha}$ *satisfies* (2.1)–(2.1)′, *then the following inequalities hold in an a priori way*:

(2.21) $$\begin{cases} |1+\omega|_T^{(0)}\leqq B_1(|y|_T^{(0)},\ T) & (\nearrow \text{ as each argument } \nearrow),\\ |(1+\omega)^{-1}|_T^{(0)}\leqq B_2(|y|_T^{(0)},\ T) & (\nearrow \text{ as each argument } \nearrow).\end{cases}$$

Finally, we obtain:

Theorem 2.1. *There exists a unique solution* $(\hat{v},\ \hat{\rho})$ *of* (2.1)–(2.1)′ *such that it belongs to* $H_{T(\bar{I})}^{2+\alpha}\times B_{T(\bar{I})}^{1+\alpha}$ *for an arbitrary* $T\in(0,\ \infty)$.

§ 3. The second problem

We express (1.2)–(1.2)′ with conventional additional conditions in the v-characteristic coordinates in the same way as in §2:

(3.1) $$\begin{cases} \hat{v}_{t_0}(x_0,\ t_0)=\dfrac{\mu^*}{\rho_0(x_0)}\left(\dfrac{\hat{v}_{x_0}}{1+\omega(x_0,\ t_0)}\right)_{x_0}-\dfrac{R}{\rho_0}\left(\dfrac{\rho_0\hat{\theta}}{1+\omega}\right)_{x_0},\\ \theta_{t_0}(x_0,\ t_0)=\dfrac{\kappa}{C_V\rho_0}\left(\dfrac{\hat{\theta}_{x_0}}{1+\omega}\right)_{x_0}-\dfrac{R}{C_V\rho_0}\dfrac{\rho_0\hat{\theta}}{1+\omega}\hat{v}_{x_0}+\dfrac{\mu^*}{C_V\rho_0}\dfrac{\hat{v}_{x_0}^2}{1+\omega},\\ \qquad (\hat{\rho}^{-1})_{t_0}=\rho_0^{-1}\hat{v}_{x_0},\ (x_0\in I,\ t_0\geqq 0),\\ \qquad \text{N.B.}: 1+\omega>0,\end{cases}$$

(3.1)′ $$\begin{cases} \hat{v}(x_0,\ 0)=v_0\in H_{(\bar{I})}^{2+\alpha},\ \hat{\theta}(x_0,\ 0)=\theta_0\in H_{(\bar{I})}^{2+\alpha}\ (\theta_0>0),\\ \hat{\rho}(x_0,\ 0)=\rho_0\in H_{(\bar{I})}^{1+\alpha}\ (0<\bar{\rho}_0\leqq\rho_0\leqq\bar{\rho}_0<\infty);\\ \hat{\theta}_{x_0}(0,\ t_0)=\hat{\theta}_{x_0}(l,\ t_0)=0,\ \hat{v}(0,\ t_0)=0,\ \hat{v}(l,\ t_0)=V(t_0)\\ \text{(which is the same as in (2.1)′)},\quad (t_0\geqq 0);\\ \mu^*v_0''(0)-R(\rho_0\theta_0)'(0)=0,\\ V'(0)=\dfrac{\mu^*}{\rho_0(l)}v_0''(l)-\dfrac{R}{\rho_0(l)}(\rho_0\theta_0)'(l).\end{cases}$$

On the basis of the same notation as mentioned in §2 we shall discuss (3.1)–(3.1)′ as an independent system of equations, derive a lemma and give a particular solution.

3.1. A lemma on (3.1)–(3.1)′,

Take an arbitrary $T\in(0,\ \infty)$ and fix it. Let $(\hat{v},\ \hat{\theta},\ \hat{\rho})\in H_{T(\bar{I})}^{2+\alpha}\times H_{T(\bar{I})}^{2+\alpha}\times B_{T(\bar{I})}^{1+\alpha}$ satisfy (3.1)–(3.1)′. We obtain easily the following equality from (3.1)–(3.1)′.

$$(3.2)\quad \int_I \rho_0 \frac{\hat{v}^2}{2}(x_0, t_0)\,dx_0 + \int_I C_V\rho_0\hat{\theta}(x_0, t_0)\,dx_0 =$$

$$= \int_I \frac{1}{2}\rho_0 v_0^2\,dx_0 + \int_I C_V\rho_0\theta_0\,dx_0 + \int_0^{t_0} \frac{\mu^*\hat{v}_{x_0} - R\rho_0\hat{\theta}}{1+\omega}\Big|_{x_0=l} V(t_0')\,dt_0'.$$

We define $Y(x_0, t_0)$ by

$$(3.3)\quad Y(x_0, t_0) \equiv -\int_0^{x_0} \frac{\rho_0}{\mu}[\hat{v}(x_0', t_0) - v_0(x_0')]\,dx_0' +$$

$$+ \int_0^{t_0} [1+\omega(0, t_0')]^{-1}[-\hat{v}_{x_0}(0, t_0') + k^*\rho_0(0)\hat{\theta}(0, t_0')]\,dt_0', \quad (k^* \equiv R/\mu^*).$$

$Y(x_0, t_0)$ satisfies

$$(3.4)\quad \begin{cases} Y_{t_0}(x_0, t_0) = \dfrac{-\hat{v}_{x_0} + k^{k*}\rho\hat{\theta}}{1+\omega(x_0, t_0)} = \dfrac{\mu^*}{1+\omega}\left(\dfrac{Y_{x_0}}{\rho_0}\right)_{x_0} + \dfrac{k^*\rho_0\hat{\theta}}{1+\omega}, \\ Y_{x_0} = -\dfrac{\rho_0}{\mu^*}\hat{v}, \\ Y(x_0, 0) = 0. \end{cases}$$

Hence, it follows that

$$(3.5)\quad \begin{cases} (1+\omega)_{t_0} + Y_{t_0}(1+\omega) = k^*\rho_0\hat{\theta}, \\ (1+\omega)(x_0, 0) = 1. \end{cases}$$

The above relation is an ordinary differential equation of the 1st degree in $1+\omega$ for each fixed x_0. Thus, we have

$$(3.6)\quad (1+\omega)(x_0, t_0) = e^{-Y(x_0, t_0)}[1 + k^*\rho_0\int_0^{t_0}\hat{\theta}(x_0, \tau)e^{Y(x_0, \tau)}\,d\tau].$$

Moreover, by (3.4), the equality (3.2) can be written in the following way:

$$(3.7)\quad \int_I (\rho_0\frac{\hat{v}^2}{2} + C_V\rho_0\hat{\theta})(x_0, t_0)\,dx_0 = E_1 - \mu^*\int_0^{t_0} Y_\tau(l, \tau)V(\tau)\,d\tau =$$

$$= E_1 - \mu^* Y(l, t_0)V(t_0) + \mu^*\int_0^{t_0} Y(l, \tau)V'(\tau\,d\tau,$$

$$(E_1 \equiv \int_I\left(\rho_0\frac{\hat{v}_0^2}{2} + C_V\rho_0\theta_0\right)dx_0).$$

In the same way as in § 2 (cf. (2.11), (2.11'), we have

$$(3.8)\quad |Y(x_0, t_0)| \leqq A_1^*\left\|\left(\frac{\rho_0}{2}\right)^{1/2}\hat{v}\right\|_{L_2(I)} + |Y(l, t_0)|,$$

$$(A_1^* \equiv \mu^{*-1}(2l\bar{\rho}_0)^{1/2}).$$

Also, in order to obtain a priori estimates for $|Y|_{T(I)}^{(0)}$, it suffices to have those for $|Y(l, t_0)|$. There follows from (3.5) an equality

(3.9) $$\int_0^l dx_0 \int_0^\tau (1+\omega)_{,t_0} dt_0 + \int_0^l dx_0 \int_0^\tau y_{,t_0}(1+\omega) dt_0 = \\ = k^* \int_0^\tau dt_0 \int_0^l \rho_0 \hat{\theta} dx_0, \quad (\forall \tau \in [0, T]).$$

Referring ourselves to (2.13) and (2.14), we have

(3.10) $$\bar{l}(\tau)|Y(l, \tau)| \leqq \bar{l}(\tau) \int_0^l \frac{\rho_0}{\mu^*} |\hat{v}(x_0, \tau)| dx_0 + \Big[\int_0^\tau |V(t_0)| dt_0 + \\ + \int_0^\tau |V(t_0)||y(l, t_0)| dt_0 + \frac{2}{\mu^*} \int_0^\tau dt_0 \int_0^l \frac{\rho_0}{2} \hat{v}^2 dx_0 + \frac{k^*}{C_V} \int_0^\tau dt_0 \int_0^l C_V \rho_0 \hat{\theta} dx_0\Big].$$

By (3.2), it holds that

(3.11) $$|Y(l, \tau)| \leqq A_1^*[E_1 + \mu^* |Y(l, \tau)| |V(\tau)| + \\ + \mu^* \int_0^\tau |Y(l, t_0)| |V'(t_0)| dt_0]^{1/2} + L(T)^{-1} \Big[\int_0^\tau |V(t_0)| dt_0 + \\ + \int_0^\tau |V(t_0)| |y(l, t_0)| dt_0 + A_2 \int_0^\tau dt_0 \int_0^l \Big(\frac{\rho_0}{2} \hat{v}^2 + C_V \rho_0 \hat{\theta} dx_0\Big],$$

$$(0 \leqq \tau \leqq T), \ \Big(A_2 \equiv \frac{2}{\mu^*} + \frac{k^*}{C_V}\Big).$$

We define $\bar{E}_1(\tau)$ by

(3.12) $$\bar{E}_1(\tau) = E_1 + \mu^* |V(\tau)| |Y(l, \tau)| + \mu^* \int_0^\tau |Y(l, t_0)| |V'(t_0)| dt_0.$$

Then, we have

(3.13) $$|Y(l, \tau)| \leqq A_1^* \bar{E}_1(\tau)^{1/2} + L(T)^{-1} \Big[T|V|_T + |V|_T \int_0^\tau |Y(l, t_0)| dt_0 + \int_0^\tau \bar{E}_1(t_0) dt_0\Big].$$

By using the inequality $\bar{E}_1(\tau)^{1/2} \leqq \beta \bar{E}_1(\tau) + \frac{1}{4\beta}$ $(\forall \beta > 0)$ as in §2, after all, we obtain

(3.14) $$|Y(l, \tau)| \leqq \bar{C}_1(T) + \bar{C}_2(T) \int_0^\tau \Big[|Y(l, t_0)| + \int_0^{t_0} |Y(l, t_0')| d t_0'\Big] dt_0, \quad (0 \leqq \tau \leqq T),$$

where $\bar{C}_1(T) (>0)$ and $\bar{C}_2(T)$ $(\geqq 0)$ are monotonically increasing in T. Thus, the following lemma holds:

Lemma 3.1. (i) *For $Y(x_0, t_0)$ defined by* (3.3), *we have in an a priori way*

(3.15) $$|Y(x_0, t_0)|_T^{(0)} \leqq \bar{B}_1(T) \ (\nearrow \text{ as } T \nearrow).$$

(ii) *For ρ and θ^{-1}, we have a priori estimates*

(3.15)′ $$\begin{cases} |\hat{\rho}|_T^{(0)} \leqq \bar{B}_2(T) \ (\nearrow \text{ as } T \nearrow) \ (cf. (3.6)), \\ |\hat{\theta}^{-1}|_T^{(0)} \leqq \bar{B}_3(T) \ (\nearrow \text{ as } T \nearrow), \ (\text{derive the equation which } \hat{\theta}^{-1} \text{ satisfies} \\ \text{from that which } \hat{\theta} \text{ does}). \end{cases}$$

We have not as yet a priori estimates from the upper for $\hat{\theta}$. If we can have such estimates, then we shall have a clue to settling the temporally global problem of (3.1)–(3.1)′.

3.2. A particular solution

Here, we add that there exists a particular solution of (3.1)–(3.1)′ such that

$$\text{(3.16)} \qquad \begin{cases} \hat{v}(x_0, t_0) = ax_0 \ (a, \text{const.}), \quad \omega = at_0, \\ \hat{\rho}(x_0, t_0) = \rho_0(1+at_0)^{-1}, \quad (\rho_0, \quad \text{const.} > 0), \\ \theta(x_0, t_0) = \bar{\theta}(t_0), \end{cases}$$

where $\bar{\theta}(t_0)$ satisfies an ordinary linear differential equation of the 1st degree

$$\text{(3.17)} \qquad \begin{cases} C_V\rho_0\bar{\theta}'(t_0) + (1+at_0)^{-1}\mu^* k^* a\rho_0\bar{\theta}(t_0) = \mu^* a^2(1+at_0)^{-1}, \\ \bar{\theta}(0) = \bar{\theta}_0 \quad (\text{const.} > 0). \end{cases}$$

We can easily solve (3.17), that is,

$$\text{(3.17)}' \qquad \bar{\theta}(t_0) = \frac{a\mu^*}{R\rho_0} + \left(\bar{\theta}_0 - \frac{a\mu^*}{R\rho_0}\right)(1+at_0)^{-R/C_V}.$$

Let us consider the three cases of $a=0$, $a>0$, and $a<0$.

(i) $a=0$.

$$\text{(3.18)} \qquad \hat{v} = 0, \quad \hat{\theta} = \bar{\theta}_0, \quad \hat{\rho} = \rho_0.$$

(ii) $a>0$.

$$\text{(3.18)}' \qquad \begin{cases} \hat{v} = ax_0 \\ \bar{\theta}(t_0) \longrightarrow \dfrac{a\mu^*}{R\rho_0} \ (t_0 \to \infty), \\ \hat{\rho} = \rho_0(1+at_0)^{-1} \longrightarrow 0 \ (t_0 \to \infty). \end{cases}$$

We note that, if $\bar{\theta}_0 = a\mu^*(R\rho_0)^{-1}$, then $\bar{\theta}(t_0) = \bar{\theta}_0$.

(iii) $a<0$.

$$\text{(3.18)}'' \qquad \begin{cases} \hat{v} = ax_0, \ \hat{v}_{x_0} = a, \ \hat{\rho}_{x_0x_0} = 0, \\ \hat{\rho} = \rho_0(1+at_0)^{-1} \longrightarrow \infty\left(t_0 \longrightarrow -\dfrac{1}{a}\right), \\ \hat{\theta} = \bar{\theta}(t_0) \longrightarrow \infty\left(t_0 \longrightarrow -\dfrac{1}{a}\right). \end{cases}$$

It is of much interest that $\hat{\rho}$ and $\hat{\theta}$, as they say, blow up in a finite time $-\dfrac{1}{a}$.

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