J. Math. Kyoto Univ. (JMKYAZ)
23-4 (1983) 617-630

Paley-Wiener type theorem for certain semidirect product groups

Dedicated to Professor Hisaaki Yoshizawa on his 60-th birthday

By

Shoichi ANDO

(Received July 6, 1982)

0. Introduction

The classical Paley-Wiener type theorem asserts that the Fourier transformation is a topological isomorphism from $C_0^\infty(\mathrm{R}^n)$ to a space of certain kind of entire functions on C^n, identified with the set of not necessarily unitary characters of R^n. As this example suggests, the study of Paley-Wiener type theorem on a Lie group is closely related to an analysis of non-unitary representations. More precisely, we may understand that the study of Paley-Wiener type theorem contains at least two main subjects:

1. To construct a family of "non-unitary" representations, which should be an "analytic continuation" of a continuous family of unitary ones, especially, those appearing in the decomposition of the regular representation.

2. To show that the Fourier transformation is a topological isomorphism from $C_0^\infty(G)$ to a space of certain kind of (operator-valued) entire functions on a complex space.

If we succeed to obtain Paley-Wiener type theorem in this sense, we may consider the constructed family as a "non-unitary" dual.

For semisimple Lie groups, we may take the so called elementary representations, which are realized through bounded operators in a Hilbert space, cf. [18, 19]. Contrary to this, non-semisimple Lie groups make us encounter a difficulty such as nonunitary representations cannot be realized through bounded operators in any Hilbert space, in general. §1 gives a simple construction of non-unitary representations for a certain kind of groups.

Examples of non-semisimple Lie groups, for which we have obtained Paley-Wiener type theorem are not so many. They are motion groups [9, 10, 11], some solvable Lie groups [1, 2, 3], and the oscillator group [14], for instance. Except the last, they are all semidirect product with normal vector groups. In the case of motion groups, Peter-Weyl theorem for compact groups plays an important role in the proof of the papers [9, 10, 11]. But we show in this paper that a certain kind of semidirect product groups, including motion groups, can be treated in a unified way without using Peter-Weyl theorem. Our group $G=N.W$ is a semidirect product of a vector group N with a connected Lie group W. We pose an assumption on the action of W.

In §1 we consider a σ-Hilbert space $\mathscr{C}$ with seminorms $\|\cdot\|_t$ $(t\in\mathbf{R})$, on which we construct a family of non-unitary representations T^λ, "induced" from $\lambda\in N^*$. We remark that they are not in general irreducible. For some familiar groups, however, almost all of them are operator-irreducible. The group of affine trnsformations of a straight line, Heisenberg groups and the group of upper triangular matrices with unities on the diagonal, these are such ones.

In §2 we consider a Sobolev space $\mathscr{C}_\infty$ on W, which turns out a subspace of C^∞-functions. Now the Fourier transform $T^\lambda(f)$ of a function $f\in C_0^\infty(G)$ has the following properties (1°)~(3°) (Proposition 8): (1°) $T^\lambda(f)$ is a continuous operator on $\mathscr{C}$ whose range is contained in $\mathscr{C}_\infty$. This is expressed by a definite scale change on $\mathscr{C}$, cf. §1 (5). (2°) Operators $T^\lambda(f)$, $\lambda\in N^*$, corresponding to the same W-orbit are similar each other, that is, $T^{\lambda\cdot w}=L_w\cdot T^\lambda\cdot L_{w^{-1}}$ by a left translation L_w, $w\in W$. (3°) Matricial coefficients of T^λ are entire functions of $\lambda\in N^*$, the complex dual of N.

Our key Proposition 9 asserts the converse, to the proof of which §3 is devoted, where Sobolev's lemma plays an essential role. It may be worth while to recall that for Euclidean spaces or compact Lie groups, Sobolev's lemma reduces to the Fourier series expansion or Peter-Weyl theorem [16, 17, 18].

In §4 we state our main theorem, an alternative formulation of Proposition 9. Now let Q be a compact set on G and $t\to\tau^\lambda(t)=\tau^\lambda(t;Q)$ be a corresponding scale change of $\mathscr{C}(t\in\mathrm{R})$, (cf. §1 (5)). Further, Let $B=B(Q)$ be the space of operator fields $T=(T^\lambda)$ on N^*, having the properties (1°)~(3°) (Proposition 9):

(1°) *for any X and Y in $U(\mathfrak{G})$ there exists a constant $C(X, Y)$ such that* $\|\partial_r(X)\cdot T^\lambda\cdot\partial_l(Y)\varphi\|_t\leqq C(X,Y)\|\varphi\|_{\tau^\lambda(t)}$ *for any $\varphi\in\mathscr{C}$ and $t\in\mathrm{R}$,*

(2°) $L_a\cdot T^\lambda\cdot L_{a^{-1}}=T^{\lambda a}$ $(a\in W)$,

(3°) Matrix coefficients of T^λ are entire functions on N^*.

Natural seminorms on B make it a Fréchet space. Final formulation states as follows:

Paley-Wiener type theorem. *Fourier transformation $f\to T^\lambda(f)$ is a to a topological isomorphism from $C_0^\infty(Q)$ onto the space $B(Q)$, and so, it is an isomorphism from $C_0^\infty(G)$ onto the inductive limit of spaces $B(Q)$.*

Now returning to the beginning, we can say that our dual object is realized as $\{T_g^\lambda,\ \lambda\in N^*/W\}$.

In §5 we show that the representations T_g^λ are operator-irreducible if the isotropy W_λ in W at $\lambda\in N^*$ is trivial.

1. Construction of representations

Let $G=N\cdot W$ be a semidirect product group of a real vector space N with a connected Lie group W, acting on N. $\hat{N}$ denotes the dual space of N and N^* the complexification of $\hat{N}$: $N^*=N\otimes_R\mathrm{C}$. For $\lambda\in N^*$ we put $\lambda(n)=\exp\sqrt{-1}\langle\lambda, n\rangle$ and in this manner N^* is identified with the set of not necessarily unitary characters of N. W acts on N^* as follows:

$$\lambda\cdot w(n)=\lambda(wnw^{-1}),\quad \lambda\in N^*,\quad n\in N.$$

First of all we realize unitary representations $T^\lambda=\mathrm{Ind}_N^G\lambda$ $(\lambda\in\hat{N})$ on $L^2(W, d_rw)$ as follows:

$$(1)\qquad T_g^\lambda\varphi(w)=\lambda(wnw^{-1})\varphi(wv),\qquad \text{for}\quad g=nv.$$

These are not in general irreducible. They appear in the process of decomposition of the regular representation ρ_G: This follows from the general theory of induced representations,

$$\rho_G=\mathrm{Ind}_e^G 1=\mathrm{Ind}_N^G\,\mathrm{Ind}_e^N 1=\mathrm{Ind}_N^G\,\rho_N=\int_{\hat{N}}\mathrm{Ind}_N^G\,\lambda d\lambda.$$

As for further procedure of decomposition into irreducible components, see [8].

The action of $w\in W$ on N coincides with the inner automorphism $n\to wnw^{-1}$ of G. In the vector space N we once for all fix an Euclidean norm $\|\cdot\|$ and $\|w\|$ denotes the norm of the automorphism relative to this norm.

Now we construct non-unitary representations after the method of [1, 2, 3]. Put $M(w)=\max(\|w\|, \|w^{-1}\|)$. Clearly it holds for any $n\in N$ and $\lambda\in N^*$

$$|\lambda\cdot w(n)|\leqq\exp(M(w)\|\mathrm{Im}\,\lambda\|\,\|n\|).$$

For every $t\in\mathrm{R}$ put $H(t)=L^2(W, \exp(tM(w))d_rw)$ and let $\|\cdot\|_t$ denote the norm in $H(t)$. If $s\leqq t$, then $\|\varphi\|_s\leqq\|\varphi\|_t$ and so $H(s)\supset H(t)$. The inclusion is continuous. The dual space of $H(t)$ is identified with $H(-t)$ via natural dual pairing:

$$\langle\varphi, \psi\rangle=\int_W\varphi(w)\psi(w)d_rw,\quad \varphi\in H(t)\quad\text{and}\quad \psi\in H(-t).$$

Proposition 1. (*i*) *The projective limit* $\mathscr{C}=\varprojlim.\ H(t)$ *is a Fréchet space with a system of seminorms* $\|\cdot\|_t$. (*ii*) *The dual space* $\mathscr{C}'$ *of* $\mathscr{C}$ *coincides with the inductive limit*: $\mathscr{C}=\varinjlim H(t)$.

Proof. (i) Let (φ_n) be a Cauchy sequence in $\mathscr{C}$, we have for every real t $\|\varphi_n-\varphi_m\|_t\to 0$ as $n, m\to\infty$. Since every $H(t)$ is complete, there exists a unique element φ_t in $H(t)$ such that $\varphi_n\to\varphi_t$. From the inclusion relation $H(s)\supset H(t)$ for $s\leqq t$, φ_s must coincide with φ_t.
(ii) Let F be an element of $\mathscr{C}'$. By the definition of the topology of $\mathscr{C}$, F is already continuous on some space $H(t)$, that is, $F\in H(-t)$ by the natural dual pairing.
Q.E.D.

Proposition 2. *In the space* $\mathscr{C}$ *the expression* (1) *gives a representation* $(T_g^\lambda, \mathscr{C})$ *also for every non-unitary character* $\lambda\in N^*$. *It holds*

$$(2)\qquad \|T_g^\lambda\varphi\|_t\leqq\|\varphi\|_{\tau^\lambda(t;g)},$$

where the scale τ^λ *is determined as follows*: *for* $g=n.w$

$$\tau^{\lambda}(t;\, g)=\begin{cases} M(w)(t+2\|\operatorname{Im}\lambda\|\,\|n\|), & \text{if} \quad t\geqq -2\|\operatorname{Im}\lambda\|\,\|n\|\,, \\ M(w)^{-1}(t+2\|\operatorname{Im}\lambda\|\,\|n\|), & \text{if} \quad t\leqq -2\|\operatorname{Im}\lambda\|\,\|n\|\,. \end{cases} \tag{3}$$

It is enough to note that

$$M(ww')\leqq M(w)M(w')\,, \quad M(w^{-1})=M(w)\,.$$

When the group W is compact, the function $M(w)$ is continuous and so the space $\mathscr{C}$ coincides with $L^2(W, d_r w)$. This is the case for the Euclidean motion groups [11], Cartan motion groups [9] and motion groups [10]. Hereafter we are concerned with the case when W is not compact. We need, however, an assumption:

Assumption. For every $\alpha\geqq 1$, *the set* $\{w\in W;\ M(w)\leqq\alpha\}$ *is compact.*
We exclude, for instance, a case when a non-compact group W acts on N through a unitary representation. The universal covering group of two dimensional motion group is an excluded one. For this group Paley-Wiener type theorem is formulated in a Hilbert space, cf. [13].

Proposition 3. *There exists an equivalence relation*:

$$L_w\cdot T_g^{\lambda}\cdot L_{w^{-1}}=T_g^{\lambda\cdot w}, \quad w\in W,\ \lambda\in N^*,$$

where L_w *is a left translation on* W: $L_w\varphi(w')=\varphi(ww')$.

Let dn and $d_r w$ denote right Haar measures on N and W respectively. Take $d_r g=dnd_r w$ $(g=nw)$ as a right Haar measure on G. Moreover $\Delta_W(\cdot)$ and $\Delta_G(\cdot)$ denote the modular functions on W and G: $d_r(vw)=\Delta_W(v)d_r w$ for $v\in W$ and so on.

Consider a set $Q_{\gamma,\alpha}=\{g=nw;\ \|n\|\leqq\gamma,\ M(w)\leqq\alpha\}$, which is compact by the Assumption.

Proposition 4. *Suppose a function* $f\in L^1(G,\, d_r g)$ *vanishes outside the set* $Q_{\gamma,\alpha}$. *Then the Fourier transform* $T^{\lambda}(f)$ *of* f: $T^{\lambda}(f)=\int_G f(g)T_g^{\lambda}d_r g$; *satisfies*

$$\|T^{\lambda}(f)\varphi\|_t\leqq\|f\|_{L^1}\|\varphi\|_{\tau^{\lambda}(t)}, \tag{4}$$

where the scale $\tau^{\lambda}(t)$ *is of the form*

$$\tau^{\lambda}(t)=\tau^{\lambda}(t;\, Q_{\gamma,\alpha})=\begin{cases} \alpha(t+2\gamma\|\operatorname{Im}\lambda\|), & \text{if} \quad t\geqq -2\gamma\|\operatorname{Im}\lambda\|\,, \\ \alpha^{-1}(t+2\gamma\|\operatorname{Im}\lambda\|), & \text{if} \quad t\leqq -2\gamma\|\operatorname{Im}\lambda\|\,. \end{cases} \tag{5}$$

Proof. It is clear that

$$\|T^{\lambda}(f)\varphi\|_t\leqq\int_G|f(g)|\,\|T_g^{\lambda}\varphi\|_t d_r g$$
$$\leqq\int_G|f(g)|\,\|\varphi\|_{\tau^{\lambda}(t;g)}d_r g \qquad \text{(by (3))}.$$

Since the support of f is contained in $Q_{\gamma,\alpha}$, we have the result. Q.E.D.

It is easy to see that $T^{\lambda}(f)$ has an integral kernel $K_f^{\lambda}(w, v)$:

$$K_f^\lambda(w, v)=\Delta_G(w)^{-1}\int_N f(w^{-1}nv)\lambda(n)dn.$$

From this or from Proposition 3 it follows immediately

$$(6)\qquad K_f^{\lambda a}(w, v)=\Delta_W(a)K_f^\lambda(aw, av), \quad a\in W.$$

2. Differential operators

Let $\mathfrak{G}$, $\mathfrak{N}$ and $\mathfrak{W}$ be the Lie algebra of G, N and W, respectively. Each element $X\in\mathfrak{G}$ defines differential operators X_l and X_r on G:

$$X_l f(g)=\frac{d}{dt}f(g.\exp tX)|_{t=0}, \quad X_r f(g)=\frac{d}{dt}f(\exp(-tX)g)|_{t=0}.$$

Moreover each element X of $\mathfrak{W}$ defines a differential operator $\partial(X)$ on W:

$$\partial(X)\varphi(w)=\frac{d}{dt}\varphi(w.\exp tX)|_{t=0}.$$

Further, a pair (λ, X), $\lambda\in N^*$, $X\in\mathfrak{N}$; gives a multiplication operator λ_X by a function $\lambda_X(w)$: $\lambda_X(w)=d/dt|_{t=0}\,\lambda(w.\exp tX.w^{-1})$.
The correspondence $X\to X_r$ (or X_l) extends to the enveloping algebra $U(\mathfrak{G})$ by associativity, so also the mapping $X\to\partial(X)$ to the whole $U(\mathfrak{W})$. For the convenience of the later use we introduce notations $\partial_l(X)$ and $\partial_r(X)$:

$$(7)\qquad \partial_r(X)=\begin{cases}\partial(X)+\Delta_G(X) & \text{for } X\in\mathfrak{W},\\ \lambda_X & \text{for } X\in\mathfrak{N},\end{cases}\qquad \partial_l(X)=\begin{cases}-\partial(X) & \text{for } X\in\mathfrak{W},\\ -\lambda_X & \text{for } X\in\mathfrak{N}.\end{cases}$$

We extend these correspondences $X\to\partial_r(X)$, $\partial_l(X)$ to the whole $U(\mathfrak{W})$ by the associativity. $(\Delta_G(X)=d/dt|_{t=0}\,\Delta_G(\exp tX))$.

Proposition 5. *For $f\in C_0^\infty(G)$ we have*
(i) $T^\lambda(X_r\cdot Y_l f)=\partial_r(X)\cdot T_f^\lambda(f)\cdot\partial_l(Y)$ for X, $Y\in U(\mathfrak{W})$,
(ii) $T^\lambda(X_r\cdot Y_l f)=\partial_r(X)\cdot T^\lambda(f)\cdot\partial_l(Y)$ for X, $Y\in U(\mathfrak{N})$.

Proof is easy. The equality (i) should be considered on a subspace $\mathscr{C}_\infty$ of $\mathscr{C}$, which consists of the functions ψ on W whose distribution derivatives $\partial(X)\psi$ also belong to $\mathscr{C}$ for any $X\in U(\mathfrak{W})$. The topology of the space $\mathscr{C}_\infty$ is endowed with the seminorms $\|\psi\|_{t,X}=\|\partial(X)\psi\|_t$, $X\in U(\mathfrak{W})$, $t\in\mathrm{R}$. We state here two properties concerning the space $\mathscr{C}_\infty$, which is crucial in the next section.

Proposition 6. $\mathscr{C}_\infty\subset C^\infty(W)$.

Proof. Choose a local chart $(U;\ x_1,\dots, x_n)$ on W such that U is relatively compact. Between two systems of tangent vectors $\partial/\partial x_1,\dots,\partial/\partial x_n$ and $\partial(X_1),\dots,\partial(X_n)$ there exist relations

$$\partial/\partial x_i=\sum_{i=1}^n a_{ij}(x)\partial(X_j), \quad i=1,\dots, n,$$

where $a_{ij}(x)$ are of C^∞-class. It is easy to show that any polynomial with constant coefficients $P(\partial/\partial x)=P(\partial/\partial x_1,\dots,\partial/\partial x_n)$ is written in a form

$$P(\partial/\partial x)=\sum_\alpha A_\alpha(x)\partial(X^\alpha),\quad X^\alpha\in U(W).$$

On the other hand two measures $d_r w$ and $dx=dx_1\dots dx_n$ are mutually equivalent, that is, $d_r w=\Delta(x)^{-1}dx$ for a suitable C^∞-function $\Delta(x)>0$. Then we have

$$\int_U |P(\partial/\partial x)\psi(x)|^2 dx \leqq \text{const.} \sum_\alpha \int_U |A_\alpha(x)|^2|\partial(x^\alpha)\psi(x)|^2\Delta(x)\Delta(x)^{-1}dx$$

$$\leqq \text{const.} \max_{x\in U,\ \alpha}(|A_\alpha(x)|^2\Delta(x)^{-1})\sum_\alpha\int_U|\partial(X^\alpha)\psi(w)|^2 d_r w,$$

abusing the notations as $\psi(x)=\psi(w)$, etc.
This means that for any polynomial $P(\partial/\partial x)$ we have $P(\partial/\partial x)\psi\in L^2(U, dx)$. According to the Sobolev's lemma on a bounded domain of Euclidean space [12], we conclude that $\psi\in C^\infty(U)$. Q. E. D.

As for the dual space of $\mathscr{C}_\infty$, we have

Proposition 7. *Every functional F on $\mathscr{C}_\infty$ has a form*

(8) $$\langle F, \psi\rangle=\sum_j\int_W h_j(w)\partial(X_j)\psi(w)d_r w,$$

by a finite number of $h_j\in\mathscr{C}$ and $X_j\in U(\mathfrak{W})$.

This is well known in the distribution theory [5, 6]. For the convenience of the readers we give a proof. It is obvious that the right hand side of (8) defines an element of $\mathscr{C}_\infty$. Now let us prove the converse. From the definition of the topology in $\mathscr{C}_\infty$, for an arbitrarily given $\varepsilon>0$ there exists a finite number of seminorms $\|\cdot\|_{t_i,X_i}$ and $\delta_i>0$ such that if ψ satisfies $\|\psi\|_{t_i,X_i}<\delta_i$, then $|F(\psi)|<\varepsilon$. Put $r=\max_j\cdot(\deg X_j)$, $t=\min_i t_i$. Let $E^r(t)$ be the space of functions f on W, whose distribution derivatives $\partial(U)f$ all belong to $H(t)$ for any differential operators $U\in U(\mathfrak{W})$ of degree $U\leqq r$. We introduce an inner product in $E^r(t)$ as follows. Take a base $X_1,\dots,X_n$ of $\mathfrak{W}$. Bikrhoff-Witt theorem asserts that monomials $X^\alpha=X_1^{\alpha_1}\dots X_n^{\alpha_n}$, $\alpha=(\alpha_1,\dots,\alpha_n)$ form a base of vector space $U(\mathfrak{W})$. We put

(9) $$(\varphi, \psi)=\sum_{|\alpha|\leqq r}\langle\partial(X^\alpha)\varphi, \partial(X^\alpha)\psi\rangle_t,$$

where $\langle\,,\rangle_t$ stands for the scalar product in $H(t)$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$. In this way $E^r(t)$ becomes a Hilbert space. Our functional F is a continuous linear form on the $E^r(t)$. According to F. Riesz' theorem F is written in a form (8) with some $\psi\in E^r(t)$. Since $\partial(X)\psi\in H(t)$ we have the result. From Propositions 3, 4 and 5 we deduce

Proposition 8. *Suppose a function $f\in C_0^\infty(G)$ has the support in a compact set $Q_{\gamma,\alpha}$. Then the Fourier transform $T^\lambda=T^\lambda(f)$ has the following properties* (1°)~(3°);

(1°). *continuity: for any X, $Y\in U(\mathfrak{G})$ there exists a constant $C(X, Y)$, inde-*

pendent of t and λ, and such that

$$\|\partial_r(X)\cdot T^\lambda\cdot\partial_l(Y)\varphi\|_t \leqq C(X,\ Y)\|\varphi\|_{\tau^\lambda(t)},\quad \text{for any } t\in \mathrm{R},$$

where the scale $\tau^\lambda(t)$ *is given by the formula* (5).

(2°). *equivalence relation*: $L_w\cdot T^\lambda\cdot L_{w^{-1}}=T^{\lambda\cdot w}$, $w\in W$, $\lambda\in N^*$.

(3°). *weak analyticity: for any* $F\in\mathscr{C}'_\infty$, $\varphi\in\mathscr{C}$, *and* X, $Y\in U(\mathfrak{G})$, *matrix element* $\langle F,\ \partial_r(X)\cdot T^\lambda\cdot\partial_l(Y)\varphi\rangle$ *is an entire function of* $\lambda\in N.^*$

The operators $\partial_r(X)\cdot T^\lambda\cdot\partial_l(Y)$ at first defined on the space $\mathscr{C}_\infty$, extend to the whole space $\mathscr{C}$ and the extended ones have these properties.

Proof. By Proposition 5 $\partial_r(X)\cdot T^\lambda(f)\cdot\partial_l(Y)=T^\lambda(X_r\cdot Y_l f)$, the right hand side is a bounded operator on $\mathscr{C}$ by Proposition 2, so we can put $C(X,\ Y)=\|X_r\cdot Y_l f\|_{L^1}$. (2°) is a consequence of Proposition 3. In fact the left translation L_w is a continuous operator on $\mathscr{C}$, because

$$\|L_a\varphi\|_t=\left(\int_W \exp[tM(w)]|\varphi(aw)|^2 d_r w\right)^{1/2}$$

$$\leqq \Delta_W(a)^{-1/2}\|\varphi\|_{M(a)t},\qquad \text{if } t\geqq 0,$$

$$\leqq \Delta_W(a)^{-1/2}\|\varphi\|_{t/M(a)}\qquad \text{if } t\leqq 0.$$

Property (3°) is reduced to a special case $\langle h,\ T^\lambda(f)\varphi\rangle$, $h\in\mathscr{C}$ by the aid of Propositions 5 and 7. Q.E.D.

Our main concern is that the properties above characterize the Fourier transforms of the functions $f\in C_0^\infty(G)$. Precisely

Proposition 9. *Suppose to each element* $\lambda\in N^*$ *there corresponds an operator* T^λ *on the space* $\mathscr{C}$ *with the properties* (1°)~(3°), *where* (1°) *is satisfied for a given scale* $\tau^\lambda(t)=\tau^\lambda(t;\ Q_{\gamma,\alpha})$, *attached to* $Q_{\gamma,\alpha}$. *Then there exists a unique functions* $f\in C_0^\infty(G)$ *such that* $T^\lambda=T^\lambda(f)$ *and the support of f is contained in the compact set* $Q_{\gamma,\alpha}$.

Remark. When W is compact, it is sufficient to require (3°) without using differential operators, [9, 10, 11].

It is easy to see that operators $\partial(X)\cdot T^\lambda\cdot\partial(Y)$ extend uniquely to the whole space $\mathscr{C}$, cf. [1, 3].

3. Proof of Proposition 9.

The key point of the proof consists in showing that the point evaluation $\psi\to\psi(w)$ is continuous on $\mathscr{C}_\infty$. Sobolev's lemma plays an essential role.

Lemma 1. (Sobolev's estimation)[12, 15]. *Let* $1\leqq p<\infty$ *and s be an integer such that* $ps>\dim W$. *For each compact neighbourhood K of e in W there exists a constant* C_K *such that*

$$|f(e)|\leqq C_K\sum_{|\alpha|\leqq s}\left(\int_K|\partial(X^\alpha)f(w)|^p d_r w\right)^{1/p} \tag{10}$$

for differentiable functions f on W.

Proposition 6 asserts that $T^{\lambda}\varphi(w)$ is a differentiable function of w for any $\varphi \in \mathscr{C}$ We can apply Lemma 1, to $T^{\lambda}\varphi$, putting $p=1$ we have

$$|T^{\lambda}\varphi(w_0)| \leqq C_K \sum_{|\alpha| \leqq s} \int_K |\partial(X^{\alpha})T^{\lambda}\varphi(w)| d_r w,$$

where K is a compact neighbourhood of w_0.
From this we deduce immediately

$$(11) \qquad |T^{\lambda}\varphi(w_0)| \leqq C_K \sum_{|\alpha| \leqq s} \|\partial(X^{\alpha}) \cdot T^{\lambda}\varphi\|_t \left(\int_K e^{-tM(w)} d_r w\right)^{1/2}$$

$$\leqq C_{K,t} \|\varphi\|_{\tau^{\lambda}(t)}. \qquad \text{(Property (1°))}.$$

The constant $C_{K,t}$ depends on t but not on $\varphi \in \mathscr{C}$. Hence for any $t \in \mathrm{R}$ point evaluation $\varphi \to T^{\lambda}\varphi(w_0)$ is a continuous linear form on $H(\tau)$, $\tau = \tau^{\lambda}(t)$. According to *F.* Riesz' theorem there exists a unique function $K^{\lambda}(w_0, \cdot) \in H(-\tau)$ such that

$$(12) \qquad T^{\lambda}\varphi(w_0) = \int_W K^{\lambda}(w_0, w)\, \varphi(w) d_r w.$$

Lemma 2. *$K^{\lambda}(w_0, w)$ is a differentiable function of w when w_0 and λ are fixed.*

Proof. By Proposition 6 $T^{\lambda} \cdot \partial(X)\varphi$ is differentiable for any $X \in U(\mathfrak{W})$. Applying (11) we have

$$|T^{\lambda} \cdot \partial(X)\varphi(w_0)| \leqq C_{K,t} \sum_{|\alpha| \leqq s} \|\partial(X^{\alpha}) \cdot T^{\lambda} \cdot \partial(X)\varphi\|_t$$

$$\leqq \text{const.} \|\varphi\|_{\tau^{\lambda}(t)}. \qquad \text{(Property (1°))}$$

Hence the linear form on $H(\tau)$: $\tau = \tau^{\lambda}(t)$, $\varphi \to T^{\lambda} \cdot \partial(X)\varphi(w_0)$ is continuous. By Riesz' theorem we have

$$(13) \qquad T^{\lambda}\partial(X)\varphi(w_0) = \int_W K_X^{\lambda}(w_0, w)\varphi(w) d_r w,$$

where $K_X^{\lambda}(w_0, \cdot)$ is unique in $H(-t)$.

On the other hand if $\varphi \in C_0^{\infty}(W)$ we have from (12)

$$(14) \qquad T^{\lambda}\partial(X)\varphi(w_0) = \int_W K^{\lambda}(w_0, w)(\partial(X)\varphi)(w) d_r w.$$

Since formulas (13) and (14) must be identical, we conclude that $K_X^{\lambda}(w_0, \cdot)$ is a distribution derivative of $K^{\lambda}(w_0, \cdot)$, the former belonging to $H(-\tau)$ for any τ, especially to $L^2(W)$. Hence, as shown in Proposition 6 $K^{\lambda}(w_0, \cdot)$ is differentiable.

Here it is worth while to note that the proof of the existence of an integral kernel and its differentiability still remains valid even when W is compact. As for the support the next lemma gives us an answer.

Lemma 3. *Suppose a function φ vanishes almost everywhere outside a* *compact set* $\{w \in W;\ \eta \leqq M(w) \leqq \zeta\}$. *Then* $\psi = T^{\lambda}\varphi$ *vanishes outside a compact set* $\{w \in W;\ \alpha^{-1}\eta \leqq M(w) \leqq \alpha\zeta\}$.

This is derived by a similar argument as in [1] from the continuity property (1°).

From this lemma it follows easily that the kernel function $K^{\lambda}(w_0, \cdot)$ vanishes outside a set $\alpha^{-1}M(w_0) \leqq M(w) \leqq \alpha M(w_0)$, in particular, the support of $K^{\lambda}(e, \cdot)$ is contained in $M(w) \leqq \alpha$, compact by our assumption.

Let $\delta_a(w)$ be the delta-function on W concentrated at $w = a$.

Lemma 4. (*i*) *There exists a finite number of functions* $\varphi_\alpha \in \mathscr{C}$ *and* $X_\alpha \in U(\mathfrak{W})$ *such that* $\delta_a = \sum_\alpha \partial(X^\alpha)\cdot\varphi_\alpha$. *(ii)* δ_a *is a continuous linear form on* $\mathscr{C}_\infty$.

Proof. Let $\psi \in C_0^\infty(W)$ and $(U;\ x_1,\dots, x_n)$ be a local coordinate system around the point a such that $x_i(a) = 0$, $i = 1,\dots, n$. In this chart consider a small set $D = \{x;\ |x_i| \leqq \delta_i,\ i = 1,\dots, n\}$ and take a C^∞-function ρ such that $\rho(a) = 1$ and the support of ρ lies in D. Put $D(\delta) = \{x;\ -\delta_i \leqq x_i \leqq 0\}$ and for multi-index $\alpha = (\alpha_1,\dots, \alpha_n)$ $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The next is obvious

$$\psi(a) = \psi(a)\rho(a) = \int_{D(\delta)} \prod_{j=1}^{n} \partial/\partial x_j(\psi(x)\rho(x))dx,$$

after differentiation we can write this as follows

$$\psi(a) = \sum_{|\alpha| \leqq n} \int_{D(\delta)} D^\alpha\psi(x) \cdot R_\alpha(x)dx, \quad R_\alpha \in C_0^\infty\text{-class}. \tag{15}$$

Let $\chi(x)$ denote the characteristic function of the set $D(\delta)$. Then in the sense of distribution derivatives we have

$$\psi(a) = \sum_{|\alpha| \leqq n} \int_{\mathbf{R}^n} \psi(x)(-1)^{|\alpha|} D^\alpha(R_\alpha(x)\chi(x)) \cdot \Delta(x)\Delta(x)^{-1}dx \tag{16}$$

The term with differentiation D^α can be transformed into a form $\sum_\beta D^\beta C_\beta(x)$, where functions C_β, are all in $\mathscr{C}$. As in the proof of Proposition 6 we substitute $\partial/\partial x_i$ by $\partial(X_j)$: $\partial/\partial x_i = \sum_j a_{ij}(x)\partial(X_j)$ or $\partial/\partial x_i = \sum_j \partial(X_j)\cdot a_{ij}(x) \ - b_i(x)$, $i = 1,\dots, n$, and by induction we can write $D^\alpha = \sum_\gamma \partial(X^\gamma)\cdot h_\gamma$ with a finite number of smooth functions h_γ. Thus

$$\psi(a) = \sum_{\beta,\gamma} \int_W \psi(w)\partial(X^\gamma)(h_\gamma C_\beta)(w)d_r w.$$

It is easy to see that $h_\gamma C_\beta, \in \mathscr{C}$. Thus we have the desired result. (ii) comes from Lemma 1 immediately.

Lemma 5. $K^{\lambda}(w, v)$ *is an analytic function of* $\lambda \in N^*$ *for fixed* w, $v \in W$.

Proof. The property (2°) yields us the relation:

$$K^{\lambda a}(w, v) = \Delta_W(a)K^{\lambda}(aw, av), \qquad \text{for} \quad a \in W,$$

so we have $K^\lambda(w, v)=\Delta_W(w)^{-1}K^{\lambda\cdot w}(e, w^{-1}v)$. Since the transformation $\lambda\to\lambda\cdot w$ ($w\in W$) is analytic, it is enough to show $K^\lambda(e, w)$ is analytic with respect to $\lambda\in N^*$. As shown already, $K^\lambda(e, w)$ is a C_0^∞-function, so we can apply Lemma 4 just proved to this function:

$$K^\lambda(e, a)=\sum_\gamma \int_W K^\lambda(e, w)\partial(X^\gamma)\varphi_\gamma(w)d_r w.$$

Each term of the right hand side is nothing but the value of a function $T^\lambda\cdot\partial(X^\gamma)\varphi_\gamma$ at e, that is,

$$K^\lambda(e, a)=\sum_\gamma \langle\delta_e, T^\lambda\cdot\partial(X^\gamma)\varphi_\gamma\rangle. \tag{17}$$

which is analytic with respect to $\lambda\in N^*$ by the Property (3°).

Remark. Operators $T^\lambda\cdot\partial(X)$, generally $\partial(X)\cdot T^\lambda\cdot\partial(Y)$, have two meanings for $\varphi\in\mathscr{C}$. The one is a unique extension firstly defined on $\mathscr{C}_\infty$ or on $C_0^\infty(W)$, the another is taken in the distribution sense as just above. The possibility of the extension is assured by the Property (1°) and the validity of the latter is assured by the fact that T^λ has a C_0^∞-kernel $K^\lambda(v, \cdot)$.

Now we examine the exponential growth of the kernel function $K^\lambda(e, a)$. To see this we treat each term $\psi_\gamma=T^\lambda\cdot\partial(X^\gamma)\varphi_\gamma$. Since $\varphi_\gamma\in\mathscr{C}$, so $\psi_\gamma\in\mathscr{C}_\infty$. We proceed as in the proof of Lemma 5. Let $(U: x_1,\dots, x_n)$ be a local chart around the identity e such that $x_i(e)=0$, $i=1,\dots, n$. Exchange in (15) $\partial/\partial x_i$ by $\partial(X_j)$ we arrive at the following expression:

$$\psi_\gamma(e)=\sum_{|\alpha|\leqq n}\int_{D(\delta)} R_\alpha(x)\cdot\partial(X^\alpha)\psi_\gamma(x)\Delta(x)\cdot\Delta(x)^{-1}dx,$$

where $R_\alpha(x)$ is smooth. By the Schwarz' inequality

$$|\psi_\gamma(e)|\leqq\sum_\alpha \|\partial(X^\alpha)\psi_\gamma\|_t\left(\int_{D(\delta)} e^{-tM(w)}d_r w\right)^{1/2}\max_{x\in D(\delta)}(R_\alpha(x)\Delta(x)).$$

Recall that $\psi_\gamma=T^\lambda\cdot\partial(X^\alpha)\varphi_\gamma$ and the property (1°), so we have

$$\|\partial(X^\alpha)\psi_\gamma\|_t=\|\partial(X^\alpha)\cdot T^\lambda\cdot\partial(X^\alpha)\varphi_\gamma\|_t\leqq C(X^\alpha, X^\gamma)\|\varphi_\gamma\|_{\tau^\lambda(t)},$$

where the constant $C(X^\alpha, X^\gamma)$ does not depend on λ. Thus we obtain

$$|\psi_\gamma(e)|\leqq C(\gamma)\|\varphi\|_{\tau^\lambda(t)}\left(\int_{D(\delta)} e^{-tM(w)}d_r w\right)^{1/2}. \tag{18}$$

Now put $t=-2\gamma\|\mathrm{Im}\lambda\|$, then $\tau^\lambda(t)=0$. As to the integral I of the right hand side we have

$$I\leqq\exp\,[\gamma\|\,\mathrm{Im}\,\lambda\|\max_{w\in D(\delta)} M(w)]\ \mathrm{vol}\ D(\delta).$$

Since $M(w)$ is continuous, we can choose for arbitrary $\varepsilon>1$ a small $\delta>0$ such that $M(w)\leqq\varepsilon$ in $D(\delta)$, so we have

(19) $$|\psi_\gamma(e)| \leqq \text{const.} \exp[\varepsilon\gamma \| \operatorname{Im} \lambda \|],$$

combining this with (17) we conclude

(20) $$|K^\lambda(e, a)| \leqq C(\varepsilon) \exp[\varepsilon\gamma \| \operatorname{Im} \lambda \|], \quad (\varepsilon > 1)$$

the constant $C(\varepsilon)$ being independent of $\lambda \in N^*$.

Repetition of the above argument shows that the operators of the form $\prod_j \langle \lambda \cdot w, Y_j \rangle^{p_j} \cdot T^\lambda$, $(Y_j \in \mathfrak{N}, w \in W)$, are integral operators with kernel $\prod_j \langle \lambda \cdot w, Y_j \rangle^{p_j} K^\lambda(w, v)$. Formula (20) is valid also in this case, so we have

$$|\prod_j \langle \lambda, Y_j \rangle^{p_j} \cdot K^\lambda(e, w)| \leqq C(\varepsilon) \exp[\varepsilon\gamma \| \operatorname{Im} \lambda \|].$$

Taking a base $Y_1, \ldots, Y_r$ of N and putting $\lambda_j = \langle \lambda, Y_j \rangle$, $\lambda^p = \lambda_1^{p_1} \ldots \lambda_n^{p_n}$

(21) $$|\lambda^p K^\lambda(e, w)| \leqq C(\varepsilon) \exp[\varepsilon\gamma \| \operatorname{Im} \lambda \|], \quad (\varepsilon > 1).$$

Moreover operators $T^\lambda \cdot \partial(X)$; $X \in U(\mathfrak{W})$, with integral kernel $-\partial(X)_v \cdot K^\lambda(w, v)$ must satisfy the inequality (21), so

(22) $$|\lambda^p \cdot \partial(X)_v \cdot K^\lambda(e, v)| \leqq C(p, X, \varepsilon) \exp[\varepsilon\gamma \| \operatorname{Im} \lambda \|].$$

Now we can restore the desired function $f(nw)$ as follows:

$$f(nw) = \int_{\hat{N}} K^\lambda(e, w) \lambda(n) d\lambda.$$

The classical Paley-Wiener theorem guarantees that the function f thus defined is infinitely differentiable and the support lies in $Q_{\gamma,\alpha}$, completing the proof of Proposition 9.

4. Main theorem

Let $B_{\gamma,\alpha}$ be the space of operator fields $T = (T^\lambda)$ on N^*, T^λ being operators on $\mathscr{C}$ with the properties (1°)∼(3°). We introduce in $B_{\gamma,\alpha}$ a system of seminorms $\| \cdot \|_{X,Y}$:

$$\| T \|_{X,Y} = \sup_{\lambda \in N^*} \sup_{t \in \mathrm{R}} \sup_{\varphi \in \mathscr{C}} \| \partial_r(X) \cdot T^\lambda \cdot \partial_l(Y) \varphi \|_t / \| \varphi \|_{\tau^\lambda(t)}.$$

Space $B_{\gamma,\alpha}$ with these seminorms becomes a Fréchet space. If $\gamma \leqq \gamma'$ and $\alpha \leqq \alpha'$ then $Q_{\gamma,\alpha} \subset Q_{\gamma',\alpha'}$ and so $\tau^\lambda(t; Q_{\gamma,\alpha}) \leqq \tau^\lambda(t; Q_{\gamma',\alpha'})$ for any $t \in \mathrm{R}$, hence $B_{\gamma,\alpha} \subset B_{\gamma',\alpha'}$ and the inclusion is continuous. Let B denote the inductive limit: $B = \bigcup_{\gamma,\alpha} B_{\gamma,\alpha}$. Our main theorem says

Paley-Wiener type theorem. *Fourier transformation $f \to T^\lambda(f)$ is a topological isomorphism from $C_0^\infty(Q_{\gamma,\alpha})$ onto $B_{\gamma,\alpha}$, so also from $C_0^\infty(G)$ onto the space B.*

Proof. First of all note that the usual topology in $C_0^\infty(G)$ coincides with the one given by the seminorms $\| X_r Y_l f \|_{L^1(G, d_r g)}$; $X, Y \in U(\mathfrak{G})$. This follows from the Sobolev's estimation (Lemma 1). Proposition 9 and its proof show that the Fourier transformation $f \to T^{\lambda}(f)$ is a surjective continuous mapping from $C_0^{\infty}(Q_{\gamma,\alpha})$ to the space $B_{\gamma,\alpha}$ and that this correspondence is injective. Hence, by the inverse mapping theorem [5], this is an isomorphism. Q.E.D.

5. On the irreducibility

The following is the main concern of this section.

Proposition 10. *Let W_λ be the stationary subgroup at $\lambda \in N^*$: $W_\lambda = \{w \in W; \lambda \cdot w = \lambda\}$. If W_λ reduces to the identity, the representation $\mathscr{D}^{\lambda} = (T^{\lambda}, \mathscr{C})$ is operator-irreducible.*

The proof rests on the kernel theorem just as in [4, 7].

Now proceed as in [1, 3]. We prepare a few lemmas.

Lemma 6. *A pair of representations $\mathscr{D}^{\lambda}$ and $\mathscr{D}^{\lambda'}$ has a $T^{\lambda} \times T^{\lambda'}$-invariant bilinear form iff $\lambda \cdot w = -\lambda'$ for some $w \in W$.*

Proof. Recall that the space $\mathscr{C}$ contains $C_0^{\infty}(G)$ and that T_w^{λ} is a right translation. Hence a bilinear form, if it exists, restricted to the functions $\varphi, \psi \in C_0^{\infty}(W)$ is of the following form by the kernel theorem:

$$B(\varphi, \psi) = \int_W (\omega(w), \varphi(ww'))\psi(w')d_r w',$$

where $\omega(w)$ is a suitable distribution, [4, 6, 7].

The invariance by $n \in N$ means

$$\langle \omega(x), f_n^{\lambda}(xy)\varphi(xy)\rangle = f_n^{-\lambda'}(y)\langle \omega(x), \varphi(xy)\rangle \qquad \text{for any} \quad y \in W,$$

where $f_n^{\lambda}(x) = \exp\langle \lambda \cdot x, n\rangle$. Now consider the set $S^{\lambda,\lambda'}(n) = \{x \in W; \exp\langle \lambda \cdot xy, n\rangle = \exp\langle \lambda' \cdot y, n\rangle$ for any $y \in W\}$. Then the distribution $\omega(w)$ is concentrated on the set $S^{\lambda,\lambda'} = \bigcap S^{\lambda,\lambda'}(n)$. The set $S^{\lambda,\lambda'}$ is not empty iff $\lambda \cdot w + \lambda' = 0$ for some $w \in W$. Moreover, if w and w' lie in $S^{\lambda,\lambda'}$, we have $\lambda \cdot w = \lambda \cdot w'$, that is, $w' \in W_\lambda \cdot w$. Thus, only the following two cases are possible;

case 1; $S^{\lambda,\lambda'}$ is empty. So λ and $-\lambda'$ are not conjugate under W. Distribution ω does not exist.

case 2; $S^{\lambda,\lambda'} = W_\lambda \cdot w_0$, a single coset, so it holds $\lambda \cdot w_0 + \lambda' = 0$. Since we assumed that $W_\lambda = (e)$, ω is concentrated at one point w_0. It is clear that the next bilinear form is certainly invariant by $T_g^{\lambda} \times T_g^{\lambda'}$ for any $g \in G$:

$$B(\varphi, \psi) = \text{const.} \int_W \varphi(w_0 w)\psi(w)d_r w, \quad (\varphi, \psi \in \mathscr{C}). \qquad \text{Q.E.D.}$$

Lemma 7. *Suppose a distribution $\omega(w)$ is concentrated on w_0; that is, $\lambda \cdot w_0 = -\lambda'$, and satisfies a relation:*

$\langle \omega(x), f_n^{\lambda}(xy)\varphi(xy)\rangle = f_n^{-\lambda'}(y)\langle \omega(x), \varphi(xy)\rangle$, for any $y \in W$, $n \in N$. Then $\omega(x) = \text{const.}\ \delta_{w_0}(x)$, Dirac's delta function.

Proof. Let $(x_1, \dots, x_p)$ be a local coordinate system around the point w_0 such that $x_i(w_0)=0$. As ω is concentrated at w_0, $\langle\omega(x), \varphi(xy)\rangle$ can be written in a form $\sum_\beta a_\beta X^\beta\varphi(y)$ by a finite number of $X^\beta \in U(\mathfrak{W})$, where X^β are considered as right invariant differential operators. Hence, we can write it as $\langle\omega(x), \varphi(xy)\rangle = \sum_\beta b_\beta D^\beta\varphi(y)$. The condition for ω becomes as follows:

$$\sum_\alpha b_\alpha D^\alpha(f_n^\lambda\varphi)(y)=f_n^{-\lambda'}(y)\sum_\beta b_\beta D^\beta\varphi(y) \qquad \text{for any} \quad y\in W \quad \text{and} \quad \varphi\in C_0^\infty(W).$$

Summation runs over the set $K=\{\alpha=(\alpha_i);\ 0\leqq i\leqq p\}$ of multi-indices.

We want to show that except $\alpha=(0,\ldots, 0)$ all the coefficients b_α are equal to 0. Let us introduce in K an order $\prec$: $\alpha\prec\beta$ iff $\alpha_i\leqq\beta_i$ for every i. As a notation put $\gamma_i=(\delta_{ij})$. Consider the set $L=\{\alpha\in K;\ b_\alpha\neq 0\}$, and let α be a maximal element in L relative to this order such that $|\alpha|$ is maximum. If $\alpha\neq 0$, then $\alpha'=\alpha-\gamma_j\in K$ for some j. For α so chosen, we once fix a function $\varphi\in C_0^\infty(\mathrm{R}^p)$ such that $D^\beta\varphi(0)=0$ for $\beta\neq\alpha'$ and $D^{\alpha'}\varphi(0)=1$. So the terms in the following are not zero only when $\beta-\gamma=\alpha'$:

$$\sum_{\beta,\gamma} b_{\beta\beta}C_\gamma D^\gamma f_n^\lambda(0)\cdot D^{\beta-\gamma}\varphi(0)=b_{\alpha'}.$$

where ${}_\beta C_\gamma$ is the number of choice. As $|\alpha|$ is maximum in L, $b_\beta=0$ if $|\beta|>|\alpha|$. So the summation runs over the set $|\gamma|=1$, that is, $\gamma=\gamma_1,\ldots, \gamma_p$. Since D^{γ_i} is the differentiation by the i-th variable, we have $D^{\gamma_i}f_n^\lambda(0)=\langle\lambda\cdot X_i, n\rangle$, X_i is the corresponding tangent vector. At last we have for any $n\in N$

$$\langle\lambda\cdot\sum_1^p b_{\alpha'+\gamma_i\alpha}C_{\gamma_i}X_i,\ n\rangle=0,$$

and so $\lambda\cdot\sum_1^p b_{\alpha'+\gamma_i\alpha'+\gamma_i}C_{\gamma_i}X_i$ belongs to W_λ, consisting of only 0. Since (X_i) forms a base of $\mathfrak{W}$, we have $b_{\alpha'+\gamma_i}=0$ for every i. Especially, $b_\alpha=b_{\alpha'+\gamma_j}=0$, contradicting the assumption. Hence, $b_\alpha=0$ except $\alpha=0$.

In the next lemma we assume that the isotropy W_λ is trivial.

Lemma 8. *Suppose a continuous linear operator A on $\mathscr{C}$ interwines representations $\mathscr{D}^\lambda$ and $\mathscr{D}^{\lambda'}$: $A\cdot T_g^\lambda=T_g^{\lambda'}\cdot A$ for any $g\in G$. Then (i) $A=0$ if λ and λ' are not conjugate under W, or (ii) when $\lambda\cdot w_0=\lambda'$ for some (unique) w_0, $A=$*cont.L_{w_0}.

Proof. Consider a bilinear form $B(\varphi, \psi)=\int_W A\varphi(x)\psi(x)d_rx$. This is invariant by $T_g^\lambda\times T_g^{-\lambda'}$ for any $g\in G$. According to Lemma 6, this form is not trivial only when $\lambda\cdot w_0=\lambda'$ for some $w_0\in W$. If this is the case, Lemma 7 asserts that $B(\varphi, \psi)=$ const.$\int_W\varphi(w_0w)\psi(w)d_rw$. Comparing two expressions, we conclude $A\varphi(w)=$ const.$\varphi(w_0w)$, the desidred result.

Proposition 10 is a special case when $\lambda=\lambda'$, so the proof is completed.

For the groups treated in [1, 2, 3] the situation is as follows: For $\lambda\in\hat{N}$ lying in general position, the isotropy W_λ at λ is trivial. Hence, if $\lambda=\lambda_1+\sqrt{-1}\lambda_2(\lambda_i \in \hat{N})$ and at elast one of λ_i lies in general position, then the isotropy W_λ is trivial.

One more example is $G=\mathrm{R}^2 \cdot SL(2, \mathrm{R})$, where $W=SL(2, \mathrm{R})$ acts on $N=\mathrm{R}^2$ through a natural linear mapping. In this case the isotropy W_λ is trivial for generic $\lambda \in N^*$.

Department of Mathematics,
Kyoto University

References

[1] S. Ando, Analogue of Paley-Wiener theorem for the group of linear transformations of a straight line (in Russian), J. Kyoto Univ., **14**–2 (1974), 195–213.
[2] S. Ando, Paley-Wiener theorem for the Heisenberg groups, Proc. Japan Acad., **52** (1976), 331–333.
[3] S. Ando, Analogue of Paley-Wiener theorem for some matricial groups (in Russian), J. Math. Kyoto Univ., **16**–2 (1976), 375–393.
[4] F. Bruhat, Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques, Bull. Soc. math. France, **89** (1961), 43–75.
[5] I. M. Gelfand, G. E. Shilov, Spaces of fundamental functions and generalized functions (Generalized Functions vol. 2), Moskow Fizmatgiz 1958.
[6] I. M. Gelfand, N. Y. Vilenkin, Some applications and harmonic analysis, rigid Hilbert spaces (Generalized Functions vol. 4), ibid 1961.
[7] I. M. Gelfand, M. I. Graev, N. Y. Vilenkin, Integral geometry and problems of representation theory related with them (Generalized Functions vol. 5), ibid 1962.
[8] A. Kleppner, R. L. Lipsman, The Plancherel formula for group extensions, Ann. Scient. Ec. Norm. Sup., **5** (1972), 459–516.
[9] K. Kumahara, Fourier transforms on the Cartan motion groups, Proc. Japan Acad., **50** (1974), 433–435.
[10] K. Kumahara, Fourier transforms on the motion groups, J. Math. Soc. Japan, **28**–1 (1976), 18–32.
[11] K. Kumahara, K. Okamoto, An anlogue of the Paley-Wiener theorem for the Euclidean motion group, Osaka J. Math., **10** (1973), 77–92.
[12] S. Narashiman, Analysis on real and complex manifolds, Amsterdam, North-Holland 1968.
[13] T. Nomura, Harmonic analysis on the universal covering group of certain semidirect product Lie group, Japanese J. Math., **6**–1 (1980), 129–145.
[14] T. Nomura, The Paley-Wiener type theorem for the oscillator group, J. Math. Kyoto Univ., **22**–1 (1982), 71–96.
[15] J. Poulsen, C^∞-vectors and intertwining bilinear forms for representations of Lie groups, J. Functional Analysis **9** (1972), 87–120.
[16] M. Sugiura, Fourier series of smooth functions on compact Lie groups, Osaka J. Math., **10** (1973), 77–92.
[17] D. P. Želobenko, Infinitely differentiable vectors in the theory of representations (in Russian), Vestnik Moskow Univ., Ser I, Math. and Mech. **20** (1965), 3–10.
[18] D. P. Želobenko, On harmonic analysis of functions on complex semisimple Lie groups. I (in Russian), Izv. AN USSR, **27** (1963), 1343–1396.
[19] D. P. Želobenko, On harmonic analysis of functions on semisimple Lie groups. II (in Russian), Izv. AN USSR, **33** (1969), 1255–1295.