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On half-homogeneous hyperbolic manifolds and Siegel domains

By

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Introduction.

Siegel domains of the second kind due to Pyatetski-Shapiro [11] are interesting objects of research not only in complex analysis but also in differential geometry. It would be desirable to characterize complex manifolds which are holomorphically isomorphic to Siegel domains among bounded domains in $\boldsymbol{C}^n$ or among hyperbolic manifolds. The present paper is an approach to this problem.

Let M be a hyperbolic manifold due to Kobayashi [3] and let $\mathfrak{g}(M)$ be the Lie algebra of $G(M)$, the identity component of the group of all holomorphic transformation of M. We say that M is *half-homogeneous* if $\mathfrak{g}(M)_c$ (=the complexification of $\mathfrak{g}(M)$) is "transitive" at every point of M (see, § 2). In this paper, we shall study half-homogeneous hyperbolic manifolds satisfying a certain condition (C). Let us denote by $\mathfrak{b}^0(p)$ the isotropy subalgebra of $\mathfrak{g}(M)_c$ at a point p of M. Then the condition (C) implies the existence of an element of $\mathfrak{b}^0(p)$ which is mapped to the identity transformation of $T_p(M)$ by the isotropy representation (see, § 3). Of course, every Siegel domain of the second kind is half-homogeneous and satisfies the condition (C) ([9]). We also introduce the notion of *pseudo-Sigel domains* in § 5. Now our main results are stated as follows:

(I) *Let M be a half-homogeneous hyperbolic manifold satisfying* (C). *Then M is holomorphically immersed in a complex vector space as a pseudo-Siegel domain D of the second kind in such a way that $G(M)$ acts on D equivariantly* (*Theorem* 7.4).

(II) *Let M be a half-homogeneous complete hyperbolic manifold satisfying* (C). *Assume further that $\mathfrak{b}^0(p)\neq\mathfrak{b}^0(q)$ if $p\neq q$. Then M is holomorphically equivalent to a Siegel domain of the second kind. Conversely, every Siegel domain of the second kind is a hyperbolic manifold having these properties* (*Theorem* 8.2).

(III) *Let M be a homogeneous hyperbolic manifold. Then M is isomophic to a homogeneous bounded domain if and only if M satisfies* (C) (*Theorem* 8.3).

Recently, Kodama and Shima [5] obtained other characterization of homogeneous bounded domains.

We now explain the various sections. In § 1, we construct for a hyperbolic manifold M and for a point p of M, a complex submanifold $M(p)$ through p by the same methods as in [10] and prove that $M(p)$ is a hermitian symmetric space of the non-compact type. We mention that once this is proved, all results in [9] for bounded domains also hold for hyperbolic manifolds.

In § 2, we recall the $G(M)$-equivariant mapping Φ of a half-homogeneous hyperbolic manifold M into a certain complex coset space G_c/B constructed in [9] and rewrite Theorem 3.3 of [9] in more detail for later use. Here G_c denote the adjoint group of $\mathfrak{g}(M)_c$. Under the assumption of the half-homogeneity and the condition (C), Φ becomes an immersion. Moreover there exists an abelian subspace θ^{-1} of $\mathfrak{g}(M)_c$ with $\dim_C \theta^{-1} = \dim_C M$ and a holomorphic imbedding h_1 of θ^{-1} into G_c/B. We show in § 3 that $\Phi(M)$ is contained in $h_1(\theta^{-1})$ and hence $h_1^{-1} \circ \Phi$ is an immersion of M onto a domain $\mathcal{M}$ of θ^{-1}.

Next in § 4, we construct a fibering of $\mathcal{M}$ with the base space $\mathcal{S}$ isomorphic to $M(p)$. We shall show in § 5 that the fiber $\mathcal{M}_0$ is isomorphic to a pseudo-Siegel domain.

Since $\mathcal{S}$ is a hermitian symmetric space of the non-compact type, $\mathcal{S}$ is realized as a symmetric Siegel domain S. After some preparations in § 6, we construct in § 7 a pseudo-Siegel domain D in a vector space $Ad\,\delta^{-1}\theta^{-1}$ with an imbedding $h_0 : Ad\,\delta^{-1}\theta^{-1} \to G_c/B_0$ in such a way that there exists a canonical fibering: $D \to S$ and that a fiber D_0 is isomorphic to $\mathcal{M}_0$ under a G_c-equivariant holomorphic diffeomorphism $\tilde{\delta} : G_c/B_0 \to G_c/B$. Here δ is an element of G_c and $B_0 = Ad\,\delta^{-1}B$. These being prepared, by taking a subgroup $A(S)$ of $Ad\,G(M)$ which acts on S transitively, we shall see $\tilde{\delta} \circ h_0(D) = h_1(\mathcal{M})$.

Finally in § 8, we shall give characterizations of Siegel domains and homogeneous bounded domains by using the results in the previous sections.

Throughout this paper, we use the following notations: For a hyperbolic manifold M, $\mathrm{Aut}(M)$ means the Lie group of all holomorphic transformations of M and $\mathfrak{g}(M)$ means its Lie algebra. For a real vector space or a real Lie algebra A, A_c denotes its complexification. For any $z \in A_c$, we denote by $\bar{z}$, $\mathrm{Re}\,z$ and $\mathrm{Im}\,z$, the complex conjugate, the real part and the imaginary part of z respectively. Let W be a vector space over $\boldsymbol{K}$ ($\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$). We denote by $Gr(W; r, \boldsymbol{K})$ the grassmann manifold consisting of all r-dimensional $\boldsymbol{K}$-subspaces of W.

§ 1. Hermitian symmetric submanifolds of hyperbolic manifolds.

Let M be a hyperbolic manifold and let $\mathfrak{g}(M)$ be the Lie algebra of $\mathrm{Aut}(M)$, the group of all holomorphic transformations of M. We denote by $\mathfrak{X}(M)$ the space of all vector fields on M. For $X \in \mathfrak{g}(M)$, X^* means the element of $\mathfrak{X}(M)$ generated by $\{\exp tX\}_{t \in \boldsymbol{R}}$. The correspondence: $X \to X^*$ can be naturally extended to a linear mapping of $\mathfrak{g}(M)_c$ to $\mathfrak{X}(M)$ by setting

$$(X + \sqrt{-1}\,Y)^* = X^* + JY^* \qquad \text{for} \quad X, Y \in \mathfrak{g}(M),$$

where J denotes the complex structure of M. By a result of Kobayashi (Theorem 1.4, Ch. III, [4]), this mapping is injective. Let p be a point of M. We define subspaces $\mathfrak{h}^k(p)$ of $\mathfrak{g}(M)_c$ for any integer k by

$$\begin{aligned} &\mathfrak{h}^k(p)=\mathfrak{g}(M)_c\,, \qquad \text{if} \quad k\leqq -1 \\ &\mathfrak{h}^k(p)=\{Z\in\mathfrak{g}(M)_c\,;\, j_p^k(Z^*)=0\} \qquad \text{if} \quad k\geqq 0, \end{aligned} \tag{1.1}$$

where $j_p^k(Z^*)$ denotes the k-jet of the vector field Z^* at p. We then have

$$\begin{aligned} &\mathfrak{h}^i(p)\supset\mathfrak{h}^j(p) \qquad \text{if} \quad i\leqq j, \\ &[\mathfrak{h}^i(p),\, \mathfrak{h}^j(p)]\subset\mathfrak{h}^{i+j}(p) \qquad \text{for} \quad i,\, j\in\boldsymbol{Z}. \end{aligned} \tag{1.2}$$

Suppose that all derivatives of a vector field Z^* at p are zeros and that $Z_p^*=0$. Then $Z^*=0$. Therefore we have

$$\bigcap_{k=0}^{\infty}\mathfrak{h}^k(p)=0\,.$$

Let K_p be the isotropy subgroup of $\mathrm{Aut}(M)$ at p and let $\mathfrak{k}_p$ be its Lie algebra. We then have

$$\mathfrak{h}^0(p)\cap\overline{\mathfrak{h}^0(p)}=(\mathfrak{k}_p)_c \tag{1.3}$$

and hence

$$[\mathfrak{h}^1(p),\, \overline{\mathfrak{h}^1(p)}]\subset(\mathfrak{k}_p)_c\,. \tag{1.4}$$

The subalgebra $\mathfrak{h}^0(p)$ may be considered as the isotropy subalgebra of $\mathfrak{g}(M)_c$ at the point p. Let us denote by ρ_p the isotropy representation of $\mathfrak{h}^0(p)$, i.e., for every $Z\in\mathfrak{h}^0(p)$, $\rho_p(Z)$ is an endomorphism of $T_p(M)$ defined by

$$\rho_p(Z)v=[Z^*,\, \xi]_p \qquad \text{for} \quad v\in T_p(M),$$

where ξ is a vector field such that $\xi_p=v$. Then an element Z of $\mathfrak{h}^0(p)$ belongs to $\mathfrak{h}^1(p)$ if and only if $\rho_p(Z)=0$. Define a subspace $\mathfrak{m}(p)$ by

$$\mathfrak{m}(p)=\{Z+\bar{Z}\,;\, Z\in\mathfrak{h}^1(p)\}\,.$$

Clearly $\mathfrak{m}(p)$ is $Ad\,K_p$-invariant.

Lemma 1.1 (cf. [10]). (1) $\mathfrak{k}_p\cap\mathfrak{m}(p)=0$.

(2) *There exists a unique complex structure I_p of $\mathfrak{m}(p)$ such that $(I_pX)^*=JX_p^*$ and the correspondence: $X\to X+\sqrt{-1}\,I_pX$ gives a linear isomorphism between $\mathfrak{m}(p)$ and $\mathfrak{h}^1(p)$.*

Proof. Let $X\in\mathfrak{m}(p)\cap\mathfrak{k}_p$. There exists $Y\in\mathfrak{g}(M)$ such that $X+\sqrt{-1}\,Y$ belongs to $\mathfrak{h}^1(p)$. Then $\rho_p(X+\sqrt{-1}\,Y)=0$. It should be noted that Y also belongs to $\mathfrak{m}(p)\cap\mathfrak{k}_p$. Since K_p is compact, there exists a hermitian inner product g of $T_p(M)$ such that both $\rho_p(X)$ and $\rho_p(Y)$ are skew-symmetric with respect to g. For any $v\in T_p(M)$, there exists a local vector field ξ around p satisfying $\xi_p=v$ and $\mathcal{L}_\xi J=0$. We then have $\rho_p(\sqrt{-1}\,Y)\,v=[JY^*,\,\xi]_p=J[Y^*,\,\xi]_p=[Y^*,\,J\xi]_p$. Therefore $\rho_p(\sqrt{-1}\,Y)=J\circ\rho_p(Y)=\rho_p(Y)\circ J$. It follows that $\rho_p(\sqrt{-1}\,Y)$ is symmetric with respect to g and hence $\rho_p(X)=\rho_p(Y)=0$. This implies $X=Y=0$. The second assertion can be verified by the same way as in [10]. q.e.d.

We set

$$\mathfrak{l}_p = \mathfrak{k}_p + \mathfrak{m}(p).$$

Using Lemma 1.1, we get from (1.4)

$$(1.5) \qquad [X, Y] + [I_pX, I_pY] \equiv 0 \pmod{\mathfrak{k}_p} \quad \text{for} \quad X, Y \in \mathfrak{m}(p).$$

On the other hand, since $[\mathfrak{b}^1(p), \mathfrak{b}^1(p)] \subset \mathfrak{b}^1(p)$, we get

$$(1.6) \qquad [X, Y] - [I_pX, I_pY] \equiv 0 \pmod{\mathfrak{m}(p)} \quad \text{for} \quad X, Y \in \mathfrak{m}(p),$$

$$(1.7) \qquad I_p([X, Y] - [I_pX, I_pY]) = [I_pX, Y] + [X, I_pY] \quad \text{for} \quad X, Y \in \mathfrak{m}(p).$$

From (1.5) and (1.6), we have $[\mathfrak{m}(p), \mathfrak{m}(p)] \subset \mathfrak{l}_p$ and hence $\mathfrak{l}_p$ is a subalgebra of $\mathfrak{g}(M)$. Let L_p be the connected subgroup of Aut(M) corresponding to $\mathfrak{l}_p$ and put

$$M(p) = L_p \cdot p.$$

By Lemma 1.1 and (1.7), $M(p)$ is a complex submanifold of M with $\dim_C M(p) = \dim_C \mathfrak{b}^1(p)$.

Let $Z \in \mathfrak{b}^2(p)$. By using Lemma 1.1, we can write $Z = X + \sqrt{-1}\, I_pX$, where $X \in \mathfrak{m}(p)$. Then $[\bar{Z}, Z] = 2\sqrt{-1}[I_pX, X]$. From (1.2), (1.4) and from Lemma 1.1 we have $[\mathfrak{b}^2(p), \overline{\mathfrak{b}^2(p)}] \subset (\mathfrak{k}_p)_c \cap \mathfrak{b}^1(p) = 0$. Therefore $[I_pX, X] = 0$ and hence $\boldsymbol{R}X + \boldsymbol{R}I_pX$ is a complex abelian subalgebra of $\mathfrak{g}(M)$. Let L' be the connected subgroup of Aut(M) corresponding to $\boldsymbol{R}X + \boldsymbol{R}I_pX$. Then the orbite $L' \cdot p$ is a complex submanifold of M with a trivial Kobayashi-distance. This means that $L' \cdot p = p$ and hence $X \in \mathfrak{k}_p$. Thus we get $X = 0$ and $\mathfrak{b}^2(p) = 0$. Consequently, $[\mathfrak{b}^1(p), \mathfrak{b}^1(p)] = 0$. We then have for any $X, Y \in \mathfrak{m}(p)$, $[X, Y] - [I_pX, I_pY] = 0$. This combined with (1.5) tells us

$$[\mathfrak{m}(p), \mathfrak{m}(p)] \subset \mathfrak{k}_p.$$

Now we can show by the same arguments as in [10] that $M(p)$ is a hermitian symmetric space. Since there is no holomorphic mapping of $\boldsymbol{C}$ into $M(p)$ except constant mappings, every component of $M(p)$ is of the non-compact type. Let us set

$$\mathfrak{k}(p) = [\mathfrak{m}(p), \mathfrak{m}(p)]$$

$$\mathfrak{l}(p) = \mathfrak{k}(p) + \mathfrak{m}(p).$$

we can show $\mathfrak{l}(p) \cong \mathfrak{g}(M(p))$ by the same way as in the proof of Proposition 1.1 of [9]. We thereby obtain the following

Theorem 1.2. *Let M be a hyperbolic manifold. Then*

(1) $\mathfrak{b}^2(p) = 0$.

(2) *$M(p)$ is a hermitian symmetric space of the non-compact type and the subalgebra $\mathfrak{l}(p)$ may be identified with $\mathfrak{g}(M(p))$. In particular, $\mathfrak{l}(p)$ is semi-simple.*

Remark 1. For any hermitian manifold M, we can construct a complex submanifold $M(p)$ by the same way, changing Aut(M) to the group of all holomorphic isometries. If M is a Kähler manifold, then we can show that

$[\mathfrak{m}(p), \mathfrak{m}(p)] \subset \mathfrak{k}_p$ and hence $M(p)$ is a hermitian symmetric space ([10]). But this does not hold for a general hermitian manifold. Indeed, let M be a connected complex Lie group with a Lie algebra $\mathfrak{m}$. With respect to any left invariant hermitian metric, M is a hermitian manifold satisfying $\dim_C \mathfrak{b}^1(p) = \dim_C M$. Moreover $\mathfrak{m}(p) = \mathfrak{m}$ and $M(p) = M$. Therefore *if the complex Lie group M admits a left invariant Kähler metric, then M must be abelian* because $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}_p \cap \mathfrak{m} = 0$. As a consequence, for any non-abelian complex Lie group M equipped with a left invariant hermitian metric, $M(p)$ $(=M)$ is not a hermitian symmetric space.

Remark 2. By virture of Theorem 1.2, all results in §§2∼5 of [9] for bounded domains also hold for hyperbolic manifolds. We shall frequently use these results in the following sections.

§2. An equivariant holomorphic mapping Φ.

Definition. A hyperbolic manifold M is called *half-homogeneous,* if for each point p, $T_p(M) = \{Z_p^*\,;\, Z \in \mathfrak{g}(M)_c\}$.

Let M be a half-homogeneous hyperbolic manifold. From the assumption, $\dim_C \mathfrak{b}^1(p)$ is constant, say r_0. Since $T_p(M) \cong \mathfrak{g}(M)_c/\mathfrak{b}^0(p)$, we have

$$\mathfrak{b}^1(p) = \{Z \in \mathfrak{b}^0(p)\,;\, [Z, \mathfrak{g}(M)_c] \subset \mathfrak{b}^0(p)\}. \tag{2.1}$$

We define a mapping Φ of M into the grassmann manifold $Gr(\mathfrak{g}(M)_c\,; r_0, \boldsymbol{C})$ by

$$\Phi(p) = \mathfrak{b}^0(p) \qquad \text{for} \quad p \in M.$$

Let G_c be the adjoint group of $\mathfrak{g}(M)_c$. The group G_c acts on $Gr(\mathfrak{g}(M)_c\,; r_0, \boldsymbol{C})$ in a natural manner and $\mathrm{Aut}(M)$ also acts on $Gr(\mathfrak{g}(M)_c\,; r_0, \boldsymbol{C})$ by the adjoint representation. It is clear that if $a \in \mathrm{Aut}(M)$ and $p \in M$, then $\Phi(a \cdot p) = Ad\, a \cdot \Phi(p)$. Moreover let $q = \mathrm{Exp}\, t_0 Z^*(p)$, where $\mathrm{Exp}\, tZ^*$ denotes the one parameter group of local transformations of M generated by Z^*. Then $\mathfrak{b}^0(q) = \exp(ad\, t_0 Z)\mathfrak{b}^0(p)$ and hence $\exp(ad\, t_0 Z) \cdot \Phi(p) = \Phi(q)$. It is easy to see that Φ is holomorphic. In fact, there exist $Z_1, \cdots, Z_n$ of $\mathfrak{g}(M)_c$ such that $\{(Z_1)_p^*, \cdots, (Z_n)_p^*\}$ forms a base of $T_p(M)$. Then there exist a neighbourhood W of 0 in $\boldsymbol{C}^n$ and a neighbourhood U of p such that the following mapping g_1 is a holomorphic diffeomorphism of W onto U;

$$g_1(z_1, \cdots, z_n) = \mathrm{Exp}\, x_1 Z_1^* \circ \mathrm{Exp}\, y_1 J Z_1^* \circ \cdots \circ \mathrm{Exp}\, x_n Z_n^* \circ \mathrm{Exp}\, y_n J Z_n^*(p)$$

$$z_i = x_i + \sqrt{-1}\, y_n\,.$$

Let g_2 be a holomorphic mapping of W to G_c defined by

$$g_2(z_1, \cdots, z_n) = \exp(ad\, z_1 Z_1) \cdot \cdots \cdot \exp(ad\, z_n Z_n)\,.$$

We then have $\Phi(q) = g_2 \circ g_1^{-1}(q) \cdot \Phi(p)$ for $q \in U$. Therefore Φ is holomorphic.

By Theorem 1.2, $\mathfrak{l}(p)$ is a semi-simple Lie algebra corresponding to the hermitian symmetric space $M(p)$ of the non-compact type. Therefore there exists a unique Z_p of $\mathfrak{k}(p)$ such that

$$(2.2)\qquad \begin{aligned} ad\, Z_p X &= I_p X \quad \text{for} \quad X \in \mathfrak{m}(p), \\ ad\, Z_p Y &= 0 \quad \text{for} \quad Y \in \mathfrak{k}(p). \end{aligned}$$

From Theorem 3.3 of [9], we know that for any $p, q \in M$, there exists f of G_c such that $f\cdot\Phi(p)=\Phi(q)$ and $f(\mathfrak{l}(p))=\mathfrak{l}(q)$. We recall its proof and verify the following

Lemma 2.1. *Let M be a half-homogeneous hyperbolic manifold and let p, q be two points of M. Then there exist contineous mappings $c_t: [0, 1]\to M$ and $f_t: [0, 1]\to G_c$ satisfying*

(a) $c_0=p$, $c_1=q$ *and* $f_0=1$.
(b) $f_t\cdot\Phi(p)=\Phi(c_t)$ *and* $f_t(\mathfrak{l}(p))=\mathfrak{l}(c_t)$.

Let $p, q\in M$. There exist $X_1, \cdots, X_m \in \mathfrak{g}(M)_c$ and $p_0, \cdots, p_m \in M$ such that $p_0=p$, $p_m=q$ and $p_i=\operatorname{Exp} X_i^*(p_{i-1})$ for $i\geqq 1$. We put for $\frac{i}{m}\leqq t\leqq\frac{i+1}{m}$

$$c_t=\operatorname{Exp}(t\,m-i)X_{i+1}^*(p_i)\,,$$

$$g_t=\exp(ad(t\,m-i)X_{i+1})\cdot\exp(ad\,X_i)\cdot\cdots\cdot\exp(ad\,X_1)\,.$$

It is clear that (c_t, g_t) satisfies the properties of the lemma except the equation $g_t(\mathfrak{l}(p))=\mathfrak{l}(c_t)$. Using (2.1), we get $g_t\cdot\mathfrak{b}^1(p)=\mathfrak{b}^1(c_t)$. We set $E_t=\sqrt{-1}\,Z_{c_t}$ and $A_\lambda=\{X\in\mathfrak{g}(M)_c\,;\,[E_0, X]=\lambda X\}$ for $\lambda\in\boldsymbol{R}$. By Lemma 3.4 of [9], $\mathfrak{g}(M)=\sum_{-1\leqq\lambda\leqq 1}A_\lambda$ and as in the proof of Theorem 3.3 of [9], we get

$$g_t^{-1}E_t=E_0+\sum_{0<\lambda\leqq 1}a_\lambda(t)\,,\quad a_\lambda(t)\in A_\lambda.$$

Let λ_1 denote the smallest positive number λ satisfying $A_\lambda\neq 0$. By a direct computation, we have

$$\exp\Big(\frac{1}{\lambda_1}ad\,a_{\lambda_1}(t)\Big)\cdot g_t^{-1}\cdot E_t=E_0+\sum_{\lambda_1<\lambda\leqq 1}a'_\lambda(t)\,,\quad a'_\lambda(t)\in A_\lambda\,.$$

Since $A_\lambda\subset\mathfrak{b}^0(p)$ for $\lambda>0$ (Lemma 3.4, [9]), repeating this process, we obtain $h_t\in G_c$ such that $h_0=1$ and

$$(2.3)\qquad h_t\cdot g_t^{-1}\cdot E_t=E_0 \quad\text{and}\quad h_t\cdot g_t^{-1}\cdot\mathfrak{b}^0(c_t)=\mathfrak{b}^0(p)\,.$$

Notice that if E_t depends contineously on t, then so does h_t. From (*b*) of Lemma 3.4 of [9] and (2.3), we have

$$h_t\cdot g_t^{-1}\cdot\mathfrak{b}^1(c_t)=\mathfrak{b}^1(p)\quad\text{and}\quad h_t\cdot g_t^{-1}\cdot\overline{\mathfrak{b}^1(c_t)}=\overline{\mathfrak{b}^1(p)}\,.$$

Hence we get

$$(2.4)\qquad h_t\cdot g_t^{-1}\cdot\mathfrak{k}(c_t)_c=\mathfrak{k}(p)_c\quad\text{and}\quad h_t\cdot g_t^{-1}\cdot\mathfrak{m}(c_t)_c=\mathfrak{m}(p)_c\,.$$

We set $\mathfrak{u}(t)=h_t\cdot g_t^{-1}(\mathfrak{k}(c_t)+\sqrt{-1}\,\mathfrak{m}(c_t))$. Then $\mathfrak{u}(t)$ is a compact real form of $\mathfrak{l}(p)_c$

for any t. Let σ_t be the conjugation of $\mathfrak{l}(p)_c$ with respect to $\mathfrak{u}(t)$. Then $\sigma_t\circ\sigma_0$ is an automorphism of $\mathfrak{l}(p)_c$. Let α_0 be the positive definite hermitian form on $\mathfrak{l}(p)_c$ defined by $\alpha_0(X, Y)=-\alpha(X, \sigma_0 Y)$ for $X, Y\in\mathfrak{l}(p)_c$, where α denotes the killing form of $\mathfrak{l}(p)_c$. By a direct calculation, we know that $\sigma_t\circ\sigma_0$ is hermitian symmetric with respect to α_0. Recall that the space of all hermitian symmetric endomorphisms is diffeomorphic to that of all hermitian symmetric linear transformations with positive eigenvalues under the exponential mapping. Therefore there corresponds a hermitian symmetric endomorphism $H(t)$ of $\mathfrak{l}(p)_c$ such that $\exp H(t)=(\sigma_t\circ\sigma_0)^2$. We set $P_t^s=\exp sH(t)$. In view of the proof of Theorem 7.2 in Ch. III of [1], we know

$$P_t^{1/4}\mathfrak{u}(0)=\mathfrak{u}(t)\,. \tag{2.5}$$

Moreover for any t, $\{P_t^s\}_{s\in R}$ is a one parameter subgroup of automorphisms of the semi-simple Lie algebra $\mathfrak{l}(p)_c$. Hence there exists uniquely $X(t)\in\mathfrak{l}(p)_c$ such that $H(t)=ad\ X(t)$. Since $E_0\in\sqrt{-1}\,\mathfrak{u}(t)$, $\sigma_t\circ\sigma_0(E_0)=E_0$. This means $P_t^sE_0=E_0$ and hence $[X(t), E_0]=0$. Form this equation, we get $X(t)\in\mathfrak{k}(p)_c$. We now set

$$f_t=g_t\cdot h_t^{-1}\cdot\exp\Big(\frac{1}{4}ad\ X(t)\Big)\,.$$

Clearly $f_t\mathfrak{b}^0(p)=\mathfrak{b}^0(c_t)$. By (2.4) and (2.5), we have

$$f_t\cdot\mathfrak{k}(p)\subset(g_t\cdot h_t^{-1}\cdot\mathfrak{u}(t))\cap\mathfrak{k}(c_t)_c=\mathfrak{k}(c_t)\,,$$

$$f_t\cdot\mathfrak{m}(p)\subset(g_t\cdot h_t^{-1}\cdot\sqrt{-1}\,\mathfrak{u}(t))\cap\mathfrak{m}(c_t)_c=\mathfrak{m}(c_t)\,.$$

We also know from (2.4), $\dim_R\mathfrak{k}(c_t)=\dim_R\mathfrak{k}(p)$ and $\dim_R\mathfrak{m}(c_t)=\dim_R\mathfrak{m}(p)$. Therefore we get $f_t\cdot\mathfrak{k}(p)=\mathfrak{k}(c_t)$ and $f_t\cdot\mathfrak{m}(p)=\mathfrak{m}(c_t)$. We thereby proved Lemma 2.1 except showing that f_t is contineous. We have already proved that for any $p, q\in M$, $\mathfrak{l}(p)$ and $\mathfrak{l}(q)$ are isomorphic to each other. In particular $\dim_R\mathfrak{m}(p)$ and $\dim_R\mathfrak{k}(p)$ are constant. To complete the proof of Lemma 2.1, it is sufficient to show the following

Proposition 2.2. (1) *The correspondences*: $p\to\mathfrak{m}(p)$ *and* $p\to\mathfrak{k}(p)$ *are differentiable mappings of* M *into* $Gr(\mathfrak{g}(M)\,;\,r_m, \boldsymbol{R})$ *and into* $Gr(\mathfrak{g}(M)\,;\,r_k, \boldsymbol{R})$ *respectively, where* $r_m=\dim_R\mathfrak{m}(p)$ *and* $r_k=\dim_R\mathfrak{k}(p)$.

(2) *The correspondence* $p\to Z_p$ *is a differentiable mapping of* M *into* $\mathfrak{g}(M)$, *where* Z_p *is the element of* $\mathfrak{k}(p)$ *defined by* (2.2).

Proof. We recall that if $f\in G_c$ and $f\cdot\mathfrak{b}^0(p)=\mathfrak{b}^0(q)$, then $f\cdot\mathfrak{b}^1(p)=\mathfrak{b}^1(q)$ because of (2.1). Therefore we can show by the same way as in the case of the mapping Φ, that the assignment: $p\to\mathfrak{b}^1(p)$ is holomorphic. Since $\dim_C\mathfrak{b}^1(p)=\frac{1}{2}\dim_R\mathfrak{m}(p)$, we can take for any point p, locally defined differentiable mappings $w_1(p'), \cdots, w_{r_m/2}(p')$ of a neighbourhood of p into $\mathfrak{g}(M)_c$ in such a way that $\{w_1(p'), \cdots, w_{r_m/2}(p')\}$ forms a base of $\mathfrak{b}^1(p')$. We write $w_1(p')=u_i(p')+\sqrt{-1}\,v_i(p')$, where $u_i(p')$, $v_i(p')\in\mathfrak{m}(p')$. Then the set $\{u_i(p'), v_j(p')\}_{1\le i,j\le r_m/2}$ forms a base of $\mathfrak{m}(p')$. Since $\mathfrak{k}(p')=[\mathfrak{m}(p'), \mathfrak{m}(p')]$ and since $\dim_R\mathfrak{k}(p')$ is constant, we can take a base $\{e_1(p'), \cdots, e_{r_k}(p')\}$ of $\mathfrak{k}(p')$ in such a way that $e_i(p')$ depends also differentiably on p'. Thus we get (1).

By using a base $\{u_i(p'), v_j(p')\}_{1\leq i,j\leq r_m/2}$, $I_{p'}$ is represented by a matrix Q which is independent to p'. Let us write $Z_{p'}=\sum_i \nu_i(p')\, e_i(p')$. As an endomorphism of $\mathfrak{m}(p')$, each $ad\, e_i(p')$ is represented by a matrix $Q_i(p')$ with respect to a base $\{u_i(p'), v_j(p')\}_{1\leq i,j\leq r_m/2}$. Then $Q_1(p'), \cdots, Q_{r_k}(p')$ are linearly independent. Now $\{\nu_1(p'), \cdots, \nu_{r_k}(p')\}$ is a unique solution of the equation $\sum_i \nu_i(p')\, Q_i(p')=Q$. This implies that $\nu_i(p')$ is differentiable and we get (2). q.e.d.

§ 3. A holomorphic immersion.

Let M be a half-homogeneous hyperbolic manifold and let ρ_p be the isotropy representation of $\mathfrak{h}^0(p)$. We consider the following condition:

(*C*) *There exists* H_p *of* $\mathfrak{h}^0(p)$ *such that* $\rho_p(H_p)=1$.

By the half-homogeneity, if M satisfies (*C*) at a point p, then M satisfies at any point of M. In what follows, we fix a point p of M and assume that M satisfies (*C*). Put

$$\theta^{\lambda}=\{X\in\mathfrak{g}(M)_c\,;\,[H_p, X]=\lambda X\}.$$

We then have (Lemma 4.1, [9])

$$\begin{aligned}&\mathfrak{g}(M)_c=\theta^{-1}+\theta^0+\theta^1, \quad [\theta^{\lambda}, \theta^{\nu}]\subset\theta^{\lambda+\nu},\\ (3.1)\qquad &\mathfrak{h}^0(p)=\theta^0+\theta^1,\\ &\mathfrak{h}^1(p)=\theta^1.\end{aligned}$$

Since M is half-homogeneous, we have $\dim_C\theta^{-1}=\dim_C M$. Moreover we may assume (Lemma 4.2, [9])

$$\overline{\mathfrak{h}^1(p)}\subset\theta^{-1}.$$

Let $\mathfrak{r}$ denotes the radical of $\mathfrak{g}(M)$ and put

$$\mathfrak{t}=[\mathfrak{r}, \mathfrak{l}(p)].$$

We also define a subalgebra $\mathfrak{a}$ of $\mathfrak{g}(M)$ by

$$\mathfrak{a}=\{X\in\mathfrak{g}(M)\,;\,[X, \mathfrak{l}(p)]=0\}.$$

If we put

$$\mathfrak{l}_+=\overline{\mathfrak{h}^1(p)}, \quad \mathfrak{l}_-=\mathfrak{h}^1(p) \quad \text{and} \quad E_p=\sqrt{-1}\,Z_p,$$

we then have from Proposition 2.4 of [9],

$$\begin{aligned}&\mathfrak{g}(M)=\mathfrak{l}(p)+\mathfrak{t}+\mathfrak{a} \quad \text{(vector space direct sum)}\\ &\theta^{-1}=\mathfrak{l}_++\mathfrak{t}_c\cap\theta^{-1}+\mathfrak{a}_c\cap\theta^{-1} \quad \text{(vector space direct sum)}\\ (3.2)\qquad &\theta^0=\mathfrak{k}(p)_c+\mathfrak{t}_c\cap\theta^0+\mathfrak{a}_c\cap\theta^0 \quad \text{(vector space direct sum)}\\ &\mathfrak{t}_c=\mathfrak{t}_c\cap\theta^{-1}+\mathfrak{t}_c\cap\theta^0\\ &\mathfrak{a}_c=\mathfrak{a}_c\cap\theta^{-1}+\mathfrak{a}_c\cap\theta^0\end{aligned}$$

and

$$\mathfrak{r}_c = \sum_{-1<\lambda<1} (\mathfrak{r}_c)_\lambda$$

$$(3.3) \qquad \mathfrak{t}_c \cap \theta^{-1} = \sum_{-1<\lambda<0} (\mathfrak{r}_c)_\lambda, \quad \mathfrak{t}_c \cap \theta^0 = \sum_{0<\lambda<1} (\mathfrak{r}_c)_\lambda$$

$$\mathfrak{a}_c \cap \theta^{-1} = (\mathfrak{r}_c)_0 \cap \theta^{-1},$$

where $(\mathfrak{r}_c)_\lambda = \{X \in \mathfrak{r}_c ; [E_p, X] = \lambda X\}$.

Let B be the subgroup of the adjoint group G_c defined by

$$B = \{f \in G_c ; f \cdot \mathfrak{b}^0(p) = \mathfrak{b}^0(p)\}.$$

Since $\mathfrak{g}(M)_c$ is centerless (Lemma 5.1, [9]), the Lie algebra of G_c is identified with $\mathfrak{g}(M)_c$. It is easy to see that under this identification, the Lie algebra of B coincides with $\mathfrak{b}^0(p)$. The homogeneous space G_c/B is the G_c-orbit of $\mathfrak{b}^0(p)$ in $Gr(\mathfrak{g}(M)_c ; r_0, \boldsymbol{C})$. Let Φ be the holomorphic mapping defined in § 2. Then Φ is an immersion of M onto an open subset of G_c/B (Propositions 3.1 and 5.2, [9]). Let h_1 be the holomorphic mapping of θ^{-1} to G_c/B defined by

$$h_1(z) = \pi_1 \cdot \exp z \qquad \text{for} \quad z \in \theta^{-1},$$

where π_1 denotes the projection of G_c onto G_c/B. It is easy to see that h_1 is a holomorphic imbedding of θ^{-1} onto an open dense subset of G_c/B (cf. Proof of Theorem 1, [8]).

Lemma 3.1. *Let $X_{-1} \in \theta^{-1}$ and $X_0 \in \theta^0$. Then* $\exp X_{-1}$ *(resp.* $\exp X_0$*) leaves $h_1(\theta^{-1})$ invariant and induces a translation (resp. a linear transformation) of θ^{-1}.*

Proof. For any $z \in \theta^{-1}$, we get

$$\exp X_{-1} \cdot h_1(z) = \pi_1(\exp X_{-1} \cdot \exp z) = h_1(X_{-1} + z),$$

$$\exp X_0 \cdot h_1(z) = \pi_1(\exp X_0 \cdot \exp z \cdot (\exp X_0)^{-1} \cdot \exp X_0)$$

$$= h_1(Ad(\exp X_0) z). \qquad \text{q.e.d.}$$

Let L, K, L_c, K_c, L_+ and L_- be the connected subgroup of G_c corresponding to the subalgebra $\mathfrak{l}(p)$, $\mathfrak{k}(p)$, $\mathfrak{l}(p)_c$, $\mathfrak{k}(p)_c$, $\mathfrak{l}_+$ and $\mathfrak{l}_-$ respectively. Recall that Φ is an imbedding on $M(p)$ (Proposition 3.2, [9]). Then $\Phi(M(p)) = L/K$. It is well known that the mapping:

$$L_+ \times K_c \times L_- \ni (a, b, c) \longrightarrow a \cdot b \cdot c \in L_c$$

is a holomorphic diffeomorphism onto an open set of L_c and that

$$L \subset L_+ \cdot K_c \cdot L_-.$$

Therefore $\Phi(M(p))$ is contained in $h_1(\mathfrak{l}_+)$. We set

$$\mathcal{S} = h_1^{-1} \circ \Phi(M(p)).$$

$\mathcal{S}$ is a symmetric bounded domain in $\mathfrak{l}_+$ and is known as the Harish-Chandra realization of $M(p)$. We now prove the following

Theorem 3.2. *Let M be a half-homogeneous hyperbolic manifold satisfying the condition (C). Then $\Phi(M)$ is contained in $h_1(\theta^{-1})$ and therefore $h_1^{-1}\circ\Phi$ gives a holomorphic immersion of M onto an open set of the vector space θ^{-1}.*

Proof. Let R_c denotes the connected subgroup of G_c corresponding to $(\mathfrak{t}+\mathfrak{a})_c$ $(=(\mathfrak{r}+\mathfrak{a})_c)$. Note that $(\mathfrak{t}+\mathfrak{a})_c$ is an ideal of $\mathfrak{g}(M)_c$. Therefore every element of G_c induces an automorphism of $\mathfrak{g}(M)_c/(\mathfrak{t}+\mathfrak{a})_c$. Since $\mathfrak{g}(M)_c=\mathfrak{l}(p)_c+(\mathfrak{t}+\mathfrak{a})_c$, the group R_c is the identity component of the kernel of this correspondence. Therefore R_c is closed. Clearly $G_c=R_c\cdot L_c$.

Let q be any point of M and let (c_t, f_t) be as in Lemma 2.1. We denote by γ the projection of G_c onto G_c/R_c. The restriction of γ to L_c gives a covering mapping of L_c onto G_c/R_c. Therefore there exists a contineous curve s_t in L_c such that $\gamma(s_t)=\gamma(f_t)$ and $s_0=1$. We set $r_t=f_t\cdot s_t^{-1}$. Then r_t is a contineous curve contained in R_c. Clearly

$$(3.4)\qquad aX\equiv X \pmod{(\mathfrak{t}+\mathfrak{a})_c} \quad \text{for} \quad X\in\mathfrak{l}(p)_c \quad \text{and} \quad a\in R_c.$$

It follows that for any $X\in\mathfrak{l}(p)_c$, $f_tX\equiv s_tX \pmod{(\mathfrak{t}+\mathfrak{a})_c}$. Since $f_tX\in\mathfrak{g}(M)$ for any $X\in\mathfrak{l}(p)$, we know

$$s_tX\in\mathfrak{g}(M)\cap\mathfrak{l}(p)_c=\mathfrak{l}(p)\,.$$

Notice that L is an identity component of the subgroup of L_c which consists of all elements of L_c leaving $\mathfrak{l}(p)$ invariant. It follows that s_t is contained in L. Consequently, $s_1\cdot\Phi(p)$ is contained in $h_1(\mathcal{S})$. Since $(\mathfrak{t}+\mathfrak{a})_c$ is contained in $\theta^{-1}+\theta^0$, we get by Lemma 3.1,

$$\Phi(q)=f_1\cdot\Phi(p)\in R_c\cdot h_1(\mathcal{S})\subset h_1(\theta^{-1})\,. \qquad \text{q.e.d.}$$

The next theorem gives a characterization of hermitian symmetric spaces of the non-compact type among half-homogeneous hyperbolic manifolds.

Theorem 3.3. *Let M be a half-homogeneous hyperbolic manifold. Then M is a hermitian symmetric space if and only if there exists H_p in $\sqrt{-1}\,\mathfrak{k}_p$ such that $\rho_p(H_p)=1$.*

Proof. Suppose that $H_p\in\sqrt{-1}\,\mathfrak{k}_p$ satisfying $\rho_p(H_p)=1$. Then $\theta^{-1}=\overline{\theta^1}$. Therefore $\dim_C\theta^1=\dim_C M$. Since $\theta^1=\mathfrak{b}^1(p)$, this implies that $M(p)=M$ and hence M is a hermitian symmetric space. The converse is clear. q.e.d.

§ 4. Fiberings of M and $h_1^{-1}\circ\Phi(M)$.

Let M be a half-homogeneous hyperbolic manifold satisfying (C). We set

$$\mathcal{M}=h_1^{-1}\circ\Phi(M)\,.$$

Every element of $\mathrm{Aut}(M)$ leaves $\Phi(M)$ invariant and hence induces an automorphism of the domain $\mathcal{M}$. In this section, we study the action of the group $G(M)$, the identity component of $\mathrm{Aut}(M)$, and construct fiberings of M and $\mathcal{M}$

with $\mathcal{S}$ as base spaces.

For convenience, let us set

$$\mathfrak{t}_+=\mathfrak{t}_c\cap\theta^{-1},\quad \mathfrak{t}_-=\mathfrak{t}_c\cap\theta^0 \quad\text{and}\quad \mathfrak{u}=\mathfrak{a}_c\cap\theta^{-1}.$$

Since $\overline{(\mathfrak{r}_c)_\lambda}=(\mathfrak{r}_c)_{-\lambda}$, we get from (3.3),

$$\overline{\mathfrak{t}_+}=\mathfrak{t}_- \tag{4.1}$$

and from Proposition 2.4 of [9],

$$\mathfrak{t}_+=[\mathfrak{t}_-,\ \mathfrak{l}_+],\quad \mathfrak{t}_-=[\mathfrak{t}_+,\ \mathfrak{l}_-]. \tag{4.2}$$

Since $[E_p,\ \mathfrak{k}(p)]=0$ and since $\mathfrak{r}_c$ is contained in $\theta^{-1}+\theta^0$, we get from (3.1) and (3.2)

$$[\mathfrak{k}(p)_c,\ \mathfrak{t}_\pm]\subset\mathfrak{t}_\pm,\quad [\mathfrak{l}_+,\ \mathfrak{t}_+]=0 \quad\text{and}\quad [\mathfrak{l}_-,\ \mathfrak{t}_-]=0. \tag{4.3}$$

Lemma 4.1. (1) $[\mathfrak{t}_-,\ \mathfrak{t}_-]=[\mathfrak{t}_+,\ \mathfrak{t}_+]=0$ *and* $[\mathfrak{t}_+,\ \mathfrak{t}_-]\subset\mathfrak{u}$.

(2) $[\mathfrak{t},\ \mathfrak{u}]=0$.

(3) $[\mathfrak{t},\ [\mathfrak{t},\ \mathfrak{t}]]=0$.

Proof. If $-1<\lambda<0$ and $0<\nu<1$, then $(\mathfrak{r}_c)_\lambda\subset\theta^{-1}$ and $(\mathfrak{r}_c)_\nu\subset\theta^0$. Therefore $[(\mathfrak{r}_c)_\lambda,\ (\mathfrak{r}_c)_\nu]\subset\theta^{-1}\cap(\mathfrak{r}_c)_{\lambda+\nu}$. On the other hand

$$[(\mathfrak{r}_c)_\lambda,\ (\mathfrak{r}_c)_\nu]\subset[\overline{(\mathfrak{r}_c)_{-\lambda}},\ \overline{(\mathfrak{r}_c)_{-\nu}}]\subset\overline{(\mathfrak{r}_c)_{-(\lambda+\nu)}\cap\theta^{-1}}.$$

If $\lambda+\nu\neq0$, then $\overline{(\mathfrak{r}_c)_{-(\lambda+\nu)}\cap\theta^{-1}}\subset\theta^0$ by (3.3). It follows that if $\lambda+\nu\neq0$, then $[(\mathfrak{r}_c)_\lambda,\ (\mathfrak{r}_c)_\nu]=0$ and $[(\mathfrak{r}_c)_\lambda,\ (\mathfrak{r}_c)_{-\lambda}]\subset(\mathfrak{r}_c)_0\cap\theta^{-1}$. Therefore $[\mathfrak{t}_+,\ \mathfrak{t}_-]\subset\mathfrak{u}$. If $0<\nu,\ \mu<1$, then $[(\mathfrak{r}_c)_\nu,\ (\mathfrak{r}_c)_\mu]=\overline{[(\mathfrak{r}_c)_{-\nu},\ (\mathfrak{r}_c)_{-\mu}]}$. Since $(\mathfrak{r}_c)_{-\nu}$ and $(\mathfrak{r}_c)_{-\mu}$ are subspaces of θ^{-1}, we get $[(\mathfrak{r}_c)_\nu,\ (\mathfrak{r}_c)_\mu]=0$. This implies $[\mathfrak{t}_-,\ \mathfrak{t}_-]=0$. If $0<\mu<1$, then $[(\mathfrak{r}_c)_\mu,\ \mathfrak{u}]\subset\theta^{-1}\cap(\mathfrak{r}_c)_\mu=0$ by (3.3). Since $[\mathfrak{t}_+,\ \mathfrak{t}_++\mathfrak{u}]=0$, we get (1) and (2). Consequently,

$$[\mathfrak{t},\ [\mathfrak{t},\ \mathfrak{t}]]\subset[\mathfrak{t},\ [\mathfrak{t}_+,\ \mathfrak{t}_-]]\subset[\mathfrak{t},\ \mathfrak{u}]=0.$$

Hence we get (3). q.e.d.

Let us denote by η_1 the projection of $\mathfrak{u}\times\mathfrak{t}_+\times\mathfrak{l}_+\ (=\theta^{-1})$ onto $\mathfrak{l}_+$. In view of the proof of Theorem 3.2, $\eta_1\circ h_1^{-1}\circ\Phi(M)=\mathcal{S}$. Therefore the domain $\mathcal{M}$ is an open set of $\mathfrak{u}\times\mathfrak{t}_+\times\mathcal{S}$. Let $g\in L$. Then g induces an automorphism of $\mathcal{S}$ which will be denoted by $g_{\mathcal{S}}$. In what follows, for every $w\in\mathfrak{t}_c$, we denote by w_+ and by w_- the $\mathfrak{t}_+$- and the $\mathfrak{t}_-$-component of w respectively.

Lemma 4.2. *Every $g\in L$ leaves $h_1(\mathfrak{u}\times\mathfrak{t}_+\times\mathcal{S})$ invariant and hence induces a holomorphic transformation $\tilde{g}$ of $\mathfrak{u}\times\mathfrak{t}_+\times\mathcal{S}$. Let $\tilde{g}(u,\ w,\ z)=(u',\ w',\ z')$. Then*

$$z'=g_{\mathcal{S}}(z),$$

$$w'=(Ad\ gw)_+-[z',\ (Ad\ gw)_-],$$

$$u'=u-\frac{1}{2}[w',\ Ad\ gw].$$

Using Lemma 4.1, this lemma can be verified similarly as Lemma 4.5 of [7].

Let R denote the connected subgroup of G_c corresponding to $\mathfrak{t}+\mathfrak{a}$. If $f\in R$, then f induces an affine transformation $\tilde{f}$ of $\mathfrak{u}\times\mathfrak{t}_+\times\mathfrak{l}_+$ by Lemma 3.1. If $\tilde{f}(u, w, z)=(u', w', z')$, then $z=z'$ by (3.4). This combined with Lemma 4.2 leads us to the following

Proposition 4.3. *Every $g\in G(M)$ induces automorphisms $g_{\mathcal{M}}$ of $\mathcal{M}$ and $g_{\mathcal{S}}$ of $\mathcal{S}$ in such a way that the following diagram commutes*:

$$\begin{array}{ccccc} M & \xrightarrow{h_1^{-1}\circ\Phi} & \mathcal{M} & \xrightarrow{\eta_1} & \mathcal{S} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_{\mathcal{M}}} & & \downarrow{\scriptstyle g_{\mathcal{S}}} \\ M & \xrightarrow{h_1^{-1}\circ\Phi} & \mathcal{M} & \xrightarrow{\eta_1} & \mathcal{S} \end{array} .$$

We set for $z\in\mathcal{S}$

$$M_z=(\eta_1\circ h_1^{-1}\circ\Phi)^{-1}(z) \quad\text{and}\quad \mathcal{M}_z=\eta_1^{-1}(z)\,. \tag{4.4}$$

Since L acts on $\mathcal{S}$ transitively, we get from Lemma 4.2 and Proposition 4.3 the following

Proposition 4.4. *Let $z, z'\in\mathcal{S}$. Then the fibers M_z (resp. $\mathcal{M}_z$) and $M_{z'}$ (resp. $\mathcal{M}_{z'}$) are holomorphically isomorphic to each other.*

Remark 3. Both $(M, \mathcal{S}, \eta_1\circ h_1^{-1}\circ\Phi)$ and $(\mathcal{M}, \mathcal{S}, \eta_1)$ give fiberings with base space $\mathcal{S}$. These are not holomorphic fiber bundles. But real analytically, we have $M\cong\mathcal{S}\times M_0$ and $\mathcal{M}\cong\mathcal{S}\times\mathcal{M}_0$. In fact let $q\in M$ and $z=\eta_1\circ h_1^{-1}\circ\Phi(q)$. Since the mapping $\beta: X\rightarrow(\exp X)_{\mathcal{S}}\cdot 0$ gives an analytic diffeomorphism of $\mathfrak{m}(p)$ onto $\mathcal{S}$, we get a real analytic diffeomorphism of M onto $\mathcal{S}\times M_0$ defined as follows:

$$M\ni q \longrightarrow (z, \exp\beta^{-1}(z)\cdot q)\in\mathcal{S}\times M_0\,.$$

The case of $\mathcal{M}$ is similar.

Remark 4. In the case where M is a Siegel domain of the second kind, the mapping Φ is an imbedding (Corollary 5, [8]) and the fibering $M\rightarrow\mathcal{S}$ corresponds to the realization of M as a Siegel domain of the third kind constructed in [7].

§ 5. The structure of the fiber $\mathcal{M}_0$.

Let H_p be as in § 3. We can decompose H_p as $H_p=H_1+H_2+H'$, where $H_1\in\mathfrak{k}(p)_c$, $H_2\in\mathfrak{t}_-$ and $H'\in\mathfrak{a}_c\cap\theta^0$. Since $[H_p, \mathfrak{l}(p)_c]\subset\mathfrak{l}(p)_c$, we have $[H_2, \mathfrak{l}(p)_c]=0$ and $ad\, H_1X=ad\, H_pX=ad\, E_pX$ for any $X\in\mathfrak{l}(p)_c$. Therefore $H_1=E_p$ and by (3.3), $H_2=0$. Thus we can write

$$H_p=E_p+H' \quad (H'\in\mathfrak{a}_c\cap\theta^0)\,. \tag{5.1}$$

Let E' (resp. A) be the real (resp. the imaginary) part of H'. We can decompose A as $A=v_r+I'$, where $v_r\in\mathfrak{a}_c\cap\theta^{-1}$ $(=\mathfrak{u})$ and $I'\in\mathfrak{a}_c\cap\theta^0$. Then $H'=E'+\sqrt{-1}(v_r+I')$.

Lemma 5.1. *Both v_r and I' are in $\mathfrak{a}$ and $[v_r, I']=0$.*

Proof. By direct calculations,

$$[E', A]=[H', A]=[H_p, A]=-v_r,$$
$$[E', v_r]=[H', v_r]-\sqrt{-1}[I', v_r]=-v_r-\sqrt{-1}[I', v_r].$$

From the first equation, we have $v_r \in \mathfrak{g}(M)$ and hence $v_r, I' \in \mathfrak{a}$. This combined with the second equation implies $[I', v_r] \subset \mathfrak{g}(M) \cap \sqrt{-1}\,\mathfrak{g}(M)=0$. q.e.d.

By Lemma 5.1 and (5.1), we have

$$Ad(\exp(-\sqrt{-1}\,v_r))H'=E'+\sqrt{-1}\,I', \tag{5.2}$$
$$Ad(\exp(-\sqrt{-1}\,v_r))H_p=E'+\sqrt{-1}(I'+Z_p).$$

If we set

$$\theta'^0=Ad(\exp(-\sqrt{-1}\,v_r))\theta^0,$$

then we have from (5.2)

$$ad(E'+\sqrt{-1}\,I')=-1 \quad \text{on } \mathfrak{u} \tag{5.3}$$
$$ad(E'+\sqrt{-1}\,I')=0 \quad \text{on } \mathfrak{a}_c \cap \theta'^0$$

Since I' is contained in $\mathfrak{k}_p$ and since $[I', v_r]=0$, $ad\,I'$ leaves $\mathfrak{u}$ and $\mathfrak{a}_c \cap \theta'^0$ invariant and every eigenvalue of $ad\,I'$ is purely imaginary. Hence if we set for $\lambda \in \boldsymbol{R}$,

$$U^\lambda=\{X \in \mathfrak{u}\,;\,[2E', X]=\lambda X\}$$
$$V^\lambda=\{X \in \mathfrak{a}_c \cap \theta'^0\,;\,[2E', X]=\lambda X\},$$

then we have

$$\mathfrak{u}=\sum_{\lambda \in \boldsymbol{R}} U^\lambda \quad \text{and} \quad \mathfrak{a}_c \cap \theta'^0=\sum_{\lambda \in \boldsymbol{R}} V^\lambda.$$

Lemma 5.2. (1) *$\mathfrak{u}=U^{-2}+U^{-1}$ and $\mathfrak{a}_c \cap \theta'^0=V^{-1}+V^0$.*
(2) *$ad(2I')=\sqrt{-1}$ on U^{-1}, $ad(2I')=-\sqrt{-1}$ on V^{-1} and $ad(2I')=0$ on $U^{-2}+V^0$.*

Proof. Let $v \in U^\lambda$. By (5.3), $ad(2I')v=(\lambda+2)\sqrt{-1}\,v$. Hence

$$ad(2I')\bar{v}=-(\lambda+2)\sqrt{-1}\,\bar{v}. \tag{5.4}$$

We write $\bar{v}=v'+X$, where $v' \in u$ and $X \in a_c \cap \theta'^0$. Clearly $v' \in U^\lambda$ and $X \in V^\lambda$. Therefore by (5.3),

$$ad(2I')v'=(\lambda+2)\sqrt{-1}\,v' \quad \text{and} \quad ad(2I')X=\lambda\sqrt{-1}\,X. \tag{5.5}$$

It follows from (5.4) and (5.5), $(\lambda+2)v'=0$ and $(\lambda+1)X=0$. Hence if $\lambda \neq -2, -1$, then $v=0$. If $\lambda=-1$, then $ad(2I')v=\sqrt{-1}\,v$ and if $\lambda=-2$, then $ad(2I')v=0$. Similarly, we have $V^\lambda=0$ if $\lambda \neq -1$, 0 and $ad(2I')=-\sqrt{-1}$ on V^{-1} and $ad(2I')=0$ on V^0. q.e.d.

As an immediate consequence of Lemma 5.2, we have

Lemma 5.3. (1) $\mathfrak{a}=\mathfrak{a}^{-2}+\mathfrak{a}^{-1}+\mathfrak{a}^{0}$, *where* $\mathfrak{a}^{\lambda}=\{X\in\mathfrak{a}\,;\,[2E',\,X]=\lambda X\}$.

(2) $ad(2I')=0$ *on* $\mathfrak{a}^{-2}+\mathfrak{a}^{0}$ *and* $(ad(2I'))^{2}=-1$ *on* $\mathfrak{a}^{-1}$.

(3) $\mathfrak{a}_{c}\cap\theta^{-1}=\mathfrak{a}_{c}^{-2}+\mathfrak{a}_{+}$ *and* $\mathfrak{a}_{c}\cap\theta'^{0}=\mathfrak{a}_{-}+\mathfrak{a}_{c}^{0}$, *where*

$$\mathfrak{a}_{\pm}=\{X\in\mathfrak{a}_{c}^{-1}\,;\,[2I',\,X]=\pm\sqrt{-1}\,X\}.$$

It is clear that the subspace $\mathfrak{t}$ is invariant by $ad\,\mathfrak{a}$. We investigate the eigenvalues of $ad\,E'$ on $\mathfrak{t}$.

Lemma 5.4. (1) $ad\,2E'=-1$ *on* $\mathfrak{t}$.

(2) $\mathfrak{t}_{\pm}=\{X\in\mathfrak{t}_{c}\,;\,[2(Z_{p}+I'),\,X]=\pm\sqrt{-1}\,X\}$.

Proof. Notice that $v_{r}\in\mathfrak{a}^{-2}$. Therefore by (2) of Lemma 4.1 and by (3) of Lemma 5.3, we have $[\mathfrak{t},\,v_{r}]=0$. It follows that $\mathfrak{t}_{+}$ and $\mathfrak{t}_{-}$ are invariant by $ad\,X$ for any $X\in\mathfrak{a}_{c}\cap\theta'^{0}$. Hence $\mathfrak{t}_{+}$ and $\mathfrak{t}_{-}$ are invariant by $ad\,E'$ and by $ad(I'+Z_{p})$. Since $I'+Z_{p}$ is in $\mathfrak{k}_{p}$, all eigenvalues of $ad\,2(I'+Z_{p})$ are purely imaginary and both $\mathfrak{t}_{+}$ and $\mathfrak{t}_{-}$ are decomposed into the sum of eigenspaces of $ad\,2(I'+Z_{p})$. Let $v\in\mathfrak{t}_{+}$ satisfying

$$ad\,2(I'+Z_{p})v=\lambda\sqrt{-1}\,v \qquad \text{for} \quad \lambda\in\boldsymbol{R}. \tag{5.6}$$

Then

$$ad\,2(I'+Z_{p})\bar{v}=-\lambda\sqrt{-1}\,\bar{v}. \tag{5.7}$$

By (5.2), (5.6) and (5.7), we have

$$ad(2E')v=(-2+\lambda)v \quad \text{and} \quad ad(2E')\bar{v}=-\lambda\bar{v},$$

because $\bar{v}\in\mathfrak{t}_{-}$. Thus we get $(\lambda-1)v=0$. Consequently, if $v\neq 0$, then $\lambda=1$ and $ad(2E')v=-v$. Considering $\mathfrak{t}_{-}=\overline{\mathfrak{t}_{+}}$, we get (1) and (2). q.e.d.

By Lemmas 5.3 and 5.4, we have

$$\begin{aligned}&[\mathfrak{a}^{-1},\,\mathfrak{a}^{-1}]\subset\mathfrak{a}^{-2}, \quad [\mathfrak{t},\,\mathfrak{t}]\subset\mathfrak{a}^{-2},\\ &[\mathfrak{a}^{-2},\,\mathfrak{t}]=[\mathfrak{a}^{-1},\,\mathfrak{t}]=[\mathfrak{a}^{-2},\,\mathfrak{a}^{-1}]=0.\end{aligned} \tag{5.8}$$

Let $\mathcal{M}_{0}$ be the fiber defined by (4.4). By Remark 3, $\mathcal{M}_{0}$ is connected and by Lemma 5.3, $\mathcal{M}_{0}$ is regarded as a domain of $\mathfrak{a}_{c}^{-2}\times(\mathfrak{a}_{+}+\mathfrak{t}_{+})$.

Lemma 5.5. *The domain* $\mathcal{M}_{0}$ *is invariant under the following transformations of* $\mathfrak{a}_{c}^{-2}\times(\mathfrak{a}_{+}+\mathfrak{t}_{+})$.

(a) $(z,\,w)\to(z+a,\,w)$ *for every* $a\in\mathfrak{a}^{-2}$.

(b) $(z,\,w)\to\left(z+\frac{1}{2}[\bar{c},\,c]+[\bar{c},\,w],\,w+c\right)$ *for every* $c\in\mathfrak{a}_{+}+\mathfrak{t}_{+}$.

Proof. Let $X=c+\bar{c}$. By using (5.8), we get $\exp X=\exp c\cdot\exp\bar{c}\cdot\exp\frac{1}{2}[\bar{c},\,c]$. We claim that $\bar{c}$ is contained in θ^{0}. In fact, by Lemma 5.3, $\mathfrak{a}_{-}\subset\theta'^{0}$. Since $[\mathfrak{a}_{-},\,v_{r}]=0$, $\mathfrak{a}_{-}\subset Ad(\exp\sqrt{-1}\,v_{r})\theta'^{0}=\theta^{0}$. Therefore $\bar{c}\in\mathfrak{a}_{-}+\mathfrak{t}_{-}\subset\theta^{0}$. It follows

$$\exp X\cdot h_1(z,\ w)=\pi_1\Big(\exp c\cdot\exp\frac{1}{2}[\bar{c},\ c]\cdot\exp z\cdot\exp Ad(\exp\bar{c})w\Big)$$
$$=h_1\Big(z+\frac{1}{2}[\bar{c},\ c]+[\bar{c},\ w],\ w+c\Big).$$

Therefore $\mathcal{M}_0$ is invariant under the transformations of the form (b). For any $a\in\mathfrak{a}^{-2}$, the following equality holds clearly:

$$\exp a\cdot h_1(z,\ w)=h_1(z+a,\ w). \qquad \text{q.e.d.}$$

We now set

(5.9)
$$\Omega_r=\{y\in\mathfrak{a}^{-2}\,;\,(\sqrt{-1}\,y,\ 0)\in\mathcal{M}_0\}+v_r,$$
$$F_r(w,\ w')=\frac{\sqrt{-1}}{2}[w,\ \bar{w}'] \qquad \text{for} \quad w,\ w'\in\mathfrak{a}_++\mathfrak{t}_+.$$

Then Ω_r is an open set of $\mathfrak{a}^{-2}$ containing v_r and F_r is an $\mathfrak{a}_c^{-2}$-valued hermitian form on $\mathfrak{a}_++\mathfrak{t}_+$.

Definition. Let Ω be an open connected cone in a real vector space V and let F be a V_c-valued hermitian form on a complex vector space W such that the condition "$F(w,\ w)=0$" implies $w=0$. *A pseudo-Siegel domain D of the second kind associated with Ω and F* is a domain in $V_c\times W$ defined as follows:

$$D=\{(z,\ w)\in V_c\times W\,;\,\mathrm{Im}\,z-F(w,\ w)\in\Omega\}.$$

A pseudo-Siegel domain D of the second kind is a Siegel domain of the second kind if Ω is a convex cone containing no entire straight lines and F is an Ω-hermitian form, i.e., $F(w,\ w)$ is contained in the closure of Ω.

Proposition 5.6. *The fiber $\mathcal{M}_0$ is holomorphically equivalent to the pseudo-Siegel domain of the second kind associated with Ω_r and F_r defined by* (5.9). *In fact, $\mathcal{M}_0$ is represented as*

$$\mathcal{M}_0=\{(z,\ w)\in\mathfrak{a}_c^{-2}\times(\mathfrak{a}_++\mathfrak{t}_+)\,;\,\mathrm{Im}\,z-F_r(w,\ w)+v_r\in\Omega_r\}.$$

Proof. Let $y\in\Omega_r$. Then $(\sqrt{-1}(y-v_r),\ 0)\in\mathcal{M}_0$. Using Lemma 5.3, we have

$$\exp tE'\cdot h_1(\sqrt{-1}(y-v_r),\ 0)$$
$$=\pi_1\cdot\exp(Ad(\exp\sqrt{-1}\,v_r)^{-1}t(H'-\sqrt{-1}\,I'))\cdot\exp\sqrt{-1}(y-v_r)$$
$$=\pi_1\cdot\exp(-\sqrt{-1}\,v_r)\cdot\exp t(H'-\sqrt{-1}\,I')\cdot\exp\sqrt{-1}\,v_r\cdot\exp\sqrt{-1}(y-v_r)$$
$$=\pi_1\cdot\exp(-\sqrt{-1}\,v_r)\cdot\exp Ad(\exp t(H'-\sqrt{-1}\,I'))\sqrt{-1}\,y$$
$$=h_1(-\sqrt{-1}\,v_r+e^{-t}\sqrt{-1}\,y,\ 0),$$

because $H'-\sqrt{-1}\,I'$ belongs to θ^0. Therefore $(\sqrt{-1}(e^{-t}y-v_r),\ 0)\in\mathcal{M}_0$. This implies $e^{-t}y\in\Omega_r$ for any $t\in\boldsymbol{R}$, proving that Ω_r is a cone.

By a suitable transformation as in Lemma 5.5, every point $(z,\ w)$ can be translated to $(\sqrt{-1}(\mathrm{Im}\,z-F_r(w,\ w)),\ 0)$. Therefore $(z,\ w)\in\mathcal{M}_0$ if and only if $\mathrm{Im}\,z-F_r(w,\ w)+v_r\in\Omega_r$. Now $\mathcal{M}_0$ is diffeomorphic to $\mathfrak{a}^{-2}\times\Omega_r\times(\mathfrak{a}_++\mathfrak{t}_+)$ under the following mapping:

$$\mathcal{M}_0 \ni (z,\ w) \longrightarrow (\mathrm{Re}\, z,\ \mathrm{Im}\, z - F_r(w,\ w) + v_r,\ w) \in \mathfrak{a}^{-2} \times \Omega_r \times (\mathfrak{a}_+ + \mathfrak{t}_+)\,.$$

Therefore Ω_r is connected.

Assume that $F_r(w,\ w)=0$. Let $X=w+\bar{w}$ and $Y=\sqrt{-1}\,w-\sqrt{-1}\,\bar{w}$. We define a mapping ψ of $\boldsymbol{C}$ into M by

$$\psi(z)=\exp xX\cdot\exp yY(p) \qquad \text{for} \quad z=x+\sqrt{-1}\,y\,.$$

Since $[w,\ \bar{w}]=0$, we get from the proof of Lemma 5.5,

$$h_1^{-1}\circ\Phi\circ\psi(z)=(0,\ xw+\sqrt{-1}\,yw)=(0,\ zw)\,.$$

Since $h_1^{-1}\circ\Phi$ is an immersion, $\psi(z)$ must be holomorphic. This means that $\psi(z)=p$ for any $z\in\boldsymbol{C}$ and hence $w=0$, completing the proof. q.e.d.

§ 6. The symmetric Siegel domain S isomorphic to $M(p)$ and the structure of $\mathfrak{t}$.

It is well known that the hermitian symmetric space $M(p)$ of the non-compact type is holomorphically isomorphic to a symmetric Siegel domain S of the second kind. Therefore by Kaup-Matsushima-Ochiai [2], there exists E_s of $\mathfrak{l}(p)$ such that

$$\text{(6.1)} \qquad \begin{aligned} &\mathfrak{l}(p)=\mathfrak{s}^{-2}+\mathfrak{s}^{-1}+\mathfrak{s}^{0}+\mathfrak{s}^{1}+\mathfrak{s}^{2},\\ &\mathfrak{s}^{\lambda}=\{X\in\mathfrak{l}(p)\,;\,[E_s,\ X]=\lambda X\}\,.\end{aligned}$$

Note that $\dim_R\mathfrak{s}^{-2}=\dim_R\mathfrak{s}^{2}$ and $\dim_R\mathfrak{s}^{-1}=\dim_R\mathfrak{s}^{1}$, because $\mathfrak{l}(p)$ is semi-simple. Moreover there exists I_s of $\mathfrak{s}^0$ such that

$$\text{(6.2)} \qquad \begin{aligned} &ad\, I_s=0 \quad \text{on} \quad \mathfrak{s}^{-2}+\mathfrak{s}^{0}+\mathfrak{s}^{2}\\ &(ad\, I_s)^2=-1 \quad \text{on} \quad \mathfrak{s}^{-1}+\mathfrak{s}^{1}.\end{aligned}$$

Define linear transformations P and $\bar{P}$ of $\mathfrak{s}_c^{-1}+\mathfrak{s}_c^{1}$ by

$$P(X)=\frac{1}{2}(X-\sqrt{-1}[I_s,\ X])$$

$$\bar{P}(X)=\frac{1}{2}(X+\sqrt{-1}[I_s,\ X])\,,$$

and set

$$\theta_s^{-1}=\mathfrak{s}_c^{-2}+P(\mathfrak{s}^{-1})$$

$$\theta_s^{0}=\bar{P}(\mathfrak{s}^{-1})+\mathfrak{s}_c^{0}+P(\mathfrak{s}^{1})$$

$$\theta_s^{1}=\bar{P}(\mathfrak{s}^{1})+\mathfrak{s}_c^{2}$$

$$H_s=\frac{1}{2}(E_s+\sqrt{-1}\,I_s)\,.$$

By (6.1) and (6.2), we have

(6.3)
$$\mathfrak{l}(p)_c=\theta_s^{-1}+\theta_s^0+\theta_s^1$$
$$ad\,H_sX=\lambda X \qquad \text{for} \quad X\in\theta_s^\lambda .$$

The Siegel domain S is regarded as a domain in $\mathfrak{s}_c^{-2}+P(\mathfrak{s}^{-1})$ defined by

$$S=\{z+w\in\mathfrak{s}_c^{-2}+P(\mathfrak{s}^{-1})\,;\,\mathrm{Im}\,z-F_s(w,\,w)\in\Omega_s\},$$

where Ω_s is an open convex cone in $\mathfrak{s}^{-2}$ containing no entire straight lines and F_s is an Ω_s-hermitian form on $P(\mathfrak{s}^{-1})$ given by $F_s(w,\,w')=\frac{\sqrt{-1}}{2}[w,\,\bar{w}']$ (Tanaka [12]). By Lemma 2.1 of [7], for every v of Ω_s, there exists a unique $\hat{v}\in\mathfrak{s}^2$ such that

(6.4)
$$[\hat{v},\,v]=E_s\,.$$

We then have (see, § 2 of [7])

(6.5)
$$\mathfrak{s}^{-2}=[v,\,[v,\,\mathfrak{s}^2]] \quad \text{and} \quad \mathfrak{s}^2=[\hat{v},\,[\hat{v},\,\mathfrak{s}^{-2}]] \qquad \text{for} \quad v\in\Omega_s.$$

We may assume that the point p of $M(p)$ corresponds to $\sqrt{-1}\,v_s$ of S where $v_s\in\Omega_s$. Then by Proposition 2.4 of [7],

(6.6)
$$Z_p=\frac{1}{2}(I_s+v_s-\hat{v}_s)\,.$$

We put

$$\delta_s=\exp\sqrt{-1}\,v_s\cdot\exp\frac{\sqrt{-1}}{2}\hat{v}_s\,.$$

Using (6.1), (6.2) and (6.4), we get

(6.7)
$$Ad\,\delta_s^{-1}Z_p=\frac{1}{2}(I_s-\sqrt{-1}\,E_s)=-\sqrt{-1}\,H_s\,.$$

Therefore by (6.3) and (6.7), we have

(6.8)
$$Ad\,\delta_s\theta_s^{-1}=\mathfrak{l}_+\,,\quad Ad\,\delta_s\theta_s^0=\mathfrak{k}(p)_c\,,\quad Ad\,\delta_s\theta_s^1=\mathfrak{l}_-\,.$$

Let B^s be the closed subgroup of L_c defined by

$$B^s=\{a\in L_c\,;\,a(\theta_s^0+\theta_s^1)=\theta_s^0+\theta_s^1\}\,.$$

It is easy to see that B^s is connected and

(6.9)
$$Ad\,\delta_sB^s=K_c\cdot L_-\,.$$

Define a holomorphic mapping h_s of θ_s^{-1} to L_c/B^s by

$$h_s(z)=\pi_s\cdot\exp z \qquad \text{for} \quad z\in\theta_s^{-1},$$

where π_s denotes the projection: $L_c\to L_c/B^s$. The mapping h_s is an imbedding of θ_s^{-1} onto an open dense subset of L_c/B^s and under the natural action of L on S, the restriction of h_s to S is L-equivariant (Tanaka [12]). Let $\tilde{\delta}_s$ be a holomorphic diffeomorphism of L_c/B^s onto $L_c/K_c\cdot L_-$ given by

$$L_c/B^s \ni gB^s \xrightarrow{\tilde{\delta}_s} g\delta_s^{-1}K_c\cdot L_- \in L_c/K_c\cdot L_-\,.$$

The mapping $\tilde{\delta}_s$ is L_c-equivariant. It should be noted that $L_c \cap B = K_c\cdot L_-$. We then have the following commutative diagram:

$$(6.10)\qquad \begin{array}{ccccc} L_c/B^s & \xrightarrow{\tilde{\delta}_s} & L_c/K_c\cdot L_- & \subset & G_c/B \\ h_s\Big\uparrow & & h_1\Big\uparrow & & h_1\Big\uparrow \\ \theta_s^{-1}\supset S\simeq\mathcal{S} & \subset & \mathfrak{l}_+ & \subset & \theta^{-1} \end{array} \qquad \tilde{\delta}_s\circ h_s(S)=h_1(\mathcal{S})\,.$$

We now consider the decomposition of $\mathfrak{t}$ $(=[\mathfrak{r}, \mathfrak{l}(p)])$ into the sum of eigenspaces of $ad\, E_s$.

Lemma 6.1. (1) $\mathfrak{t}=\mathfrak{t}^{-1}+\mathfrak{t}^0+\mathfrak{t}^1$, *where* $\mathfrak{t}^\lambda=\{X\in\mathfrak{t}\,;\,[E_s, X]=\lambda X\}$.
(2) $\mathfrak{t}^{-1}=[\mathfrak{t}^1, v_s]$ *and* $\mathfrak{t}^1=[\mathfrak{t}^{-1}, \hat{v}_s]$.
(3) $[\mathfrak{t}^0, \mathfrak{t}^{-1}+\mathfrak{t}^1]=0$ *and* $[\mathfrak{t}^0, \mathfrak{s}^{-2}+[\mathfrak{s}^{-2}, \mathfrak{s}^2]+\mathfrak{s}^2]=0$.

Proof. Since E_s is a real diagonal element of the semi-simple Lie algebra $\mathfrak{l}(p)$, we have by Lemma 1.5 of [6],

$$\mathfrak{t}=\sum_{\lambda\in R}\mathfrak{t}^\lambda,\quad \mathfrak{t}^\lambda=\{X\in\mathfrak{t}\,;\,[E_s, X]=\lambda X\}.$$

Let $w\in\mathfrak{t}^\lambda$. Then by Lemma 5.4, we have $w=w_++w_-$, where

$$w_\pm=\frac{1}{2}(w\mp 2\sqrt{-1}[I'+Z_p, w])\,.$$

For any $X\in\mathfrak{s}^2$,

$$Ad\,\delta_s X=X+\sqrt{-1}[v_s, X]-\frac{1}{2}[v_s, [v_s, X]]\,.$$

By using (6.6),

$$2w_-=w+2\sqrt{-1}[I', w]+\sqrt{-1}[I_s, w]+\sqrt{-1}[v_s, w]-\sqrt{-1}[\hat{v}_s, w]\,.$$

Since $[Ad\,\delta_s X, w_-]=0$ by (4.3) and (6.8), we get by considering the $\mathfrak{t}^{\lambda\pm4}$-component of $[Ad\,\delta_s X, w_-]$, $[X, [\hat{v}_s, w]]=0$ and $[[v_s, [v_s, X]], [v_s, w]]=0$. These show that $[[\hat{v}_s, w], \mathfrak{s}^2]=0$ and $[[v_s, w], \mathfrak{s}^{-2}]=0$, because of (6.5). Therefore we have proved $[[\mathfrak{t}, \hat{v}_s], \mathfrak{s}^2]=0$ and $[[\mathfrak{t}, v_s], \mathfrak{s}^{-2}]=0$. From (6.5), we also know that $\mathfrak{s}^{-2}=[v_s, \mathfrak{s}^0]$ and $\mathfrak{s}^2=[\hat{v}_s, \mathfrak{s}^0]$. It follows that

$$\begin{aligned}[][[\mathfrak{t}, \mathfrak{s}^2], \mathfrak{s}^2]&=[[\mathfrak{t}, \mathfrak{s}^2], [\hat{v}_s, \mathfrak{s}^0]]\\ &\subset[[[\mathfrak{t}, \mathfrak{s}^2], \mathfrak{s}^0], \hat{v}_s]\subset[[\mathfrak{t}, \mathfrak{s}^2], \hat{v}_s]=0\,.\end{aligned}$$

Similarly we get $[[\mathfrak{t}, \mathfrak{s}^{-2}], \mathfrak{s}^{-2}]=0$. Let λ_1 (resp. λ_2) be the maximal (resp. the minimal) λ such that $\mathfrak{t}^\lambda\neq0$. Then $[\mathfrak{t}^{\lambda_1}, \mathfrak{s}^2]=0$. Therefore if we set $\mathfrak{s}=\mathfrak{s}^{-2}+[\mathfrak{s}^{-2}, \mathfrak{s}^2]+\mathfrak{s}^2$, then $\mathfrak{t}^{\lambda_1}+[\mathfrak{t}^{\lambda_1}, \mathfrak{s}^{-2}]$ is an $ad\,\mathfrak{s}$-invariant subspace. Since $E_s\in[\mathfrak{s}^{-2}, \mathfrak{s}^2]$, the trace of $ad\,E_s$ on $\mathfrak{t}^{\lambda_1}+[\mathfrak{t}^{\lambda_1}, \mathfrak{s}^{-2}]$ is equal to zero. Therefore

(6.11) $$\lambda_1(\dim_R \mathfrak{t}^{\lambda_1}+\dim_R[\mathfrak{t}^{\lambda_1}, \mathfrak{s}^{-2}])=2\dim_R[\mathfrak{t}^{\lambda_1}, \mathfrak{s}^{-2}].$$

It follows that $\lambda_1<2$. Similarly we have $\lambda_2>-2$. Consequently, $[\mathfrak{t}^{\lambda_1-2}, \mathfrak{s}^{-2}]=0$ because $\lambda_1-4<-2$. Then $\mathfrak{t}^{\lambda_1}+\mathfrak{t}^{\lambda_1-2}$ and $[\mathfrak{t}^{\lambda_1-2}, \mathfrak{s}^2]+\mathfrak{t}^{\lambda_1-2}$ are $ad\,\mathfrak{s}$-invariant. Hence we have

(6.12) $$\lambda_1(\dim_R \mathfrak{t}^{\lambda_1}+\dim_R \mathfrak{t}^{\lambda_1-2})=2\dim_R \mathfrak{t}^{\lambda_1-2}.$$
$$\lambda_1(\dim_R[\mathfrak{t}^{\lambda_1-2}, \mathfrak{s}^2]+\dim_R \mathfrak{t}^{\lambda_1-2})=2\dim_R \mathfrak{t}^{\lambda_1-2}.$$

From (6.11) and (6.12), we get $\mathfrak{t}^{\lambda_1-2}=[\mathfrak{t}^{\lambda_1}, \mathfrak{s}^{-2}]$ and if $\lambda_1\neq 0$, then $\mathfrak{t}^{\lambda_1}=[\mathfrak{t}^{\lambda_1-2}, \mathfrak{s}^2]$. Assume $\lambda_1\neq 0$. Then using (6.5), we have

$$\begin{aligned}\mathfrak{t}^{\lambda_1-2}=[\mathfrak{t}^{\lambda_1}, [v_s, [v_s, \mathfrak{s}^2]]]&\subset[[\mathfrak{t}^{\lambda_1}, v_s], [v_s, \mathfrak{s}^2]]+[v_s, \mathfrak{t}^{\lambda_1}]\\ &\subset[[[\mathfrak{t}^{\lambda_1}, v_s], \mathfrak{s}^2], v_s]+[v_s, \mathfrak{t}^{\lambda_1}]\\ &=[v_s, \mathfrak{t}^{\lambda_1}].\end{aligned}$$

Therefore $\mathfrak{t}^{\lambda_1-2}=[v_s, \mathfrak{t}^{\lambda_1}]$. Similarly, $\mathfrak{t}^{\lambda_1}=[\mathfrak{t}^{\lambda_1-2}, \bar{v}_s]$. It follows that $\dim_R \mathfrak{t}^{\lambda_1}=\dim_R \mathfrak{t}^{\lambda_1-2}$ and hence $\lambda_1=1$. By the same way, we can show $\lambda_2=-1$ or 0. We have proved that if $\mathfrak{t}^{\lambda}\neq 0$, then $-1\leqq\lambda\leqq 1$ and $\mathfrak{t}^1=[\bar{v}_s, \mathfrak{t}^{-1}]$ and $\mathfrak{t}^{-1}=[v_s, \mathfrak{t}^1]$. Consequently, if $-1<\lambda<1$, then $[\mathfrak{t}^{\lambda}, \mathfrak{s}^2+\mathfrak{s}^{-2}]=0$ and hence $[\mathfrak{t}^{\lambda}, \mathfrak{s}]=0$. This means $\lambda=0$.

It remains to show that $[\mathfrak{t}^0, \mathfrak{t}^1+\mathfrak{t}^{-1}]=0$. Recall that $[\mathfrak{t}, \mathfrak{t}]\subset\mathfrak{a}$ by (1) of Lemma 4.1. Since $ad\,E_s=\pm 1$ on $[\mathfrak{t}^0, \mathfrak{t}^{\pm 1}]$, this means $[\mathfrak{t}^0, \mathfrak{t}^{-1}+\mathfrak{t}^1]=0$. q.e.d

Next we shall prove the following

Lemma 6.2. *Let* $\mathfrak{t}_+$ *and* $\mathfrak{t}_-$ *be as in* § 4. *Then*

$$\mathfrak{t}_+=Ad\,\delta_s\mathfrak{t}_c^{-1}+\mathfrak{t}_+\cap\mathfrak{t}_c^0$$
$$\mathfrak{t}_-=Ad\,\delta_s\mathfrak{t}_c^1+\mathfrak{t}_-\cap\mathfrak{t}_c^0.$$

Proof. Every element of $\mathfrak{s}^0$ leaves $\mathfrak{t}^{\lambda}$ invariant because $[E_s, \mathfrak{s}^0]=0$. In particular, $[I_s, \mathfrak{t}^{\lambda}]\subset\mathfrak{t}^{\lambda}$. Since I_s is contained in the isotropy subalgebra at $\sqrt{-1}\,v_s$, we can decompose $\mathfrak{t}_c^{-1}$ and $\mathfrak{t}_c^1$ as follows:

$$\mathfrak{t}_c^{-1}=\sum_{\lambda\in R}T_{\lambda}^{-1},\quad T_{\lambda}^{-1}=\{X\in\mathfrak{t}_c^{-1}\,;\,[I_s, X]=\lambda\sqrt{-1}\,X\},$$
$$\mathfrak{t}_c^{1}=\sum_{\lambda\in R}T_{\lambda}^{1},\quad T_{\lambda}^{1}=\{X\in\mathfrak{t}_c^{1}\,;\,[I_s, X]=\lambda\sqrt{-1}\,X\}.$$

Note that $\overline{T_{\lambda}^{-1}}=T_{-\lambda}^{-1}$ and $\overline{T_{\lambda}^{1}}=T_{-\lambda}^{1}$. From (6.7), we have

$$E_p=\sqrt{-1}\,Z_p=\frac{1}{2}Ad\,\delta_s(E_s+\sqrt{-1}\,I_s).$$

Let $u\in T_{\lambda}^{-1}$. Then

$$[E_p, Ad\,\delta_s u]=-\frac{1}{2}(\lambda+1)Ad\,\delta_s u$$

and

$$[E_p,\ Ad\ \delta_s\bar{u}]=\frac{1}{2}(\lambda-1)Ad\ \delta_s\bar{u}\ .$$

We know from (3.2) and (3.3) that if $u\neq 0$, then $-1<\frac{1}{2}(\lambda-1)$, $-\frac{1}{2}(\lambda+1)<1$. This implies $-1<\lambda<1$ and hence $-1<\frac{1}{2}(\lambda-1)$, $-\frac{1}{2}(\lambda+1)<0$. It follows from (3.3) that u and $\bar{u}$ are contained in $\mathfrak{t}_+$. Therefore $Ad\ \delta_s\mathfrak{t}_c^{-1}\subset\mathfrak{t}_+$. We can show $Ad\ \delta_s\mathfrak{t}_c^1\subset\mathfrak{t}_-$ similarly. Let $w\in\mathfrak{t}_c^0$. Then $w=w_++w_-$ where

$$w_\pm=\frac{1}{2}(w\mp 2\sqrt{-1}[Z_p+I',\ w])\ .$$

By Lemma 6.1, $w=Ad\ \delta_s w$. Therefore

$$[Z_p,\ w]=\frac{1}{2}Ad\ \delta_s[I_s-\sqrt{-1}\ E_s,\ w]=\frac{1}{2}[I_s,\ w]\in\mathfrak{t}_c^0\ .$$

This shows that $w_\pm\in\mathfrak{t}_c^0$. Hence we get $\mathfrak{t}_c^0=\mathfrak{t}_+\cap\mathfrak{t}_c^0+\mathfrak{t}_-\cap\mathfrak{t}_c^0$. q. e. d.

§7. Relization of $\mathcal{M}$ as a pseudo-Siegel domain.

We put

$$V=\mathfrak{s}^{-2}+\mathfrak{t}^{-1}+\mathfrak{a}^{-2}$$

and define an open set Ω of V by

$$\Omega=\left\{a+b+c\ ;\ a\in\Omega_s,\ b\in\mathfrak{t}^{-1},\ c\in\mathfrak{a}^{-2},\ c-\frac{1}{2}[[b,\ \hat{a}],\ b]\in\Omega_r\right\},$$

where $\hat{a}\in\mathfrak{s}^2$ given by (6.4). Making use of the uniqueness, we know that $\widehat{ta}=\frac{1}{t}\hat{a}$ for any $t>0$. Hence Ω is a cone. Next we put

$$W=P(\mathfrak{s}^{-1})+\mathfrak{t}_c^0\cap\mathfrak{t}_++\mathfrak{a}_+\ .$$

Since $[\mathfrak{t}^0,\ \mathfrak{s}^{-1}]\subset\mathfrak{t}^{-1}$, we have $[W,\ \overline{W}]\subset V_c$. Thus we can define a V_c-valued hermitian form F on W by

$$F(w,\ w')=\frac{\sqrt{-1}}{2}[w,\ \bar{w}']\qquad \text{for}\quad w,\ w'\in W\ .$$

By using Ω and F, define an open set D of $V_c\times W$ by

$$D=\{(z,\ w)\in V_c\times W\ ;\ \mathrm{Im}\ z-F(w,\ w)\in\Omega\}\ .$$

Let $\delta=\exp\sqrt{-1}\,v_r\cdot\exp\sqrt{-1}v_s\cdot\exp\frac{\sqrt{-1}}{2}\hat{v}_s$, where v_r and v_s are the elements of Ω_r and Ω_s as in §5 and §6 respectively. It is easy to see that

$$Ad\ \delta(V_c+W)=\theta^{-1}.$$

Therefore V_c+W is abelian. Let us put

$$B_0=Ad\ \delta^{-1}B$$

and define a holomorphic mapping h_0 of V_c+W to G_c/B_0 by

$$h_0(z)=\pi_0\cdot\exp z,$$

where π_0 denotes the projection: $G_c\to G_c/B_0$. Then h_0 is an imbedding. It is clear that $B^s=L_c\cap B_0$ and the restriction of h_0 to $\mathfrak{s}_c^{-2}+P(\mathfrak{s}^{-1})$ coincides with h_s. We shall examine the action of $A(S)$ on $h_0(D)$, where $A(S)$ means the connected subgroup of L corresponding to $\mathfrak{s}^{-2}+\mathfrak{s}^{-1}+\mathfrak{s}^0$. In what follows, we represent a vector of V_c+W by a system of vectors $(z_1, z_2, z_3, w_1, w_2, w_3)$ under the identification:

$$V_c+W\cong\mathfrak{s}_c^{-2}\times\mathfrak{t}_c^{-1}\times\mathfrak{a}_c^{-2}\times P(\mathfrak{s}^{-1})\times(\mathfrak{t}_c^0\cap\mathfrak{t}_+)\times\mathfrak{a}_+ .$$

Lemma 7.1. *Let $f_{-2}=\exp c_{-2}$, $f_{-1}=\exp c_{-1}$ and $f_0=\exp c_0$, where $c_{-2}\in\mathfrak{s}^{-2}$, $c_{-1}\in\mathfrak{s}^{-1}$ and $c_0\in\mathfrak{s}^0$. Then f_j leaves $h_0(V_c+W)$ invariant and induces an affine transformation $\tilde{f}_j$ of V_c+W as follows $(j=-2, -1, 0)$:*

$$\tilde{f}_{-2}(z_1, z_2, z_3, w_1, w_2, w_3)=(z_1+c_{-2}, z_2, z_3, w_1, w_2, w_3),$$

$$\tilde{f}_{-1}(z_1, z_2, z_3, w_1, w_2, w_3)=(z_1', z_2', z_3', w_1', w_2', w_3')$$

$$z_1'=z_1+\frac{1}{2}[\bar{P}(c_{-1}), P(c_{-1})]+[\bar{P}(c_{-1}), w_1]$$

$$z_2'=z_2+[\bar{P}(c_{-1}), w_2]$$

$$z_3'=z_3$$

$$w_1'=w_1+P(c_{-1})$$

$$w_2'=w_2$$

$$w_3'=w_3,$$

$$\tilde{f}_0(z_1, z_2, z_3, w_1, w_2, w_3)=(Ad\, f_0z_1, Ad\, f_0z_2, z_3, Ad\, f_0w_1, w_2, w_3).$$

Proof. The assertions for f_{-2} and f_0 are obvious. We shall prove for the case of f_{-1}. Since $c_{-1}=P(c_{-1})+\bar{P}(c_{-1})$ and since $[\mathfrak{s}^{-1}, [\mathfrak{s}^{-1}, \mathfrak{s}^{-1}]]=0$, $f_{-1}=\exp P(c_{-1})\cdot\exp\bar{P}(c_{-1})\cdot\exp\frac{1}{2}[\bar{P}(c_{-1}), P(c_{-1})]$. Therefore

$$\begin{aligned} &f_{-1}\cdot h_0(z_1, z_2, z_3, w_1, w_2, w_3)\\ &\quad=\pi_0\cdot\exp\Big(z_1+\frac{1}{2}[\bar{P}(c_{-1}), P(c_{-1})]\Big)\cdot\exp P(c_{-1})\cdot\exp z_2\cdot\exp z_3\cdot\exp w_3\\ &\qquad\times\exp\bar{P}(c_{-1})\cdot\exp w_1\cdot\exp w_2 .\end{aligned}$$

Here we used $[\bar{P}(c_{-1}), z_2]\in[\mathfrak{s}_c^{-1}, \mathfrak{t}_c^{-1}]\subset\mathfrak{t}_c^{-2}=0$. Since $\exp\bar{P}(c_{-1})\in B_0$ and since $[\bar{P}(c_{-1}), [\bar{P}(c_{-1}), w_2]]\in\mathfrak{t}_c^{-2}=0$, we have

$$\begin{aligned} &\exp\bar{P}(c_{-1})\cdot\exp w_1\cdot\exp w_2\\ &\quad\equiv\exp(w_1+[\bar{P}(c_{-1}), w_1])\cdot\exp(w_2+[\bar{P}(c_{-1}), w_2])\\ &\quad\equiv\exp w_1\cdot\exp[\bar{P}(c_{-1}), w_1]\cdot\exp w_2\cdot\exp[\bar{P}(c_{-1}), w_2] \qquad (\mathrm{mod}\ B_0).\end{aligned}$$

It follows

$$f_{-1}\cdot h_0(z_1, z_2, z_3, w_1, w_2, w_3)$$
$$\equiv\exp\Big(z_1+\frac{1}{2}[\bar{P}(c_{-1}), P(c_{-1})]+[\bar{P}(c_{-1}), w_1])\cdot\exp(w_1+P(c_{-1})\Big)$$
$$\times\exp(z_2+[\bar{P}(c_{-1}), w_2])\cdot\exp w_2\cdot\exp z_3\cdot\exp w_3 \qquad (\mathrm{mod}\ B_0).$$

q. e. d.

Next we verify

Lemma 7.2. *Let* f_{-2}, f_{-1} *and* f_0 *be as in Lemma* 7.1. *Then* $\tilde{f}_j(D)=D$ $(j=-2, -1, 0)$.

Proof. (a) The case of f_{-2}. Clear.

(b) The case of f_{-1}. Let $z=z_1+z_2+z_3$ and let $w=w_1+w_2+w_3$. Then $F(w, w)=F(w_1, w_1)+F(w_2, w_2)+F(w_3, w_3)+F(w_1, w_2)+F(w_2, w_1)$. By Lemma 7.1, $F(w_1', w_2')+F(w_2', w_1')=F(w_1, w_2)+F(w_2, w_1)+F(P(c_{-1}), w_2)+F(w_2, P(c_{-1}))$ and $\mathrm{Im}\, z_2'=\mathrm{Im}\, z_2+\mathrm{Im}[\bar{P}(c_{-1}), w_2]=\mathrm{Im}\, z_2+F(P(c_{-1}), w_2)+F(w_2, P(c_{-1}))$. It is clear that $\mathrm{Im}\, z_1'-F(w_1', w_1')=\mathrm{Im}\, z_1-F(w_1, w_1)$. Since $z_3'=z_3$, $w_2'=w_2$ and $w_3'=w_3$, combining the above equalities we get $\mathrm{Im}\, z'-F(w', w')=\mathrm{Im}\, z-F(w, w)$, where $(z', w')=\tilde{f}_{-1}(z, w)$.

(c) The case of f_0. Let $(z', w')=\tilde{f}_0(z, w)$. It is clear that $\mathrm{Im}\, z'-F(w', w')=Ad\, f_0(\mathrm{Im}\, z-F(w, w))$. Therefore it is sufficient to show that $Ad\, f_0\Omega=\Omega$. Let $v=v_1+v_2+v_3$, where $v_1\in\mathfrak{s}^{-2}$, $v_2\in\mathfrak{t}^{-1}$ and $v_3\in\mathfrak{a}^{-2}$. We set $Ad\, f_0v_i=v_i'$. Since f_0 is an automorphism of S, v_1' belongs to Ω_s if v_1 belongs to Ω_s and by (6.4), $\hat{v}_1'=Ad\, f_0\hat{v}_1$. Therefore if $v_1\in\Omega_s$, then

$$v_3'-\frac{1}{2}[[v_2', \hat{v}_1'], v_2']=v_3-\frac{1}{2}[[Ad\, f_0v_2, Ad\, f_0\hat{v}_1], Ad\, f_0v_2]$$
$$=v_3-\frac{1}{2}[[v_2, \hat{v}_1], v_2].$$

This means $Ad\, f_0v\in\Omega$ if $v\in\Omega$. q. e. d.

We set

$$D_0=\{(z_1, z_2, z_3, w_1, w_2, w_2)\in D\,;\, z_1=\sqrt{-1}\,v_s \text{ and } w_1=0\}.$$

Let η_0 denote the projection: $(z_1, z_2, z_3, w_1, w_2, w_3)\to(z_1, w_1)$. Then $\eta_0(D)=S$. By Lemmas 7.1 and 7.2, every $f\in A(S)$ induces an automorphism f_D of D such that $f_S\circ\eta_0=\eta_0\circ f_S$, where f_S denotes the automorphism of S corresponding to f. It is well known that $A(S)$ acts on S transitively. Since $D_0=\eta_0^{-1}(\sqrt{-1}\,v_s, 0)$, we have

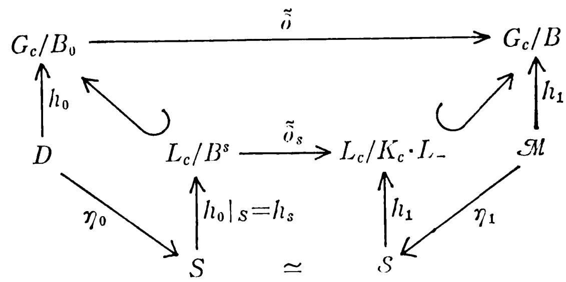
$$A(S)\cdot h_0(D_0)=h_0(D)\,. \tag{7.1}$$

We define a G_c-equivariant holomorphic diffeomorphism $\tilde{\delta}$ of G_c/B_0 onto G_c/B by

$$G_c/B_0\ni gB_0 \xrightarrow{\ \tilde{\delta}\ } g\delta^{-1}B\in G_c/B\,.$$

Considering (6.10), we get the following diagram:

(7.2)



Lemma 7.3. $\tilde{\delta}\circ h_0(D_0)=h_1(\mathcal{M}_0)$.

Proof. Let $u=(\sqrt{-1}\,v_s,\, z_2,\, z_3,\, 0,\, w_2,\, w_3)$. Then

$$\tilde{\delta}h_0(u)=\pi_1\cdot\exp(-\sqrt{-1}\,v_r)\cdot\exp z_3\cdot\exp w_2\cdot\exp w_3\cdot\exp z_2\cdot\exp\sqrt{-1}\,v_s\cdot\delta_s^{-1}$$
$$=\pi_1\cdot\exp(-\sqrt{-1}\,v_r)\cdot\exp z_3\cdot\exp w_2\cdot\exp w_3\cdot\exp z_2\,.$$

Here we used the fact $\exp\sqrt{-1}\,v_s\delta_s^{-1}\in K_c\cdot L_-$. Notice that

$$Ad\,\delta_s z_2=\frac{1}{2}(z_2+\sqrt{-1}[\hat{v}_s,\, z_2])$$

and

$$\frac{1}{2}(z_2-\sqrt{-1}[\hat{v}_s,\, z_2])=Ad\,\delta_s\Big(-\frac{\sqrt{-1}}{2}[\hat{v}_s,\, z_2]\Big)\in Ad\,\delta_s\mathfrak{t}_c^1\,.$$

Hence by Lemma 6.2,

$$\exp z_2=\exp\frac{1}{2}(z_2+\sqrt{-1}[\hat{v}_s,\, z_2])\cdot\exp\frac{1}{2}(z_2-\sqrt{-1}[\hat{v}_s,\, z_2])$$
$$\times\exp\frac{1}{8}[z_2-\sqrt{-1}[\hat{v}_s,\, z_2],\, z_2+\sqrt{-1}[\hat{v}_s,\, z_2]]$$
$$\equiv\exp(Ad\,\delta_s z_2)\cdot\exp\frac{\sqrt{-1}}{4}[z_2,\, [\hat{v}_s,\, z_2]] \qquad (\text{mod } B).$$

From the above equalities, we get $\tilde{\delta}\circ h_0(u)=h_1(u')$ for some $u'\in\mathfrak{a}_c^{-2}\times(\mathfrak{a}_++\mathfrak{t}_+)$. If we write $u'=(z',\, w')$, then

$$z'=-\sqrt{-1}\,v_r+z_3+\frac{\sqrt{-1}}{4}[z_2,\, [\hat{v}_s,\, z_2]]$$
(7.3)
$$w'=w_2+Ad\,\delta_s z_2+w_3\,.$$

It is not difficult to see that conversely for any $u'\in\mathfrak{a}_c^{-2}\times(\mathfrak{a}_++\mathfrak{t}_+)$, there exists u of the form $u=(\sqrt{-1}\,v_s,\, z_2,\, z_3,\, 0,\, w_2w_3)$ such that $\tilde{\delta}\circ h_0(u)=h_1(u')$. Note that $[\bar{w}_2,\, Ad\,\delta_s z_2]=[w_2,\, \overline{Ad\,\delta_s z_2}]=0$ by Lemma 6.2. Using this equation and using $Ad\,\delta_s z_2=\frac{1}{2}[z_2+\sqrt{-1}[\hat{v}_s,\, z_2]]$, we get from (7.3),

$$\text{Im}\, z'-F_r(w',\, w')+v_r=\text{Im}\, z_3-\frac{1}{2}[[\text{Im}\, z_2,\, \hat{v}_s],\, \text{Im}\, z_2]-F(w_2+w_3,\, w_2+w_3)\,.$$

This shows $u\in D_0$ if and only if $u'\in\mathcal{M}_0$. q.e.d.

Since $A(S)$ acts transitively on S, so on $\mathcal{S}$. It follows from Proposition 4.3, $h_1(\mathcal{M})=A(S)\cdot h_1(\mathcal{M}_0)$. Since the mapping $\tilde{\delta}$ commutes with the action of $A(S)$, considering the diagram (7.2), we get by (7.1) and Lemma 7.3,

$$\tilde{\delta}\circ h_0(D)=h_1(\mathcal{M}).$$

We thereby proved that D is holomorphically isomorphic to $\mathcal{M}$. In particular, D is connected and so is Ω (cf. Proof of Proposition 5.6). Let $w=w_1+w_2+w_3$ and suppose $F(w, w)=0$, where $w_1\in P(\mathfrak{s}^{-1})$, $w_2\in\mathfrak{t}_+\cap\mathfrak{t}_c^0$ and $w_2\in\mathfrak{a}_+$. Then $F_s(w_1, w_1)=0$. Since F_s is an Ω_s-hermitian form, we have $w_1=0$. Hence $F(w, w)=F_r(w, w)$. Therefore $w_2=w_3=0$. We have proved the following

Theorem 7.4. *Let M be half-homogeneous hyperbolic manifold satisfying the condition (C) in § 3. Then M is immersed in a complex vector space as a pseudo-Siegel domain D of the second kind in such a way that $G(M)$ acts on D equivariantly.*

§ 8. Characterizations of Siegel domains.

Let D be a pseudo-Siegel domain of the second kind associated with a connected cone Ω in a real vector space V and a V_c-valued hermitian form F on W. We prove

Proposition 8.1. *A pseudo-Siegel domain D of the second kind is a Siegel domain of the second kind if and only if D is a complete hyperbolic manifold.*

Proof. Assume that D is complete hyperbolic. Let $(z, w)\in D$, where $z\in V_c$ and $w\in W$. Then $(\sqrt{-1}(\operatorname{Im} z-F(w, w)), 0)\in D$. Therefore if we set $D_1=D\cap(V_c\times\{0\})$, then $D_1\neq\phi$ and $D_1\cong\{z\in V_c\,;\operatorname{Im} z\in\Omega\}$. By Theorem 3.4, Ch. V of [3], D is holomorphically convex. We then have from the proof of Proposition 1.1 in [13],

(a) $(z, w)\in D$ implies $\in(z, 0)\in D_1$.

(b) Ω is convex.

Suppose that Ω contains a line. Then D_1 contains a complex line which contradicts the assumption that D is hyperbolic. Therefore Ω is a convex cone containing no entire straight lines. For any $v\in\Omega$ and $w\in W$, we know $(\sqrt{-1}(v+F(w, w)), w)\in D$. Therefore from (a), $v+F(w, w)\in\Omega$. Since v is arbitrary, $F(w, w)$ is in the closure of Ω. This implies F is an Ω-hermitian form. Hence D is a Siegel domain of the second kind. The converse follows from Theorem 4.15, Ch. IV of [3]. q.e.d.

As an immediate consequence of Theorem 7.4 and Proposition 8.1, we have

Theorem 8.2. *Let M be a half-homogeneous hyperbolic manifold. Assume* *the followings*:

(a) *M is complete hyperbolic.*
(b) *M satisfies the condition* (C) *in* § 3.
(c) $\mathfrak{b}^0(p) \neq \mathfrak{b}^0(q)$ *if* $p \neq q$.

Then M is holomorphically isomorphic to a Siegel domain of the second kind. Conversely, every Siegel domain of the second kind is a half-nomogeneous hyperbolic manifold satisfying (a), (b) *and* (c).

The converse follows from Theorem 4.15, Ch. IV of [3], Corollary 5 of [8] and from Proposition 6.2 of [9].

In the case where M is a homogeneous hyperbolic manifold satisfying (C), the equivariant immersion Φ becomes a covering mapping. Then by Theorem 4.7, Ch. IV of [3], the corresponding pseudo-Siegel domain D is complete hyperbolic. Applying Proposition 8.1, D becomes a homogeneous Siegel domain of the second kind. In particular, D is simply connected. Consequently, Φ is a holomorphic diffeomorphism of M onto $\mathcal{M}$. Therefore M is holomorphically isomorphic to D and hence isomorphic to a homogeneous bounded domain. Thus we obtain the following

Theorem 8.3. *A homogeneous bounded domain in* $\boldsymbol{C}^n$ *is a homogeneous hyperbolic manifold satisfying the condition* (C) *in* § 3.

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