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On singular 4-manifolds of the homology type of CP^2

By

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Introduction. We continue our study of compact 4-dimensional manifolds-with-singularities which have global topological features in common with the complex projective plane. In [5] an algorithm was developed for the systematic construction of certain such spaces X which in addition support a complex structure, and in [8] this procedure was carried out in detail to produce a list of examples of spaces of the rational homology or cohomology type of $\boldsymbol{CP}^2$ whose singularities are rational double points. These spaces turn out to be degenerate Del Pezzo surfaces of the type studied by Du Val in [11], and in [2] all representations of such spaces as modified projective planes are characterized in terms of certain "global extensions" of Dynkin diagrams (see also [7]). In the present paper we exploit more thoroughly the relations between the global topological properties of X and the analytic structure of the singular points to obtain results that are much sharper locally than those of [5] and [4], yet apply to a more general class of objects than that considered in [8] and [2], and without requiring the tedious case by checking which the proofs of [8] and [2] entailed.

There are two kinds of results. On the one hand, we show how global hypotheses impose conditions on the singular points. For example, we have the following theorems.

Theorem A. *Let X be a singular 4-manifold of the integral cohomology type of $\boldsymbol{CP}^2$ (that is, $H^*(X, \boldsymbol{Z}) \cong H^*(\boldsymbol{CP}^2, \boldsymbol{Z})$ as rings). Suppose that X supports a complex structure $\mathcal{O}_X$ with vanishing geometric genus. Then X has only one singular point x, and X has the local structure at x of the space underlying the rational double point E_8. That is, x admits a neighborhood U in X which is homeomorphic to the quotient of $\boldsymbol{C}^2$ modulo the action of the binary icosahedral group $SL(2, 5)$. (Indeed, the normalization of $\mathcal{O}_{X,x}$ is biholomorphic to E_8.)*

Theorem B. *Let X be a singular 4-manifold of the integral homology type of $\boldsymbol{CP}^2$ (that is, $H_i(X, \boldsymbol{Z}) \cong H_i(\boldsymbol{CP}^2, \boldsymbol{Z}) \forall i$). Suppose that X supports a complex Gorenstein structure $\mathcal{O}_X$ which admits an effective anticanonical divisor K^*. Then $(X, \mathcal{O}_X)$ is a projective algebraic variety birationally equivalent to $\boldsymbol{CP}^2$ whose*

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singular point(s) is (are) the rational double point(s) associated to the Dynkin diagram "E_k", where $k=9-K^2$ and where "E_k" is the graph obtained from E_8 by deleting $8-k$ consecutive vertices, starting at the end of the longest arm.

On the other hand, beginning with assumptions about the singularities, we determine the global settings in which they occur.

Theorem C. *Let $(X, \mathcal{O}_X)$ be a normal complex surface of the rational homology type of $\boldsymbol{CP}^2$. Suppose that each singular point of X is a rational double point. Then either*

(a) *X is biholomorphic to $\boldsymbol{CP}^2$ or to the singular complex quadric hypersurface $\boldsymbol{Q}_0^2 \subset \boldsymbol{CP}^3$;*

(b) *X is a rational projective surface derived from $\boldsymbol{CP}^2$ by blowing up some number $s \leqq 8$ points in relatively general position, then blowing down s non-singular rational curves, each with self-intersection -2; or*

(c) *X is derived from a minimal non-singular Enriques surface $\tilde{X}$ or from a minimal non-singular projective surface $\tilde{X}$ of general type, with $q(\tilde{X})=p_g(\tilde{X})=0$ and admitting $s=b_2(\tilde{X})-1$ non-singular rational curves C_i with $C_i^2=-2$, by blowing down $\bigcup C_i$.*

Theorem D. *Let r be an integer, $1 \leqq r \leqq 4$, and let $k_1, \cdots, k_r$ be positive integers satisfying*

$$(*) \qquad 4\prod_{i=1}^{r}(k_i+1)=\left(9-\sum_{i=1}^{r}k_i\right)n^2 \quad \text{for some integer } n.$$

Then there exists a normal rational complex surface X of the rational homology type of $\boldsymbol{CP}^2$ possessing exactly r singular points $x_1, \cdots, x_r$, with x_i the cone on a lens space of type (k_{i+1}, k_i). Conversely, (*) holds for every such space X. X has the rational cohomology type of $\boldsymbol{CP}^2$ if and only if $\sum_{i=1}^{r} k_i=8$.

We also take the opportunity below to correct an error that appeared in [5] (see the remarks following theorem 9).

I. The structure of the singular points.

By a *singular 4-manifold* we shall mean in this paper a second countable Hausdorff topological space X of which each point x admits a neighborhood U_x, called a *spherical neighborhood*, such that ∂U_x is a topological 3-manifold M_x, and such that $\bar{U}$ is homeorphic to the cone on ∂U. If U can be chosen so that ∂U_x is a 3-sphere, then x is a *regular point* of X; otherwise x is called *singular*.

It is clear that under this definition the set S of singular points of a singular 4-manifold is discrete. Thus we may assume that $\bar{U}_x \cap \bar{U}_y = \emptyset$ for distinct points $x, y \in X$. A singular 4-manifold is *orientable* if $X-\bigcup_{x\in S} U_x$ is orientable as a manifold with boundary. This condition is clearly independent of choice of (sufficiently small) neighborhoods U_x. In this paper all singular 4-manifolds will be compact, connected, and oriented. Thus $H_4(X, \boldsymbol{Z}) \cong \boldsymbol{Z}$ and is presumed to have a preferred generator μ, the *fundamental class* of X. In case X is also a complex analytic space, μ is chosen to be compatible with the complex structure. All homology and cohomology groups in this paper will be assumed to have integer coefficients, unless otherwise indicated.

We will have need of the following purely algebraic facts about Abelian groups.

1. Lemma. *Let* $0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \overset{h}{\longrightarrow} D \longrightarrow 0$ *be an exact sequence of finitely generated Abelian groups, with* D *finite of order* n *and expotent* e, *and let* $\langle\, ,\, \rangle$: $B \times B \to \boldsymbol{Z}$ *be a unimodular symmetric bilinear form.*

(a) *Suppose that* A *and* B *have equal rank* t, *and let* $\alpha_1, \cdots, \alpha_t$ *and* $\beta_1, \cdots, \beta_t$ *be bases over* $\boldsymbol{Z}$ *for the free parts of* A *and* B *respectively* (*that is,* $\alpha_1, \cdots, \alpha_t$ *are elements of* A *such that the cosets* $\bar{\alpha}_1, \cdots, \bar{\alpha}_t$ *mod the torsion subgroup Tors* A *form a basis of the free group* $A/\mathrm{Tors}\, A$, *and similarly for* B). *Then*

$$|C| = |\det(\langle f\alpha_i, \beta_j \rangle)| \frac{|\mathit{Tors}\, B|}{|\mathit{Tors}\, A|} \cdot n\,,$$

where $\det(\langle f\alpha_i, \beta_j\rangle)$ *is the determinant of the non-singular matrix of integers* $\langle f\alpha_i, \beta_j\rangle$, *and* $|\ |$ *denotes the order of a finite group or the absolute value of a real number.*

(b) *Suppose instead that* C *is free with basis* $\gamma_1, \cdots, \gamma_s$. *Suppose further that* B *contains a free subgroup* S *with basis* $\sigma_1, \cdots, \sigma_s$, *with* $\det(\langle \sigma_i, \sigma_j \rangle) \neq 0$ *and such that the map* g *is given by* $g(\beta) = \sum_{i=1}^{s} \langle \beta, \sigma_i \rangle \gamma_i \forall \beta \in B$. *Then for each* $\beta \in B$ *we have an equation*

$$ep\beta = f\alpha + \sum_{i=1}^{s} m_i \sigma_i$$

for some integers m_i *and for some* $\alpha \in A$, *where*

$$p = \frac{\det(\langle f\alpha_j, f\alpha_k \rangle)}{\text{g.c.d.}\, \{\det(\langle f\alpha_j, f\alpha_k\rangle), \langle \beta, f\alpha_1\rangle, \cdots, \langle \beta, f\alpha_t \rangle\}}$$

for any basis $\alpha_1, \cdots, \alpha_t$ *of the free part of* A (*g.c.d. is the greatest common divisor*).

Proof. For part (a), clearly

$$|C| = |\mathrm{Im}\, g| \cdot |D| = |B/\mathrm{Im}\, f| \cdot n\,.$$

To compute $|B/\mathrm{Im}\, f|$ we use the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Tors}\, A & \longrightarrow & \mathrm{Tors}\, B & \longrightarrow & \mathrm{Tors}\, B/\mathrm{Tors}\, A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & B/\mathrm{Im}\, f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/\mathrm{Tors}\, A & \longrightarrow & B/\mathrm{Tors}\, B & \longrightarrow & \dfrac{B/\mathrm{Tors}\, B}{A/\mathrm{Tors}\, A} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

to conclude $|B/\mathrm{Im}\, f| = |\det(b_{ij})| \dfrac{|\mathrm{Tors}\, B|}{|\mathrm{Tors}\, A|}$, where $f\alpha_j = \sum_j b_{ij}\beta_j$. Part (a) now follows from the condition that $|\det\langle\beta_i, \beta_j\rangle| = 1$.

For part (b), since C is free f is an isomorphism on the torsion parts of A and B. Thus we may assume without loss of generality that A and B are free. Also, $\det(\langle\sigma_i, \sigma_j\rangle) \neq 0$ implies that $S \cap S^\perp = \{0\}$, where $S^\perp = \{\beta \in B \mid \langle\rho, \sigma\rangle = 0\ \forall \sigma \in S\}$. But $\mathrm{Im}\, f \subset S^\perp$, for if $\alpha \in A$ then $0 = gf\alpha = \sum_j \langle f\alpha, \sigma_j\rangle \gamma_j$, whence independence of the γ_j in C guarantees that $\langle f\alpha, \sigma_j\rangle = 0\ \forall j$. From this and injectivity of f we have that the $t+s$ elements $f\alpha_1, \cdots, f\alpha_t, \sigma_1, \cdots, \sigma_s$ are independent in B and form a basis of $B \otimes \boldsymbol{Q}$ over $\boldsymbol{Q}$. Moreover $\langle\ ,\ \rangle$ extends to a bilinear form on $B \otimes \boldsymbol{Q}$ and is non-singular on $(\mathrm{Im}\, f) \otimes \boldsymbol{Q}$ and $S \otimes \boldsymbol{Q}$ separately. If $\beta \in B$, write $\beta = \sum_i q_i f\alpha_i + \sum_j r_j \sigma_j$ in $B \otimes \boldsymbol{Q}$ for some rational numbers q_i, r_j. Then $\langle\beta, f\alpha_k\rangle = \sum_i q_i \langle f\alpha_i, f\alpha_k\rangle$. Put $a_{ik} = \langle f\alpha_i, f\alpha_k\rangle$ and denote by (a_{ik}^*) the matrix of rational numbers inverse to $a = (a_{ki})$. Then $q_i = \sum_k a_{ik}^* \langle\beta, f\alpha_k\rangle$ and so $\forall i$ the expression

$$pq_i = \sum_k \det(a) a_{ik}^* \frac{\langle\beta, f\alpha_k\rangle}{\mathrm{g.c.d.}\{\det(a), \langle\beta, f\alpha_1\rangle, \cdots, \langle\beta, f\alpha_t\rangle\}}$$

is an integer. Thus

$$p\beta = \sum_i l_i f\alpha_i + \sum_j u_j \sigma_j \tag{*}$$

for l_i the integer pq_i and for u_j the rational number pr_j.

Now D has exponent e, so $h(e\gamma_k) = 0\ \forall k$ and thus $\exists \beta_k \in B$ with $g(\beta_k) = e\gamma_k$. That is, for $j \neq k$ $\langle\sigma_j, \beta_k\rangle = 0$, while $\langle\sigma_j, \beta_j\rangle = e\ \forall j$. Applying $\langle\cdot, \beta_k\rangle$ to (*), yields

$$\langle p\beta - \sum_i l_i f\alpha_i, \beta_k\rangle = \sum_j u_j \langle\sigma_j, \beta_k\rangle = eu_k .$$

Since $p\beta - \sum_i l_i f\alpha_i$ and β_k are both in B, we conclude that $eu_k \in \boldsymbol{Z}$ for all k. Hence

$$ep\beta = f\alpha + \sum_j m_j \sigma_j$$

for $\alpha=\sum el_i\alpha_i\in A$ and for m_j the integer eu_j. Q. E. D.

2. Lemma. *Let X be a (compact, connected, oriented) singular 4-manifold with Betti numbers $b_3=0$, $b_2=1$, and such that the cup product pairing $H^2(X)\times H^2(X)\rightarrow H^4(X)\cong\boldsymbol{Z}$ is not identically zero. Let $g\in H^2(X)$ generate the free part of $H^2(X)$ (that is, its image in $H^2(X)/\mathrm{Tors}(H^2(X))$ generates). As above denote by $S=\{x_1,\cdots,x_r\}$ the set of singular points of X and by U_{x_i} a spherical neighborhood of x_i. Then $H_1(\partial U_{x_i})$ is a finite group for each i, and*

$$\prod_{i=1}^{r}|H_1(\partial U_{x_i})|=|g\cdot g|\frac{|H^3(X)|^2}{|\mathrm{Tors}\,H^2(X)|^2}.$$

Proof. We may assume that the U_{x_i}'s have disjoint closures, so that $Y=X-\cup U_{x_i}$ is a manifold with boundary. Since Y is a deformation retract of $X-S$, the inclusion induces the isomorphisms $H_*(Y,\partial Y)\cong H_*(X,S)$ and $H^*(Y,\partial Y)\cong H^*(X,S)$, and hence by dimension

$$H_k(X)\cong H_k(Y,\partial Y)\quad\text{and}\quad H^k(X)\cong H^k(Y,\partial Y)$$

for $k>1$.

Using Poincaré duality and the universal coefficient theorem we have $H^1(\partial Y)\cong\mathrm{Free}(H^2(\partial Y))$, $\mathrm{Tors}(H^2(Y))\cong\mathrm{Tors}(H^3(X))=H^3(X)$, and $\mathrm{Free}(H^2(Y))\cong\mathrm{Free}(H^2(X))\cong\boldsymbol{Z}$. Thus the relevant part of the long exact sequence for the pair $(Y,\partial Y)$ is

$$\begin{array}{ccccccc} H^1(\partial Y) & \longrightarrow & H^2(Y,\partial Y) & \xrightarrow{f} & H^2(Y) & \xrightarrow{h} & H^2(\partial Y)\longrightarrow \\ \wr\| & & \wr\| & & \wr\| & & \\ \mathrm{Free}(H^2(\partial Y)) & & \boldsymbol{Z}\oplus\mathrm{Tors}(H^2(X)) & & \boldsymbol{Z}\oplus H^3(X) & & \end{array}$$

$$\begin{array}{ccccc} H^3(Y,\partial Y) & \xrightarrow{i} & H^3(Y) & \longrightarrow & H^3(\partial Y) \\ \wr\| & & & & \\ H^3(X) & & & & \end{array}$$

Furthermore, since $H^3(\partial Y)$ is free, i maps $H^3(Y,\partial Y)$ onto $\mathrm{Tors}(H^3(Y))\cong\mathrm{Tors}(H^2((X))$.

To examine the map f, let $g'\in H^3(Y,\partial Y)$ and $\mu'\in H_4(Y,\partial Y)$ correspond to g and μ. Then

$$0\neq g^2=\langle g'\cdot g',\mu'\rangle=\langle fg',g'\cap\mu'\rangle,$$

so fg' has infinite order in $H^2(Y)$. Thus the image of h is finite, and since $H^3(Y,\partial Y)\cong H^3(X)$ is also finite, it follows that $H^2(\partial Y)$ is finite and hence that $H^1(\partial Y)=0$. Thus our sequence is

$$\begin{array}{l} 0\longrightarrow H^2(Y,\partial Y)\xrightarrow{f}H^2(Y)\longrightarrow H^2(\partial Y)\longrightarrow H^3(X)\xrightarrow{i}\mathrm{Tors}(H^2(X))\longrightarrow 0. \\ \qquad\qquad\wr\|\qquad\qquad\qquad\wr\| \\ \quad\boldsymbol{Z}\oplus\mathrm{Tors}(H^2(X))\quad\boldsymbol{Z}\oplus H^3(X) \end{array}$$

Of course, $|H^3(X)|=|\ker i|\,|\mathrm{Tors}\,H^2(X)|$, and applying Lemma 1(a) to

$$0\longrightarrow H^2(Y,\partial Y)\xrightarrow{f}H^2(Y)\longrightarrow H^2(\partial Y)\longrightarrow\ker i\longrightarrow 0,$$

leads to the equality

$$|H^2(\partial Y)|=|g^2|\frac{|H^3(X)|^2}{|\mathrm{Tors}\,H^2(X)|^2}.$$

Finally, the observation

$$H^2(\partial Y)=\prod_{i=1}^{r}|H^2(\partial U_{x_i})|=\prod_{i=1}^{r}|H_1(\partial U_{x_i})|$$

completes the proof.

A rich and particularly interesting source of examples of singular 4-manifolds is provided by the class of compact two (-complex-) dimensional complex analytic spaces (*complex surfaces*) with isolated singularities. The reader is referred to [5] or [3] for a review of the terminology of global invariants of complex surfaces X and for some facts relating their global properties to the structure of the singular points. In particular, in the lemma below $q=\dim_C H^1(X, \mathcal{O}_X)$ and $p_g=\dim_C H^2(X, \mathcal{O}_X)$ are respectively the irregularity and geometric genus, while b^+ and b^- denote the dimensions of the positive and negative eigenspaces of the intersection pairing.

3. Lemma. *Let* $(X, \mathcal{O}_X)$ *be a normal compact complex surface with Betti numbers* $b_3=0$, $b_2=1$, *and with singular points* $x_1, \cdots, x_r$. *Let* $\pi:\tilde{X}\to X$ *be a resolution of singularities with exceptional curve* $C=\pi^{-1}(\{x_1, \cdots, x_r\})$. *Assume without loss of generality that the components* $C_1, \cdots, C_s$ *of* C *are non-singular, that* C_i *and* C_j *meet, if at all, transversally in a single point, and that there are no triple intersections. Denote by* Γ *the dual intersection graph of* C *and by* $\det(\Gamma)$ *the determinant of the positive definite matrix* $(-C_i\cdot C_j)$. *Then*

(a) $b_1(X)=b_1(\tilde{X})=b_3(\tilde{X})=q(X)=q(\tilde{X})=p_g(\tilde{X})=0$, $b^+(\tilde{X})=1$, $b^-(\tilde{X})=s$, $g^2>0$ *for* g *a generator of* $H^2(X)/\mathrm{Tors}(H^2(X))$, $\tilde{c}_1^2=9-s$ *for* $\tilde{c}_1$ *the first Chern class of* $\tilde{X}$;

(b) C_i *is rational* $\forall i$, Γ *has no cycles,* $\sum_{i=1}^{r}\dim(R^1\pi_*\mathcal{O}_{\tilde{X}})_{x_i}=p_g(X)$, *and* $\det(\Gamma)=g^2n^2$ *for* $n=$*the integer* $\dfrac{|H^3(X)|}{|\mathrm{Tors}\,H^2(X)|}$; *and*

(c) *If* X *admits a holomorphic line bundle* L *with* $L^2\neq 0$, *then* X *is a projective algebraic variety and* g *may be chosen to be the Chern class* $c(L_0)$ *of an ample holomorphic line bundle* L_0. *Furthermore, every line bundle* $\tilde{L}$ *on* $\tilde{X}$ *satisfies*

$$\tilde{L}^{\otimes p}\cong\pi^*(L_0^{\otimes m}\otimes T)\otimes\bigotimes_{i=1}^{s}[C_i]^{\otimes m_i}$$

for some integers m_i *and some torsion bundle* T, *where* $p=\dfrac{g^2n}{d}$ *and* $m=\dfrac{\tilde{c}(\tilde{L})\cdot\pi^*g}{d}n$, *for* d *the greatest common divisor of* g^2 *and* $\tilde{c}(\tilde{L})\cdot\pi^*g$, *and where* $[C_i]$ *denotes the bundle of the divisor of* C_i *on* X. *Also,* $\tilde{c}_1\cdot\pi^*g\equiv g^2 \bmod 2$.

Proof. For (a) and (b), all but the last claim of (a) and the last claim of (b) are proved in [5], Proposition 3. The relation $\tilde{c}_1^2=9-s$ is then immediate from the Todd-Noether formula (or from the Hirzebruch index theorem). To compute $\det(\Gamma)$, first note that by normality of $\mathcal{O}_X$ the set of analytic singularities coincides with the set S of topological singularities ([17]). Since the curves C_i are non-singular rational meeting normally with no 3 in a point and with no cycles in their graph, we may take for the contractible neighborhood U_{x_i} of x_i the image under π of a tubular neighborhood of the curve $\pi^{-1}(x_i)\subset C$. By Mumford's calculation [17] of the fundamental group $\pi_1(\partial U_{x_i})$ we have, $\forall i$, $|H_1(\partial U_{x_i})|=\det(\Gamma_i)$, where Γ_i is the component of Γ corresponding to $\pi^{-1}(x_i)$. Since $g^2>0$, Lemma 2 applies to give

$$\det(\Gamma)=\prod_{i=1}^{r}\det(\Gamma_i)=\prod_{i=1}^{r}|H_1(\partial U_{x_i})|=g^2\cdot n^2$$

as specified.

Part (c) is an exercise in chasing the diagram

$$\begin{array}{ccccccccccc}
 & & H^2(X,\mathcal{O}_X) & \to & H^2(\tilde{X},\mathcal{O}_{\tilde{X}}) & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
H^1(C,\boldsymbol{Z}) & \to & H^2(X,\boldsymbol{Z}) & \xrightarrow{\pi^*} & H^2(\tilde{X},\boldsymbol{Z}) & \to & H^2(C,\boldsymbol{Z}) & \xrightarrow{\delta} & H^3(X,\boldsymbol{Z}) & \to & H^3(\tilde{X},\boldsymbol{Z})\to 0 \\
 & & \uparrow & & \uparrow & & & & & & \\
 & & H^1(X,\mathcal{O}_X^*) & \to & H^1(\tilde{X},\mathcal{O}_{\tilde{X}}^*) & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
 & & H^1(X,\mathcal{O}_X) & \to & H^1(\tilde{X},\mathcal{O}_{\tilde{X}})\,, & & & & & &
\end{array}$$

(see [3], Lemma 1), where the rows are induced by π and by the inclusion of C into $\tilde{X}$, and the columns by the exponential maps $\mathcal{O}\to\mathcal{O}^*$, and where $H^1(X,\mathcal{O}_X^*)$, $H^1(\tilde{X},\mathcal{O}_{\tilde{X}}^*)$ are identified with the Picard groups of (isomorphism classes of) holormorphic line bundles on X, $\tilde{X}$. By part (a) above, $H^1(C,\boldsymbol{Z})=H^1(X,\mathcal{O}_X)=H^1(\tilde{X},\mathcal{O}_{\tilde{X}})=H^2(\tilde{X},\mathcal{O}_{\tilde{X}})=0$, so the Chern class map $\tilde{c}$ is an isomorphism, while c and π^* are injections. The existence of $L\in H^1(X,\mathcal{O}_{\tilde{X}})$ with $L^2\neq 0$ guarantees that $\operatorname{rank}(H^1(X,\mathcal{O}_X^*))=\operatorname{rank}(H^2(X,\boldsymbol{Z}))=1$, whence, since $H^2(X,\mathcal{O}_X)$ is torsion free, c is in fact an isomorphism. Put $L_0=c^{-1}(g)$. X is projective algebraic by [3], Proposition 6, so X admits a positive (=ample) line bundle. Indeed, since L_0 generates $H^1(X,\mathcal{O}_X^*)$ mod torsion, either L_0 or its dual is positive. Replacing g by $-g$ if necessary we may assume that L_0 is positive.

Now $H^3(X,\boldsymbol{Z})$ is finite, so $H^3(\tilde{X},\boldsymbol{Z})\cong\operatorname{Tors}(H^2(\tilde{X},\boldsymbol{Z}))\cong\operatorname{Tors}(H^2(X,\boldsymbol{Z}))$, the last isomorphism ($\pi^*$) holding because $H^2(C,\boldsymbol{Z})\cong\boldsymbol{Z}^s$ is torsion free. Thus $\operatorname{im}(\delta)$ is a group N of order $n=\dfrac{|H^3(X)|}{|\operatorname{Tors}H^2(X)|}$, and we have

$$0\longrightarrow H^2(X,\boldsymbol{Z})\xrightarrow{\pi^*}H^2(\tilde{X},\boldsymbol{Z})\xrightarrow{i^*}H^2(C,\boldsymbol{Z})\xrightarrow{\bar{\delta}}N\longrightarrow 0\,.$$

This diagram satisfies the hypotheses of Lemma 1, part (b), for S=the subgroup of $H^2(X,\boldsymbol{Z})$ generated by the Poincaré duals of the curves C_i and for $\langle\,,\rangle$ the intersection pairing. By that lemma, and using the isomorphisms c and $\tilde{c}$, we conclude that every line bundle $\tilde{L}$ on $\tilde{X}$ satisfies

$$\tilde{L}^{\otimes p} \cong \pi^*(L') \otimes \bigotimes_{i=1}^{s} [C_i]^{\otimes m_i},$$

where

$$p = \frac{\langle \pi^* g \cdot \pi^* g \rangle}{\text{g.c.d.}\{\langle \pi^* g, \pi^* g \rangle, \langle \pi^* g, \tilde{c}(\tilde{L}) \rangle\}} \cdot n = \frac{g^2 n}{d},$$

for some integers m_i and some line bundle $L' \cong L_0^{\otimes m} \otimes T$ (T torsion, $m \in \boldsymbol{Z}$) on X. Applying $\langle \cdot, \pi^* g \rangle$ to this equation shows that $m = \frac{\tilde{c}(\tilde{L}) \cdot \pi^* g}{d} n$ as required in (c) above.

Finally, to check the relation $g^2 \equiv \tilde{c}_1 \cdot \pi^* g \mod 2$, apply Riemann-Roch to the bundle $\pi^* L_0$:

$$\chi(\mathcal{O}_{\tilde{X}}(\pi^* L_0)) - \chi(\mathcal{O}_{\tilde{X}}) = \frac{1}{2}(\pi^* g \cdot \pi^* g + \tilde{c}_1 \cdot \pi^* g) = \frac{1}{2}(g^2 + \tilde{c}_1 \cdot \pi^* g)$$

to conclude that $g^2 + \tilde{c}_1 \cdot \pi^* g$ is an even integer. This completes the proof.

4. Corollary. *Let* $(X, \mathcal{O}_X)$ *and* $\pi: \tilde{X} \to X$ *be as in part* (c) *of Lemma* 3. *Suppose further that* $H^3(X, \boldsymbol{Z}) = 0$. *Then if* $g^2 | \tilde{c}_1 \cdot \pi^* g$, $\mathcal{O}_X$ *is Gorenstein* (*that is, the canonical dualizing sheaf* Ω_X *on* X *is locally trivial*). *In particular, this is true if* $g^2 = 1$ *or* 2. (*The notation throughout is as in Lemma* 3 *and its proof*).

Proof. First note that a normal surface is always Cohen-Macauley, so Ω_X is defined. Also, $H^3(X, \boldsymbol{Z}) = 0 \Rightarrow H^2(X, \boldsymbol{Z})$ is torsion free. Denote by $K_{\tilde{X}}$ the canonical line bundle of holomorphic differential 2-forms on $\tilde{X}$. If we take $\tilde{L} = K_{\tilde{X}}$ in part (c) of Lemma 3, the condition $g^2 | \tilde{c}_1 \cdot \pi^* g$ (together with $H^3(X, \boldsymbol{Z}) = 0$), implies that for this bundle the integer p of Lemma 3 is equal to 1, and thus that

$$K_{\tilde{X}} = \pi^*(L_0^{\otimes m}) \otimes \bigotimes_{i=0}^{s} [C_i]^{\otimes m_i}$$

for $m = \frac{-\tilde{c}_1 \cdot \pi^* g}{g^2}$, $m_i \in \boldsymbol{Z}$. It follows that $\Omega_X \cong \mathcal{O}_X(L_0^{\otimes m})$ and so is locally trivial.

Now if $g^2 = 1$ then surely $g^2 | \tilde{c}_1 \cdot \pi^* g$ whatever $\tilde{c}_1 \cdot \pi^* g$ may be. If $g^2 = 2$, the last part of Lemma 3 shows that $\tilde{c}_1 \cdot \pi^* g$ is even, so $g^2 | \tilde{c}_1 \cdot \pi^* g$ in this case as well. This completes the proof.

We turn now to an examination of the singular points. For a discussion of the rational double points A_k, D_k and E_k the reader is referred to [1], [9], or [10]. We have found it a useful notational convenience to introduce, for any integer $k \geqq 0$, the label "E_k" for the Dynkin diagram ●—●—●(●)—···—● (k vertices in all, each carrying weight 2). Thus "E_5" $= D_5$: ●—●—●(●)—●, "E_4" $= A_4$: ●—●—●(●), "E_3" $= A_2 + A_1$: ●—● ●, "E_2" $= 2A_1$: ● ●, "E_1" $= A_1$: ●, and "E_0" = the empty graph (cf. [7], where a slightly different convention was adopted for $k = 1$ and 2. The graph "E_2" will not occur in this paper.)

Rational double points $x \in X$ are defined by the property $(R^1 \pi_* \mathcal{O}_{\tilde{X}})_x = 0$ and $Z^2 = -2$ for $\pi: \tilde{X} \to X$ the minimal resolution, R^1 the first right derived functor, and Z the fundamental cycle ([1]). They are characterized by the condition that the canonical bundle $K_{\tilde{X}}$ is trivial in a neighborhood of the exceptional curve, and thus $K_{\tilde{X}}=\pi^*K_X$ for some holomorphic line bundle on X ([11]). If $x\in X$ is a rational double point with Dynkin diagram $\Gamma=A_k$, D_k or E_k, then, as in the proof of Lemma 3, the order of the first homology group of the boundary of a contractible neighborhood U_x is determined by $|H_1(\partial U_x)|=\det(\Gamma)$. Direct calculation (or an examination of presentations of the corresponding finite subgroups of $SL(2, \boldsymbol{C})$) establishes the well-known facts $\det(A_k)=k+1$, $\det(D_k)=4$, $\det(E_k)=9-k$. This last relation continues to hold for the graphs "E_k", $k=3$, 4, and 5. Conversely,

5. Lemma. *Let Γ be a k-point Dynkin diagram, not necessarily connected, with no multiple edges and satisfying* $\det(\Gamma)=9-k$. *Then* $3\leqq k\leqq 8$ *and* $\Gamma=$"E_k".

The proof is by inspection of the graphs A_k, D_k, and E_k.

6. Corollary. *Let $(X, \mathcal{O}_X)$ be a complex surface with $b_3=0$, $b_2=1$, and admitting an ample holomorphic line bundle L_0 whose Chern class g generates $H^2(X, \boldsymbol{Z})$* mod *torsion. Suppose that each singularity x_i of X is a rational double point, denote by Γ_i the Dynkin diagram associated to x_i, and put $\Gamma=\cup\Gamma_i$ and $s=|\Gamma|$. Let $\pi:\tilde{X}\to X$ be the minimal resolution of singularities and write $K_{\tilde{X}}=\pi^*(L_0^{\otimes m}\otimes T)$ for some integer m and some torsion bundle T_0. As above, put* $n=\dfrac{|H^3(X)|}{|\mathrm{Tors}\, H^2(X)|}$. *Then we have*

(a) $m^2g^2=9-s\geqq 0$

(b) $\det(\Gamma)=g^2n^2$, *and*

(c) $mg^2\equiv g^2 \bmod 2$.

The only possibilities for the integers $g^2>0$, $s\geqq 0$, and m allowed by these equations are

(—) $m=0$, $s=9$, g^2 *even*;

(i) $m=\pm 1$, $s\leqq 8$, $g^2=9-s$;

(ii) $m=\pm 2$, $s=1$, $g^2=2$; *and*

(iii) $m=\pm 3$, $s=0$, $g^2=1$.

Proof. (a) $m^2g^2=K_{\tilde{X}}^2=\tilde{c}_1^2=9-s$, (b) $\det(\Gamma)=g^2n^2$, and (c) $mg^2=K_{\tilde{X}}\cdot\pi^*L_0=-\tilde{c}_1\cdot\pi^*g\equiv g^2 \bmod 2$ follow respectively from parts (a), (b) and (c) of Lemma 3. Verifying the list (—)-(iii) of integral solutions is trivial.

It is now easy to establish the main results of this section, which extend and augment the ideas of [5] and [4].

7. Theorem (main theorem on cohomology $\boldsymbol{CP}^2$'s).

Let X be a singular 4-manifold whose integral cohomology ring is isomorphic to that of the complex projective plane. Then

(1) *Each singular point x of X has perfect local fundamental group* $\pi_1(\partial U_x)$.

(2) *If X supports a normal complex structure $\mathcal{O}_X$ admitting a non-trivial holomorphic line bundle, then $(X, \mathcal{O}_X)$ is a projective algebraic veriety and each singular point is Gorenstein.*

(3) *(Theorem A of the introduction.) If X supports any complex structure $\mathcal{O}_X$ with vanishing geometric genus $p_g = \dim_C H^2(X, \mathcal{O}_X)$, then either X is non-singular, or else X has exactly one singular point x and the germ of x in X is homeomorphic to that of the rational double point E_8.*

Proof. (1) is immediate from Lemma 2. Namely, by hypothesis $H^3(X) = \mathrm{Tors}(H^2(X)) = 0$ and $g^2 = 1$ for g a generator of $H^2(X)$, so

$$\prod_{x \in S(X)} |H_1(\partial U_x)| = g^2 \frac{|H^3(X)|^2}{|\mathrm{Tors}(H^2(X))|^2} = 1 .$$

Thus $\forall x$ $\pi_1(\partial U_x)/(\text{commutor subgroup}) = H_1(\partial U_x) = 0$, and $\pi_1(\partial U_x)$ is perfect.

(2) follows from Lemma 3 and Corollary 4, for the sequence $0 \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \boldsymbol{Z}) \cong \boldsymbol{Z}$, together with $g^2 \neq 0$, shows that a non-trivial line bundle L on X necessarily satisfies $L^2 \neq 0$.

For (3), let $\pi' : X' \to X$ be the normalization of X. By the definition of singular 4-manifold, $X - S(X)$ is locally connected, hence X is locally irreducible at each singular point, so π' is a homeomorphism ([14], Prop. 3.3, [12], Theorem 19, page 116). Also, $p_g(X') \leqq p_g(X)$, so $p_g(X) = 0 \Rightarrow p_g(X') = 0$, which in turn implies the existence of a non-trivial line bundle on X' by the sequence $H^1(X', \mathcal{O}_{X'}^*) \to H^2(X', \boldsymbol{Z}) \to 0$. Thus X' is Gorenstein by Corollary 4, while the relation $\sum (R^1 \pi_* \mathcal{O}_{\tilde{X}})_x = p_g(X') = 0$ of Lemma 3 shows that each singularity is rational. Since among rational singularities only double points are Gorenstein ([15]), and since among rational double points only E_8 has perfect local fundamental group, each analytic singularity of X' is an E_8. But then the number s of exceptional curves in the minimal resolution is a multiple of 8. Since $s \leqq 9$ by Corollary 6, there are only two possibilities: either $s = 0$ and X' is non-singular or $s = 8$ and X' has one point of type E_8 and no other singularities. Since the topological singularities of X are exactly the analytic singularities of X', this completes the proof.

8. Theorem (main theorem on homology $\boldsymbol{CP}^2$'s).

Let X be a singular 4-manifold of the integral homology type of $\boldsymbol{CP}^2$, with $H^2(X)$ generated, say, by g. Suppose that X supports a normal complex structure $\mathcal{O}_X$ with vanishing geometric genus. Then

(1) *If $g^2 = 1$ we are in the situation of the preceeding theorem and either*
  (a) *X is non-singular (the singularities are of type "E_0") or*
  (b) *X has one singular point, a rational double point of type E_8.*

(2) *If $g^2 = 2$ then X has precisely one singular point, either*
  (a) *a rational double point of type A_1 ("E_1"), or*
  (b) *a rational double point of type E_7.*

(3) *If $g^2 \neq 1, 2$, and if $\mathcal{O}_X$ is Gorenstein, then $3 \leqq g^2 \leqq 6$ and X has singularity type "E_k" for $k = 9 - g^2$. That is, if $g^2 = 3, 4,$ or 5, then X has one singular point*

of type "E_k" and no other singularities, while if $g^2=6$, X has two singular points, one of type A_2 and one of type A_1.

Proof. In each case (by Corollary 4 if $g^2=1$ or 2 and by assumption otherwise) X is Gorenstein. As above, then, $p_g(X)=0$ implies that each singularity of X is a rational double point. Also, since $\text{Tors}(H^2(X))=0$ the integer m of Corollary 9 cannot be 0, lest the canonical sheaf $\Omega_X=\mathcal{O}_X(L_0^{\otimes m})$ be trivial, contradicting $p_g=0$ via Serre duality. Putting $n=1$ in Corollary 6 shows that the only possibilities are

(i) $m=\pm 1$, $\det(\Gamma)=g^2=9-s$
(ii) $m=\pm 2$, $g^2=2$, $s=1$, and
(iii) $m=\pm 3$, $g^2=1$, $s=0$.

(iii) and (ii) are respectively cases (1) (a) and (2) (a) above, while Lemma 5 shows that (i)$\Rightarrow$(1) (b), (2) (b), or (3), This completes the proof.

Theorem B of the introduction is an immediate corollary, for if the anti-canonical bundle $K_X^*=L_0^{\otimes(-m)}$ admits a section, then $m<0$ and K_X is negative. Thus $p_g(X)=0$ and X is rational ([5]). If $m=-1$ (so that K_X generates $H^1(X, \mathcal{O}_X^*)$) we are in the case where the singularities are of type "E_s", $3\leqq s\leqq 8$ (cf. [8] and the last sentence of [5]). If $m=-2$ then X is biholomorphic to the singular complex quadric hypersurface $\boldsymbol{Q}_0^2\subset \boldsymbol{CP}^3$, while if $m=-3$ (the non-singular case) X is biholomorphic to $\boldsymbol{CP}^2$ ([5]).

II. Global constructions.

The question of actually producing complex surfaces of the type under discussion, and with specified singular points, was undertaken in [5] (see also [8] and [7]). Somewhat restated, the main result of [5] is as follows:

9. Theorem (main theorem on surfaces of the rational cohomology type of $\boldsymbol{CP}^2$).

Let $X\neq \boldsymbol{CP}^2$ be a compact two (-complex-) dimensional complex space each of whose singularities is a rational double point. Suppose that $H^*(X, \boldsymbol{Q})\cong H^*(\boldsymbol{CP}^2, \boldsymbol{Q})$ as rings, with $H^2(Y, \boldsymbol{Z})$ torsion free and generated by an analytic cocycle (that is, by an effective Cartier divisor). Then X is a rational projective variety derived from $\boldsymbol{CP}^2$ by blowing up 8 points (including infinitely near points), then blowing down 8 non-singular rational curves, each with self-intersection -2. X is homotopy equivalent to $\boldsymbol{CP}^2\Leftrightarrow H^3(X, \boldsymbol{Z})=0$.

In [1] and [2], Daniel Drucker, Geert C. E. Prins, and the present writers derived the complete list of all such constructions. The singularities that occur are E_8, E_7+A_1, E_6+A_2, D_5+A_3, $2A_4$, $A_5+A_2+A_1$, A_7+A_1, D_8, A_8, D_6+2A_1, $2D_4$, $2A_3+2A_1$, and $4A_2$ (cf. DuVal [11]). Each of the spaces X so constructed is regular with vanishing geometric genus, with $H^1(X, \boldsymbol{Z})=0$, and with $H^2(X, \boldsymbol{Z})\cong \boldsymbol{Z}$

and generated by an effective anti-canonical divisor K^*. References [8] and [2] also give similar constructions for surfaces X of the rational *homology* type of $\boldsymbol{CP}^2$, but with $g^2>1$, for g a generator of $H^2(X, \boldsymbol{Z})$. These spaces are examples of "degenerate Del Pezzo surfaces", which arise in connection with the study of 3-dimensional rational singularities and simple elliptic singularities, and whose singular points have been investigated in such works as [11], [6], [18], [16], and [19]. In [5], however, it was assumed in error that these surfaces all have $H^3(X, \boldsymbol{Z})=0$ and thus are homotopy equivalent to $\boldsymbol{CP}^2$. But Lemma 3 above shows that in fact $|H^3(X, \boldsymbol{Z})|=\det(\Gamma)$, for Γ the associated Dynkin diagram. Of the three examples of [5], then, only the second has the correct homotopy type; examples 1 and 3 are surfaces of the rational cohomology type of $\boldsymbol{CP}^2$ but with third integral cohomology groups isomorphic respectively to $\boldsymbol{Z}_2$ and to $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$.

As an application of these results we insert here a theorem on singular 4-manifolds whose singularities are cones on lens spaces—a topic of some independent interest. Since the rational double point A_k is topologically the cone on a lens space of type $(k+1, k)$, those examples of [2] which have only A_k singularities are spaces of this type.

10. Theorem (Theorem D of the introduction). *Let $r, k_1, \cdots, k_r$ be positive integers, with $r\leqq 4$. Then the relation*

$$(*)\qquad 4\prod_{i=1}^{r}(k_i+1)=(9-\sum_{i=1}^{r}k_i)N^2$$

for some integer N, is necessary and sufficient for the existence of a normal rational complex surface X of the rational homology type of $\boldsymbol{CP}^2$ possessing exactly r singular points $x_1, \cdots, x_r$, with x_i the cone on a lens space of type (k_i+1, k_i). Such a space X has the rational cohomology type of $\boldsymbol{CP}^2 \Leftrightarrow \sum_{i=1}^{r}k_i=8$.

Proof. Suppose that X exists. Then the singular point x_i is a rational double point of type A_{k_i}, for A_{k_i} is the only normal complex structure on a lens space of type (k_i+1, k_i) ([9]). Since X is rational with only rational double points as singularities, the plurigenera $P_l(X)=\dim H^0(X, \mathcal{O}_X(K_X^{\otimes l}))$ identically vanish for $l>0$, where $K_X=L_0^{\otimes m}\otimes T_0$ is the "canonical bundle" as in Corollary 6. (In fact, $T_0=0$ since $H^2(X)$ is torsion free for normal rational surfaces.) Thus $m\neq 0$. Since X has at least one singular point, Corollary 6 shows that $m=-1$ or -2 and $m^2\det(\Gamma)=(9-s)n^2$, with notation as in that Corollary. Since $\det(\Gamma)=\prod_{i=1}^{r}(k_i+1)$ and $s=\sum_{i=1}^{r}k_i$, the equation (*) holds for $N=n$ or $N=2n$ according as to whether $m=-2$ or -1.

Conversely, an enumeration of all choices of unordered r-tuples $(k_1, \cdots, k_r)$, $r\leqq 4$, produces the following list of possibities for which the condition (*) is satisfied: (1), (2, 1), (4), (3, 1, 1), (5, 1), (2, 2, 2), (7), (5, 2), (3, 3, 1), (8), (7, 1), (5, 2, 1), (4, 4), (3, 3, 1, 1), and (2, 2, 2, 2). To complete the proof we must produce, for each $(k_1, \cdots, k_r)$ on the list, a rational surface of the rational homology type of $\boldsymbol{CP}^2$ with singularities $A_{k_1}+\cdots+A_{k_r}$. For $(k_1, \cdots, k)=(1)$, the singular quadric surface Q_0^2 provides an example. The remaining choices on the list all appear among the examples of [11], or of [2].

Finally, for rational surfaces of this type it is easy to show that $m=2$ only for $X=Q_0^2$. In all other cases $g^2=9-\sum_{i=1}^{r} k_i$ for g a generator of $H^2(X)$, so X is a rational cohomology $CP^2 \Leftrightarrow \sum_{i=1}^{r} k_i=8$, as claimed.

So far all of the spaces considered in this section have been rational. To complete the program we must consider the non-rational case. This turns out to be surprisingly easy:

11. Theorem (main theorem on surfaces of the rational homology type of CP^2, Theorem C of the introduction). *Let $(X, \mathcal{O}_X)$ be a normal complex surface with only rational double points as singularities and with $H^i(X, \boldsymbol{Q}) \cong H^i(\boldsymbol{CP}^2, \boldsymbol{Q})$ $\forall i$. Then either*

(a) *$X=\boldsymbol{CP}^2$ or $\boldsymbol{Q}_0^2$,*

(b) *X is a rational surface obtained from $\boldsymbol{CP}^2$ by blowing up some number $s \leqq 8$ points, then blowing down s curves, as in theorem 9, or*

(c) *X is derived from a minimal non-singular Enriques surface $\tilde{X}$, or from a minimal non-singular projective surface $\tilde{X}$ of general type with $q(\tilde{X})=p_g(\tilde{X})=0$, by blowing down $s=b_2(\tilde{X})-1$ non-singular rational curves, each with self-intersection -2.*

Proof. Put $K_X=L_0^{\otimes m}\otimes T_0$ as above, for L_0 a positive generator of $H^1(X, \mathcal{O}_X^*)$ mod torsion and for T_0 a torsion bundle. If $m<0$ then K_X is negative and either (1) or (2) obtains by [5], Theorem 1. Otherwise, let $\pi: \tilde{X} \to X$ be the minimal resolution of singularities. Then $K_{\tilde{X}}=\pi^*(K_X)=\pi^*(L_0^{\otimes m}\otimes T_0)$. If $m=0$, $K_{\tilde{X}}$ is torsion. Since $q(\tilde{X})=p_g(\tilde{X})=0$ by Lemma 3, the classification of non-singular surfaces ([13]) shows that $\tilde{X}$ is a projective Enriques surface (i.e., that $K_{\tilde{X}} \neq 0$ but $K_{\tilde{X}}^{\otimes 2}=0$). If $m>0$, then (since positivity of line bundles on an algebraic surface is a topological property) K_X is ample and $P_l(K_{\tilde{X}})=P_l(K_X)\sim l^2$ for large l. Hence $\tilde{X}$ is of general type.

To show that $\tilde{X}$ is minimal, assume contrariwise that $\tilde{X}$ admits an exceptional curve D of the first kind. Then by adjunction,

$$(*) \qquad -1=K_{\tilde{X}}\cdot D=m(\pi^*L_0)\cdot D=mL_0\cdot\pi_*D\,.$$

Since $L_0\cdot\gamma$ is positive for any effective 2-cycle γ, (*) is an absurdity if $m\neq -1$. Thus in particular $\tilde{X}$ is minimal if $\tilde{X}$ is an Enriques surface or of general type. Since π is the minimal resolution of rational double points, each curve C_i blown down by π is non-singular rational with $C_i^2=-2$, and by Lemma 3 there are $s=b^-(\tilde{X})=b_2(\tilde{X})-1$ of them. This completes the proof.

The results of this paper can conveniently be summarized by the following chart. The notation is as in Corollary 6.

Complex surfaces of the rational homology type of $\boldsymbol{CP}^2$ with only rational double points as singularities

m	r	s	g^2	$\det(\Gamma)$	$\tilde{X}$
-3	0 (X is non-singular)	0	1	1 ($\Gamma=\emptyset$)	$X=\tilde{X}=\boldsymbol{CP}^2$
-2	1	1	2	2 ($\Gamma=A_1$)	$X=\boldsymbol{Q}_0^2$ $\tilde{X}$=the rational ruled surface S_2.
-1	$\leqq 4$	$3\leqq s\leqq 8$	$9-s$	$(9-s)\lvert H^3(X)\rvert^2$ ($s=3$, 4, or $5\Rightarrow H^3(X, \boldsymbol{Z})=0$. $H^3(X, \boldsymbol{Z})=0\Rightarrow \Gamma=$"$E_s$".)	Rational Del Pezzo surface of degree $9-s$
0	$1\leqq r\leqq 9$	9	even, >0	g^2n^2	Minimal Enriques surface ($K\neq 0$ *but* $K^{\otimes 2}=0$)
1	$1\leqq r\leqq 8$	$3\leqq s\leqq 8$	$9-s$	$(9-s)n^2$ ($s=3$, 4, or $5\Rightarrow n=1\Rightarrow\Gamma=$"$E_s$")	Minimal of General Type
2	1	1	2	2 ($\Gamma=A_1$, $n=1$)	
3	0 (X is non-singular)	0	1	1 ($\Gamma=\emptyset$, $n=1$)	

X has the rational *cohomology* type of $\boldsymbol{CP}^2\Leftrightarrow X$ is non-singular or $m=\pm 1$ and $s=8$.

X has the *integral* homology type of $\boldsymbol{CP}^2\Rightarrow m\neq 0$, $\det(\Gamma)=\dfrac{9-s}{m^2}$, and $\Gamma=$"E_s".

X has the *integral cohomology* type of $\boldsymbol{CP}^2\Rightarrow X$ is non-singular, or $m=\pm 1$ and $\Gamma=E_8$.

If $H^2(X)$ is torsion free, then the last two implications are equivalences. In particular, this is the case if X is rational. X is rational$\Leftrightarrow m<0\Leftrightarrow H^2(X)$ is torsion free and generated by an effective divisor. $m<0$, $n=1$ and $s=8\Rightarrow X$ is homotopy equivalent to $\boldsymbol{CP}^2$. The assertion $r\leqq 4$ for $m=-1$ is shown in [19] and in [2].

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