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On the syzygy part of Koszul homology on certain ideals

By

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1. Introduction.

Let A be a Noetherian local ring, m the maximal ideal of A and M a finitely generated A-module. a will always denote an ideal in A. Let $a_1, \cdots, a_r$ be a set of generators for a. Then we denote by $K.(a; M)$ the Koszul complex associated to a. Furthermore, $Z.(a; M)$ and $B.(a; M)$ denote the cycle and boundary of the Koszul complex respectively. For an arbitrary positive integer n we set

$$\tilde{H}_n(a; M)=Z_n(a; M)/[Z_n(a; M)\cap aK_n(a; M)]$$

and name this module the syzygy part of the homology $H_n(a; M)$.

The purpose of this paper is to study some properties of the syzygy part.

Obviously there exists a canonical homomorphism of A-modules

$$H_n(a; M) \longrightarrow \tilde{H}_n(a; M) \longrightarrow 0.$$

If the canonical map is injective for some integer n, then we call that $a_1, \cdots, a_r$ is $\tilde{H}_n$-faithful (cf. [5]). A sequence of elements $a_1, \cdots, a_r$ is called a d-sequence for M if

$$(a_1, \cdots, a_{i-1})M: a_ia_j=(a_1, \cdots, a_{i-1})M: a_j$$

for every $1\leqq i\leqq j\leqq r$ and an unconditioned d-sequence for M if any permutation of $a_1, \cdots, a_r$ is a d-sequence for M (C. Huneke has defined a d-sequence for $M=A$ in [2]).

A. Simis and W.V. Vasconcelos [6] has defined $\delta(a)=[Z_1(a)\cap aA^r]/B_1(a)$ for arbitrary ideal a generated by r elements and shown that $\delta(a)=0$ if and only if the canonical homomorphism $\mathrm{Symm}(a)\rightarrow R(a)$ from the symmetric algebra to the Rees algebra is the isomorphism in degree two part of both algebras.

On the other hand, C. Huneke has discussed in [2] that if $a_1, \cdots, a_r$ is an unconditioned d-sequence for A, then $\mathrm{Symm}((a_1, \cdots, a_r))\cong R((a_1, \cdots, a_r))$ (see also [3]). Thus we can immediately see that if $a_1, \cdots, a_r$ is an unconditioned d-sequence for A, then it is $\tilde{H}_1$-faithful.

Our first result is

Theorem 1.1. *Let $a_1, \cdots, a_r$ be an unconditioned d-sevuence for M, then*

$a_1, \cdots, a_r$ is $\tilde{H}_n$-faithful for every positive integer n.

Now, M is called a Buchsbaum A-module if every system of parameters is d-sequence for M. Then we have the another result as follows:

Theorem 1.2. *The following conditions are equivalent:*

(i) *M is a Buchsbaum A-module of dimension d,*
(ii) *$m\tilde{H}_n(a_1, \cdots, a_d; M)=0$ for every system of parameters $a_1, \cdots, a_d$ for M and every positive integer n,*
(iii) *$m\tilde{H}_1(a_1, \cdots, a_d; M)=0$ for every system of parameters $a_1, \cdots, a_d$ for M.*

Recently N. Suzuki [7] has proved that M is a Buchsbaum A-module if and only if $mH_1(a_1, \cdots, a_d; M)=0$ for any system of parameters $a_1, \cdots, a_d$ for M. Theorem 1.2 says the above result is valid for the syzygy part.

2. The proof of Theorem 1.1.

In this section we wish to prove Theorem 1.1. For this purpose we need a definition and a few lemmas.

For a sequence of elements $a_1, \cdots, a_r$ of A we define $I_j=(a_1, \cdots, a_j)$ and $U(I_jM)=I_jM: a_{j+1}$ $(a_0=0,\ a_{r+1}=1)$ for $0\leqq j\leqq r$.

Lemma 2.1. *If $a_1, \cdots, a_r$ is an unconditioned d-sequence for M, then $U(I_{i-1}M) =I_{i-1}M: a_j$ for $1\leqq i\leqq j\leqq r$.*

Proof. By definition

$$U(I_{i-1}M)\subseteqq(a_1, \cdots, a_{i-1})M: a_ia_j=(a_1, \cdots, a_{i-1})M: a_j\,.$$

On the other hand, as $a_1, \cdots, a_{i-1}, a_j, a_i$ is also a d-sequence for M, we have

$$I_{i-1}M: a_j\subset I_{i-1}M: a_ja_i=U(I_{i-1}M)\,.$$

Lemma 2.2. *If $a_1, \cdots, a_r$ is a d-sequence for M, then $U(I_nM)\cap I_rM=I_nM$ for $0\leqq n\leqq r$.*

Proof. This assertion is similar as Lemma 4.2 in [1]. Let x be an element of $U(I_nM)\cap I_rM$, and express

$$x=\sum_{i=1}^{r} a_ix_i$$

for some $x_i\in M$. Then we can see

$$a_{n+1}x=\sum_{i=1}^{r} a_{n+1}a_ix_i=\sum_{j=1}^{n} a_jy_j$$

for some $y_j\in M$. Thus $a_{n+1}a_rx_r\in I_nM$, which implies $x_r\in I_nM: a_{n+1}a_r$. But as $a_1, \cdots, a_r$ is a d-sequence for M, $x_r\in I_nM: a_r$. Therefore, $x\in I_{r-1}M$. Repeating the above argument, we have the desired result.

Proposition 2.3. *Suppose that $a_1, \cdots, a_r$ is an unconditioned d-sequence for M. Then $Z_n(a_1, \cdots, a_{r-1}; M) = B_n(a_1, \cdots, a_{r-1}; M) : a_r^m$ for positive integers n, m.*

Proof. We prove this assertion by induction on r. If $r=1$, there is nothing to prove. Suppose that $r=2$. Obviously we may prove this assertion in case $n=1$. Since $Z_1(a_1; M)=0 : a_1$ and $B_1(a_1; M)=0$, we have the following equalities from Lemma 2.1.

$$B_1(a_1; M) : a_2^m = 0 : a_2^m = 0 : a_2 = 0 : a_1 = Z_1(a_1; M).$$

Now, suppose that $r>2$ and the assertion holds for $r-1$. Let $K. = K.(a_1, \cdots, a_{r-1}; M)$ and $L. = K.(a_1, \cdots, a_{r-2}; M)$. Let $d.$ (resp. $e.$) denote the differential of $K.$ (resp. $L.$). Then, we can see that $K_n = L_n \oplus L_{n-1}$ for every $n \geqq 1$ by the definition of the Koszul complex. Thus the differential $d.$ is induced from $e.$ as follows:

$$d_n(u, v) = (e_n(u) + a_{r-1}v, \ -e_{n-1}(v)) \qquad \text{(cf. [7])}.$$

With notation as above, let (u, v) be an element of $B_n(K) : a_r^m$. Then we have

$$\text{(2.3.a)} \qquad a_r^m u = e_{n+1}(t) + a_{r-1}w$$

$$\text{(2.3.b)} \qquad a_r^m v = -e_n(w)$$

where $t \in L_{n+1}$, $w \in L_n$.

Since both $a_1, \cdots, a_{r-2}, a_r$ and $a_1, \cdots, a_{r-1}$ are the unconditioned d-sequences for M of length $r-1$, we get $v \in B_{n-1}(L) : a_r^m = Z_{n-1}(L) = B_{n-1}(L) : a_{r-1}$ by induction. This implies $e_{n-1}(v)=0$ and $a_{r-1}v = e_n(w')$, where $w' \in L$. Using (2.3.a) and (2.3.b), we have the following equalities;

$$\begin{aligned} 0 &= a_{r-1}a_r^m v + a_{r-1}e_n(w) \\ &= a_r^m a_{r-1}v + e_n(a_{r-1}w) \\ &= a_r^m e_n(w') + e_n(a_r^m u) \\ &= a_r^m [e_n(w'+u)]. \end{aligned}$$

This leads

$$a_r^m w' + a_r^m u \in Z_n(L) = B_n(L) : a_r$$

by induction. Hence

$$w' + u \in B_n(L) : a_r^{m+1} = Z_n(L).$$

This implies

$$0 = e_n(w'+u) = e_n(u) + a_{r-1}v.$$

Thus $(u, v) \in Z_n(K)$.

Conversely, let (u, v) be an element of $Z_n(K)$. The equation

$$\text{(2.3.c)} \qquad 0 = d_n(u, v) = (e_n(u) + a_{r-1}v, \ -e_{n-1}(v)).$$

Then $v \in Z_{n-1}(L) = B_{n-1}(L) : a_r^m$, since $a_1, \cdots, a_{r-2}, a_r$ is a d-sequence of length $r-1$. Thus there exists $w \in L_n$ such that

(2.3.d) $$a_r^m v=e_n(w).$$

On the other hand, $e_n(u)+a_{r-1}v=0$ shows that $e_n(u)=0$ in $K.(a_1, \cdots, a_{r-2}; M/a_{r-1}M)$.
As $a_1, \cdots, a_{r-2}, a_r$ is an unconditioned d-sequence for $M/a_{r-1}M$, by induction we get

$$u\in B_n(a_1, \cdots, a_{r-2}; M/a_{r-1}M): a_r^m.$$

Hence there exist $x\in L_n$ and $t\in L_{n+1}$ such that

(2.3.e) $$a_r^m u=e_{n+1}(t)+a_{r-1}x.$$

From (2.3.d) and (2.3.e), we have

$$0=e_n(a_r^m u)+a_r^m a_{r-1}v=e_n(a_{r-1}x)+e_n(a_{r-1}w).$$

Thus we get

$$a_{r-1}x+a_{r-1}w\in Z_n(L).$$

Therefore, as $a_1, \cdots, a_{r-2}, a_{r-1}$ is an unconditioned d-sequence for M,

$$a_{r-1}x+a_{r-1}w\in B_n(L): a_{r-1}^m.$$

This implies that

$$x+w\in B_n(L): a_{r-1}^{m+1}=B_n(L): a_{r-1}.$$

Hence, there exists $t'\in L_{n+1}$ such that

(2.3.f) $$a_{r-1}x+a_{r-1}w=e_{n+1}(t').$$

Combining the above equations (2.3.d), (2.3.e) and (2.3.f), we get

$$a_r^m u=e_{n+1}(t)+a_{r-1}x=e_{n+1}(t+t')+a_{r-1}(-w)$$

$$a_r^m v=e_n(w)=-e_n(-w).$$

Therefore, $(u, v)\in B_n(K): a_r^m$. q.e.d.

Corollary 2.4. *Suppose that $a_1, \cdots, a_r$ is an unconditioned d-sequence for M and put $K.=K.(a_1, \cdots, a_{r-1}; M)$. Then*

$$I_{r-1}K_n\cap Z_n(K)=I_rK_n\cap Z_n(K)$$

for an arbitrary positive integer n.

Proof. Let u be an element of $I_rK_n\cap Z_n(K)$, then $u=y+a_rx\in B_n(K): a_r$ by Proposition 2.3, where $y\in I_{r-1}K_n$ and $x\in K_n$. This implies that

$$a_r^2x+a_ry\in B_n(K)\subset I_{r-1}K_n.$$

Hence,

$$x\in I_{r-1}K_n \underset{K_n}{:} a_r^2=\overset{n}{\wedge} A^r\otimes(I_{r-1}M: a_r^2)$$
$$=\overset{n}{\wedge} A^r\otimes(I_{r-1}M: a_r)$$
$$=I_{r-1}K_n \underset{K_n}{:} a_r,$$

because $a_1, \cdots, a_r$ is an unconditioned d-sequence for M. Thus, $a_r x \in I_{r-1}K_n$. Therefore, $u=y+a_r x \in I_{r-1}K_n$, as desired.

Proof of Theorem 1.1. Let $K'=K.(a_1, \cdots, a_r; M)$ and $K.=K.(a_1, \cdots, a_{r-1}; M)$. First, we show that

$$Z_n(K')\cap I_r K'_n = B_n(K')$$

for any positive integer n. We prove this by induction on n. We may assume that $n \leqq r$.

If $r=1$, then $B_1(K')=0$ and $Z_1(K')=0: a_1$. Let x be an element of $Z_1(K')\cap(a_1)K'_1$, then there exists $y \in K'_1=M$ such that $x=a_1 y$. Thus we have

$$y \in 0: a_1^2 = 0: a_1 .$$

Hence $x \in B_1(K')$.

Suppose that $r \geqq 2$ and that the assertion holds for $r-1$. As $K'_n=K_n \oplus K_{n-1}$, d', the differential of K', is induced from the differential d. Now, let (u, v) be an element of $Z_n(K')\cap I_r K'_n$, where $u \in I_r K_n$ and $v \in I_r K_{n-1}$. Then

$$\text{(a)} \quad 0=d'_n(u, v)=(d_n(u)+a_r v, -d_{n-1}(v)).$$

Thus, by Corollary 2.4

$$v \in Z_{n-1}(K)\cap I_r K_{n-1} = Z_{n-1}(K)\cap I_{r-1}K_{n-1} = B_{n-1}(K) .$$

Hence there exists $t \in K_n$ such that $v=d_n(t)$. On the other hand, from (a), we have

$$0=d_n(u)+a_r v = d_n(u)+a_r d_n(t) = d_n(u+a_r t) .$$

Thus, by Corollary 2.4

$$u+a_r t \in Z_n(K)\cap I_r K_n = Z_n(K)\cap I_{r-1}K_n = B_n(K) .$$

Hence, there exists $w \in K_{n+1}$ such that

$$u+a_r t = d_{n+1}(w) , \quad \text{i.e.,} \quad u=d_{n+1}(w)+a(-t) .$$

Therefore, $(u, v)=d'_{n+1}(w, -t) \in B_n(K')$. This completes the proof of Theorem 1.1.

Now, we show some corollaries which are immediate from Theorem 1.1.

Corollary 2.5. *Let A be a Noetherian local ring and $\mathfrak{m}$ the maximal ideal of A. Suppose that $a_1, \cdots, a_r$ is an unconditioned d-sequence for A. Then $a_1, \cdots, a_r$ is $\tilde{H}_n$-faithful for an arbitrary positive integer n.*

Corollary 2.6. *Let M be a Buchsbaum A-module and $a_1, \cdots, a_r$ a subsystem of parameter for M. Then $a_1, \cdots, a_r$ is $\tilde{H}_n$-faithful for an arbitrary positive integer n.*

Proof. This follows from the fact that $a_1, \cdots, a_r$ is an unconditioned d-sequence for M.

Now, assume that $l(H_{\mathfrak{m}}^i(M))<\infty$ for every $i \neq d$ ($d=\dim M$). Then by [4], there exists an $\mathfrak{m}$-primary ideal q such that any system of parameters $a_1, \cdots, a_d$ for M contained in q forms an unconditioned d-sequence for M. Thus

Corollary 2.7. *If $a_1, \cdots, a_r$ is contained in q and a subsystem of parameters for M, then it is $\widetilde{H}_n$-faithful for an arbitrary positive integer n.*

3. The proof of Theorem 1.2.

In this section we will prove Theorem 1.2. In proving this theorem, we need the following key proposition.

Now let $a_1, \cdots, a_r$ be an arbitrary sequence of elements of A. We put $I=(a_1, \cdots, a_r)$ and let J be any ideal such that $I \subseteqq J \subseteqq m$. We call that $a_1, \cdots, a_r$ is a strong d-sequence for M if $a_1^{k_1}, \cdots, a_r^{k_r}$ is a d-sequence for M for positive integers k's. Then we have

Proposition 3.1. *If $J\widetilde{H}_1(a_1^{k_1}, \cdots, a_r^{k_r}; M)=0$ for every positive integer k_j $(1 \leqq j \leqq r)$, then $a_1, \cdots, a_r$ is a strong d-sequence for M.*

Proof. First we show that

$$(a_1, \cdots, a_i)M : a_k^2=(a_1, \cdots, a_i)M : a_k$$

for every $i \geqq 0$ and $k \geqq i+1$.

Indeed, let x be an element of $(a_1, \cdots, a_i)M : a_k^2$. Then there exists the following equation

$$a_k^2 x=\sum_{j=1}^{i} a_j x_j\,,$$

where $x_j \in M$. Let $[\cdot, \cdots, \cdot]$ denote an element of a free module in a Koszul complex. Now, let n be an arbitrary positive integer and fix this number. Then, as

$$\begin{aligned}&[x_1, \cdots, x_i, 0, \cdots, 0, -x, 0, \cdots, 0]\\ &\in Z_1(a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_{k-1}^n, a_k^2, a_{k+1}^n, \cdots, a_r^n; M)\end{aligned}$$

and as $a_k \in J$, we have

$$\begin{aligned}&a_k[x_1, \cdots, x_i, 0, \cdots, 0, -x, 0, \cdots, 0]\\ &\in(a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_{k-1}^n, a_k^2, a_{k+1}^n, \cdots, a_r^n)\cdot\\ &\quad K_1(a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_{k-1}^n, a_k^2, a_{k+1}^n, \cdots, a_r^n; M)\,.\end{aligned}$$

Thus we conclude that

$$a_k x \in(a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_{k-1}^n, a_k^2, a_{k+1}^n, \cdots, a_r^n)M\,.$$

Claim.

$$a_k x \in(a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_{k-1}^n, a_k^p, a_{k+1}^n, \cdots, a_r^n)M$$

for every $p \geqq 2$.

We prove this by induction on p. If $p=2$, there is nothing to prove. Suppose that $p>2$ and that the assertion holds for $p-1$. Hence we may assume that

$$a_k x \in(a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_k^{p-1}, \cdots, a_r^n)M\,.$$

Then $a_k x$ may be written as

$$a_k x = \sum_{j=1}^{i} a_j t_j + \sum_{j \neq k} a_j^n t_j + a_k^{p-1} t,$$

where t_j, $t \in M$. On the other hand, as $a_k^2 x \in (a_1, \cdots, a_i)M$, we have

$$a_k^p t = a_k^2 x - \sum_{j=1}^{i} a_k a_j t_j - \sum_{j \neq k} a_k a_j^n t_j$$
$$\in (a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_{k-1}^n, a_{k+1}^n, \cdots, a_r^n)M.$$

Thus we get $a_k^p t = \sum_{j=1}^{i} a_j s_j + \sum_{j \neq k} a_j^n s_j$, where $s_j \in M$. Since

$$[s_1, \cdots, s_{k-1}, t, s_{k+1}, \cdots, s_r]$$
$$\in Z_1(a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_{k-1}^n, a_k^p, a_{k+1}^n, \cdots, a_r^n; M).$$

we know that

$$a_k^{p-1} t \in (a_1, \cdots, a_{i+1}^n, \cdots, a_{k-1}^n, a_k^p, a_{k+1}^n, \cdots, a_r^n)M,$$

which completes the proof of the claim.

Let us continue the proof of Proposition 3.1. By the above claim we know that

$$a_k x \in \bigcap_{n,p} (a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_{k-1}^n, a_k^p, a_{k+1}^n, \cdots, a_r^n)M$$
$$= (a_1, \cdots, a_i)M,$$

which shows $x \in (a_i, \cdots, a_i)M : a_k$.

To establish the proof of Proposition 3.1, we only need to show that

$$(a_1, \cdots, a_i)M : a_j a_k = (a_1, \cdots, a_i)M : a_j$$

for every $0 \leqq i < k \leqq j \leqq r$. Now let x be an element of $(a_1, \cdots, a_i)M : a_j a_k$ and n be an arbitrary positive integer. Then we have

$$\text{(b)} \qquad a_j a_k x + \sum_{p=1}^{i} a_p x_p = 0,$$

where $x_p \in M$. Multiplying a_k^{n-1} to the above equation (b),

$$a_j a_k^n x + \sum_{p=1}^{i} a_k^{n-1} a_p x_p = 0.$$

This shows that

$$[x_1, \cdots, x_i, 0, \cdots, a_j x, 0, \cdots, 0] \in Z_1(a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_r^n; M).$$

As $a_j \in J \subseteqq I$, we have $a_j^2 x \in (a_1, \cdots, a_i, a_{i+1}^n, \cdots, a_r^n)M$. Therefore,

$$a_j^2 x \in (a_1, \cdots, a_i)M + \bigcap_n (a_{i+1}^n, \cdots, a_r^n)M = (a_1, \cdots, a_i)M$$

by Kull's intersection theorem. This implies that $x \in (a_1, \cdots, a_i)M : a_j^2$. But as $(a_1, \cdots, a_i)M : a_j^2 = (a_1, \cdots, a_i)M : a_j$ by virtue of the first assertion, we have $x \in (a_1, \cdots, a_i)M : a_j$. Thus we have proved that $a_1, \cdots, a_r$ is a d-sequence for M.

Finally, if we put $b_i = a_i^{k_i}$, then it is easy to see that $b_1, \cdots, b_r$ is also a d-sequence for M by the same routine in the previous proof.

Proof of Theorem 1.2. If M is a Buchsbaum A-module, then

$$mH_1(a_1, \cdots, a_d; M)=0$$

for every system of parameters $a_1, \cdots, a_d$ for M by the main Theorem in [7]. On the other hand, by Corollary 2.6 we have

$$H_1(a_1, \cdots, a_d; M)=\tilde{H}_1(a_1, \cdots, a_d; M).$$

Hence (i) implies (ii). (ii) implies (iii) is trivial. (iii) implies (i) follows from Proposition 3.1 in case $J=m$.

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