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Variations of meromorphic differentials under quasiconformal deformations

Dedicated to Professor M. Ozawa on his sixtieth birthday

By

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Introduction.

The motivations of this paper come from our previous study (Kusunoki-Maitani [7]) and the recent results due to I. Guerrero [5] and H. Yamaguchi [14]. In [7] we gave the first variational formulas for fundamental meromorphic differentials on open Riemann surfaces induced by quasiconformal deformations, where those differentials should have the boundary behavior subject to (complex) behavior spaces and the "first" above suggests the first derivatives. While, Guerrero [5] discussed the firft variational formula of Green's functions on finite Riemann surfaces by using the quasiconformal mappings and Fuchsian groups, and he asked its generalization to arbitrary hyperbolic Riemann surfaces. And Yamaguchi [14] showed the second variational formulas for Robin's constants and some other quantities under variational consideration for a certain analytic family of Stein manifolds.

In this paper we shall study the variational formulas of various differentials under quasiconformal deformations of arbitrary open Riemann surfaces, and give an answer to Guerrero's question and also show the second variational formulas for various meromorphic differentials under quasiconformal deformations. Practically we develop our previous method by using the (real) behavior spaces of Shiba's type and obtain the similar formulas for wider classes of meromorphic differentials than those in [9], which are applicable for Green's functions, Neumann's functions and so forth. We also show a certain differentiability property of their meromorphic differentials, which allows us to establish the second variational formulas for those differentials under quasiconformal deformations. If we take a specific kind of behavior space, we can obtain the second variational formulas for Green's functions, Robin's constants and some others, which have the similar forms as those due to Yamaguchi.

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§ 1. Quasiconformal deformation and spaces of differentials.

1. We shall investigate the deformation of a Riemann surface R as follows. Consider Beltrami differentials $\mu(z, t)\frac{d\bar{z}}{dz}$ on R with a complex parameter t (t may be the set of parameters $(t_1, \cdots, t_n)$) varying in a domain about 0. We shall assume the following condition A;

1. $\mu(z, t)$ is measurable, $\mu(z, 0)\equiv 0$ and
$$\|\mu(\cdot, t)\|_\infty = \text{ess sup}|\mu(z, t)| < 1,$$
2. for every t there exists a constant M_t such that
$$\|\mu(z, t+h)-\mu(z, t)\|_\infty \leqq |h| M_t$$
for sufficiently small h,
3. for almost all $z\in R$, $t\to\mu(z, t)$ is holomorphic.

For each t, denote by R^t ($R^0=R$) the Riemann surface which has basic surface R and the conformal structure induced by $\mu(z, t)$. Let f_t be the quasiconformal homeomorphism from R to R^t with Beltrami coefficient $\mu(z, t)$. We express f_t sometimes as $\zeta=f_t(z)$ in terms of respective generic local parameter z and ζ of R and R^t. Then $\mu(z, t)=\zeta_{\bar{z}}/\zeta_z$.

In the present paper, such a family $\{R^t\}$ is treated as a deformation of R. Now $f_t: R\to R^t$ defines the homeomorphism of differentials as follows; for any first order differential $\lambda=a dz+b d\bar{z}$ on R, we denote by $f_t^\#(\lambda)$ the pull back $\lambda\circ f_t^{-1}$, that is,

$$f_t^\#(\lambda)=[(a\circ f_t^{-1})z_\zeta+(b\circ f_t^{-1})(\bar{z})_\zeta]d\zeta+[(a\circ f_t^{-1})z_{\bar{\zeta}}+(b\circ f_t^{-1})(\bar{z})_{\bar{\zeta}}]d\bar{\zeta},$$

where the derivatives are taken in the sense of distribution. Note that $(f_t^{-1})^\#$ and $(f_{t'}\circ f_t^{-1})^\#$ are defined similarly and that $(f_{t'}\circ f_t^{-1})^\#=(f_{t'})^\#\circ(f_t^{-1})^\#$ and $f_t^\#\circ(f_t^{-1})^\#$ is an identity mapping. The $f_t^\#$ will induce a deformation of spaces of differentials.

Let $\tilde{\Lambda}=\tilde{\Lambda}(R)$ be the Hilbert space of square integrable complex differentials whose inner product is given by

$$(\lambda_1, \lambda_2)_R=\iint_R \lambda_1\wedge *\overline{\lambda_2}=i\iint_R (a_1\bar{a}_2+b_1\bar{b}_2)dz d\bar{z},$$

where $\lambda_i=a_i dz+b_i d\bar{z}\in\tilde{\Lambda}(R)$, $i=1, 2$ and $*\lambda_2=-ia_2dz+ib_2d\bar{z}$ is the conjugate differential of λ_2, z being a local parameter. We regard the same set $\tilde{\Lambda}(R)$ as a Hilbert space over the real number field with another inner product

$$\langle\lambda_1, \lambda_2\rangle=\text{Re}(\lambda_1, \lambda_2),$$

where Re means the real part (cf. Shiba [13]). Hereafter we use this space and write it $\Lambda=\Lambda(R)$. The following subspaces of Λ will be used:

$\Lambda_c=\Lambda_c(R)=\{\lambda\in\Lambda;\ \lambda \text{ is a closed differential}\}$,
$\Lambda_h=\Lambda_h(R)=\{\lambda\in\Lambda;\ \lambda \text{ is a harmonic differential}\}$,
$\Lambda_{eo}=\Lambda_{eo}(R)=\{\lambda\in\Lambda_c;\ \langle\lambda, \omega\rangle=0 \text{ for any } \omega\in\Lambda_h\}$.

2. We return to the mapping $f_t^{\#}$. The $f_t^{\#}$ gives an isomorphism from $\Lambda(R)$ to $\Lambda(R^t)$, because

$$\|f_t^{\#}(\lambda)\|_{R^t}^2 \leqq \frac{1+k}{1-k}\|\lambda\|_R^2 \qquad \text{for any} \quad \lambda\in\Lambda(R),$$

where $\|\mu(\cdot,\, t)\|_\infty \leqq k<1$. Further, for any Dirichlet potential W_0 on R^t the composite $W_0\circ f_t$ is also a Dirichlet potential (cf. [4]), henceforce $f_t^{\#}$ gives an isomorphism from $\Lambda_{eo}(R)$ to $\Lambda_{eo}(R^t)$. Let P_h denote the projection from Λ to Λ_h and $(f_t)_h^{\#}$ the composite mapping $P_h\circ f_t^{\#}$ from $\Lambda(R)$ to $\Lambda_h(R^t)$. Then, clearly $(f_t^{-1})_h^{\#}=\{(f_t)_h^{\#}\}^{-1}$ on $\Lambda_h(R^t)$ and $(f_t)_h^{\#}$ gives an isomorphism from $\Lambda_h(R)$ to $\Lambda_h(R^t)$. The following lemma shows a correspondence by $f_t^{\#}$ between the inner products in $\tilde{\Lambda}(R)$ ($\Lambda(R)$) and $\tilde{\Lambda}(R^t)$ ($\Lambda(R^t)$).

Lemma 1. (see, [7], [9])

$$(f_t^{\#}(\omega_1),\, -{}^*f_t^{\#}({}^*\omega_2))_{R^t}=(\omega_1,\, \omega_2)_R \qquad \textit{for any} \quad \omega_1,\, \omega_2\in\Lambda(R)\,,$$

$$((f_t)_h^{\#}(\sigma_1),\, -{}^*(f_t)_h^{\#}({}^*\sigma_2))_{R^t}=(\sigma_1,\, \sigma_2)_R \qquad \textit{for any} \quad \sigma_1,\, \sigma_2\in\Lambda_h(R)\,.$$

We know $f_t^{\#}(\Lambda_c(R))=\Lambda_c(R^t)$, because by Lemma 1

$$\langle f_t^{\#}(\sigma),\, {}^*\omega\rangle_{R^t}=\langle\sigma,\, {}^*(f_t^{-1})^{\#}(\omega)\rangle_R=0$$

for $\sigma\in\Lambda_c(R)$, $\omega\in\Lambda_{eo}(R^t)$. Further we have the following.

Lemma 2. (cf. [10], [11]) *The $f_t^{\#}$ and $(f_t)_h^{\#}$ preserve the periods of closed differentials.*

Proof. Let $\gamma\subset R$ be a Jordan closed curve and V_γ be a ring domain such that γ is a component of the boundary ∂V_γ and is oriented so that V_γ is seen on the left hand of γ. Take a C^∞-function S_γ on $R-\gamma$ such that the support of S_γ is in V_γ, $S_\gamma=1$ on a neighbourhood of γ in V_γ. Note that $(\omega,\, {}^*\mathrm{d}S_\gamma)=\int_\gamma\omega$ for C^1-closed differential ω and C^1-curve γ. Similarly, take a $S_{f_t(\gamma)}$ on R_t for the Jordan curve $f_t(\gamma)$. Then $S_{f_t(\gamma)}\circ f_t-S_\gamma$ is a Dirichlet potential on R, hence $\mathrm{d}(S_{f_t(\gamma)}-S_\gamma\circ f_t^{-1})\in\Lambda_{eo}(R^t)$. Thus for a closed differential ω on R,

$$\begin{aligned}(f_t^{\#}(\omega),\, {}^*\mathrm{d}S_{f_t(\gamma)})_{R^t}&=(f_t^{\#}(\omega),\, {}^*\mathrm{d}(S_{f_t(\gamma)}-S_\gamma\circ f_t^{-1})+{}^*\mathrm{d}(S_\gamma\circ f_t^{-1}))_{R^t}\\ &=(f_t^{\#}(\omega),\, {}^*f_t^{\#}(\mathrm{d}S_\gamma))_{R^t}=(\omega,\, {}^*\mathrm{d}S_\gamma)_R\,.\end{aligned}$$

Symbolically we can write as

$$\int_{f_t(\gamma)} f_t^{\#}(\omega)=\int_\gamma\omega\,.$$

A closed differential ω is said to be exact (semiexact) if it has a vanishing period along every cycle (dividing cycle), i.e.,

$$(\omega,\, {}^*\mathrm{d}S_\gamma)=0 \quad \text{for any cycle } \gamma \text{ (dividing cycle).}$$

Let Λ_e, Λ_{se}, Λ_{he} and Λ_{hse} be the spaces of exact, semiexact, harmonic exact and harmonic semiexact differentials respectively. By Lemma 2, these spaces are preserved by $f_t^\sharp$ or $(f_t)_h^\sharp$. As was already shown, the locally exactness is preserved by $f_t^\sharp$. On the other hand $f_t^\sharp(^*\omega)$ does not always coincide with $^*f_t^\sharp(\omega)$. This comes from the difference of conformal structures. We shall observe the distortion by $f_t^\sharp$.

Lemma 3.

$$\|f_t^\sharp(^*\omega)-{}^*f_t^\sharp(\omega)\|\leqq\frac{2k}{\sqrt{1-k^2}}\|\omega\| \qquad for\ any\quad \omega\in\Lambda,$$

where $|\mu(\cdot, t)|\leqq k<1$. *The equality holds if and only if* $|\mu(\cdot, t)|=k$ *almost everywhere on the support of* ω.

Proof. For any $\omega=a\mathrm{d}z+b\mathrm{d}\bar{z}\in\Lambda$, we have

$$\|f_t^\sharp(^*\omega)-{}^*f_t^\sharp(\omega)\|_{R^t}^2=4i\iint_{R^t}(|a|^2+|b|^2)|z_{\bar\zeta}|^2\mathrm{d}\zeta\mathrm{d}\bar\zeta$$

$$=4i\iint_R(|a|^2+|b|^2)\frac{|z_{\bar\zeta}|^2}{|z_\zeta|^2-|z_{\bar\zeta}|^2}\mathrm{d}z\mathrm{d}\bar{z}$$

$$\leqq 4i\frac{k^2}{1-k^2}\iint_R(|a|^2+|b|^2)\mathrm{d}z\mathrm{d}\bar{z}=4\frac{k^2}{1-k^2}\|\omega\|^2.$$

The equality holds if and only if $|z_{\bar\zeta}/z_\zeta|=k$ on the support of ω.

Proposition 1.

(i) $|(f_t^\sharp(\omega_1), f_t^\sharp(\omega_2))_{R^t}-(\omega_1, \omega_2)_R|\leqq\dfrac{2k}{1-k}\|\omega_1\|\|\omega_2\|$ *for* $\omega_1, \omega_2\in\Lambda(R)$,

(ii) $|((f_t)_h^\sharp(\sigma), f_t^\sharp(\omega))_{R^t}-(\sigma, \omega)_R|\leqq\dfrac{2k}{1-k}\|\sigma\|\|\omega\|$ *for* $\sigma\in\Lambda_h(R)$, $\omega\in\Lambda(R)$,

where $|\mu(\cdot, t)\|\leqq k$.

Proof. (i) By Lemma 1,

$$|(f_t^\sharp(\omega_1), f_t^\sharp(\omega_2))_{R^t}-(\omega_1, \omega_2)_R|=|(f_t^\sharp(\omega_1), {}^*(f_t^\sharp(^*\omega_2)-{}^*f_t^\sharp(\omega_2)))_{R^t}|$$

$$\leqq\|f_t^\sharp(\omega_1)\|\|f_t^\sharp(^*\omega_2)-{}^*f_t^\sharp(\omega_2)\|$$

$$\leqq\frac{2k}{1-k}\|\omega_1\|\|\omega_2\|.$$

(ii) Take the orthogonal decomposition of ω;

$$\omega=\omega_1+\omega_2+{}^*\omega_3, \qquad \omega_1\in\Lambda_h,\ \omega_2,\ \omega_3\in\Lambda_{eo}.$$

Then we have

$$|((f_t)_h^\sharp(\sigma), f_t^\sharp(\omega))_{R^t}-(\sigma, \omega)_R|=|((f_t)_h^\sharp(\sigma), (f_t)_h^\sharp(\omega_1+{}^*\omega_3)+{}^*(f_t)_h^\sharp(^*(\omega_1+{}^*\omega_3)))|$$

$$=\|(f_t)_h^\sharp(\sigma)\|\|(f_t)_h^\sharp(\omega_1+{}^*\omega_3)+{}^*(f_t)_h^\sharp(^*(\omega_1+{}^*\omega_3))\|$$

$$\leqq\frac{2k}{1-k}\|\sigma\|\|\omega\|.$$

3. We investigate the variations of Green's function, reproducing differentials and some other fundamental differentials under our deformations. For the purpose of systematic investigation of these differentials, we introduce behavior spaces.

Let $\Gamma=\Gamma(R)$ be the subspace of $\Lambda(R)$ which consists of real differentials and $\Gamma_h=\Gamma_h(R)=\Gamma(R)\cap\Lambda_h(R)$. For any subspace $\Gamma_x(R)$ of $\Gamma_h(R)$, we set

$$\Lambda_x(R)=\Gamma_x(R)+i^*\Gamma_x(R)^{\perp},$$

where $^*\Gamma_x(R)^{\perp}=\{^*\lambda\in\Gamma_h\,;\,\langle\lambda,\omega\rangle=0 \text{ for any } \omega\in\Gamma_x\}$, $i=\sqrt{-1}$. We call such a space Λ_x a behavior space (cf. [7], [13]). Clearly $\Lambda_x(R)$ is a subspace of $\Lambda_h(R)$ and $i^*\Lambda_x(R)$ is the orthogonal complement of $\Lambda_x(R)$ in $\Lambda_h(R)$, i.e., $\Lambda_h(R)=\Lambda_x(R)+i^*\Lambda_x(R)$. Now $(f_t)_h^\#[\Lambda_x(R)]$ is a subspace of $\Lambda_h(R^t)$ and is written as $\Lambda_x(R^t)$. Then we have the following.

Proposition 2.

$$(f_t)_h^\#[^*\Gamma_x(R)^{\perp}]={}^*((f_t)_h^\#[\Gamma_x(R)])^{\perp},$$

$$\Lambda_h(R^t)=\Lambda_x(R^t)\dot{+}i^*\Lambda_x(R^t)\,,$$

and $\Lambda_x(R^t)$ is a behavior space.

Proof. For an $\omega\in\Gamma_x(R)$ and a $\sigma\in\Gamma_x(R)^{\perp}$, by Lemma 1,

$$((f_t)_h^\#(\omega),\ -^*(f_t)_h^\#(^*\sigma))_{R^t}=(\omega,\ \sigma)_R=0\,.$$

Hence $^*(f_t)_h^\#[\Gamma_x(R)]$ is orthogonal to $(f_t)_h^\#[^*\Gamma_x(R)^{\perp}]$. If $\tau\in\Gamma_h(R^t)$ is orthogonal to $^*(f_t)_h^\#[\Gamma_x(R)]+(f_t)_h^\#[^*\Gamma_x(R)^{\perp}]$, then for an $\omega\in\Gamma_x(R)^{\perp}$ $0=((f_t)_h^\#(^*\omega),\tau)_{R^t}=(^*\omega,\ -^*(f_t^{-1})_h^\#(^*\tau))_R$. Therefore $(f_t^{-1})_h^\#(^*\tau)\in\Gamma_x(R)$ and $^*\tau\in(f_t)_h^\#[\Gamma_x(R)]$. Thus $\tau=0$ and the assertion follows.

Since $\Gamma_{hse}=\Lambda_{hse}\cap\Gamma_h$ and $\Gamma_{he}=\Lambda_{he}\cap\Gamma_h$ are preserved by $(f_t)_h^\#$, the spaces $\Gamma_{hm}={}^*\Gamma_{hse}{}^{\perp}$ and $\Gamma_{ho}={}^*\Gamma_{he}{}^{\perp}$ are also preserved by $(f_t)_h^\#$. Set $\Lambda_{-1}=\{0\}+i\Gamma_h$, $\Lambda_0=\Gamma_{he}+i\Gamma_{ho}$ and $\Lambda_1=\Gamma_{hm}+i\Gamma_{hse}$. Then $(f_t)_h^\#[\Lambda_i(R)]=\Lambda_i(R^t)$, $i=-1, 0, 1$. These are important behavior spaces which are related to fundamental functions and differentials on the surfaces. A canonical differential (a meromorphic differential whose real part is a distinguished differential) has Λ_1-behavior (cf. [13]). A meromorphic differential whose real part is a differential of difference of the Green's (resp. Neumann's) functions with different poles has Λ_{-1}-behavior (resp. Λ_0-behavior).

§2. Variational formulas of certain meromorphic differentials.

4. We shall show some variational formulas of specific kind of meromorphic differentials. We begin with showing the continuity property of certain meromorphic differentials.

Lemma 4. *A meromorphic differential ϕ^t on R^t satisfing the condition $(f_t^{-1})^*(\phi^t)-\phi^0\in\Lambda_x(R)+\Lambda_{eo}(R)$ is uniquely determined by ϕ^0.*

Proof. Let a meromorphic differential $\tilde{\phi}^t$ also satisfy the above condition. Then $(f_t^{-1})^{\#}(\phi^t-\tilde{\phi}^t)\in\Lambda_x(R)+\Lambda_{eo}(R)$ and $\phi^t-\tilde{\phi}^t\in\Lambda_x(R^t)+\Lambda_{eo}(R^t)$. Therefore analytic differential $\phi^t-\tilde{\phi}^t$ has no poles and belongs to $\Lambda_x(R^t)$. Thus $(\phi^t-\tilde{\phi}^t)=i^*(\phi^t-\tilde{\phi}^t)\in\Lambda_x(R^t)\cap i^*\Lambda_x(R^t)$ and $\phi^t=\tilde{\phi}^t$.

Proposition 3. *Let $\{\phi^t\}$ be meromorphic differentials such that $(f_t^{-1})^{\#}(\phi^t)-\phi^0\in\Lambda_x(R)+\Lambda_{eo}(R)$. If f_t is conformal in a neighbourhood V of the poles of ϕ^0, then*

$$\|(f_t^{-1})^{\#}(\phi^t)-\phi^0\|_R\leqq\frac{\sqrt{2}\,k(t)}{1-k(t)}\|\phi^0\|_{R-V},$$

where the Beltrami coefficient $\mu(\cdot,t)$ of f_t has absolute values less than $k(t)(<1)$.

Proof. Since $\Lambda_x+\Lambda_{eo}$ is orthogonal to $i^*(\Lambda_x+\Lambda_{eo})$, we have

$$\langle(f_t^{-1})^{\#}(\phi^t)-\phi^0,\ i^*((f_t^{-1})^{\#}(\phi^t)-\phi^0)\rangle=0.$$

Write

$$((f_t^{-1})^{\#}(\phi^t)+i^*(f_t^{-1})^{\#}(\phi_t))/2=\omega,$$

$$((f_t^{-1})^{\#}(\phi^t)-i^*(f_t^{-1})^{\#}(\phi^t))/2=\sigma.$$

Then $(f_t^{-1})^{\#}(\phi^t)-\phi^0=\omega-\phi^0+\sigma$. Note that $(\omega-\phi^0,\sigma)=0$ and $\sigma=\omega\mu\dfrac{d\bar{z}}{dz}$. It follows that

$$\|\omega-\phi^0\|=\|\sigma\|\leqq k(t)\|\omega\|_{R-V},$$

$$\|\omega\|_{R-V}\leqq\frac{1}{1-k(t)}\|\phi^0\|_{R-V},$$

and

$$\|\omega-\phi^0\|=\|\sigma\|\leqq\frac{k(t)}{1-k(t)}\|\phi^0\|_{R-V}.$$

Therefore

$$\|(f_t^{-1})^{\#}(\phi^t)-\phi^0\|_R^2=\|\omega-\phi^0\|^2+\|\sigma\|^2$$
$$\leqq2\Big(\frac{k(t)}{1-k(t)}\Big)^2\|\phi^0\|_{R-V}^2.$$

This proposition convince us of the smoothness of ϕ^t.

Theorem 1. *Let $\{\phi^t\}$ be meromorphic differentials such that $(f_t^{-1})^{\#}(\phi^t)-\phi^0\in\Lambda_x(R)+\Lambda_{eo}(R)$. Assume that the Beltrami coefficient $\mu(z,t)$ of f_t satisfies condition A and the support of $\mu(z,t)$ does not meet an open set V including poles of ϕ^0. Then for $t=u+iv$ there exist differentials ϕ_u^t and ϕ_v^t in $\Lambda_x(R^t)+\Lambda_{eo}(R^t)$ such that*

$$\lim_{\tilde{u}\to0}\left\|\frac{(f_t\circ f_{t+\tilde{u}}^{-1})^{\#}(\phi^{t+\tilde{u}})-\phi^t}{\tilde{u}}-\phi_u^t\right\|_{R^t}=0,$$

$$\lim_{\tilde{v}\to0}\left\|\frac{(f_t\circ f_{t+i\tilde{v}}^{-1})^{\#}(\phi^{t+i\tilde{v}})-\phi^t}{\tilde{v}}-\phi_v^t\right\|_{R^t}=0,$$

where $\tilde{u}$ and $\tilde{v}$ are real. Further,

$$\phi_u^t - i^*\phi_u^t = -i(\phi_v^t - i^*\phi_v^t)$$

$$= \phi^t \frac{2}{1-|\mu(z(\zeta),\ t)|^2} \cdot \frac{\zeta_z}{\bar{\zeta}_{\bar{z}}} \cdot \frac{\partial}{\partial t}\mu(z(\zeta),\ t)\frac{d\bar{\zeta}}{d\zeta}.$$

Proof. First note that the Beltrami differential $\nu(\zeta,\ \tau)\frac{d\bar{\zeta}}{d\zeta}$ on R^t of the quasiconformal homeomorphism $f_{t+\tau}\circ f_t^{-1}$ (from R^t to $R^{t+\tau}$) is

$$\nu(\zeta,\ \tau)\frac{d\bar{\zeta}}{d\zeta} = \frac{\mu(z,\ t+\tau)-\mu(z,\ t)}{1-\overline{\mu(z,\ t)}\mu(z,\ t+\tau)} \cdot \frac{\zeta_z}{\bar{\zeta}_{\bar{z}}} \cdot \frac{d\bar{\zeta}}{d\zeta}$$

and satisfies the similar condition as in A. Since

$$(f_{t+\tilde{u}}^{-1})^\#(\phi^{t+\tilde{u}}) - (f_t^{-1})^\#(\phi^t) \in \Lambda_x(R) + \Lambda_{eo}(R),$$

we have

$$((f_t\circ f_{t+\tilde{u}}^{-1})^\#(\phi^{t+\tilde{u}}) - \phi^t)/\tilde{u} \in \Lambda_x(R^t) + \Lambda_{eo}(R^t).$$

We show that $((f_t\circ f_{t+\tilde{u}}^{-1})^\#(\phi^{t+\tilde{u}}) - \phi^t)/\tilde{u}$ converges in $\Lambda_x(R^t)+\Lambda_{eo}(R^t)$ as $\tilde{u}$ tends to 0. Write

$$\omega_{\tilde{u}} = \{(f_t\circ f_{t+\tilde{u}}^{-1})^\#(\phi^{t+\tilde{u}}) + i^*(f_t\circ f_{t+\tilde{u}}^{-1})^\#(\phi^{t+\tilde{u}})\}/2,$$

$$\sigma_{\tilde{u}} = \{(f_t\circ f_{t+\tilde{u}}^{-1})^\#(\phi^{t+\tilde{u}}) - i^*(f_t\circ f_{t+\tilde{u}}^{-1})^\#(\phi^{t+\tilde{u}})\}/2,$$

then $\omega_{\tilde{u}}+\sigma_{\tilde{u}}-\phi^t \in \Lambda_x(R^t)+\Lambda_{eo}(R^t)$. For real $\tilde{u}$ and $\tilde{v}$, we have

$$0 = \langle(\omega_{\tilde{u}}+\sigma_{\tilde{u}}-\phi^t)/\tilde{u} - (\omega_{\tilde{v}}+\sigma_{\tilde{v}}-\phi^t)/\tilde{v},\ i^*((\omega_{\tilde{u}}+\sigma_{\tilde{u}}-\phi^t)/\tilde{u} - (\omega_{\tilde{v}}+\sigma_{\tilde{v}}-\phi^t)/\tilde{v})\rangle.$$

By the same way as the proof of Proposition 3,

$$\|(\omega_{\tilde{u}}-\phi^t)/\tilde{u} - (\omega_{\tilde{v}}-\phi^t)/\tilde{v}\|_{R^t} = \|\sigma_{\tilde{u}}/\tilde{u} - \sigma_{\tilde{v}}/\tilde{v}\|_{R^t}$$

$$\leqq \left\|((\omega_{\tilde{u}}-\phi^t)/\tilde{u} - (\omega_{\tilde{v}}-\phi^t)/\tilde{v})\nu(\zeta,\ \tilde{v})\frac{d\bar{\zeta}}{d\zeta}\right\|_{R^t}$$

$$+ \left\|(\omega_{\tilde{u}}-\phi^t)(\nu(\zeta,\ \tilde{u})-\nu(\zeta,\ \tilde{v}))/\tilde{u}\frac{d\bar{\zeta}}{d\zeta}\right\|_{R^t}$$

$$+ \left\|\phi^t(\nu(\zeta,\ \tilde{u})/\tilde{u} - \nu(\zeta,\ \tilde{v})/\tilde{v})\frac{d\bar{\zeta}}{d\zeta}\right\|_{R^t}.$$

For the second term, from condition A,

$$|(\nu(\zeta,\ \tilde{u})-\nu(\zeta,\ \tilde{v}))/\tilde{u}| \leqq \left(1+\left|\frac{\tilde{v}}{\tilde{u}}\right|\right)\frac{M_t}{1-\|\mu(z,\ t)\|_\infty},$$

and by Proposition 3,

$$\|\omega_{\tilde{u}}-\phi^t\| \leqq \frac{k_t(\tilde{u})}{1-k_t(\tilde{u})}\|\phi^t\|_{R^t-V^t},$$

where $k_t(\tilde{u}) = \sup_{\zeta\in R^t}|\nu(\zeta,\ \tilde{u})| \leqq \frac{M_t|\tilde{u}|}{1-\|\mu(z,\ t)\|_\infty}$, $V^t = f_t(V)$ and

$$\|\phi^t\|_{R^t-V^t} \leqq \|\phi^t - f_t^\#(\phi^0)\|_{R^t-V^t} + \|f_t^\#(\phi^0)\|_{R^t-V^t}.$$

For the third term we have also an estimation. Let $\nu(\zeta,\ \tau)$ be holomorphic on

$|\tau|\leqq\varepsilon$. Then for $|\tilde{v}|$, $|\tilde{u}|\leqq\varepsilon/2$,

$$|\nu(\zeta,\ \tilde{u})/\tilde{u}-\nu(\zeta,\ \tilde{v})/\tilde{v}|=\left|\frac{1}{2\pi i}\int_{|\tau|=\varepsilon}\left(\frac{\nu(\zeta,\ \tau)}{\tau(\tau-\tilde{u})}-\frac{\nu(\zeta,\ \tau)}{\tau(\tau-\tilde{v})}\right)\mathrm{d}\tau\right|\quad(\nu(\zeta,\ 0)=0)$$

$$=\left|\frac{\tilde{u}-\tilde{v}}{2\pi i}\int_{|\tau|=\varepsilon}\frac{\nu(\zeta,\ \tau)}{\tau(\tau-\tilde{u})(\tau-\tilde{v})}\mathrm{d}\tau\right|$$

$$\leqq|\tilde{u}-\tilde{v}|\frac{4M_t}{(1-\|\mu(z,\ t)\|_\infty)\varepsilon}.$$

Thus, if $|\tilde{v}|\leqq|\tilde{u}|\leqq\varepsilon/2$,

$$\|(\omega_{\tilde{u}}+\sigma_{\tilde{u}}-\phi^t)/\tilde{u}-(\omega_{\tilde{v}}+\sigma_{\tilde{v}}-\phi^t)/\tilde{v}\|_{R^t}$$

$$=\sqrt{2}\,\|(\omega_{\tilde{u}}-\phi^t)/\tilde{u}-(\omega_{\tilde{v}}-\phi^t)/\tilde{v}\|_{R^t}$$

$$\leqq\frac{\sqrt{2}}{1-k_t(\tilde{v})}\cdot\frac{M_t}{1-\|\mu(z,\ t)\|_\infty}\left(\frac{2k_t(\tilde{u})}{1-k_t(\tilde{u})}+\frac{4}{\varepsilon}|\tilde{u}-\tilde{v}|\right)\|\phi^t\|_{R^t-V^t}.$$

This proves that $((f_t\circ f_{t+\tilde{u}}^{-1})^{\#}(\phi^{t+\tilde{u}})-\phi^t)/\tilde{u}$ converges to a differential ϕ_u^t in $\Lambda_x(R^t)+\Lambda_{eo}(R^t)$ as $\tilde{u}$ tends to 0. In the same way we can get a differential ϕ_v^t in $\Lambda_x(R^t)+\Lambda_{eo}(R^t)$ as

$$\lim_{\tilde{v}\to 0}((f_t\circ f_{t+i\tilde{v}}^{-1})^{\#}(\phi^{t+i\tilde{v}})-\phi^t)/\tilde{v}.$$

From our notations, $\phi_u^t=\lim_{\tilde{u}\to 0}(\omega_{\tilde{u}}+\sigma_{\tilde{u}}-\phi^t)/\tilde{u}$, hence

$$\phi_u^t-i{}^*\phi_u^t=2\lim_{\tilde{u}\to 0}\frac{\sigma_{\tilde{u}}}{\tilde{u}}$$

$$=2\lim_{\tilde{u}\to 0}\left(\frac{\omega_{\tilde{u}}-\phi^t}{\tilde{u}}\nu(\zeta,\ \tilde{u})\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta}+\phi^t\frac{\nu(\zeta,\ \tilde{u})}{\tilde{u}}\cdot\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta}\right).$$

By the way

$$\left\|\frac{\omega_{\tilde{u}}-\phi^t}{\tilde{u}}\nu(\zeta,\ \tilde{u})\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta}\right\|\leqq\frac{1}{|\tilde{u}|}\cdot\frac{k_t(\tilde{u})^2}{1-k_t(\tilde{u})}\|\phi^t\|_{R^t-V^t}$$

$$\leqq\frac{|\tilde{u}|}{1-k_t(\tilde{u})}\left(\frac{M_t}{1-\|\mu(z,\ t)\|_\infty}\right)^2\|\phi^t\|_{R^t-V^t}.$$

Therefore

$$\phi_u^t-i{}^*\phi_u^t=2\phi^t\frac{\partial}{\partial\tau}\nu(\zeta,\ \tau)|_{\tau=0}\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta}$$

$$=2\phi^t\frac{1}{1-|\mu(z,\ t)|^2}\cdot\frac{\zeta_z}{\bar{\zeta}_{\bar{z}}}\cdot\frac{\partial}{\partial t}\mu(z,\ t)\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta}.$$

Similarly

$$\phi_v^t-i{}^*\phi_v^t=2\phi^t\lim_{\tilde{v}\to 0}\frac{1}{\tilde{v}}\nu(\zeta,\ i\tilde{v})\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta}$$

$$=2i\phi^t\frac{\partial}{\partial\tau}\nu(\zeta,\ \tau)|_{\tau=0}\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\zeta}.$$

Thus

$$\phi_u^t - i^*\phi_u^t = -i(\phi_v^t - i^*\phi_v^t)$$

$$= \phi^t \frac{2}{1-|\mu(z,t)|^2} \cdot \frac{\partial}{\partial t}\mu(z,t)\frac{\zeta_z}{\bar{\zeta}_{\bar{z}}} \cdot \frac{d\bar{\zeta}}{d\zeta}.$$

We shall write

$$\frac{\partial}{\partial t}\phi^t = \frac{1}{2}(\phi_u^t - i\phi_v^t) \quad \text{and} \quad \frac{\partial}{\partial \bar{t}}\phi^t = \frac{1}{2}(\phi_u^t + i\phi_v^t).$$

Then by Theorem 1 $\frac{\partial}{\partial \bar{t}}\phi^t = i^*\frac{\partial}{\partial \bar{t}}\phi^t$ and this is a holomorphic differential on R^t. Here we give one of the first variational formulas.

Theorem 2. *Let ϕ^t and ψ^t be meromorphic differentials on R^t such that $(f_t^{-1})^\#(\phi^t)-\phi^0$ and $(f_t^{-1})^\#(\psi^t)-\psi^0$ belong to $\Lambda_x(R)+\Lambda_{eo}(R)$. Assume that the Beltrami coefficient $\mu(z,t)$ of f_t satisfies the condition A and the support of $\mu(z,t)$ does not meet a neighbourhood of poles of ϕ^0 and ψ^0. Then*

$$\frac{\partial}{\partial t}\langle (f_t^{-1})^\#(\phi^t)-\phi^0, \overline{\psi^0}\rangle_R = \frac{1}{2}\left(\frac{\partial}{\partial t}\phi^t, \overline{\psi^t}\right)_{R^t}$$

$$= \frac{i}{2}\iint_R \hat{\phi}^t\hat{\psi}^t \frac{\partial}{\partial t}\mu(z,t)\zeta_z^2 dz d\bar{z},$$

where $\phi^t = \hat{\phi}^t d\zeta$ and $\psi^t = \hat{\psi}^t d\zeta$.

Proof. Observe that

$$\frac{\partial}{\partial u}\langle (f_t^{-1})^\#(\phi^t)-\phi^0, \overline{\psi^0}\rangle_R$$

$$= \lim_{\tilde{u}\to 0}\frac{1}{\tilde{u}}\langle (f_{t+\tilde{u}}^{-1})^\#(\phi^{t+\tilde{u}})-(f_t^{-1})^\#(\phi^t), \overline{\psi^0}\rangle_R$$

$$= \lim_{\tilde{u}\to 0}\left\langle \frac{(f_t \circ f_{t+\tilde{u}}^{-1})^\#(\phi^{t+\tilde{u}})-\phi^t}{\tilde{u}}, -{}^*f_t^\#({}^*\overline{\psi^0})\right\rangle_{R^t}$$

$$= \langle \phi_u^t, -i^*f_t^\#(\overline{\psi^0})\rangle_{R^t}.$$

Since ϕ_u^t and $f_t^\#(\overline{\psi^0})-\overline{\psi^t}$ belong to $\Lambda_x(R^t)+\Lambda_{eo}(R^t)$, we know

$$\frac{\partial}{\partial u}\langle (f_t^{-1})^\#(\phi^t)-\phi^0, \overline{\psi^0}\rangle_R = \langle \phi_u^t, \overline{\psi^t}\rangle_{R^t}.$$

Similary

$$\frac{\partial}{\partial v}\langle (f_t^{-1})^\#(\phi^t)-\phi^0, \overline{\psi^0}\rangle_R = \langle \phi_v^t, \overline{\psi^t}\rangle_{R^t}.$$

Therefore

$$\frac{\partial}{\partial t}\langle (f_t^{-1})^\#(\phi^t)-\phi^0, \overline{\psi^0}\rangle_R = \frac{1}{2}(\langle \phi_u^t, \overline{\psi^t}\rangle_{R^t} - i\langle \phi_v^t, \overline{\psi^t}\rangle_{R^t}).$$

On the other hand,

$$\left(\frac{\partial}{\partial t}\phi^t, \overline{\psi^t}\right)_{R^t} = \left\langle \frac{\partial}{\partial t}\phi^t, \overline{\psi^t}\right\rangle_{R^t} - i\left\langle i\frac{\partial}{\partial t}\phi^t, \overline{\psi^t}\right\rangle_{R^t}.$$

Since $\frac{\partial}{\partial \bar{t}}\phi^t$ is holomorphic, we have $\left(\frac{\partial}{\partial \bar{t}}\phi^t, \overline{\psi^t}\right)_{R^t}=0$. Thus

$$\left\langle\frac{\partial}{\partial t}\phi^t, \overline{\psi_t}\right\rangle_{R^t}=\left\langle\frac{\partial}{\partial t}\phi^t+\frac{\partial}{\partial \bar{t}}\phi^t, \overline{\psi^t}\right\rangle_{R^t}=\langle\phi_u^t, \overline{\psi^t}\rangle_{R^t},$$

$$\left\langle i\frac{\partial}{\partial t}\phi^t, \overline{\psi_t}\right\rangle_{R^t}=\left\langle i\left(\frac{\partial}{\partial t}\phi^t-\frac{\partial}{\partial \bar{t}}\phi^t\right), \overline{\psi^t}\right\rangle_{R^t}=\langle\phi_v^t, \overline{\psi^t}\rangle_{R^t},$$

and

$$\frac{\partial}{\partial t}\langle(f_t^{-1})^*(\phi^t)-\phi^0, \overline{\psi^0}\rangle_R=\frac{1}{2}\left(\frac{\partial}{\partial t}\phi^t, \overline{\psi^t}\right)_{R^t}.$$

Further we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}\phi^t, \overline{\psi^t}\right)_{R^t}&=(\phi_u^t, \overline{\psi^t})_{R^t}=\frac{1}{2}(\phi_u^t+i^*\phi_u^t+\phi_u^t-i^*\phi_u^t, \overline{\psi^t})_{R^t}\\&=\frac{1}{2}(\phi_u^t-i^*\phi_u^t, \overline{\psi^t})_{R^t}=\left(\phi^t\frac{\partial}{\partial\tau}\nu(\zeta, \tau)|_{\tau=0}\frac{d\bar{\zeta}}{d\zeta}, \overline{\psi^t}\right)_{R^t}\\&=i\iint_{R^t}\hat{\phi}^t\hat{\psi}^t\frac{\partial}{\partial\tau}\nu(\zeta, \tau)|_{\tau=0}d\zeta d\bar{\zeta}\\&=i\iint_R\hat{\phi}^t\hat{\psi}^t\frac{\partial}{\partial t}\mu(z, t)\zeta_z^2 dz d\bar{z}.\end{aligned}$$

This completes the proof.

Remark. Although ψ^0 has poles, $\langle(f_t^{-1})^*(\phi^t)-\phi^0, \overline{\psi^0}\rangle_R$ is defined by principal value because the integral vanishes in a neighbourhood of poles. The notation of the inner product will be used also in such a case.

Let $p\in R$ and V_1 be a parametric disc about p with a local variable z. We set $V_r=\{p'\in V_1; |z(p')|<r\}$ $(0<r\leqq 1)$ and $P_n=\{p, q\}$ for $n=0$, $\{p\}$ for $n\geqq 1$. Take a $q\in V_{1/2}$. Then there exist functions $s_n\in C^2(R-P_n)$ such that

$$s_0=\begin{cases}\log\left|\dfrac{z}{z-z(q)}\right| & \text{on } \overline{V}_{1/2}\\ 0 & \text{on } R-V_1,\end{cases}$$

$$s_n=\begin{cases}-\dfrac{1}{n}\mathrm{Re}\dfrac{1}{z^n} & \text{on } \overline{V}_{1/2}\\ 0 & \text{on } R-V_1\end{cases}\quad (n\geqq 1).$$

Denote $ds_n=\sigma_n$. Now since $\int_{|z|=1/2}{}^*\sigma_n=\int_{|z|=1}{}^*\sigma_n=0$, there exists a C^1-closed differential $\tilde{\sigma}_n$ such that $\tilde{\sigma}_n={}^*\sigma_n$ on $(R-V_1)\cup\overline{V}_{1/2}$. Then $\sigma_n+{}^*\tilde{\sigma}_n\in\Lambda$ and $\sigma_n+{}^*\tilde{\sigma}_n=0$ on $(R-V_1)\cup\overline{V}_{1/2}$. By the orthogonal decomposition we can write

$$\sigma_n+{}^*\tilde{\sigma}_n=\lambda_n+\tilde{\lambda}_n,\quad \lambda_n\in\Lambda_x+\Lambda_{eo},\quad \tilde{\lambda}_n\in i^*\Lambda_x+{}^*\Lambda_{eo}.$$

Set $\phi_n=\sigma_n-\lambda_n=\tilde{\lambda}_n-{}^*\tilde{\sigma}_n$. Then ϕ_n is closed and coclosed, hence ϕ_n is harmonic in $R-P_n$. Since $i^*\phi_n=i^*\tilde{\lambda}_n$ on $R-V_1$, the meromorphic differential $\psi_n=\phi_n+i^*\phi_n$

has Λ_x-behavior (cf. [7], [13]). The ψ_0 (resp. ψ_n, $n\geqq 1$) has singularities $\frac{dz}{z}-\frac{dz}{z-z(q)}$ $\left(\text{resp. } \frac{dz}{z^{n+1}}\right)$. Further note that

$$\psi_n-(\sigma_n+i\tilde{\sigma}_n)=\phi_n-\sigma_n+i(*\phi_n-\tilde{\sigma}_n)=-\lambda_n+i*\tilde{\lambda}_n),$$

and $\psi_n-(\sigma_n+i\tilde{\sigma}_n)$ belongs to $\Lambda_x+\Lambda_{eo}$. Assume that $\bar{V}_1$ does not meet the support of μ. Similarly we can construct a meromorphic differential on R^t as ψ_n and denote it $\psi^t_{x,n}$. Note that $(f_t^{-1})^{\#}(\psi^t_{x,n})-\psi_n\in\Lambda_x(R)+\Lambda_{eo}(R)$. For a meromorphic differential ψ^t with $(f_t^{-1})^{\#}(\psi^t)-\psi^0\in\Lambda_x(R)+\Lambda_{eo}(R)$,

$$\begin{aligned}
&\langle (f_t^{-1})^{\#}(\psi^t)-\psi^0,\ \bar{\psi}_n\rangle_{R-V_1}\\
&=-\langle (f_t^{-1})^{\#}(\psi^t)-\psi^0,\ i*(\overline{\psi_n-(\sigma_n+i\tilde{\sigma}_n)})\rangle_{R-V_1}\\
&=\langle (f_t^{-1})^{\#}(\psi^t)-\psi^0,\ i*(\overline{\psi_n-(\sigma_n+i\tilde{\sigma}_n)})\rangle_{V_1}\\
&=\mathrm{Re}\, i\int_{\partial V_1}(\Psi^t\circ f_t-\Psi^0)\psi_n\\
&=\begin{cases} -2\pi\,\mathrm{Re}\{\Psi^t\circ f_t(p)-\Psi^t\circ f_t(q)-(\Psi^0(p)-\Psi^0(q))\} & \text{for } n=0\\ -\dfrac{2\pi}{n!}\mathrm{Re}\left\{\dfrac{d^n}{dz^n}\Psi^t\circ f_t(p)-\dfrac{d^n}{dz^n}\Psi^0(p)\right\} & \text{for } n\geqq 1,\end{cases}
\end{aligned}$$

where Ψ^t is a primitive function of ψ^t on a neighbourhood of $f_t(V_1)$. Now let $\tilde{V}_\varepsilon=\{z\,;\,|z|<\varepsilon\}\cup\{z\,;\,|z-z(q)|<\varepsilon\}$. Then

$$\begin{aligned}
&\lim_{\varepsilon\to 0} i\int_{\partial\tilde{V}_\varepsilon}(\Psi^t\circ f_t-\Psi^0)\psi_0\\
&=-2\pi\{(\Psi^t\circ f_t(p)-\Psi^0(p))-(\Psi^t\circ f_t(q)-\Psi^0(q))\}.
\end{aligned}$$

Hence, even if the support of μ contains p and q, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\langle (f_t^{-1})^{\#}(\psi^t)-\psi^0,\ \bar{\psi}_0\rangle_{R-\tilde{V}_\varepsilon}\\
&=-2\pi\,\mathrm{Re}\{\Psi^t\circ f_t(p)-\Psi^0(p)-(\Psi^t\circ f_t(q)-\Psi^0(q))\}.
\end{aligned}$$

Hereafter, the singular integral $(\omega,\sigma)_R$ means the principal value $\lim(\omega,\sigma)_{R-\tilde{V}_\varepsilon}$ if it has a finite value. The space $i\Lambda_x={}^*\Gamma_x{}^{\perp}+i\Gamma_x$ is also a behavior space which is denoted by $\Lambda_{*x^\perp}$. If ψ_x has Λ_x-behavior, then $i\psi_x$ has $\Lambda_{*x^\perp}$-behavior.

Proposition 4. *Let* $(f_t^{-1})^{\#}(\psi^t)-\psi^0\in\Lambda_x(R)+\Lambda_{eo}(R)$. *Then*

$$\langle (f_t^{-1})^{\#}(\psi^t)-\psi^0,\ \overline{\psi^0_{x,0}}\rangle_R=-2\pi\,\mathrm{Re}\{\Psi^t\circ f_t(p)-\Psi^0(p)-(\Psi^t\circ f_t(q)-\Psi^0(q))\},$$

$$\langle (f_t^{-1})^{\#}(i\psi^t)-i\psi^0,\ \overline{\psi^0_{*x^\perp,0}}\rangle_R=2\pi\,\mathrm{Im}\{\Psi^t\circ f_t(p)-\Psi^0(p)-(\Psi^t\circ f_t(q)-\Psi^0(q))\}.$$

If the support of μ *does not meet* $V_\varepsilon=\{z\,;\,|z|<\varepsilon\}$,

$$\begin{aligned}
&\langle (f_t^{-1})^{\#}(\psi^t)-\psi^0,\ \overline{\psi^0_{x,n}}\rangle_R\\
&=-\frac{2\pi}{n!}\mathrm{Re}\left\{\frac{d^n}{dz^n}\Psi^t\circ f_t(p)-\frac{d^n}{dz^n}\Psi^0(p)\right\}\quad \textit{for}\quad n\geqq 1,
\end{aligned}$$

$$\langle (f_t^{-1})^{\#}(i\psi^t)-i\psi^0,\ \overline{\psi^0_{*x^\perp,n}}\rangle_R$$
$$=\frac{2\pi}{n!}\mathrm{Im}\left\{\frac{d^n}{dz^n}\Psi^t\circ f_t(p)-\frac{d^n}{dz^n}\Psi^0(p)\right\} \qquad for\quad n\geq 1.$$

Thus, applying Theorem 2 we can get the variational formulas with respect to $\Psi^t\circ f_t(p)$ and $\frac{d^n}{dz^n}\Psi^t\circ f_t(p)$.

Proposition 5. *Let* $(f_t^{-1})^{\#}(\psi^t)-\psi^0\in\Lambda_x(R)+\Lambda_{eo}(R)$. *Under the similar condition as in Theorem 2,*

$$\frac{\partial}{\partial t}\{\Psi^t\circ f_t(p)-\Psi^t\circ f_t(q)\}=-\frac{i}{4\pi}\iint_R\hat{\psi}^t(\hat{\psi}^t_{x,0}+\hat{\psi}^t_{*x^\perp,0})\frac{\partial}{\partial t}\mu(z,\ t)\zeta_z^2 dzd\bar{z}\,,$$

$$\frac{\partial}{\partial \bar{t}}\{\Psi^t\circ f_t(p)-\Psi^t\circ f_t(q)\}=\frac{i}{4\pi}\iint_R\overline{\hat{\psi}(\hat{\psi}^t_{x,0}-\hat{\psi}^t_{*x^\perp,0})\frac{\partial}{\partial t}\mu(z,\ t)\zeta_z^2 dzd\bar{z}},$$

$$\frac{\partial}{\partial t}\cdot\frac{d^n}{dz^n}\Psi^t\circ f_t(p)=-\frac{n!}{4\pi}i\iint_R\hat{\psi}^t(\hat{\psi}^t_{x,n}+\hat{\psi}^t_{*x^\perp,n})\frac{\partial}{\partial t}\mu(z,\ t)\zeta_z^2 dzd\bar{z}\,,$$

$$\frac{\partial}{\partial \bar{t}}\cdot\frac{d^n}{dz^n}\Psi^t\circ f_t(p)=\frac{n!}{4\pi}i\iint_R\overline{\hat{\psi}^t(\hat{\psi}^t_{x,n}-\hat{\psi}^t_{*x^\perp,n})\frac{\partial}{\partial t}\mu(z,\ t)\zeta_z^2 dzd\bar{z}}.$$

If $\Gamma_x={}^*\Gamma_{x^\perp}$, then $\psi^t_{x,n}=\psi^t_{*x^\perp,n}$. Hence we have the following by Proposition 5.

Corollary. *Let* $\Lambda_x=i\Lambda_x$ *and* $(f_t^{-1})^{\#}(\psi^t)-\psi^0\in\Lambda_x(R)+\Lambda_{eo}(R)$. *Then* $\Psi^t\circ f_t(p)-\Psi^t\circ f_t(q)$ *and* $\frac{d^n}{dz^n}\Psi^t\circ f_t(p)$ *are holomorphic with respect to* t (cf. [7]).

5. Next we show one of the second variational formulas.

Theorem 3. *Under the similar condition as in Theorem 2,*

$$\frac{\partial^2}{\partial\bar{t}\partial t}\langle (f_t^{-1})^{\#}(\phi^t)-\phi^0,\ \overline{\psi^0}\rangle_R=\frac{1}{2}\left\{\left(\frac{\partial}{\partial\bar{t}}\phi^t,\ \overline{\frac{\partial}{\partial t}\psi_t}\right)_{R^t}+\left(\frac{\partial}{\partial\bar{t}}\psi^t,\ \overline{\frac{\partial}{\partial t}\phi^t}\right)_{R^t}\right\}$$
$$=\frac{i}{2}\iint_R(\hat{\phi}^t_{\bar{t}}\hat{\psi}^t+\hat{\psi}^t_{\bar{t}}\hat{\phi}^t)\frac{\partial}{\partial t}\mu(z,\ t)\zeta_z^2 dzd\bar{z}\,,$$

where $\frac{\partial}{\partial\bar{t}}\phi^t=\hat{\phi}^t_{\bar{t}}d\zeta$ *and* $\frac{\partial}{\partial\bar{t}}\psi^t=\hat{\psi}^t_{\bar{t}}d\zeta$.

Proof. Let $\omega^t=(f_t^{-1})^{\#}(\phi^t)+i^*(f_t^{-1})^{\#}(\phi^t)=2\hat{\phi}^t\zeta_z dz$, $\sigma^t=(f_t^{-1})^{\#}(\psi^t)+i^*(f_t^{-1})^{\#}(\psi^t)=2\hat{\psi}^t\zeta_z dz$. By Theorem 2, observe that

$$\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial t}\langle (f_t^{-1})^{\#}(\phi^t)-\phi^0,\ \overline{\psi^0}\rangle_R$$
$$=\frac{1}{8}\lim_{\tilde{u}\to 0}\frac{1}{\tilde{u}}\left\{(\omega^{t+\tilde{u}}\mu_t(z,\ t+\tilde{u})\frac{d\bar{z}}{dz},\ \overline{\sigma^{t+\tilde{u}}})_R-(\omega^t\mu_t(z,\ t)\frac{d\bar{z}}{dz},\ \overline{\sigma^t})_R\right\}$$
$$=\frac{1}{8}\lim_{\tilde{u}\to 0}\left\{\left(\frac{\omega^{t+\tilde{u}}-\omega^t}{\tilde{u}}\mu_t(z,\ t+\tilde{u})\frac{d\bar{z}}{dz},\ \overline{\sigma^{t+\tilde{u}}}\right)_R\right.$$

$$+\left(\omega^t\frac{\mu_t(z,t+\tilde{u})-\mu_t(z,t)}{\tilde{u}}\cdot\frac{d\bar{z}}{dz},\ \overline{\sigma^{t+\tilde{u}}}\right)_R$$

$$+\left.\left(\omega^t\mu_t(z,t)\frac{d\bar{z}}{dz},\ \overline{\frac{\sigma^{t+\tilde{u}}-\sigma^t}{\tilde{u}}}\right)_R\right\}.$$

We know

$$\lim_{\tilde{u}\to 0}\frac{\omega^{t+\tilde{u}}-\omega^t}{\tilde{u}}=(f_t^{-1})^{\#}(\phi_u^t)+i^*(f_t^{-1})^{\#}(\phi_u^t),$$

$$\lim_{\tilde{u}\to 0}\frac{\sigma^{t+\tilde{u}}-\sigma^t}{\tilde{u}}=(f_t^{-1})^{\#}(\psi_u^t)+i^*(f_t^{-1})^{\#}(\psi_u^t)$$

and write them $2\omega_u^t$ and $2\sigma_u^t$ respectively. Hence

$$\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial t}\langle(f_t^{-1})^{\#}(\phi^t)-\phi^0,\ \overline{\psi^0}\rangle_R$$

$$=\frac{1}{8}\left\{\left(2\omega_u^t\mu_t(z,t)\frac{d\bar{z}}{dz},\ \overline{2\hat{\psi}^t\zeta_z dz}\right)_R+\left(2\hat{\phi}^t\zeta_z\frac{\partial}{\partial t}\mu_t(z,t)d\bar{z},\ \overline{2\hat{\psi}^t\zeta_z dz}\right)_R\right.$$

$$\left.+(2\hat{\phi}^t\zeta_z\mu_t(z,t)d\bar{z},\ \overline{2\sigma_u^t})_R\right\}.$$

Similarly we can get

$$\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial t}\langle(f_t^{-1})^{\#}(\phi^t)-\phi^0,\ \overline{\psi^0}\rangle_R$$

$$=\frac{1}{8}\left\{\left(2\omega_v^t\mu_t(z,t)\frac{d\bar{z}}{dz},\ \overline{2\hat{\psi}^t\zeta_z dz}\right)_R+i\left(2\hat{\phi}^t\zeta_z\frac{\partial}{\partial t}\mu_t(z,t)d\bar{z},\ \overline{2\hat{\psi}^t\zeta_z dz}\right)_R\right.$$

$$\left.+(2\hat{\phi}^t\zeta_z\mu_t(z,t)d\bar{z},\ \overline{2\sigma_v^t})_R\right\},$$

where $2\omega_v^t=(f_t^{-1})^{\#}(\phi_v^t)+i^*(f_t^{-1})^{\#}(\phi_v^t)$, $2\sigma_v^t=(f_t^{-1})^{\#}(\psi_v^t)+i^*(f_t^{-1})^{\#}(\psi_v^t)$. Therefore

$$\frac{\partial^2}{\partial\bar{t}\partial t}\langle(f_t^{-1})^{\#}(\phi^t)-\phi^0,\ \overline{\psi^0}\rangle_R$$

$$=\frac{1}{4}\left\{\left((\omega_u^t+i\omega_v^t)\mu_t(z,t)\frac{d\bar{z}}{dz},\ \overline{\hat{\psi}^t\zeta_z dz}\right)_R+(\hat{\phi}^t\zeta_z\mu_t(z,t)d\bar{z},\ \overline{\sigma_u^t+i\sigma_v^t})_R\right\}$$

$$=\frac{i}{2}\iint_R(\hat{\phi}_{\bar{t}}^t\hat{\psi}^t+\hat{\psi}_{\bar{t}}^t\hat{\phi}^t)\mu_t(z,t)\zeta_z^2 dz d\bar{z}$$

$$=\frac{i}{2}\iint_{R^t}(\phi_{\bar{t}}^t\psi^t+\psi_{\bar{t}}^t\phi^t)\frac{\mu_t(z,t)}{1-|\mu(z,t)|^2}\cdot\frac{\zeta_z}{\bar{\zeta}_{\bar{z}}}d\zeta d\bar{\zeta}$$

$$=\frac{1}{4}\left\{\left(\frac{\partial}{\partial\bar{t}}\phi^t,\ \overline{\psi_u^t-i^*\psi_u^t}\right)_{R^t}+\left(\frac{\partial}{\partial\bar{t}}\psi^t,\ \overline{\phi_u^t-i^*\phi_u^t}\right)_{R^t}\right\}\quad\text{(by Theorem 1)}$$

$$=\frac{1}{2}\left\{\left(\frac{\partial}{\partial\bar{t}}\phi^t,\ \overline{\frac{\partial}{\partial t}\psi^t-i^*\frac{\partial}{\partial\bar{t}}\psi^t}\right)_{R^t}+\left(\frac{\partial}{\partial\bar{t}}\psi^t,\ \overline{\frac{\partial}{\partial t}\phi^t-i^*\frac{\partial}{\partial\bar{t}}\phi^t}\right)_{R^t}\right\}$$

$$=\frac{1}{2}\left\{\left(\frac{\partial}{\partial\bar{t}}\phi^t,\ \overline{\frac{\partial}{\partial t}\psi^t}\right)_{R^t}+\left(\frac{\partial}{\partial\bar{t}}\psi^t,\ \overline{\frac{\partial}{\partial t}\phi^t}\right)_{R^t}\right\}.$$

Corollary *For* $t=(t_1,\cdots,t_n)$

$$\begin{aligned}&\frac{\partial^2}{\partial\bar{t}_j\partial t_i}\langle(f_{\bar{t}}^{-1})^{\#}(\phi^t)-\phi^0,\ \overline{\psi^0}\rangle_R\\&=\frac{1}{2}\left\{\left(\frac{\partial}{\partial\bar{t}_j}\phi^t,\ \overline{\frac{\partial}{\partial t_i}\psi^t}\right)_{R^t}+\left(\frac{\partial}{\partial\bar{t}_j}\psi^t,\ \overline{\frac{\partial}{\partial t_i}\phi^t}\right)_{R^t}\right\}\\&=\frac{i}{2}\iint_R(\hat{\phi}^t_{\bar{t}_j}\hat{\psi}^t+\hat{\psi}^t_{\bar{t}_j}\hat{\phi}^t)\frac{\partial}{\partial t_i}\mu(z,\ t)\zeta_z^2 dz d\bar{z}\,.\end{aligned}$$

§ 3. Remarks for the case of behavior space Λ_{-1}.

6. Here we observe Theorem 3 for $\Lambda_x=\Lambda_{-1}=\{0\}+i\Gamma_h$. Then our formulas have similar forms as Yamaguchi's corresponding ones (cf. [14]). First note that ψ^t_u (resp. ψ^t_v) belongs to $\Lambda_{-1}+\Lambda_{eo}$ and can be written $\psi^t_u=i\omega+\omega_0$, $\omega\in\Gamma_h$, $\omega_0\in\Lambda_{eo}$ (resp. $\psi^t_v=i\sigma+\sigma_0$, $\sigma\in\Gamma_h$, $\sigma_0\in\Lambda_{eo}$). Hence

$$\overline{\frac{\partial}{\partial t}\psi^t}+\frac{\partial}{\partial\bar{t}}\psi^t=\frac{1}{2}(\omega_0+\overline{\omega_0}+i\sigma_0+i\overline{\sigma_0})\in\Lambda_{eo})\,.$$

Therefore $\left(\frac{\partial}{\partial\bar{t}}\phi^t,\ \overline{\frac{\partial}{\partial t}\psi^t}\right)=-\left(\frac{\partial}{\partial\bar{t}}\phi^t,\ \frac{\partial}{\partial\bar{t}}\psi^t\right)$. Thus we have the following.

Theorem 4. *Let $(f_{\bar{t}}^{-1})^{\#}(\phi^t)-\phi^0$ and $(f_{\bar{t}}^{-1})^{\#}(\psi^t)-\psi^0$ belong to $\Lambda_{-1}(R)+\Lambda_{eo}(R)$. Under the similar condition as in Theorem 2,*

$$\frac{\partial^2}{\partial\bar{t}_j\partial t_i}\langle(f_{\bar{t}}^{-1})^{\#}(\phi^t)-\phi^0,\ \overline{\psi^0}\rangle_R=-\frac{1}{2}\left\{\left(\frac{\partial}{\partial\bar{t}_j}\phi^t,\ \frac{\partial}{\partial\bar{t}_i}\psi^t\right)_{R^t}+\left(\frac{\partial}{\partial\bar{t}_j}\psi^t,\ \frac{\partial}{\partial\bar{t}_i}\phi^t\right)_{R^t}\right\}\,.$$

Further, if ψ^t is holomorphic in R^t, then

$$\frac{\partial}{\partial t_i}\langle(f_{\bar{t}}^{-1})^{\#}(\phi^t)-\phi^0,\ \overline{\psi^0}\rangle_R=-\frac{1}{2}\left(\psi^t,\ \frac{\partial}{\partial\bar{t}_i}\phi^t\right)_{R^t},\qquad t=(t_1\cdots t_n)\,.$$

Next we have a variational formula with respect to the inner product.

Theorem 5. *Let ϕ^t and ψ^t be holomorphic differentials such that $(f_{\bar{t}}^{-1})^{\#}(\phi^t)-\phi^0$ and $(f_{\bar{t}}^{-1})^{\#}(\psi^t)-\psi^0$ belong to $\Lambda_{-1}(R)+\Lambda_{eo}(R)$. Under the similar condition as in Theorem 2,*

$$\frac{\partial}{\partial t_i}\langle\phi^t,\ \psi^t\rangle_{R^t}=\frac{1}{2}\left\{\left(\phi^t,\ \frac{\partial}{\partial\bar{t}_i}\psi^t\right)_{R^t}+\left(\psi^t,\ \frac{\partial}{\partial\bar{t}_i}\phi^t\right)_{R^t}\right\}\,,$$

$$\frac{\partial^2}{\partial\bar{t}_j\partial t_i}\langle\phi^t,\ \psi^t\rangle_{R^t}=\left(\frac{\partial}{\partial\bar{t}_j}\phi^t,\ \frac{\partial}{\partial\bar{t}_i}\psi^t\right)_{R^t}+\left(\frac{\partial}{\partial\bar{t}_j}\psi^t,\ \frac{\partial}{\partial\bar{t}_i}\phi^t\right)_{R^t},\qquad t=(t_1\cdots t_n).$$

Proof. By Lemma 1 and the property of behavior space,

$$\begin{aligned}&\langle\phi^t,\ \psi^t\rangle_{R^t}-\langle\phi^0,\ \psi^0\rangle_R\\&=\langle(f_{\bar{t}}^{-1})^{\#}(\phi^t)-\phi^0,\ i^*(f_{\bar{t}}^{-1})^{\#}(\psi^t)\rangle_R+\langle\phi^0,\ i^*(f_{\bar{t}}^{-1})^{\#}(\psi^t)-\psi^0\rangle_R\\&=\langle(f_{\bar{t}}^{-1})^{\#}(\phi^t)-\phi^0,\ i^*\psi^0\rangle_R+\langle(f_{\bar{t}}^{-1})^{\#}(\psi^t)-\psi^0,\ i^*\phi^0\rangle_R\,.\end{aligned}$$

Since $\phi^0+\overline{\phi^0}$ and $\psi^0+\overline{\psi^0}$ belong to Γ_h, it holds that

$$\langle\phi^t, \psi^t\rangle_{R^t}-\langle\phi^0, \psi^0\rangle_R$$
$$=-\langle(f_t^{-1})^\#(\phi^t)-\phi^0, \overline{\psi^0}\rangle_R-\langle(f_t^{-1})^\#(\psi^t)-\psi^0, \overline{\phi^0}\rangle_R\,.$$

Thus by Theorem 4 the assertion follows.

7. Now we will apply these theorems to some specific kind of differentials. Then we can get some variational formulas with similar forms as Yamaguchi's. Let C^0 be a cycle on R, $C^t=f_t(C^0)$ and σ_{C^t} be the period reproducing harmonic differentials on R^t, i.e., $(\omega, {}^*\sigma_{C^t})_{R^t}=\int_{C'}\omega$ for C^1-differential $\omega\in\Gamma(R^t)$, where C' is a closed curve which is homologous to C^t. Then σ_{C^t} is represented as $\sigma_{C^t}=dS_{C^t}+\sigma_0$, where S_{C^t} is the one as in the proof of Lemma 2 and $\sigma_0\in\Gamma_{e0}$. Hence $\psi_c^t=\sigma_{C^t}+i^*\sigma_{C^t}$ satisfies that $(f_t^{-1})^\#(\psi_c^t)-\psi_c^0\in\Lambda_{-1}(R)+\Lambda_{e0}(R)$.

Formula 1.
$$\frac{\partial}{\partial t_i}\|\psi_c^t\|^2=\left(\psi_c^t, \frac{\partial}{\partial \bar{t}_i}\psi_c^t\right),$$
$$\frac{\partial^2}{\partial \bar{t}_j\partial t_i}\|\psi_c^t\|^2=2\left(\frac{\partial}{\partial \bar{t}_j}\psi_c^t, \frac{\partial}{\partial \bar{t}_i}\psi_c^t\right)_{R^t},$$

and hence $\|\psi_c^t\|^2$ *is plurisubharmonic.*

Note that $\|\sigma_{C^t}\|^2$ is equal to the extremal length $\lambda(C^t)$ of the family of curves homologous to C^t. Thus

Formula 2.
$$\frac{\partial}{\partial t_i}\lambda(C^t)=-\frac{i}{2}\iint_R(\psi_c^t)^2\frac{\partial}{\partial t_i}\mu(z, t)\zeta_z^2 dz d\bar{z}\,,$$
$$\frac{\partial^2}{\partial \bar{t}_j\partial t_i}\lambda(C^t)=\left(\frac{\partial}{\partial \bar{t}_j}\psi_c^t, \frac{\partial}{\partial \bar{t}_i}\psi_c^t\right)_{R^t},$$

and hence $\lambda(C^t)$ *is plurisubharmonic.*

Let G_p^t be Green's function on R^t with pole at $f_t(p)$ and $\psi_p^t=dG_p^t+i^*dG_p^t$ $=2\dfrac{\partial}{\partial\zeta}G_p^t d\zeta$. If a neighbourhood V of p does not meet the support of $\mu(z, t)$, then $(f_t^{-1})^\#(\psi_p^t)-\psi_p^0\in\Lambda_{-1}(R)+\Lambda_{e0}(R)$. We know

$$\langle(f_t^{-1})^\#(\psi_p^t)-\psi_p^0, \overline{\psi_q^0}\rangle_R=2\pi(G_p^t(f_t(q))-G_p^0(q)) \qquad \text{for} \quad q(\neq p)\in V,$$
$$\langle(f_t^{-1})^\#(\psi_p^t)-\psi_p^0, \overline{\psi_p^0}\rangle_R=2\pi(\gamma^t(p)-\gamma^0(p))\,.$$

where $\gamma^t(p)=\dfrac{1}{2\pi i}\displaystyle\int_{|z|=\varepsilon}G_p^t(f_t(z))\frac{dz}{z}$ (z is a local variable in V about p, for which the singularity of ψ_p^t is written as $-\dfrac{dz}{z}$). The $\gamma^t(p)$ is called the Robin's constant at p on R^t.

Formula 3. *For* $p\neq q$

$$\frac{\partial}{\partial t_i}G_p^t(f_t(q))=\frac{i}{\pi}\iint_R\frac{\partial}{\partial\zeta}G_p^t\frac{\partial}{\partial\zeta}G_q^t\frac{\partial}{\partial t_i}\mu(z,\ t)\zeta_z^2 dz d\bar{z}$$

$$=\frac{1}{4\pi}\Big(\frac{\partial}{\partial t_i}\phi_p^t,\ \overline{\phi_q^t}\Big)_{R^t},$$

$$\frac{\partial^2}{\partial\bar{t}_j\partial t_i}G_p^t(f_t(q))=-\frac{1}{4\pi}\Big\{\Big(\frac{\partial}{\partial\bar{t}_j}\phi_p^t,\ \frac{\partial}{\partial\bar{t}_i}\phi_q^t\Big)_{R^t}+\Big(\frac{\partial}{\partial\bar{t}_j}\phi_q^t,\ \frac{\partial}{\partial\bar{t}_i}\phi_p^t\Big)_{R^t}\Big\}.$$

Remark. This formula is valid on an arbitrary hyperbolic Riemann surface, and so gives an extension of Guerrero's result on a finite Riemann surface (cf. [5]).

Formula 4.

$$\frac{\partial}{\partial t_i}\gamma^t(p)=\frac{i}{\pi}\iint_R\Big(\frac{\partial}{\partial\zeta}G_p^t\Big)^2\frac{\partial}{\partial t_i}\mu(z,\ t)\zeta_z^2 dz d\bar{z}$$

$$=\frac{1}{4\pi}\Big(\frac{\partial}{\partial t_i}\phi_p^t,\ \overline{\phi_p^t}\Big)_{R^t}$$

$$\frac{\partial^2}{\partial\bar{t}_j\partial t_i}\gamma^t(p)=-\frac{1}{2\pi}\Big(\frac{\partial}{\partial\bar{t}_j}\phi_p^t,\ \frac{\partial}{\partial\bar{t}_i}\phi_p^t\Big)_{R^t}$$

and $t\rightarrow\gamma^t(p)$ *is plurisuperharmonic.*

Let R_*^t be Royden's compactification of R^t (see [4]) and $h^t(\zeta)$ be real valued continuous Dirichlet function. We know that f_t has an extension to a homeomorphism $\hat{f}_t$ from R_*^0 to R_*^t (see [12]) and $h^t(\zeta)$ has a continuous extension $\hat{h}^t$ to R_*^t. Let $H_{\hat{h}^t}^t$ be the solution of the Dirichlet problem on R_*^t with boundary value $\hat{h}^t$ (cf. [4]). For $\hat{h}^t=\hat{h}^0\circ\hat{f}_t^{-1}$, set $\phi_h^t=dH_{\hat{h}^t}^t+i{}^*dH_{\hat{h}^t}^t$. Since $H_{\hat{h}^t}^t\circ f_t-H_{\hat{h}^0}^0$ is a Dirichlet potential, the holomorphic differential ϕ_h^t satisfies that $(f_t^{-1})^\#(\phi_h^t)-\phi_h^0\in\Lambda_{-1}+\Lambda_{eo}$.

Formula 5.

$$\frac{\partial}{\partial t_i}\|dH_{\hat{h}^t}^t\|^2=-i\iint_R\Big(\frac{\partial}{\partial\zeta}H_{\hat{h}^t}^t\Big)^2\frac{\partial}{\partial t_i}\mu(z,\ t)\zeta_z^2 dz d\bar{z}$$

$$=\frac{1}{2}\Big(\phi_h^t,\ \frac{\partial}{\partial\bar{t}_i}\phi_h^t\Big)_{R^t}$$

$$\frac{\partial^2}{\partial\bar{t}_j\partial t_i}\|dH_{\hat{h}^t}^t\|^2=\Big(\frac{\partial}{\partial\bar{t}_j}\phi_h^t,\ \frac{\partial}{\partial\bar{t}_i}\phi_h^t\Big)_{R^t}$$

and $\|dH_{\hat{h}^t}^t\|^2$ *is plurisubharmonic.*

Remark. If $\hat{h}^t$ is a characteristic function on the boundary, then $H_{\hat{h}^t}^t$ is called a harmonic measure. Hence Formula 5 gives variational formulas of harmonic measures.

Further note that

$$\int_{C^t} {}^*dH^t_{\hat{h}^t} = \frac{1}{2}\langle\phi^t_h, \phi^t_c\rangle_{R^t},$$

$$\int_{C^t} {}^*dG^t_p - \int_C {}^*dG^0_p = -\langle (f_t^{-1})^*(\phi^t_p) - \phi^0_p, \overline{\phi^0_c}\rangle_R$$

$$H^t_{\hat{h}^t}(f_t(p)) - H^0_{\hat{h}^0}(p) = \frac{1}{2\pi}\langle (f_t^{-1})^*(\phi^t_h) - \phi^0_h, \overline{\phi^0_p}\rangle_R .$$

Formula 6.

$$\frac{\partial^2}{\partial \bar{t}_j \partial t_i}\int_{C^t} {}^*dH^t_{\hat{h}^t} = \frac{1}{2}\left\{\left(\frac{\partial}{\partial \bar{t}_j}\phi^t_h, \frac{\partial}{\partial \bar{t}_i}\phi^t_c\right)_{R^t} + \left(\frac{\partial}{\partial \bar{t}_j}\phi^t_c, \frac{\partial}{\partial \bar{t}_i}\phi^t_h\right)_{R^t}\right\}.$$

Formula 7.

$$\frac{\partial^2}{\partial \bar{t}_j \partial t_i}\int_{C^t} {}^*dG^t_p = \frac{1}{2}\left\{\left(\frac{\partial}{\partial \bar{t}_j}\phi^t_p, \frac{\partial}{\partial \bar{t}_i}\phi^t_c\right)_{R^t} + \left(\frac{\partial}{\partial \bar{t}_j}\phi^t_c, \frac{\partial}{\partial \bar{t}_i}\phi^t_p\right)_{R^t}\right\}.$$

Formula 8.

$$\frac{\partial}{\partial t_i} H^t_{\hat{h}^t}(f_t(p)) = \frac{i}{\pi}\iint_R \frac{\partial}{\partial \zeta} H^t_{\hat{h}^t} \frac{\partial}{\partial \zeta} G^t_p \frac{\partial}{\partial t}\mu(z, t)\zeta_z^2 dz d\bar{z}$$

$$= \frac{1}{4\pi}\left(\phi^t_p, \overline{\frac{\partial}{\partial \bar{t}_i}\phi^t_h}\right)_{R^t},$$

$$\frac{\partial^2}{\partial \bar{t}_j \partial t_i} H^t_{\hat{h}^t}(f_t(p)) = -\frac{1}{4\pi}\left\{\left(\frac{\partial}{\partial \bar{t}_j}\phi^t_h, \frac{\partial}{\partial \bar{t}_i}\phi^t_p\right)_{R^t} + \left(\frac{\partial}{\partial \bar{t}_j}\phi^t_p, \frac{\partial}{\partial \bar{t}_i}\phi^t_h\right)_{R^t}\right\}.$$

At last we refer to Bergmann kernels. Let $\Lambda_x = \Lambda_{-1}$, $\Lambda_{*x^\perp} = i\Lambda_{-1}$, and $\phi_{x,1}$ and $\phi_{*x^\perp,1}$ be the meromorphic differentials with pole $\frac{dz}{z^2}$ at p as in § 2, 4. Set $\kappa^t = \frac{1}{4\pi}(\phi^t_{x,1} - \phi^t_{*x^\perp,1})$ $(=\hat{\kappa}^t dz)$. It is known that κ^t is a Bergmann kernel and for $\omega(=\hat{\omega}dz) \in \tilde{\Lambda}_a(R^t)$ $(\omega, \kappa^t) = \hat{\omega}(p)$, particularly $\hat{\kappa}^t(p) = (\kappa^t, \kappa^t) > 0$. Then we have the following formula.

Formula 9.

$$\frac{\partial}{\partial t_i}\hat{\kappa}^t(p) = \left(\kappa^t, \frac{\partial}{\partial \bar{t}_i}\kappa^t\right)_{R^t},$$

$$\frac{\partial^2}{\partial \bar{t}_j \partial t_i}\hat{\kappa}^t(p) = \frac{1}{8\pi^2}\left\{\left(\frac{\partial}{\partial \bar{t}_j}\phi^t_{x,1}, \frac{\partial}{\partial \bar{t}_i}\phi^t_{x,1}\right)_{R^t} + \left(\frac{\partial}{\partial \bar{t}_j}\phi^t_{*x^\perp,1}, \frac{\partial}{\partial \bar{t}_i}\phi^t_{*x^\perp,1}\right)_{R^t}\right\}$$

$$= \left(\frac{\partial}{\partial \bar{t}_j}\kappa^t, \frac{\partial}{\partial \bar{t}_i}\kappa^t\right)_{R^t} + \left(\frac{\partial}{\partial \bar{t}_j}L^t, \frac{\partial}{\partial \bar{t}_i}L^t\right)_{R^t},$$

(*where* $L^t = \frac{1}{4\pi}(\phi^t_{x,1} + \phi^t_{*x^\perp,1})$),

$$\frac{\partial^2}{\partial \bar{t}_j \partial t_i}\log \hat{\kappa}^t(p) = \frac{1}{\hat{\kappa}^t(p)}\left\{\left(\frac{\partial}{\partial \bar{t}_j}\kappa^t, \frac{\partial}{\partial \bar{t}_i}\kappa^t\right)_{R^t} + \left(\frac{\partial}{\partial \bar{t}_j}L^t, \frac{\partial}{\partial \bar{t}_i}L^t\right)_{R^t}\right\}$$

$$-\frac{1}{\hat{\kappa}^t(p)^2}\left(\frac{\partial}{\partial \bar{t}_j}\kappa^t,\ \kappa^t\right)_{R^t}\left(\kappa^t,\ \frac{\partial}{\partial \bar{t}_i}\kappa^t\right)_{R^t}.$$

Hence $\sum t_j\bar{t}_i\dfrac{\partial^2}{\partial\bar{t}_j\partial t_i}\hat{\kappa}^t(p)\geqq 0$ *and* $\sum t_j\bar{t}_i\dfrac{\partial^2}{\partial\bar{t}_j\partial t_i}\log\hat{\kappa}^t(p)\geqq 0.$

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