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A necessary and sufficient condition for the hyperbolicity of second order equations with two independent variables

By

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1. Introduction and Result.

In this paper we shall give a necessary and sufficient condition in order that the Cauchy problem for second order equations with two independent variables is C^∞-well posed.

Let us consider the following operator.

$$L=D_t^2-A(t,\ x)D_x^2+B(t,\ x)D_x+C(t,\ x)D_t+R(t,\ x) \tag{1.1}$$

where we assume that the coefficients are real analytic in a neighborhood of the origin in $\boldsymbol{R}^2$. We are concerned with the following Cauchy problem,

$$Lu(t,\ x)=f(t,\ x),\quad D_t^j u(t_0,\ x)=u_j(x),\quad j=0,\ 1. \tag{1.2}$$

We say that the Cauchy problem (1.2) is C^∞-well posed in a neighborhood of the origin if there is a neighborhood W of the origin in $\boldsymbol{R}^2$ such that for any $(t_0,\ x_0)\in W$ and for any given C^∞-data $f(t,\ x)\in C^\infty(W)$, $u_j(x)\in C^\infty\ (W\cap\{t=t_0\})$, the problem (1.2) has a C^∞-solution $u(t,\ x)$ in a neighborhood of $(t_0,\ x_0)$.

Before formulating the condition of the hyperbolicity, we state some remarks and notations. If we consider the second order operator for which $\{t=\text{const.}\}$ is non-characteristic and the Cauchy problem is C^∞-well posed in a neighborhood of the origin, it follows from the Lax-Mizohata theorem [5] that the operator is reduced to the one having the form (1.1) with non-negative $A(t,\ x)$. Therefore we always assume that $A(t,\ x)\geqq 0$ in a neighborhood of the origin.

Suppose that $A(t,\ x)$ does not vanish identically, then from the Weierstrass preparation theorem and the non-negativity of $A(t,\ x)$, $A(t,\ x)$ is written as follows,

$$A(t,\ x)=x^{2n}\tilde{A}(t,\ x)E(t,\ x),\quad \tilde{A}(t,\ x)=\prod_{\nu=1}^{2m}(t-t_\nu(x)), \tag{1.3}$$

where $E(0,\ 0)>0$ and $\tilde{A}(t,\ x)$ is the Weierstrass polynomial in t. If $m=0$, we mean that $\tilde{A}(t,\ x)\equiv 1$. We set

$$\mathcal{T}(A)=\{\mathrm{Re}\, t_1(x),\ \cdots,\ \mathrm{Re}\, t_{2m}(x)\},$$

where $\operatorname{Re} t_\nu(x)$ denotes the real part of $t_\nu(x)$. If $\tilde{A}(t, x)\equiv 1$, we set $\mathcal{F}(A)=\{0\}$. Then $\operatorname{Re} t_\nu(x)$ is expressed by the Puiseux series of the real variable $x>0$, $x<0$.

$$\operatorname{Re} t_\nu(x)=\sum_{j\geq 0} C^{\pm}_{\nu,j}(\pm x)^{j/p(\nu)}, \quad C^{\pm}_{\nu,j}\in \boldsymbol{R}, \ p(\nu)\in \boldsymbol{N}$$

where the coefficient $C^{+}_{\nu,j}$ (resp. $C^{-}_{\nu,j}$) corresponds to the expansion in $x>0$ (resp. $x<0$).

Now we define the Newton polygon of $f(t+\operatorname{Re} t_\nu(x), x)$ at $(0, \pm 0)$. Let $f(t, x)$ be an analytic function defined in a neighborhood of the origin in $\boldsymbol{R}^2$. For sufficiently small $|x|$, $x\in \boldsymbol{R}$, we have

$$f(t+\operatorname{Re} t_\nu(x), x)=\sum_{i,j\geq 0} C^{\pm}_{\nu,i,j} t^i (\pm x)^{j/p(\nu)}.$$

We define

$$\Gamma_{\pm}(f(t+\operatorname{Re} t_\nu(x), x))=\text{convex hull of } \{\bigcup_{C^{\pm}_{\nu,i,j}\neq 0} (i, j/p(\nu))+\boldsymbol{R}^2_+\}. \tag{1.4}$$

For convenience sake, we set $\Gamma_{\pm}(f(t+\operatorname{Re} t_\nu(x), x))=\emptyset$, if $f(t, x)$ vanish identically. We also denote by $\Gamma^{1/2}_{\pm}(f(t+\operatorname{Re} t_\nu(x), x))$ the set

$$\{(\alpha, \beta)\in \boldsymbol{R}^2_+ ; (2\alpha, 2\beta)\in \Gamma_{\pm}(f(t+\operatorname{Re} t_\nu(x), x))\}.$$

Using these notations, we have

Theorem 1.1. *In order that the Cauchy problem* (1.2) *is C^∞-well posed in a neighborhood of the origin, it is necessary and sufficient that the following two conditions are fulfilled.*

$$A(t, x)\geqq 0 \ \textit{in a neighborhood of the origin,} \tag{1.5}$$

$$\Gamma_{\pm}(tB(t+\phi(x), x))\subset \Gamma^{1/2}_{\pm}(A(t+\phi(x), x)), \quad \textit{for all} \quad \phi(x)\in \mathcal{F}(A). \tag{1.6}$$

Remark 1.1. From the Weierstrass preparation theorem, one can decompose $f(t, x)$ in the following form uniquely.

$$f(t, x)=x^n \tilde{f}(t, x) e(t, x)$$

where $e(0, 0)\neq 0$ and $\tilde{f}(t, x)$ is Weierstrass polynomial or $\tilde{f}(t, x)\equiv 1$. Then it is easy to see that

$$\Gamma_{\pm}(f(t+\operatorname{Re} t_\nu(x), x))=\Gamma_{\pm}(x^n \tilde{f}(t+\operatorname{Re} t_\nu(x), x)).$$

Remark 1.2. Formally, the condition (1.6) is similar to a necessary and sufficient condition of the hyperbolicity of the operator $A^{1/2}(D_t, D_x)+B(D_t, D_x)$. See [8]. Especially, consider the case when $A(t, x)$, $B(t, x)$ has the following form,

$$A(t, x)=\left\{\prod_{j=1}^{m}(t-\lambda_j(x))\right\}^2, \quad B(t, x)=\sum_{j=0}^{m-1} B_j(x) t^j,$$

where $\lambda_j(x)$ (real valued), $B_j(x)$ is real analytic at $x=0$ and $\lambda_j(0)=0$. Then, applying the same reasoning in [4] (Proposition 5.1), we can conclude from (1.6) that $B(t, x)$ is expressed

$$B(t, x)=\sum_{j=1}^{m} c_j(x)\prod_{\substack{\nu=1\\ \nu\neq j}}^{m}(t-\lambda_\nu(x))$$

with analytic $c_j(x)$. In the case when $\lambda_j(x)=\alpha_j x$, $\alpha_j\in\boldsymbol{R}$, $B_j(x)=b_j x^{m-1-j}$, $b_j\in\boldsymbol{C}$, this condition for B is a necessary and sufficient condition of the hyperbolicity of the operator

$$\prod_{j=1}^{m}(D_t-\alpha_j D_x)+B(D_t, D_x).$$

2. Proof of the sufficiency.

From [6], one can represent $A(t, x)$ in the form

$$A(t, x)=x^{2n}\left\{\prod_{\nu=1}^{m}\Lambda_\nu(t, x)\right\}^2 e(t, x)^2, \quad \Lambda_\nu(t, x)=\{(t-\lambda_\nu(x))^2+\psi_\nu(x)\}^{1/2} \tag{2.1}$$

where $e(0, 0)\neq 0$, $\lambda_\nu(x)$, $\psi_\nu(x)$ is analytic in $0<|x|<\delta$, $x\in\boldsymbol{R}$ and $\psi_\nu(x)\geqq 0$. If $\psi_\nu(x)$ does not vanish identically, $\lambda_\nu(x)$ is real valued. Here, we note that $\mathcal{T}(A)$ coincides with the set $\{\mathrm{Re}\,\lambda_1(x), \cdots, \mathrm{Re}\,\lambda_m(x)\}$. From lemma 1.1 in [6], the function

$$a(t, x)=x^n\left\{\prod_{\nu=1}^{m}\Lambda_\nu(t, x)\right\}e(t, x) \tag{2.2}$$

is analytic in $U\backslash(0, 0)$ and continuous in U, where U is a neighborhood of the origin in $\boldsymbol{R}^2$.

Following [6], we introduce some notations. Renumbering, if necessary, we may assume that

$$\mathrm{Re}\,\lambda_1(x)\leqq\mathrm{Re}\,\lambda_2(x)\leqq\cdots\leqq\mathrm{Re}\,\lambda_m(x), \qquad \text{in} \quad 0<x<\delta.$$

Let us set

$$s_j(x)=2^{-1}(\mathrm{Re}\,\lambda_j(x)+\mathrm{Re}\,\lambda_{j+1}(x)), \quad j=1, \cdots, m-1, \quad s_0(x)=-\hat{\lambda}(x),$$

$$s_m(x)=\hat{\lambda}(x), \quad \hat{\lambda}(x)^2=4\sum_{j=1}^{m}(|\lambda_j(x)|^2+\psi_j(x)).$$

By ω_j, $\omega(T)$ we denote the following region,

$$\omega_j=\{(t, x)\,;\,0<x<\delta,\ s_{j-1}(x)\leqq t\leqq s_j(x)\}, \qquad j=1, \cdots, m,$$

$$\omega(T)=\{(t, x)\,;\,0<x<\delta,\ \hat{\lambda}(x)\leqq t\leqq T\}.$$

Our aim in this section is to derive the following inequalities from the condition (1.6).

$$|(t-\mathrm{Re}\,\lambda_j(x))B(t, x)|\leqq C|a(t, x)| \qquad \text{in} \quad \omega_j,\ j=1, \cdots, m, \tag{2.3}$$

$$\begin{cases} |(t-\mathrm{Re}\,\lambda_m(x))B(t, x)|\leqq C|a(t, x)| & \text{in} \quad \omega(T) \text{ if } n\geqq 1, \\ |B(t, x)|\leqq C|D_t a(t, x)| & \text{in} \quad \omega(T) \text{ if } n=0, \end{cases} \tag{2.4}$$

where n is the non-negative integer in (1.3). If this is done, using the inequalities (2.3), (2.4) and the inequalities of the same type obtained in $x<0$ (which shall be proved by the same way), we can proceed following [6] and prove the sufficiency of (1.6).

Now we shall proceed to the proof of (2.3). Fix j_0 $(1\leqq j_0\leqq m)$ arbitrarily and suppose that

$$\operatorname{Re}\lambda_{j_0-k-1}(x)<\operatorname{Re}\lambda_{j_0-k}(x)=\cdots=\operatorname{Re}\lambda_{j_0}(x)=\cdots=\operatorname{Re}\lambda_{j_0+l}(x)<\operatorname{Re}\lambda_{j_0+l+1}(x)$$

in $0<x<\delta$. We set $\lambda(x)=\operatorname{Re}\lambda_{j_0}(x)$, $\phi^+(x)=2^{-1}(\operatorname{Re}\lambda_{j_0+l+1}(x)-\lambda(x))$, $\phi^-(x)=2^{-1}(\lambda(x)-\operatorname{Re}\lambda_{j_0-k-1}(x))$. If $\operatorname{Re}\lambda_{j_0}(x)=\operatorname{Re}\lambda_m(x)$, we put $\phi^+(x)=2^{-1}(\hat{\lambda}(x)-\lambda(x))$. Similarly, $\phi^-(x)=2^{-1}(\lambda(x)+\hat{\lambda}(x))$ if $\operatorname{Re}\lambda_{j_0}(x)=\operatorname{Re}\lambda_1(x)$.

Since j_0 is arbitrary, to prove (2.3) it suffice to show that

$$|(t-\lambda(x))B(t,x)|\leqq C|a(t,x)| \quad \text{in } \tilde{\omega}_{j_0}, \tag{2.5}$$

where $\tilde{\omega}_{j_0}=\{(t,x)\,;\,0<x<\delta,\ \lambda(x)-\phi^-(x)\leqq t\leqq\lambda(x)+\phi^+(x)\}$.

For two functions $f_1(x)$, $f_2(x)$ we write $f_1(x)\approx f_2(x)$ if and only if the following inequalities are valid in $0<x<\delta$, with some positive constants C_i, δ.

$$C_1|f_1(x)|\geqq|f_2(x)|\geqq C_2|f_1(x)|.$$

Proposition 2.1. *For all* ν, $1\leqq\nu\leqq m$, *we have*

$$C_1\{|\lambda(x)-\lambda_\nu(x)|+|\psi_\nu(x)|^{1/2}+\delta|\phi^\pm(x)|\}\geqq|\Lambda_\nu(\lambda(x)+\delta\phi^\pm(x),x)|$$
$$\geqq C_2\{|\lambda(x)-\lambda_\nu(x)|+|\psi_\nu(x)|^{1/2}+\delta|\phi^\pm(x)|\},$$

where positive constants C_i *do not depend on* δ, $0\leqq\delta\leqq1$.

Remark 2.1. If $\lambda(x)=\operatorname{Re}\lambda_m(x)$ (resp. $=\operatorname{Re}\lambda_1(x)$) the above estimate with $+\delta\phi^+$ (resp. $-\delta\phi^-$) is valid uniformly in $\delta\geqq0$.

Proof. We prove this proposition for $\phi^+(x)$. If $\psi_\nu(x)$ does not vanish identically, $\lambda_\nu(x)$ is real valued, and then this means that

$$|\Lambda_\nu(\lambda(x)+\delta\phi^+(x),x)|^2=|\lambda(x)-\lambda_\nu(x)+\delta\phi^+(x)|^2+\psi_\nu(x). \tag{2.6}$$

First consider the case when $\operatorname{Re}\lambda_{j_0}(x)<\operatorname{Re}\lambda_m(x)$. If $\operatorname{Re}\lambda_\nu(x)\geqq\operatorname{Re}\lambda_{j_0+l+1}(x)$, the following is valid uniformly in δ, $0\leqq\delta\leqq1$,

$$|\lambda(x)-\lambda_\nu(x)+\delta\phi^+(x)|\approx|\lambda(x)-\lambda_\nu(x)|.$$

Noting the inequality $C|\lambda(x)-\lambda_\nu(x)|\geqq|\phi^+(x)|$, (2.6) proves the desired inequality.

Next, if $\operatorname{Re}\lambda_\nu(x)\leqq\lambda(x)$, the non-negativity of $\phi^+(x)$ shows that

$$|\lambda(x)-\lambda_\nu(x)+\delta\phi^+(x)|\approx|\lambda(x)-\lambda_\nu(x)|+\delta|\phi^+(x)|.$$

Then, (2.6) gives the desired inequality. When the case $\operatorname{Re}\lambda_{j_0}(x)=\operatorname{Re}\lambda_m(x)$, we have $\operatorname{Re}\lambda_\nu(x)\leqq\lambda(x)$, for all ν, and then the proof is the same as those of the second case.

If we note that $\Gamma_+^{1/2}(A(t+\phi(x),x))=\Gamma_+(a(t+\phi(x),x))$, (1.6) is reduced to $\Gamma_\pm(tB(t+\phi(x),x))\subset\Gamma_\pm(a(t+\phi(x),x))$. Moreover, in virtue of Remark 1.1, we may suppose that

$$(2.7)_{\pm} \qquad \Gamma_{\pm}(tx^{\bar{n}}\tilde{B}(t+\phi(x),\ x))\subset\Gamma_{\pm}\Big(x^{n}\prod_{\nu=1}^{m}\Lambda_{\nu}(t+\phi(x),\ x)\Big)$$

where $B(t,\ x)=x^{\bar{n}}\tilde{B}(t,\ x)\tilde{E}(t,\ x)$ is the decomposition corresponding to (1.3).

We shall derive (2.5) from the condition $(2.7)_{+}$. We define $\varepsilon(\nu)$, $(1\leqq\nu\leqq m)$, ε by

$$|\lambda(x)-\lambda_{\nu}(x)|+\psi_{\nu}(x)^{1/2}\approx x^{\varepsilon(\nu)},\quad \phi^{+}(x)\approx x^{\varepsilon}.$$

If $|\lambda(x)-\lambda_{\nu}(x)|+\psi_{\nu}(x)^{1/2}$ vanish identically, we set $\varepsilon(\nu)=\infty$.

Let

$$(2.8) \qquad \varepsilon(\nu(1))\geqq\cdots\geqq\varepsilon(\nu(l))>\varepsilon\geqq\varepsilon(\nu(l+1))\geqq\cdots\geqq\varepsilon(\nu(m)).$$

From (2.8) and proposition 2.1, it follows that

$$(2.9) \qquad \prod_{j=l+1}^{m}|\Lambda_{\nu(j)}(\lambda(x)+\delta\phi^{+}(x),\ x)|\geqq C\prod_{j=l+1}^{m}x^{\varepsilon(\nu(j))},$$

where C does not depend on δ, $0\leqq\delta\leqq 1$. Similarly, from Proposition 2.1, we have

$$(2.10) \qquad \prod_{j=1}^{l}|\Lambda_{\nu(j)}(\lambda(x)+\delta\phi^{+}(x),\ x)|\approx\prod_{j=1}^{l}(x^{\varepsilon(\nu(j))}+\delta x^{\varepsilon})\geqq C\delta^{p}x^{p\varepsilon+\varepsilon(\nu(1+p))+\cdots+\varepsilon(\nu(l))},$$

for $p=0,\ 1,\ \cdots,\ l$.

The other hand, the condition (2.7) implies that

$$\mathrm{Order}\,\{D_{t}^{j}(x^{\bar{n}}\tilde{B}(\lambda(x),\ x))\}\geqq n+\sum_{i=j+2}^{m}\varepsilon(i),\qquad \text{for}\quad j\geqq 0,$$

where $\sum_{i=j+2}^{m}\varepsilon(i)=0$ if $j\geqq m-1$. Especially, we get $\bar{n}\geqq n$.

Proposition 2.2. *There is a positive constant C which does not depend on δ, $0\leqq\delta\leqq 1$, such that*

$$|\delta\phi^{\pm}(x)x^{\bar{n}}\tilde{B}(\lambda(x)+\delta\phi^{\pm}(x),\ x)|\leqq C|x^{n}\prod_{\nu=1}^{m}\Lambda_{\nu}(\lambda(x)+\delta\phi^{\pm}(x),\ x)|.$$

Proof. We prove this proposition for $\phi^{+}(x)$. First we rewrite $B(\lambda(x)+\delta\phi^{+}(x),\ x)$ as a polynomial in δ.

$$\tilde{B}(\lambda(x)+\delta\phi^{+}(x),\ x)=\sum_{j=0}^{\bar{m}}B_{j}(x)\delta^{j},\quad B_{j}(x)=(j\,!)^{-1}\phi^{+}(x)^{j}\partial_{t}^{j}\tilde{B}(\lambda(x),\ x).$$

From (2.7), it follows that

$$(2.11) \qquad \delta|\phi^{+}(x)|\,|\delta^{j}\phi^{+}(x)^{j}x^{\bar{n}}\partial_{t}^{j}B(\lambda(x),\ x)|\leqq C\delta^{j+1}x^{\varepsilon(\nu(j+2))+\cdots+\varepsilon(\nu(m))+(j+1)\varepsilon+n}.$$

In the case when $j+2\leqq l$, taking into account of (2.9) and (2.10), the second term of (2.11) is estimated by

$$C\delta^{j+1}x^{n+\varepsilon(\nu(j+2))+\cdots+\varepsilon(\nu(l))}\prod_{j=l+1}^{m}x^{\varepsilon(\nu(j))}\leqq C|x^{n}\prod_{\nu=1}^{m}\Lambda_{\nu}(\lambda(x)+\delta\phi^{+}(x),\ x)|.$$

In the case when $j+2>l$, noting the inequalities (2.9), (2.10) and

$$(j+1)\varepsilon+\varepsilon(\nu(j+2))+\cdots+\varepsilon(\nu(m))\geqq l\varepsilon+\varepsilon(\nu(l+1))+\cdots+\varepsilon(\nu(m)),$$

we can estimate the second term of (2.11) by

$$C\delta^l x^{n+l\varepsilon}\prod_{j=l+1}^{m} x^{\varepsilon(\nu(j))}\delta^{(j+1)-l}\leqq C|x^n\prod_{\nu=1}^{m}\Lambda_\nu(\lambda(x)+\delta\phi^+(x),\ x)|.$$

This completes the proof.

Now we derive the estimate (2.5). Let $(t,\ x)\in\tilde{\omega}_{j_0}\cap\{t\geqq\lambda(x)\}$, then there is a δ, $0\leqq\delta\leqq 1$, such that $t=\lambda(x)+\delta\phi^+(x)$. Hence Proposition 2.2 implies (2.5). In the region $\tilde{\omega}_{j_0}\cap\{t\leqq\lambda(x)\}$, the proof is the same. Next we derive the following proposition from $\Gamma_+(tx^{\bar{n}}B(t+\lambda(x),\ x))\subset\Gamma_+\left(x^n\prod_{\nu=1}^{m}\Lambda_\nu(t+\lambda(x),\ x)\right)$, $\lambda(x)=\mathrm{Re}\,\lambda_m(x)$.

Proposition 2.3. *We have in $\omega(T)$, with small T,*

$$|B(t,\ x)|\leqq C\sum_{q=1}^{m}|x^n\prod_{\nu\neq q}^{m}\Lambda_\nu(t,\ x)|,\quad |(t-\mathrm{Re}\,\lambda_m(x))B(t,\ x)|\leqq C|x^n\prod_{\nu=1}^{m}\Lambda_\nu(t,\ x)|.$$

Proof. From Remark 2.1 and the proof of Proposition 2.2, it is clear that

$$|\delta\phi^+(x)x^{\bar{n}}B(\lambda(x)+\delta\phi^+(x),\ x)|\leqq C|x^n\prod_{\nu=1}^{m}\Lambda_\nu(\lambda(x)+\delta\phi^+(x),\ x)|$$

is valid for $\delta\geqq 0$, $\lambda(x)=\mathrm{Re}\,\lambda_m(x)$. Let $(t,\ x)\in\omega(T)$, then one can write with some $\delta\geqq 0$ so that $t=\lambda(x)+\delta\phi^+(x)$. Hence this shows the second inequality in proposition 2.3 immediately. Moreover, it is easy to see that

$$t-\mathrm{Re}\,\lambda_m(x)\approx t\approx|t-\lambda_\nu(x)|+\phi_\nu(x)^{1/2},$$

for all ν, $1\leqq\nu\leqq m$, in $\omega(T)$. Thus, the second inequality in Proposition 2.3 implies the first one in $\omega(T)$.

Proposition 2.4. *In $\omega(T)$ with small T, the following is valid.*

$$\sum_{q=1}^{m}|x^n\prod_{\nu\neq q}^{m}\Lambda_\nu(t,\ x)|\approx|D_t a(t,\ x)|.$$

Proof. Since the inequality $|D_t a(t,\ x)|\leqq C\sum_{q=1}^{m}|x^n\prod_{\nu\neq q}^{m}\Lambda_\nu(t,\ x)|$, is easy, it suffice to show the inverse inequality. It follows from the expression (2.2) that

$$a\partial_t a=x^{2n}\sum_{\nu=1}^{m}(t-\mathrm{Re}\,\lambda_\nu(x))\{\prod_{\mu\neq\nu}|\Lambda_\mu(t,\ x)|^2\}|e|^2+x^{2n}(\partial_t e\cdot e)\prod_{\mu=1}^{m}|\Lambda_\mu(t,\ x)|^2.$$

The other hand, in $\omega(T)$, we know that

$$t-\mathrm{Re}\,\lambda_\nu(x)\geqq c|\Lambda_\nu(t,\ x)|,\qquad 1\leqq\nu\leqq m,$$

with positive c. Then remarking $\Lambda_\nu(0,\ 0)=0$, we get this proposition.

Now, combining Proposition 2.3 and 2.4, the estimate (2.4) follows immediately.

3. Some remarks on the dependence domain.

In this section, we limit our considerations in the region $\{t>0,\ x>0\}$. In another region, there is no difference in the reasoning.

Let $|\hat{t}(x)|\approx x^{\gamma}$, $\gamma\in \boldsymbol{Q}^{+}$ and we assume that $\hat{t}(x)$ does not vanish identically. We denote by $\mathcal{G}_{+}(A)$ the set of all functions $\phi(r)$ which is real valued and expressed by the Puiseux series of the real positive variable r, satisfying the estimate $|\phi(r)|\leqq Cx^{\gamma}$, in a some interval $(0, r(\phi))$ with a constant C.

$$\phi(r)=\sum_{j\geqq 0} c_j r^{j/p}, \quad c_j=c_j(\phi)\in \boldsymbol{R}, \quad p=p(\phi)\in \boldsymbol{N}.$$

Also we define $\sigma^{+}(\phi)$ for $\phi\in\mathcal{G}_{+}(A)$, $\phi\not\equiv 0$, by $\phi(r)\approx x^{\sigma^{+}(\phi)}$. The definition of $\Gamma_{+}(f(t+\phi(x), x))$ with analytic $f(t, x)$ is clear.

We set

$$D(r, M)=\{(t, x)\,;\ 0<x<r,\ 0<t<Mx^{\gamma}\},$$

$$\Delta(\hat{t}, \hat{x}\,; c)=\{(t, x)\,;\ (t-\hat{t})+c^{-1}|x-\hat{x}|\leqq 0,\ 0\leqq t\leqq \hat{t}\},$$

then from the proof of lemma 2.2 in [6], it follows that

$$|A(t, x)|\leqq C(M)^2 r^2 \quad \text{for} \quad (t, x)\in D(r, M). \tag{3.1}$$

Let us put $\mu(M, \gamma, \hat{x})=C(M)^{-1}(2M)^{-1}$ if $\gamma\geqq 1$ and $C(M)^{-1}(2M)^{-1}\hat{x}^{1-\gamma}$ if $\gamma<1$, then we get

Proposition 3.1. *One can find a positive constant $T(M, \gamma)$ so that if*

$$(\hat{t}, \hat{x})\in D(\mu(M, \gamma, \hat{x}), M), \quad 0<\hat{x}<T(M, \gamma)$$

then we have

$$\Delta(\hat{t}, \hat{x}\,; C(M)\mu(M, \gamma, \hat{x}))\subset D(\mu(M, \gamma, \hat{x}), M).$$

Proof. By a simple calculation.

Remark 3.1. From (3.1), we know that $|A(t, x)|\leqq C(M)^2\mu(M, \gamma, \hat{x})^2$ for $(t, x)\in D(\mu(M, \gamma, \hat{x}), M)$, and then Proposition 3.1 implies that $\Delta(\hat{t}, \hat{x}\,; C(M)\mu(M, \gamma, \hat{x}))$ is the dependence domain of $(\hat{t}, \hat{x})$ provided that $0<\hat{x}<T(M, \gamma)$, $0<\hat{t}<M\hat{x}^{\gamma}$. In other words, from $Lu(t, x)=0$ in $\Delta(\hat{t}, \hat{x}\,; C(M)\mu(M, \gamma, \hat{x}))$, $u(t, x)=0$ for $t\leqq 0$, we can conclude that $u(\hat{t}, \hat{x})=0$.

Next, we consider the coordinate transformation associated with $\phi\in\mathcal{G}_{+}(A)$. Let $T_{\phi}\,; U\cap\{x>0\}\rightarrow W\cap\{x_2>0\}$, be a diffeomorphism defined by

$$x_1=t-\phi(x), \quad x_2=x$$

where U, W is a neighborhood of the origin in $\boldsymbol{R}^2_{t,x}$, $\boldsymbol{R}^2_{x_1,x_2}$ respectively. Denote by L_{ϕ} the operator transformed by T_{ϕ} which is defined in $W\cap\{x_2>0\}$. Then

Proposition 3.2. *Let $\phi(x)\in\mathcal{G}_{+}(A)$, $\phi(x)>0$. Then there are positive constants M_0, $T(M, \gamma, \phi)$ $(\leqq T(M, \gamma))$ such that for any $(\hat{t}, \hat{x})$ satisfying $0<\hat{x}<T(M, \gamma, \phi)$,*

$\phi(\hat{x})<\hat{t}<M\hat{x}^{\gamma}$, *with* $M\geqq M_0$, *one has*

$$T_{\phi}(\Delta(\hat{t}, \hat{x}\,; C(M)\mu(M, \gamma, \hat{x})))\subset\{(x_1, x_2)\,; 0\leqq x\leqq\hat{t}-\phi(\hat{x})\}.$$

Proof. Let us set $\sigma=\sigma^{+}(\phi)>0$. First we note that the following inequality, $|\phi(x)-\phi(\hat{x})|\leqq C|x-\hat{x}||\hat{x}|^{\sigma-1}$, for $0<x$, $\hat{x}<\delta=\delta(\phi)$, $|x-\hat{x}|\leqq 2^{-1}|\hat{x}|$. The other hand, if $\gamma<1$, $(t, x)\in\Delta(\hat{t}, \hat{x}\,; C(M)\mu(M, \gamma, \hat{x}))$ implies that $|x-\hat{x}|\leqq 2^{-1}|\hat{x}|$, and thus, taking into account of $\sigma\geqq\gamma$, we get this proposition. When the case $\gamma\geqq 1$, the inequalities $|\phi(x)-\phi(\hat{x})|\leqq C|x-\hat{x}|$, for $0<x$, $\hat{x}<\delta=\delta(\phi)$, and $\sigma\geqq\gamma$ show this proposition.

Denote by $E(M, \gamma, \phi)$ the set

$$\{(x_1, x_2)\,;\ 0<x_2<\delta_0,\ 0<x_1<Mx_2^{\gamma}-\phi(x_2)\}.$$

Proposition 3.3. *Suppose that the Cauchy problem* (1.2) *for* L *is* C^{∞}*-well posed in a neighborhood of the origin. Then there are a neighborhood of the origin* $\widetilde{W}$ *in* $\boldsymbol{R}^2_{x_1,x_2}$, *a constant* C *and an integer* l *such that the inequality*

$$\sup_{0\leq x_1\leq T}|u|\leqq C\sup_{0\leq x_1\leq T,\,|\alpha|\leq l}|D^{\alpha}L_{\phi}u|, \tag{3.2}$$

is valid for any $T>0$ *and for any* $u\in C_0^{\infty}(\widetilde{W}\cap E(M, \gamma, \phi))$. *Where* $M\geqq M_0$, *and* $D^{\alpha}=D_1^{\alpha_1}D_2^{\alpha_2}$, $D_j=D_{x_j}$.

Next consider the coordinate transformation T_{ρ},

$$T_{\rho}\,;\ y_1=\rho^{\delta p}x_1,\quad y_2=\rho^{\delta q}x_2.$$

We assume that δ, p, $q\in\boldsymbol{Q}^{+}$ and

$$p\geqq\gamma q. \tag{3.3}$$

Let $u\in C_0^{\infty}(\{y_1>0,\ y_2>0\})$, then one can find M $(\geqq M_0)$ and ρ_0 so that

$$\mathrm{supp}(u\circ T_{\rho}^{-1})\subset\widetilde{W}\cap E(M, \gamma, \phi)\qquad\text{for}\quad \rho\geqq\rho_0.$$

We denote by $L_{\phi,\rho}$ the operator obtained from L_{ϕ} by the transformation T_{ρ}, then, from (3.2), it follows that

$$\sup_{0\leq y_1\leq\bar{y}_1}|u|\leqq C\rho^{k(\delta,p,q,l)}\sup_{0\leq y_1\leq\bar{y}_1,\,|\beta|\leq l}|D_y^{\beta}(L_{\phi,\rho}u)|, \tag{3.4}$$

for any $\bar{y}_1>0$.

For the later use, we prove the following simple proposition.

Proposition 3.4. *For any* $\phi(x)\in\mathcal{G}_{+}(A)$, *we have*

$$\Gamma_{\pm}^{1/2}(A(t+\phi(x), x))\subset\textit{convex hull of }\{((m, n)+\boldsymbol{R}_{+}^2)\cup((0, n+1)+\boldsymbol{R}_{+}^2)\}.$$

Proof. From Remark 2.1 and (1.3), it follows that

$$\Gamma_{\pm}^{1/2}(A(t+\phi(x), x))=\Gamma_{\pm}^{1/2}\Bigl(x^{2n}\prod_{\nu=1}^{2m}(t+\phi(x)-t_{\nu}(x))\Bigr).$$

One can find $t_{\nu_0}(x)$, from the definition of $\hat{t}(x)$ such that $t_{\nu_0}(x)\approx\hat{t}(x)$, and this shows that $C|t_{\nu_0}(x)|\geqq|\phi(x)-t_\nu(x)|$ for all ν, $1\leqq\nu\leqq m$. Consequently we have

$$\Gamma_{\pm}^{1/2}\Big(x^{2n}\prod_{\nu=1}^{2m}(t+\phi(x)-t_\nu(x))\Big)\subset\Gamma_{\pm}^{1/2}\Big(x^{2n}\prod_{\nu=0}^{2m}(t-t_{\nu_0}(x))\Big).$$

The other hand, from the proof of lemma 2.2 in [6], we know that $|t_{\nu_0}(x)^{2m}|\leqq C|x^2|$, and this completes the proof.

4. Proof of the necessity.

Let

$$L_\phi=\sum_{i,j=1}^{2}A^{(i,j)}(x)D_iD_j+\sum_{i=1}^{2}B^{(i)}(x)D_i+F(x),$$

where $D_j=D_{x_j}$, and $\phi\in\mathcal{G}_+(A)$. First we write down the coefficients of L_ϕ explicitly. If we denote by $f_\phi(x)$ the function $f(x_1+\phi(x_2), x_2)$, the coefficients are written as follows

$$A^{(1,1)}(x)=1-A_\phi(x)\{\phi^{(1)}(x_2)\}^2,\ A^{(1,2)}(x)=2A_\phi(x)\phi^{(1)}(x_2),\ A^{(2,2)}(x)=-A_\phi(x),$$

$$B^{(2)}(x)=B_\phi(x),\ B^{(1)}(x)=-iA_\phi(x)\phi^{(2)}(x_2)-B_\phi(x)\phi^{(1)}(x_2)+C_\phi(x),\ F(x)=R_\phi(x),$$

where $\phi^{(j)}(x_2)=(d/dx_2)^j\phi(x_2)$.

From (1.3), $A(t, x)$ and $B(t, x)$ are expressed as follows

$$A(t, x)=x^{2n}\{t^{2m}+a_1(x)t^{2m-1}+\cdots+a_{2m}(x)\}E(t, x)$$

$$B(t, x)=x^{\bar{n}}\{t^{\bar{m}}+b_1(x)t^{\bar{m}-1}+\cdots+b_{\bar{m}}(x)\}\tilde{E}(t, x).$$

Therefore if $\phi(x_2)\in\mathcal{G}_+(A)$, one can write

$$A_\phi(x)=\sum_{(\alpha,\beta)\in M(\phi)}A_{\alpha,\beta}(x)x_1^\alpha x_2^\beta(1+O(x_2^{1/p})),$$

$$B_\phi(x)=\sum_{(\alpha,\beta)\in\mathcal{M}(\phi)}B_{\alpha,\beta}(x)x_1^\alpha x_2^\beta(1+O(x_2^{1/p})),\qquad p\in\boldsymbol{N}.$$

Here we note that

$$\lim_{x_2\to+0,\,x_1\to0}A_{\alpha,\beta}(x)=\bar{A}_{\alpha,\beta}(\neq0),\qquad\text{if}\quad(\alpha,\beta)\in M(\phi),$$

$$\lim_{x_2\to+0,\,x_1\to0}B_{\alpha,\beta}(x)=\bar{B}_{\alpha,\beta}(\neq0),\qquad\text{if}\quad(\alpha,\beta)\in\mathcal{M}(\phi).$$

Consequently, it follows that

$$(4.1)\qquad\begin{aligned}&\Gamma_+(A(t+\phi(x), x))=\text{convex hull of }\{\bigcup_{(\alpha,\beta)\in M(\phi)}(\alpha,\beta)+\boldsymbol{R}_+^2\},\\&\Gamma_+(B(t+\phi(x), x))=\text{convex hull of }\{\bigcup_{(\alpha,\beta)\in\mathcal{M}(\phi)}(\alpha,\beta)+\boldsymbol{R}_+^2\}.\end{aligned}$$

Put $\sigma=\sigma^+(\phi)>0$, then from the expression of the coefficients, we obtain with some $\tau\in\boldsymbol{N}$

$$
\begin{aligned}
&A^{(2,2)}(x)=-\sum_{(\alpha,\beta)\in M(\phi)} A_{\alpha,\beta}(x)x_1^{\alpha}x_2^{\beta}(1+O(x_2^{1/\tau}))\,,\\
&A^{(1,2)}(x)=2\sum_{(\alpha,\beta)\in M(\phi)} c_1A_{\alpha,\beta}(x)x_1^{\alpha}x_2^{\beta+(\sigma-1)}(1+O(x_2^{1/\tau}))\,,\\
&A^{(1,2)}(x)=1-\sum_{(\alpha,\beta)\in M(\phi)} c_1^2A_{\alpha,\beta}(x)x_1^{\alpha}x_2^{\beta+2(\sigma-1)}(1+O(x_2^{1/\tau}))\,,\\
&B^{(2)}(x)=\sum_{(\alpha,\beta)\in \mathcal{M}(\phi)} B_{\alpha,\beta}x_1^{\alpha}x_2^{\beta}(1+O(x_1^{2/\tau}))\,,\\
&B^{(1)}(x)=-i\sum_{(\alpha,\beta)\in M(\phi)} c_2A_{\alpha,\beta}(x)x_1^{\alpha}x_2^{\beta+(\sigma-2)}(1+O(x_2^{1/\tau}))\\
&\qquad\quad-\sum_{(\alpha,\beta)\in \mathcal{M}(\phi)} c_1B_{\alpha,\beta}(x)x_1^{\alpha}x_2^{\beta+(\sigma-1)}(1+O(x_2^{1/\tau}))+C_{\phi}(x)\,.
\end{aligned}
\tag{4.2}
$$

Next, we consider the operator $L_{\phi,\rho}$.

$$
\begin{aligned}
\rho^{-2\delta p}L_{\phi,\rho}=&A_{\rho}^{(1,1)}(y)D_1^2+A_{\rho}^{(1,2)}(y)\rho^{\delta q-\delta p}D_1D_2+A_{\rho}^{(2,2)}(y)\rho^{2\delta q-2\delta p}D_2^2\\
&+B_{\rho}^{(1)}(y)\rho^{-\delta p}D_1+B_{\rho}^{(2)}(y)\rho^{\delta q-2\delta p}D_2+F_{\rho}(y)\rho^{-2\delta p},
\end{aligned}
$$

where we denote by $f_{\rho}(y)$ the function $f(\rho^{-\delta p}y_1,\ \rho^{-\delta q}y_2)$. Let us set

$$
\begin{aligned}
&\mathcal{M}(\phi)(p,\ q)=\{(\alpha,\ \beta)\in\mathcal{M}(\phi)\,;\,(\alpha+1)p+\beta q<1\},\\
&M(\phi)(p,\ q)=\{(\alpha,\ \beta)\in M(\phi)\,;\,\alpha p+\beta q<2\},\\
&\theta(p,\ q)=\min_{(\alpha,\beta)\in\mathcal{M}(\phi)(p,q)}\{2^{-1}\{\alpha p+\beta q+(1+p)\}\},\\
&\sigma_1(p,\ q)=\delta(p,\ q)(1-\theta(p,\ q))\,,\quad \delta(p,\ q)=(1+p-q)^{-1}.
\end{aligned}
$$

Then we have

Proposition 4.1. *Suppose that*

$$
\mathcal{M}(\phi)(p,\ q)\neq\emptyset\,,\quad M(\phi)(p,\ q)=\emptyset\,,\quad p\geqq\sigma q,\quad 1>q(1-\sigma)\,.
$$

Then we can find $\hat{p}\in\boldsymbol{Q}^{+}$, $p\leqq\hat{p}\leqq 1$, *so that*

$$
\begin{aligned}
&\mathcal{M}(\phi)(\hat{p},\ q)\neq\emptyset\,,\quad M(\phi)(\hat{p},\ q)=\emptyset\,,\quad 0<\sigma_1(\hat{p},\ q)<1\,,\\
&\sigma_1(\hat{p},\ q)-\delta(\hat{p},\ q)q\sigma-1+\delta(\hat{p},\ q)\hat{p}<0\,.
\end{aligned}
$$

Proof. We follow the proof of theorem 7.1 in [3]. Set

$$
\mathcal{K}(p,\ q)=\{(\alpha,\ \beta)\,;\,2q(1-\sigma)<\alpha p+\beta q+(1+p)\}\,.
$$

In the case when $\mathcal{M}(\phi)(p,\ q)\subset\mathcal{K}(p,\ q)$, we take $\hat{p}=p$. The inequality $0<\theta(p,\ q)<1$ is trivial. Remarking the equality

$$
\sigma_1(p,\ q)-\delta(p,\ q)q\sigma-1+\delta(p,\ q)p=\delta(p,\ q)\{q(1-\sigma)-\theta(p,\ q)\}\,,
$$

the desired inequality follows from $\mathcal{M}(\phi)(p,\ q)\subset\mathcal{K}(p,\ q)$, and $\delta(p,\ q)\geqq 1-q(1-\sigma)>0$. The other hand, since $p\geqq\sigma q$, we have $\theta(p,\ q)>q(1-\sigma)\geqq q-p$, and this implies that $0<\delta(p,\ q)(1-\theta(p,\ q))<1$.

Next consider the case when $\mathcal{M}(\phi)(p,\ q)\not\subset\mathcal{K}(p,\ q)$. Denote

$$f(p, \beta)=p^{-1}(1-\beta q-p), \quad g(p, \beta)=p^{-1}\{2q(1-\sigma)-1-p-\beta q\},$$

then we see $f(p', \beta)-g(p', \beta)\geqq 2\{1-q(1-\sigma)\}>0$, for $\sigma q\leqq p'\leqq 1$. Take $Z_1=(\alpha_1, \beta_1)\in\mathcal{M}(\phi)(p, q)\backslash\mathcal{K}(p, q)$. Since $g(p, \beta)$, $f(p, \beta)$ depend continuously on p and $g(1, \beta)<0$, one can find $p_1\in\boldsymbol{Q}$, $p<p_1\leqq 1$ so that $Z_1\in\mathcal{M}(\phi)(p_1, q)\cap\mathcal{K}(p_1, q)$. If $\mathcal{M}(\phi)(p_1, q)\subset\mathcal{K}(p_1, q)$, it suffice to take $\hat{p}=p_1$. Otherwise there is $Z_2=(\alpha_2, \beta_2)\in\mathcal{M}(\phi)(p_1, q)\backslash\mathcal{K}(p_1, q)$, and by the same reasoning, one finds $p_2\in\boldsymbol{Q}$, $p_1<p_2\leqq 1$ such that $Z_2\in\mathcal{M}(\phi)(p_2, q)\cap\mathcal{K}(p_2, q)$. This procedure ends after the finite times. If not, there is a sequence $\{Z_j\}_{j=1}^{\infty}\subset\mathcal{M}(\phi)(p, q)$ satisfying

$$Z_{j+1}\in\mathcal{M}(\phi)(p_j, q)\backslash\mathcal{K}(p_j, q), \quad Z_{j+1}\in\mathcal{M}(\phi)(p_{j+1}, q)\cap\mathcal{K}(p_{j+1}, q). \tag{4.3}$$

If we show that $Z_i\neq Z_j$ for $i\neq j$, the proof is complete, because $\mathcal{M}(\phi)(p, q)$ is a finite set. Suppose that $Z_i=Z_j$ and $i>j$. Then from (4.3) it follows that $Z_i=Z_j\in\mathcal{M}(\phi)(p_{j-1}, q)\backslash\mathcal{K}(p_{j-1}, q)$. The inclusion $\mathcal{K}(p_j, q)\subset\mathcal{K}(p_{i-1}, q)$ implies that $Z_j=Z_i\in\mathcal{K}(p_j, q)\cap\mathcal{M}(p_j, q)\subset\mathcal{K}(p_{i-1}, q)$, but this contradicts to (4.3). The proof of the rest part is the same as in the first case.

Take $\phi\in\mathcal{G}_+(A)$, $p, q\in\boldsymbol{Q}^+$ which satisfies the hypotheses of proposition 4.1. Then from this proposition one can find $\hat{p}$, $\sigma_1=\sigma_1(\hat{p}, q)$, $\delta=\delta(\hat{p}, q)$, $\theta=\theta(\hat{p}, q)$. In the following, we write $\hat{p}=p$. Choose $\tau\in\boldsymbol{N}$ so that $\tau\delta$, τp, τq, $\tau\sigma$, $\tau\sigma_1$, $\tau\beta\in\boldsymbol{N}$, where $(\alpha, \beta)\in\mathcal{M}(\phi)\cup M(\phi)$. With this τ, (4.1) is valid clearly.

Denote $\sigma_1=\hat{\tau}/\tau$, $1\leqq\hat{\tau}\leqq\tau-1$, and define $\sigma_j=(\hat{\tau}+1-j)/\tau$, $j=1, \cdots, \hat{\tau}$, $u^n(y)=\exp\left\{i\left(\mu\rho y_2+\sum_{j=1}^{n}l^j(y)\rho^{\sigma_j}\right)\right\}$, $0\leqq n\leqq\hat{\tau}$, $u^0(y)=\exp\{i\mu\rho y_2\}$, where μ is a real parameter of which signature will be determined in later.

Here we remark that if $(\alpha, \beta)\in\mathcal{M}(\phi)^0(p, q)$ the following equality holds,

$$1-\alpha\delta p-\beta\delta q+\delta q-2\delta p=2\sigma_1. \tag{4.4}$$

If $(\alpha, \beta)\in\mathcal{M}(\phi)(p, q)\backslash\mathcal{M}(\phi)^0(p, q)$, we have $1-\alpha\delta p-\beta\delta q+\delta q-2\delta p<2\sigma_1$. Where $\mathcal{M}(\phi)^0(p, q)=\{(\alpha, \beta)\in\mathcal{M}(\phi)(p, q)\,;\,\alpha p+\beta q+(1+p)=2\theta\}$.

Proposition 4.2. *Suppose that p, q, δ, σ, σ_1 satisfy the hypotheses of proposition* 4.1. *Then we have*

$$(u^1)^{-1}\rho^{-2\delta p}L_{\phi,\rho}(u^1)=\rho^{2\sigma_1}[\Phi_1(y, \mu\,;\,l^1_{y_1})+O(\rho^{-1/\tau})],$$

where $\Phi_1(y, \mu\,;\,l^1_{y_1})=l^1_{y_1}(y)^2+\mu\sum_{(\alpha,\beta)\in\mathcal{M}(\phi)^0(p,q)}\bar{B}_{\alpha,\beta}y_1^{\alpha}y_2^{\beta}$.

Proof. We consider each term separately. Since $M(\phi)(p, q)=\emptyset$ and $1>q(1-\sigma)$, it follows that $-\delta\alpha p-\delta\beta q+2\delta q(1-\sigma)=-\delta(\alpha p+\beta q-2q(1-\sigma))<0$. This shows that

$$\rho^{2\sigma_1}A_{\rho}^{(1,1)}(y)=\rho^{2\sigma_1}(1+O(\rho^{-1/\tau})). \tag{4.5}$$

From $M(\phi)(p, q)=\emptyset$ and proposition 4.1, we get

$$\begin{aligned}&-\delta\alpha p-\delta\beta q+\delta q-\delta p+1+\sigma_1-\delta q(\sigma-1)\\&=2\delta(2-\alpha p-\beta q)+(\sigma_1-\delta q\sigma-1+\delta p)<0,\end{aligned}$$

and this implies that

$$\rho^{\delta q-\delta p+\sigma_1+1}A_\rho^{(1,2)}(y)=O(\rho^{-1/\tau}). \tag{4.6}$$

The other hand, it is easy to see that

$$\rho^{2\delta q-2\delta p+2}A_\rho^{(2,2)}(y)=O(1). \tag{4.7}$$

Next consider the term $\rho^{q-2\delta p+1}B_\rho^{(2)}(y)$. From (4.3) and (4.4), it follows

$$\rho^{\delta q-2\delta p+1}B_\rho^{(2)}(y)=\rho^{2\sigma_1}\Big[\sum_{(\alpha,\beta)\in\mathcal{M}(\phi)^0(p,q)}\bar{B}_{\alpha,\beta}y_1^\alpha y_2^\beta+O(\rho^{-1/\tau})\Big]. \tag{4.8}$$

In virtue of $M(\phi)(p, q)=\emptyset$ and proposition 4.1, it is easy to see that

$$\begin{aligned}&-\delta\alpha p-\delta\beta q-\delta q(\sigma-2)-\delta p+\sigma_1\\&=-\delta(\alpha p+\beta q-2)+(\sigma_1-\delta q\sigma-1+\delta p)-1<-1,\end{aligned}$$

for $(\alpha, \beta)\in M(\phi)$, and from (4.3), (4.4) and proposition 4.1 it follows that

$$\begin{aligned}&-\delta\alpha p-\delta\beta q-\delta q(\sigma-1)-\delta p+\sigma_1\\&=(-\delta\alpha p-\delta\beta q+\delta q-2\delta p+1)+(\delta p-\delta q\sigma+\sigma_1-1)<2\sigma_1\end{aligned}$$

for $(\alpha, \beta)\in\mathcal{M}(\phi)$. These inequalities show that

$$\rho^{-\delta p+\sigma_1}B_\rho^{(1)}(y)=O(\rho^{2\sigma_1-1/\tau}). \tag{4.9}$$

(4.6) through (4.9) complete the proof.

Starting from proposition 4.2, by the standard method of constructing an asymptotic solution, we can get the following lemma (See [3] and [2]).

Lemma 4.1. *Suppose that $\phi\in\mathcal{G}_+(A)$, $p, q\in\boldsymbol{Q}^+$, $p\geqq\sigma^+(\phi)q$, $1>q(1-\sigma^+(\phi))$ and $M(\phi)(p, q)=\emptyset$, $\mathcal{M}(\phi)(p, q)\neq\emptyset$. Then for any given $\hat{y}=(\hat{y}_1, \hat{y}_2)$, $\hat{y}_i>0$, a neighborhood $U(\hat{y})$ of $\hat{y}$ and an integer N, one can find $\bar{y}\in U(\hat{y})$, a neighborhood Y of $\bar{y}$ $(Y\subset U(\hat{y}))$ and analytic functions $l^j(y)$, $v_n(y)$, $1\leqq j\leqq\hat{\tau}$, $0\leqq n\leqq N$ defined in Y so that*

$$(E(y,\rho))^{-1}\rho^{-2\delta p}L_{\phi,\rho}u_\rho=O(\rho^{2\sigma_1-(\hat{\tau}+N+1)/\tau})\qquad in\quad Y.$$

Where $E(y, \rho)=\exp i\Big[\sum_{j=0}^{\hat{\tau}}l^j(y)\rho^{\sigma j}\Big]$, $u_\rho(y)=E(y, \rho)\sum_{n=0}^{N}v_n(y)\rho^{-n/\tau}$. *Moreover, one can assume that* $\operatorname{Im} l^1(y)\geqq(y_2-\bar{y}_2)^2+\delta_0(y_1-\bar{y}_1)$ *in* $Y\cap\{y_1\leqq\bar{y}_1\}$ *with* $\delta_0>0$, $v_0(\bar{y})=1$, $v_n(\bar{y})=0$, $n\geqq1$.

Proposition 4.3. *Suppose that $\phi\in\mathcal{G}_+(A)$, $\phi>0$, $p, q\in\boldsymbol{Q}^+$, $p\geqq\sigma^+(\phi)q$, $1>q(1-\sigma^+(\phi))$ and the Cauchy problem for L is C^∞-well posed in a neighborhood of the origin. Then we have*

$$\mathcal{M}(\phi)(p, q)=\emptyset\qquad if\quad M(\phi)(p, q)=\emptyset.$$

Proof. First note that $p\geqq\sigma^+(\phi)q$, $\phi\in\mathcal{G}_+(A)$ imply $p\geqq\gamma q$. Suppose that $M(\phi)(p, q)=\emptyset$ and $\mathcal{M}(\phi)(p, q)\neq\emptyset$. Then from lemma 4.1, we can construct the asymptotic solution u_ρ for $L_{\phi,\rho}$. Now take $\chi(y)\in C_0^\infty(Y)$ which is identically equal to 1 in a neighborhood of $\bar{y}$, and consider $U_\rho(y)=\chi(y)u_\rho(y)$. Then it is easy to see that

$$(4.10)\qquad \sup_{0\leq y_1\leq \bar{y},\, |\alpha|\leq l} |D^\alpha(L_{\phi,\rho}U_\rho)|\leqq C_l\rho^{2\sigma_1+2\delta p+l+1-(\hat{\tau}+N+1)/\tau}, \quad |U_\rho(\bar{y})|=1,$$

when $\rho\to\infty$. For sufficiently large N, this inequality contradicts to (3.4).

Remark 4.1. In the case when ϕ is identically zero and $p, q\in \boldsymbol{Q}^+$ satisfy $1+p>q$, we have the same conclusion as that of proposition 4.3, from theorem 7.1 in [3].

5. Final remarks.

In this section, from proposition 4.3 and Remark 4.1, we shall prove the next lemma and complete the proof of the necessity.

Lemma 5.1. *Suppose that the Cauchy problem* (1.2) *is C^∞-well posed in a neighborhood of the origin. Then we have*

$$\Gamma_+(tB(t+\phi(x), x))\subset\Gamma_+^{1/2}(A(t+\phi(x), x)) \qquad \textit{for all} \quad \phi\in\mathcal{Q}_+(A).$$

Proof. Denote by $\{(j, \beta(\phi, j))\}_{j=1}^{m}$, $(\beta(\phi, m)=n)$, $\{(j, \gamma(\phi, j))\}_{j=1}^{\bar{m}}$, $(\gamma(\phi, \bar{m})=\bar{n})$ the set of vertices of $\Gamma_+\left(x^n\prod_{\nu=1}^{m}\Lambda_\nu(t+\phi(x), x)\right)$, $\Gamma_+(B(t+\phi(x), x))$, respectively.

Set

$$\varepsilon(\phi, j)=\beta(\phi, j-1)-\beta(\phi, j),\ 1\leqq j\leqq m,\ \delta(\phi, j)=\gamma(\phi, j-1)-\gamma(\phi, j),\ 1\leqq j\leqq \bar{m}.$$

Since the set of vertices of $\Gamma_+(tB(t+\phi(x), x))$ consists of $\{(j+1, \gamma(\phi, j))\}_{j=0}^{\bar{m}}$, to show this lemma, it will be suffice to prove that

$$(5.1)\qquad \gamma(\phi, j)\geqq\beta(\phi, j+1), \qquad \text{for} \quad j\geqq 0.$$

Let

$$(5.2)\qquad \varepsilon(\phi, 1)\geqq\cdots\geqq\varepsilon(\phi, l)\geqq\sigma^+(\phi)>\varepsilon(\phi, l+1)\geqq\cdots\geqq\varepsilon(\phi, m),$$

and let $\alpha p(j)+\beta q(j)=1$ be the equation of the line through $(j, \beta(\phi, j))$ and $(j-1, \gamma(\phi, j-1))$. Then it follows from (5.2) that $p(j)/q(j)=\varepsilon(\phi, j)\geqq\sigma^+(\phi)$, $1\leqq j\leqq l$, and consequently one of the hypotheses of proposition 4.1 is satisfied.

The other hand, from proposition 3.4, it is easy to see that $\beta=1/q(j)\geqq n+1$, at $\alpha=0$, and this implies that $q(j)\leqq 1$. Taking into account of $\sigma^+(\phi)>0$, we have $1>q(j)(1-\sigma^+(\phi))$. Therefore, from proposition 4.3, it follows that the $\Gamma_+(B(t+\phi(x), x))$ lies in the right side of the lines $(\alpha+1)p(j)+\beta q(j)=1$, $1\leqq j\leqq l$, and this fact shows that

$$(5.3)\qquad \gamma(\phi, j)\geqq\beta(\phi, j+1), \qquad 0\leqq j\leqq l-1.$$

In the case when $n\geqq 1$, noting Remark 4.1, we apply proposition 4.3 with $\phi\equiv 0$, $q=s/n$, $p=(1-s)/m$, $(s\uparrow 1,\ s\in\boldsymbol{Q})$. Then we get $\bar{n}\geqq n$, and from this inequality, it follows that

$$\gamma(\phi, j)\geqq\bar{n}\geqq n=\beta(\phi, j+1), \qquad m-1\leqq j. \tag{5.4}$$

It remains to show that

$$\gamma(\phi, j)\geqq\beta(\phi, j+1), \qquad \text{for} \quad l\leqq j\leqq m-2. \tag{5.5}$$

Now assume that there is at least one j, $l\leqq j\leqq m-2$, such that $\gamma(\phi, j)<\beta(\phi, j+1)$. Let $j_0=\max\{j\,;\gamma(\phi, j)<\beta(\phi, j+1)\}$, then from the definition, we see that $\gamma\leqq\delta(\phi, j_0+1)<\varepsilon(\phi, j_0+2)<\sigma^+(\phi)$. Take $\psi\in\mathcal{G}_+(A)$ so that $\sigma^+(\psi)=\varepsilon(\phi, j_0+2)$. Since $\sigma^+(\psi-\phi)=\sigma^+(\psi)$, $\delta(\phi, j_0+1)<\sigma^+(\psi)$, the following equalities are easily verified that $\delta(\psi, j+1)=\delta(\phi, j+1)$, $j\geqq j_0$. Thus we have

$$\gamma(\psi, j_0)=\sum_{j=j_0+1}^{\bar{m}}\delta(\psi, j)+\bar{n}=\sum_{j=j_0+1}^{\bar{m}}\delta(\phi, j)+\bar{n}=\gamma(\phi, j_0). \tag{5.6}$$

Next, inequalities $\sigma^+(\psi)=\varepsilon(\phi, j_0+2)\geqq\cdots\geqq\varepsilon(\phi, m)$, imply that $\varepsilon(\psi, j)\geqq\varepsilon(\phi, j)$, $j_0+2\leqq j\leqq m$. Then it follows that

$$\beta(\psi, j_0+1)=\sum_{j=j_0+2}^{m}\varepsilon(\psi, j)+n\geqq\sum_{j=j_0+2}^{m}\varepsilon(\phi, j)+n=\beta(\psi, j_0+1). \tag{5.7}$$

From the inequalities $\varepsilon(\phi, j)\geqq\sigma^+(\psi)=\sigma^+(\psi-\phi)$, $0\leqq j\leqq j_0+2$, we have $\varepsilon(\psi, j)\geqq\sigma^+(\psi)$ for $0\leqq j\leqq j_0+2$, and then the same reasoning obtaining (5.1) shows that

$$\gamma(\psi, j)\geqq\beta(\psi, j+1), \qquad 0\leqq j\leqq j_0+1. \tag{5.8}$$

Now, combining (5.6), (5.7) and (5.8), we have

$$\gamma(\phi, j_0)=\gamma(\psi, j_0)\geqq\beta(\psi, j_0+1)\geqq\beta(\phi, j_0+1),$$

but this contradicts to the assumption, and the proof is complete.

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