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Primeness in divisor sense for certain entire functions

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Introduction. For entire and meromorphic functions, the primeness and pseudo-primeness etc. have been considerably studied. In this paper, taking a new point of view, we shall introduce the concept of *primeness in divisor sense* for an entire function and that of primeness for a divisor (discrete countable set), as well (see § 1, Definitions 1 and 2).

Now, for a divisor, the concept of *NPS* (non-trivial pre-image set) was defined together with some other concepts ([9]). Note that, according to their definitions, a divisor is not NPS if it is (left-) prime.

Relating to the above concepts, it is one of the main purposes of this paper to prove that the function; $P(z)+Q(e^z)$ (P, Q : non-constant polynomials), which is known to be prime, is further prime in divisor sense (under certain conditions, Theorems 1 and 2). In the proof of this fact, we shall need some additional results (Theorems 3 and 4) which show that certain entire functions are prime or left-prime.

Also, we shall prove a result (Theorem 5) which tells us explicitely that the primeness in divisor sense is not necessarily q. c. invariant. More precisely, this means that a divisor D which is not pseudo-prime may be mapped by a quasi-conformal mapping of $\boldsymbol{C}$ onto itself to a divisor $\tilde{\mathrm{D}}$ which is prime (see § 6).

§ 1. Preliminaries.

For a meromorphic function $F(z)$, the factorization under composition operation such as

$$F(z)=f\circ g(z)=f(g(z)) \tag{1}$$

has been considered, where f is meromorphic and g is entire (g may be meromorphic, when f is a rational function). Then, by definition, F is called to be *prime* (*pseudo-prime*; *right-prime*; *left-prime*), if, for every factorization as above, we can always deduce the following assertion: f or g is linear (f or g is rational; g is linear whenever f is transcendental; f is linear whenever g is transcendental, respectively).

When F is entire and both factors f (left-factor) and g (right-factor) of F under (1) are restricted to entire functions, then it is called that the factorization is to be in entire sense. Thus we may use the phrase "prime in entire sense" instead of "prime" etc. It is known and easily proved that any non-periodic entire function is prime if it is prime in entire sense (cf. [8]).

Now, in constrast to the above, we define the primeness in divisor sense (for entire functions) as follows.

Definition 1. An entire function $F(z)$, with zeros, is called to be *prime in divisor sense* (*pseudo-prime in divisor sense*; *right-prime in divisor sense*; *left-prime in divisor sense*), if, for every identical relation such as

$$F(z)=f(g(z))\cdot e^{A(z)}, \tag{2}$$

where f, g ($\not\equiv$const.) and A are entire functions, we can deduce the following assertion: f has just one simple zero or g is a linear polynomial (f has only a finite number of zeros or g is a polynomial; g is a linear polynomial whenever f has an infinite number of zeros; f has just one simple zero whenever g is transcendental, respectively).

Definition 2. A divisor D, a discrete countable set in $\boldsymbol{C}$, is called to be *prime* (*pseudo-prime*; *right-prime*; *left-prime*), if an entire function $F(z)$ whose zero-set is identical with D is prime (pseudo-prime; right-prime; left-prime, resp.) in divisor sense, as in Definition 1.

Remarks. 1). Prime entire functions need not be prime in divisor sense. For example, even if p is a prime number ($\geqq 2$), the function z^p (which is clearly prime) is not prime in divisor sense:

$$z^p=(z\cdot e^{B(z)})^p\cdot e^{-pB(z)}$$

for any entire function $B(z)$.

Further, letting $h(z)$, with $h(0)\neq 0$, be an entire function such that the order of $h(e^z)$ is finite and that $h(z)$ has zeros of order n for all sufficiently large natural number n and letting m be an integer, $P(z)$ and $Q(z)$ ($\not\equiv$const.) be polynomials, then the function $F(z)$ defined by

$$F(z)=h(e^z)\cdot\exp[mz+P(e^z)+Q(e^{-z})]$$

is known to be prime (cf. [18]). While $F(z)$ is clearly not pseudo-prime in divisor sense (composite in divisor sense).

Also, entire functions which are prime in divisor sense need not be prime. This is seen, for instance, by

$$F(z)=z\cdot\exp[z(e^z+1)]=(ze^z)\circ(ze^z).$$

2). If $F(z)$ is an entire function whose zeros are all contained in a straight line (a half line), then F is pseudo-prime (right-prime, resp.) in divisor sense. These will follow from a Theorem due to Edrei ([3]);

Theorem A. *Let $f(z)$ be an entire function. Assume that there exists an* *unbounded sequence* $\{a_n\}$ *such that all the roots of the equations*

$$f(z)=a_n \qquad (n=1, 2, \cdots)$$

be real. Then $f(z)$ *is a polynomial of degree not greater than* 2.

2′). Let $F(z)$ be an entire function with zeros $\{z_n\}$ such that the order of $\mathrm{N}(r, \{z_n\})$ (cf. [10] around p. 16) is less than one and that there exist constants γ and ω with $|\arg z_n-\gamma|<\omega<\frac{\pi}{2}$ for all n. Then F is right-prime in divisor sense. This is seen by applying a Theorem due to Kobayashi (Lemma 3 in §4) or Tuzuki.

3). There exist periodic entire functions which are prime in divisor sense. In fact, prime periodic entire functions exhibited by Ozawa in [15] give these examples.

4). Certain prime entire functions which are constructed by Liverpool ([12]) in connection with Picard set are at the same time prime in divisor sense.

§ 2. Statement of main results.

Theorem 1. *Let* $P(z)$ *be a polynomial. Then the function*

$$F(z)=z+P(e^z)$$

is prime in divisor sense.

Theorem 2. *Let* $P(z)$ *and* $Q(z)$ *be two non-constant polynomials. Assume that, for any natural number* k *and constant* c, *the function* $e^{-kz}\cdot[Q(e^z)+c]$ *is non-constant. Then*

$$F(z)=P(z)+Q(e^z)$$

is prime in divisor sense.

Remark. The latter condition in Theorem 2 cannot be dropped, otherwise the statement is not valid generally. To see this, it is enough to consider the following identity: $[R(z)]^k+e^z=\{(w^k+1)\circ(R(z)\cdot e^{-z/k})\}\cdot e^z$, where $R(z)$ is a polynomial.

The following facts are also valid. Now, we call that $f(z)$ is c-even, if $f(z+c)$ is an even function (i.e. $f(-z+c)=f(z+c)$ for any z).

Proposition 1. *Let* $P(z)$ *and* $Q(z)$ *be two non-constant polynomials. Assume that* $[P(z)]^2$ *is not* c*-even for any constant* c *and* $Q(z)$, *with* $Q(0)\neq 0$, *has only simple zeros. Then the function*

$$F(z)=P(z)\cdot Q(e^z)$$

is prime in divisor sense.

This is proved by the similar argument as in the proof of Theorems 1 and 2.

Remark. Let

$$F(z)=z^{2}\cdot(e^{2z}+1), \quad \text{or} \quad F(z)=z^{3}\cdot(e^{2z}-1).$$

Then F is not prime in divisor sense. In fact, if we put $G(z)=F(z)\cdot e^{-z}$, we have $G(z)=h(z^{2})$ for some entire function $h(z)$ with infinitely many zeros, since $G(z)$ is 0-even.

Proposition 2. *Let F be an entire function of two variables z and w defined by*

$$F(z, w)=(z+w)^{2}-(e^{z}+e^{w})^{2}.$$

Then $F(z, w)$ is prime in divisor sense. (*However, F is reducible.*)

This means that, under

$$F(z, w)=f(g(z, w))\cdot e^{A(z, w)},$$

where $f(z)$, $g(z, w)$ and $A(z, w)$ are entire functions, if $f(z)$ has at least two zeros, we can conclude that $g(z, w)=a+bz+cw$; linear.

Note that, for any fixed w, $F(z, w)$ is prime in divisor sense (considered as the function of z) by Theorem 2. This is valid if we change z and w. Further, putting $w=2z$ (say), the function $F(z, 2z)$ is also prime in divisor sense. Using these facts, we can show the above assertion.

In connection with above Theorems, we may recall the following result, proved in [19]: Let $F(z)=h(z)+H(z)$ and $G(z)=k(z)+K(z)$, where h and k are non-constant entire functions of order less than one and that H and K are non-constant, periodic, entire functions with periods b_1 and b_2 (resp.). Assume that the identical relation

$$F(z)=R(z)\cdot G(z)\cdot e^{A(z)}$$

holds, where $R(z)$ is a meromorphic function of order less than one and $A(z)$ is an entire function. Then we have necessarily that A and R are constant (and b_1/b_2 is a rational number) so that

$$F(z)\equiv c\cdot G(z)$$

for some constant $c\neq 0$.

Hence, if the functions $F(z)$ and $G(z)$ as above have the (essentially) same divisor, then they are identical up to a non-zero multiplicative constant.

§3. We shall show the following fact, to which the proof of Theorems 1 and 2 will be reduced, as is seen in §4.

Theorem 3. *Let $P(z)$ be a polynomial and $H(z)$ be a periodic entire function of exponential type with period $2\pi i$ (i.e. $H(z+2\pi i)\equiv H(z)$). Assume that, for any integer k and constant c, $P(z)$ is not c-even and $e^{kz}\cdot[H(z)+c]$ is non-constant. Then, for any constant a, the function*

$$F(z)=e^{az}\cdot[P(z)+H(z)]$$

is prime.

Remark. From the proof given below, it will be seen that the assertion of Theorem 3 (primeness of F) is valid, even if the condition that $P(z)$ is not c-even is dropped, when $H(z)=Q(e^z)$ for a polynomial $Q(z)$. Also, if $P(z)\equiv z$, then the non-constancy of $e^{kz}\cdot[H(z)+c]$ is needless.

Proof of Theorem 3. At first, we observe that the right-factor g of F cannot be a non-linear polynomial. To see this, it will be sufficient to consider the case where g is a polynomial of degree two, since the zeros of $F(z)$ are distributed very near to the imaginary axis.

Let $g(z)=b(z-z_0)^2+d$. Then, using $w=z-z_0$, we have that $F(-w)=F(w)$ which is rewritten as

$$e^{-2aw}\cdot[P(-w+z_0)+H(-w+z_0)]=P(w+z_0)+H(w+z_0).$$

Since P is a polynomial and H is a periodic entire function with period $2\pi i$, by comparing the growth of the both sides along the imaginary axis, we see at first that a is a real number and further $a=0$. Then the above relation can be written as

$$P(-w+z_0)-P(w+z_0)=H(w+z_0)-H(-w+z_0).$$

This implies that the left-hand side which is a polynomial is bounded on the imaginary axis so that it is constant. Further, since it becomes zero at $w=0$, it must be identically zero. Hence $P(-w+z_0)\equiv P(w+z_0)$, that is, $P(w)$ is z_0-even, contrary to the assumption. Note that, if $H(z)=Q(e^z)$, where Q is a non-constant polynomial, then the identity $H(-w+z_0)=H(w+z_0)$ cannot hold (a contradiction, as is to be shown. cf. Remark below Theorem 3).

Next, we must prove that $F(z)$ is left-prime (in entire sense). To do so, we'll consider the following two cases separately:

(i) a is an integer

(ii) a is not an integer.

It may be noted that for the case $a=0$ the primeness of F is already known.

The case (i). Setting $a=m$ for an integer m, assume that

$$F(z)=e^{mz}\cdot[P(z)+H(z)]=f(g(z)), \tag{3}$$

where f is a non-constant entire function (and g is a transcendental entire function, as is shown above). By the periodicity of e^{mz} and $H(z)$, we have

$$f(g(z+2\pi i))-f(g(z))=[P(z+2\pi i)-P(z)]\cdot e^{mz}.$$

Hence we see that

$$g(z+2\pi i)-g(z)=Q(z)\cdot e^{bz},$$

where $Q(z)$ ($\not\equiv 0$, since F is non-periodic) is a polynomial and b is a constant (note that g is of exponential type). Here, taking a polynomial $R(z)$ such that

$$R(z+2\pi i)\cdot e^{2b\pi i}-R(z)=Q(z),$$

we have

$$g(z)=R(z)\cdot e^{bz}+K(z), \tag{4}$$

where $K(z+2\pi i)=K(z)$ is an entire function of exponential type. Now, $R(z)$ may be constant, if $Q(z)$ is constant and b is not an integer. Otherwise, $R(z)$ is non-constant.

By (3) and (4), we have the following identity

$$e^{mz}\cdot[P(z)+H(z)]=f(R(z)\cdot e^{bz}+K(z)). \tag{5}$$

Since g, under (4), is transcendental, we see that the order of f is zero, by a well-known Polya's Lemma, so that, if f is transcendental, by using the minimum modulus Theorem and comparing the growth of the both sides of (5) along the imaginary axis, we obtain a contradiction (cf. [17], p. 108-109).

If f is a polynomial, we put

$$f(z)=c_0z^n+c_1z^{n-1}+\cdots+c_n, \qquad c_0\neq 0.$$

Then, from (5), we see at first that $nb=m$ and that $|P(z)|\sim|c_0||R(z)|^n$ along the imaginary axis. Hence $R(z)$ is non-constant. Also, if $P(z)\equiv z$, the latter is possible only if $n=1$.

Assume now that $e^{-bz}\cdot[K(z)+c']$ is constant for some $c'\in\boldsymbol{C}$. Then, we may write the right hand side of (5) as $f(R_1(z)\cdot e^{bz})$ with $R_1(z)=R(z)+\text{const}$. In view of $nb=m$, the identity (5) can be written as

$$[P(z)-c_0R_1(z)^n]\cdot e^{nbz}-c_1R_1(z)^{n-1}e^{(n-1)bz}-\cdots-c_{n-1}R_1(z)\cdot e^{bz}$$
$$-c_n+e^{nbz}\cdot H(z)=0.$$

Here, $b\neq 0$ (since g is transcendental) and $R_1(z)$ is non-constant ($R(z)$ is so). By applying Borel-type unicity Theorem, we conclude that this identity holds only if $P(z)-c_0R_1(z)^n=\text{const.}=c''$ and $c_1=\cdots=c_{n-1}=0$. This implies that $e^{nbz}\cdot[H(z)+c'']$ $(nb=m)$ is constant, contrary to the assumption. Hence $e^{-bz}\cdot[K(z)+c']$ is non-constant for any $c'\in\boldsymbol{C}$.

Now, we can rewrite the identity (5) such as

$$e^{nbz}\cdot\{P(z)-c_0R(z)^n+e^{-bz}\cdot(nc_0K(z)+c_1)\cdot R(z)^{n-1}[1+o(1)]\}$$
$$=c_0K(z)^n+c_1K(z)^{n-1}+\cdots+c_n-e^{nbz}\cdot H(z), \tag{6}$$

considering the equation on the imaginary axis. As noted above, the periodic function $e^{-bz}\cdot(nc_0K(z)+c_1)$ is non-constant. Hence, if $n\geqq 2$, we can conclude that the left hand side of (6) is unbounded on the imaginary axis. While, the right hand side (periodic, with period $2\pi i$) remains bounded there. This is not in reason. Hence we must have $n=1$, which shows that $f(z)$ is linear. Thus $F(z)$ is left-prime in entire sense.

The case (ii). In this case, we can prove the following fact, which shows that F is also left-prime in entire sense.

Theorem 4. *Let $P(z)$ be a non-constant polynomial and $H(z)$ be a non-constant, periodic, entire function of exponential type with period $2\pi i$. Assume that $a \in \boldsymbol{C}$ is not an integer, then the function*

$$F(z)=e^{az}\cdot[P(z)+H(z)]$$

is left-prime in entire sense.

To prove Theorem 4, we shall need the following result due to Ozawa.

Lemma 1. ([14]). *Let $F(z)$ be an entire function of finite order whose derivative $F'(z)$ has an infinite number of zeros. Assume that the number of common roots of the equations*

$$F(z)=c \quad \text{and} \quad F'(z)=0 \tag{7}$$

is finite for any constant c. Then F is left-prime in entire sense.

We wish to check the conditions in Lemma 1.

Now, for $F(z)$ in Theorem 4, we have

$$F'(z)=\{aP(z)+P'(z)+aH(z)+H'(z)\}\cdot e^{az}.$$

Since a is not an integer, both $aP(z)+P'(z)$ and $aH(z)+H'(z)$ are non-constant. Hence it will be clear that $F'(z)$ has infinitely many zeros.

Assume that there exists an unbounded sequence $\{z_n\}_1^\infty$ each member of which satisfies (7). In what follows, we consider the case $c\neq 0$. (If $c=0$, then we have that $P(z_n)=-H(z_n)$ and $P'(z_n)=-H'(z_n)$ so that $P'(z_n)/P(z_n)=H'(z_n)/H(z_n)$, whose left hand side tends to zero while the right hand side tends to a non-zero value as n tends to infinity, a contradiction.) By assumption, we have

$$F(z_n)=e^{az_n}[P(z_n)+H(z_n)]=c \tag{8}$$

$$\begin{aligned} F'(z_n)&=e^{az_n}\{a[P(z_n)+H(z_n)]+P'(z_n)+H'(z_n)\} \\ &=c\left\{a+\frac{P'(z_n)+H'(z_n)}{P(z_n)+H(z_n)}\right\}=0\,. \end{aligned}$$

Hence

$$\frac{P'(z_n)+H'(z_n)}{P(z_n)+H(z_n)}=-a \tag{9}$$

so that

$$H'(z_n)+aH(z_n)=-aP(z_n)-P'(z_n)\to\infty \qquad (\text{as } n\to\infty),$$

because a is not an integer (in particular $a\neq 0$) and $P(z)$ is a non-constant polynomial. Since $H(z)$ is periodic with period $2\pi i$ and of exponential type, we can write

$$H(z)=\sum_{k=-m}^{m} c_k e^{kz} \qquad (c_m\neq 0 \text{ or } c_{-m}\neq 0)$$

and

$$H'(z)=\sum_{k=-m}^{m} kc_k e^{kz}$$

for a natural number m. Hence we conclude, taking a subsequence if necessary, that

$$e^{z_n}\to\infty \quad\text{or}\quad e^{z_n}\to 0 \qquad (\text{as } n\to\infty).$$

In either case, we have

$$H(z_n)\to\infty \qquad (\text{as } n\to\infty). \tag{10}$$

Then, from (9), we have

$$\lim_{n\to\infty}\frac{P'(z_n)/H(z_n)+H'(z_n)/H(z_n)}{P(z_n)/H(z_n)+1}=-a\,. \tag{11}$$

From this, we see that the following limit exists;

$$\lim_{n\to\infty} P(z_n)/H(z_n)=b \quad (\neq -1). \tag{12}$$

In fact, noting that the non-zero limit of $H'(z_n)/H(z_n)$ exists, we see at first that $P(z_n)/H(z_n)$ remains bounded w.r.t. n and further that the limit exists (note $a\neq 0$). Then it is clear that the limit value b cannot be -1.

Returning to (8), we have

$$P(z_n)/H(z_n)+1=c/(e^{az_n}\cdot H(z_n))\,. \tag{13}$$

In view of (12) and (13), we have

$$\lim_{n\to\infty} e^{az_n}\cdot H(z_n)=A \quad (\neq 0).$$

Hence, if $\exp[z_n]\to\infty$ $(n\to\infty)$, then

$$\lim_{n\to\infty} e^{(m+a)z_n}=B \quad (=A/c_m\neq 0) \tag{14}$$

or if $\exp[z_n]\to 0$ $(n\to\infty)$, then

$$\lim_{n\to\infty} e^{(-m+a)z_n}=B \quad (=A/c_{-m}\neq 0). \tag{15}$$

Using these, we wish to deduce a contradiction.

Assume that (14) is valid. Then

$$\lim_{n\to\infty} \mathrm{Re}\,\{(m+a)z_n\}=\log|B|\,. \tag{16}$$

While, we have

$$\mathrm{Re}\,\{(m+a)z_n\}=(m+\mathrm{Re}\,a)\cdot\mathrm{Re}\,z_n-\mathrm{Im}\,a\cdot\mathrm{Im}\,z_n \tag{17}$$

Note here that, in this case, $\mathrm{Re}\,z_n\to\infty$ $(n\to\infty)$ and that $\mathrm{Im}\,z_n$ is unbounded (the latter will be seen by using the equation just under (9)). If $m+\mathrm{Re}\,a=0$, then by (16) and (17) we have also $\mathrm{Im}\,a=0$, which implies that a is an integer, a contradiction.

Now assume that $m+\mathrm{Re}\,a\neq 0$. Then $\mathrm{Im}\,a\neq 0$ and

$$\mathrm{Im}\,z_n/\mathrm{Re}\,z_n\to(m+\mathrm{Re}\,a)/\mathrm{Im}\,a \quad (=c_1 \text{ (say)},\ \neq 0),$$

also in view of (16) and (17). Hence $|\mathrm{Im}\, z_n| \sim |c_1| \cdot |\mathrm{Re}\, z_n|$ so that $|z_n| \sim (1+c_1^2)^{1/2} \cdot |\mathrm{Re}\, z_n|$. Then

$$|P(z_n)/H(z_n)| \sim c \cdot |z_n|^p / (|c_m| \cdot \exp[m(\mathrm{Re}\, z_n)]) \to 0 \quad (n \to \infty),$$

where $\deg \mathrm{P} = p$.

Hence from (12), $b=0$, while from (11) and (12), we have $m/(b+1) = -a$. Thus $a = -m$; an integer, which contradicts the assumption.

In the case where (15) is valid, we have a similar contradiction.

Thus we've checked that the conditions of Lemma 1 are all satisfied. Therefore F is left-prime in entire sense.

Hence we have that $F(z)$ (in Theorem 3) is prime in entire sense. Since F is non-periodic (as is easily seen), F is also prime (cf. [8]). Thus the proof of Theorem 3 is now complete.

Remark. A discrete countable set D is called *NPS* if there exists at least one non-linear function $f(z)$ and a finite range set T of distinct values with $|T| \geqq 2$ (where $|T|$ denotes the cardinality of T) such that $f^{-1}(T) = D$, including multiplicities ([9]).

Then, Theorem 3 tells us that the set: $D = \{z \mid P(z) + H(z) = 0\}$ cannot be *NPS* (non-trivial pre-image set), if $P(z)$ is a polynomial and $H(z)$ is a periodic entire function of exponential type such that, for any integer k and constant c, $P(z)$ is not c-even and the function $e^{kz} \cdot [H(z)+c]$ is non-constant.

§4. Proof of Theorem 1 and 2.

We use the following Lemmas. Now, for an entire function $f(z)$, we shall denote by $\rho(f)$ ($\underline{\rho}(f)$) the order (lower order, resp.) of f and by $\rho^*(f)$ the exponent of convergence of the zeros of $f(z)$ (this is equal to the order of $\mathrm{N}(r, 0, f)$).

Lemma 2 ([4]). *Let $f(z)$ and $g(z)$ be two transcendental entire functions. If $\rho^*(f)$ is positive, then $\rho^*(f(g))$ is infinite.*

Lemma 3 ([11]). *Let $f(z)$ be an entire function of lower order less than* 1 (*more precisely,* $\liminf T(r, f)/r = 0$). *Assume that there exists an unbounded sequence $\{w_n\}$ such that all the roots of the equations*

$$f(z) = w_n \qquad (n \geqq 1)$$

lie in a half plane, then $f(z)$ is a polynomial of degree not greater than 2.

Lemma 4 ([20]). *Let $f(z)$ be an entire function of finite order and of positive lower order. Suppose that $g(z)$ is an entire function of order not greater than* 1/2 *and $g \notin \underline{\boldsymbol{H}}$ i.e.*

$$\liminf_{r \to \infty} \frac{\log \log M(r, g)}{\log \log r} = \infty . \tag{20}$$

Then $\rho^*(g(f(z)))=\infty$.

Lemma 5 (cf. [20]). *Let $f(z)$ be an entire function of finite order. Then if $g\in \underline{\boldsymbol{H}}$ (i.e. the left hand side of* (20) *is finite), the lower order $\underline{\rho}(g(f(z))$ is finite.*

Lemma 6 (cf. [20]). *Let $f(z)$ be an entire function. If the lower order $\underline{\rho}(e^{f(z)})$ is finite, then f is a polynomial.*

Lemma 7 ([5]). *Let $F(z)$ be an entire function of finite order such that $\delta(a, F)=1$ for some $a\neq\infty$, where $\delta(\ , F)$ denotes the Nevanlinna deficiency. Then $F(z)$ is pseudo-prime.*

Now, consider the generalized factorization:

$$F(z)=P(z)+Q(e^z)=f(g(z))\cdot e^{A(z)}, \tag{21}$$

where f, g ($\not\equiv$const.) and A are entire functions. Note that F has an infinite number of zeros which are all simple except at most a finite number of them. Also note that the zeros of F are all contained in a half plane such as $\{\mathrm{Re}\, z\geqq c\}$. Further, since we have $\mathrm{Re}\, z_n\to\infty$ $(n\to\infty)$ with the zeros $\{z_n\}$ of F, $g(z)$ under (21) cannot be a non-linear polynomial.

If f has a finite number of zeros, as we may assume that f is a polynomial with at least two zeros, we see $\rho(f(g))=1$ so that $A(z)$ is linear.

If f has an infinite number of zeros, assuming f is the canonical product constructed by the zeros (we may do so), by Lemma 2 we have $\rho(f)=\rho^*(f)=0$, since g is transcendental. Naturally we have $\rho(g)\leqq 1$. Further, by applying Lemma 3, we see $\underline{\rho}(g)\geqq 1$. Then, noting $\rho^*(F)=1$, we have by Lemma 4 that $f\in\underline{\boldsymbol{H}}$ so that by Lemma 5 we have $\underline{\rho}(f(g))$ is finite. Hence, by Lemma 6, $A(z)$ is a polynomial. In this case, rewriting the identity (21) as

$$[P(z)+Q(e^z)]\cdot e^{-A(z)}=f(g(z)), \tag{22}$$

and then applying Lemma 7 to the above (noting f and g are transcendental), we conclude that $e^{A(z)}$ is of order 1 so that $A(z)$ must be linear.

As we may put $-A(z)=az$, from Theorem 3 (Remark there), we have that the left hand side of (22) is prime. This implies that $f(z)$ has just one simple zero or $g(z)$ is a linear polynomial, which is to be proved. Thus the proof of Theorems 1 and 2 are complete.

§5. Here we wish to note the following fact.

Theorem 5. *For a sequence of point-sets $\{A_n \mid n=1, 2, \cdots\}$ with $A_n=\{a_{nj}\}_{j=1}^{m_n}$ $(1\leqq m_n\leqq\infty)$ and a sequence of mutually distinct prime numbers $\{p_n\}_1^\infty$ with $p_n\geqq 3$, let $F(z)$ be an entire function whose zero-set be $\bigcup_{n=1}^{\infty} A_n$. We assume the following conditions;*

(i) *the order of the zero-point a_{nj} of $F(z)$ is equal to p_n for $j=1, \cdots, m_n$.*

(ii) *there exists a half plane H such that $A_n \subset H$ $(n \geqq 1)$.*

(iii) *the order of $N(r, A_n)$ is less than 1 $(n \geqq 1)$, while the order of $N(r, \bigcup_{n=1}^{\infty} A_n)$ is not less than 1.*

Then $F(z)$ is left-prime in divisor sense.

Proof. Let

$$F(z)=f(g(z))\cdot e^{A(z)},$$

where f ($\not\equiv$const.), g (transcendental) and A are entire functions.

Assume that f has at least two zeros. We deduce a contradiction. Now, we have

$$\rho(g) \geqq 1 . \tag{23}$$

Indeed, if f has an infinite number of zeros, then noting the assumption (ii) and applying Lemma 3, we have $\underline{\rho}(g) \geqq 1$ and if f has a finite number of zeros, then noting the latter condition of (iii), we see (23) is valid.

We show at first that the number of the multiple zeros of $f(z)$ is at most one. Indeed, if f has two zeros b_1 and b_2 with multiplicities q_1 and q_2 (>1, resp.), then

$$q_1 = p_{n_1} \quad \text{and} \quad q_2 = p_{n_2}$$

for some n_1 and n_2 such that zeros of $g(z)-b_j$ are all contained in $A(n_j)$ and simple $(j=1, 2)$. Thus

$$N(r, b_1, g)+N(r, b_2, g) \leqq N(r, A_{n_1})+N(r, A_{n_2}) . \tag{24}$$

The left hand side of (24) is of order at least 1 by (23) except possibly for a set E of finite linear measure, while the right hand side is of order less than 1 by the assumption. This is a contradiction.

Next, we show that the number of the simple zeros of $f(z)$ is also at most one. In fact, if f has two simple zeros b_1 and b_2, then the zeros of $g(z)-b_j$ have multiplicities at least 3 by the assumption so that $\Theta(b_j, g)$ is not less than 2/3. Hence

$$\Theta(b_1, g)+\Theta(b_2, g)>1 . \tag{25}$$

This is impossible, since g is entire ([10], p. 43).

Thus it is only necessary to rule out the case where f has one multiple zero b_1 and one simple zero b_2. Note the facts (23) and that the order of $N(r, b_1, g)$ is less than 1 (cf. (24)), whence we have that b_1 is a Borel exceptional value of $g(z)$. Hence the lower order of g is not less than 1. Then clearly $\Theta(b_1, g)=1$ and, naturally, $\Theta(b_2, g) \geqq 2/3$ so that the inequality (25) is also valid, a contradiction.

Therefore $f(z)$ must have just one simple zero, which shows that F is left-prime in divisor sense.

Remark. If we put certain additional assumption on A_n, then $F(z)$ in Theorem 5 becomes to be prime in divisor sense. Hence, it will be clear that

$F(z)$ considered in § 6 is prime in divisor sense.

§ 6. We note here that the primeness of a divisor need not be quasiconformally invariant. This is shown explicitely as follows.

First of all, we recall the following fact due to A. Beurling and L. V. Ahlfors (cf. [1]);

Theorem B. *There exists a quasi-conformal automorphism of the upper half plane with the boundary function* $h(x)$ $(x\in\boldsymbol{R})$ *if and only if*

$$\frac{1}{\rho}\leqq\frac{h(x+t)-h(x)}{h(x)-h(x-t)}\leqq\rho$$

for some constant ρ $(\geqq 1)$ *and for all* x *and* t $(\neq 0)$.

Actually, if the condition (26) is satisfied, there exists a mapping whose maximal dilatation is not greater than ρ^2. This mapping is given, for instance, by the function;

$$\tilde{f}(z)=\frac{1}{2y}\int_{-y}^{y}h(x+s)\mathrm{d}s+i\cdot\frac{r_h}{2y}\int_0^y[h(x+s)-h(x-s)]\mathrm{d}s \tag{27}$$

with $z=x+iy$, $y>0$, and a certain positive constant r_h.

Now, we consider the function $h(x)=x^3$ on the real axis. Then $h(x)$ satisfies the condition (26) for some ρ. Hence h is the boundary function of a quasi-conformal mapping $\tilde{f}$ defined by (27). In this case, we have

$$\tilde{f}(z)=x^3+xy^2+i\cdot r_h(6x^2+y^2)y/4 \tag{28}$$

with $z=x+iy$ $(y>0)$. We extend this function to the lower half plane by the right hand side (replacing y by $-y$). Now, these are suggested by the work of Shiga ([16]).

Note that the point set $\{a+2n\pi i\,|\,n\in\boldsymbol{Z}\}$ with $a\in\boldsymbol{C}$ is mapped by $\tilde{f}$ to the point set whose exponent of convergence is less than one.

Here, we consider the composite function;

$$G(z)=h(e^z),$$

where

$$h(z)=\prod_{n=1}^{\infty}\Big(1-\frac{z}{a_n}\Big)^{p_n}$$

or

$$h(z)=\prod_{n=1}^{\infty}\Big(1-\frac{z}{a_n}\Big)^{p_n}\cdot\exp\Big[\frac{z}{a_n}+\cdots+\frac{1}{q_n}\cdot\Big(\frac{z}{a_n}\Big)^{q_n}\Big]$$

with $q_n\geqq 1$ such that $\{a_n\}$ with $a_n\to\infty$ $(n\to\infty)$ is an increasing sequence of positive numbers and $\{p_n\}$ is a sequence of mutually distinct prime numbers $\geqq 3$ and assume that the exponent of convergence of $\{\log a_n\}$ is not less than 3. Then the divisor;

$$E_n=\{z\,|\,e^z=a_n\}\quad\text{(including multiplicities } p_n)$$

is mapped to the point set A_n $(n \geqq 1)$ which satisfy the condition (iii) in Theorem 5. Thus an entire function $F(z)$ with zero set $\bigcup_{n=1}^{\infty} A_n$ considered there is prime in divisor sense, while $G(z)$ above is naturally composite in divisor sense.

Hence, the divisor $D=\bigcup_{n=1}^{\infty} E_n$ is composite, while the divisor $\tilde{D}=\bigcup_{n=1}^{\infty} A_n$, which is the image of D under a quasi-conformal automorphism $\tilde{f}: \boldsymbol{C} \rightarrow \boldsymbol{C}$, is prime. This shows that the primeness of a divisor is not q.c. invariant.

Remark. It will be easily seen that, for an entire function, the primeness in divisor sense (as well as the primeness) is either not invariant under locally uniform convergence.

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References

[1] L.V. Ahlfors, Lectures on quasi-conformal mappings, Van Nostrand, 1966.
[2] I.N. Baker and F. Gross, Further results on factorization of entire functions, Proc. Symp. Pure Math. (Amer. Math. Soc. Providence R.I.), **11** (1968), 30-35.
[3] A. Edrei, Meromorphic functions with three radially distributed values, Trans. Amer. Math. Soc., **78** (1955), 276-293.
[4] A. Edrei and W.H.J. Fuchs, On the zeros of $f(g(z))$ where f and g are entire functions, J. Analyse Math., **12** (1964), 243-255.
[5] R. Goldstein, On factorization of certain entire functions, J. London Math. Soc. (2), **2** (1970), 221-224.
[6] F. Gross, On the distribution of values of meromorphic functions, Trans. Amer. Math. Soc., **131** (1968), 199-214.
[7] F. Gross, On factorization of meromorphic functions, Ibid., 215-222.
[8] F. Gross, Factorization of entire functions which is periodic mod g, Indian J. of Pure and Appl. Math., **2** (1971), 561-571.
[9] F. Gross and C.-C. Yang, On preimage and range sets of meromorphic functions, to appear.
[10] W.K. Hayman, Meromorphic functions, Clarendon Prsss, Oxford, 1964.
[11] T. Kobayashi, Distribution of values of entire functions of lower order less than one, Kodai Math. Sem. Rep., **28** (1976), 33-37.
[12] L.S.O. Liverpool, Some remarks on factorization of entire functions, J. London Math. Soc. (2), **7** (1973), 272-276.
[13] R. Nevanlinna, Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthier Villars, 1929.
[14] M. Ozawa, On certain criterion for left-primeness of entire functions, Kodai Math. Sem. Rep., **26** (1975), 304-317.
[15] M. Ozawa, On the existence of prime periodic entire functions, Ibid., **29** (1978), 308-321.
[16] H. Shiga, A remark on the boundary behaviour of quasi-conformal mappings and the classification of Riemann surfaces, Proc. Japan Acad., **57** Ser. A, No. 7 (1981), 356-358.
[17] H. Urabe, Uniqueness of the factorization under composition of certain entire functions, J. Math. Kyoto Univ., **18** (1978), 95-120.
[18] H. Urabe, Infinite order, periodic, entire functions which together with their derivatives are prime, Bull. Kyoto Univ. of Education, Ser. B, No. 55 (1979), 21-32.

[19] H. Urabe and C.-C. Yang, On a characteristic property of periodic entire functions, Kodai Math., J., **3** (1980), 253-286.
[20] C.-C. Yang and H., Urabe, On permutability of certain entire functions, J. London Math. Soc., (2), **14** (1976), 153-159.